Derivation of first term in equation (4.2) [Stiassnie(2010)]

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Starting with the contribution to bottom pressure arising from the surface wave i.e. the first integral in [Stiassnie(2010)] (3.14).

$$\frac{8\rho\tilde{\zeta}_0}{\pi\tau}\int_0^\infty \frac{\mu_0\sin(\omega\tau/2)\sin(k_0 b)}{k_0^2\left[2\mu_0 h+\sinh(2\mu_0 h)\right]}\cos\left(k_0 x-\omega t+\frac{\omega\tau}{2}\right)d\omega \tag{0.1}$$

- $\hat{x}=\frac{x}{h}, \quad \Rightarrow x=\hat{x}h$
- $\hat{b}=\frac{b}{h}, \quad \Rightarrow b=\hat{b}h$
- $\hat{t}=\tilde{t}-\frac{\hat{\tau}}{2}=\sqrt{\frac{g}{h}}t-\frac{1}{2}\sqrt{\frac{g}{h}}\tau=\sqrt{\frac{g}{h}}\left(t-\frac{\tau}{2}\right), \quad \Rightarrow t-\frac{\tau}{2}=\sqrt{\frac{h}{g}}\hat{t}$
- $\hat{\tau}=\sqrt{\frac{g}{h}}\tau, \quad \Rightarrow \tau=\sqrt{\frac{h}{g}}\hat{\tau}$
- $\hat{k}_0=k_0 h, \quad \Rightarrow k_0=\frac{\hat{k}_0}{h}$
- $\hat{\omega}=\sqrt{\frac{h}{g}}\omega, \quad \Rightarrow \omega=\sqrt{\frac{g}{h}}\hat{\omega}$ also $d\omega=\sqrt{\frac{g}{h}}d\hat{\omega}$
- $k_0=\sqrt{\mu_0^2+\frac{\omega^2}{c^2}}, \qquad$ [Stiassnie(2010)] (3.7a) & (3.8)

Shallow water approximation implies $\frac{\omega^2}{c^2}\ll 1$, so that $\mu_0\approx k_0$.

$$\frac{8\rho\tilde{\zeta}_0}{\pi\sqrt{\frac{h}{g}}\hat{\tau}}\mathrm{Re}\int_0^\infty \frac{\sin(\sqrt{\frac{g}{h}}\hat{\omega}\sqrt{\frac{h}{g}}\frac{\hat{\tau}}{2})\sin(\frac{\hat{k}_0}{h}\hat{b}h)}{\frac{\hat{k}_0}{h}\left[2\frac{\hat{k}_0}{h}h+\sinh(2\frac{\hat{k}_0}{h}h)\right]}\mathrm{e}^{i\left(\frac{\hat{k}_0}{h}\hat{x}h-\sqrt{\frac{g}{h}}\hat{\omega}\sqrt{\frac{h}{g}}\hat{t}\right)}\sqrt{\frac{g}{h}}d\hat{\omega} \tag{0.2}$$

$$\frac{8\rho g\tilde{\zeta}_0}{\pi\hat{\tau}}\mathrm{Re}\int_0^\infty \frac{\sin(\hat{\omega}\hat{\tau}/2)\sin(\hat{k}_0\hat{b})}{\hat{k}_0\left[2\hat{k}_0+\sinh(2\hat{k}_0)\right]}\mathrm{e}^{i\left(\hat{k}_0\hat{x}-\hat{\omega}\hat{t}\right)}d\hat{\omega} \tag{0.3}$$

The phase is given by,

$$g_0\left(\hat{\omega}\right)=\hat{k}_0\left(\hat{\omega}\right)\frac{x}{t}-\hat{\omega}. \tag{0.4}$$

Differentiation of the phase term leads to,

$$\frac{\partial g_0\left(\hat{\omega}\right)}{\partial\hat{\omega}}=\frac{d\hat{k}_0\left(\hat{\omega}\right)}{d\hat{\omega}}\frac{x}{t}-1=0 \quad \text{at stationary point,} \tag{0.5}$$

with $\frac{\mathrm{d}\hat{k}_0(\hat{\omega})}{\mathrm{d}\hat{\omega}}$ obtained by differentiation of the dispersion relation $\hat{\omega}^2 = \hat{k}_0 \tanh \hat{k}_0$,

$$\frac{\mathrm{d}\hat{k}_0\left(\hat{\omega}\right)}{\mathrm{d}\hat{\omega}} = \frac{-2\hat{\omega}}{k_0\left(\hat{\omega}\right)\tanh^2 k_0\left(\hat{\omega}\right) - \tanh k_0\left(\hat{\omega}\right) - k_0\left(\hat{\omega}\right)}. \tag{0.6}$$

The stationary phase calculation requires the second derivative of $g_0\left(\hat{\omega}\right)$ so differentiate 0.6 again to give,

$$\begin{aligned}\frac{\mathrm{d}^2\hat{k}_0\left(\hat{\omega}\right)}{\mathrm{d}\hat{\omega}^2} = &\frac{-2}{\hat{k}_0\left(\hat{\omega}\right)\tanh^2 \hat{k}_0\left(\hat{\omega}\right) - \tanh \hat{k}_0\left(\hat{\omega}\right) - \hat{k}_0\left(\hat{\omega}\right)} \\ &+ \frac{\left(4\hat{\omega}\hat{k}_0 \tanh \hat{k}_0\left(\hat{\omega}\right)\frac{\mathrm{d}\hat{k}_0(\hat{\omega})}{\mathrm{d}\hat{\omega}}\right)\left(1-\tanh^2 \hat{k}_0\left(\hat{\omega}\right)\right) + 2\hat{\omega}\tanh^2 \hat{k}_0\left(\hat{\omega}\right)\frac{\mathrm{d}\hat{k}_0(\hat{\omega})}{\mathrm{d}\hat{\omega}}}{\left(k_0\left(\hat{\omega}\right)\tanh^2 k_0\left(\hat{\omega}\right) - \tanh k_0\left(\hat{\omega}\right) - k_0\left(\hat{\omega}\right)\right)^2} \\ &- \frac{2\hat{\omega}\frac{\mathrm{d}\hat{k}_0(\hat{\omega})}{\mathrm{d}\hat{\omega}}\left(1-\tanh^2 \hat{k}_0\left(\hat{\omega}\right)\right) + 2\hat{\omega}\frac{\mathrm{d}\hat{k}_0(\hat{\omega})}{\mathrm{d}\hat{\omega}}}{\left(k_0\left(\hat{\omega}\right)\tanh^2 k_0\left(\hat{\omega}\right) - \tanh k_0\left(\hat{\omega}\right) - k_0\left(\hat{\omega}\right)\right)^2}.\end{aligned} \tag{0.7}$$

The expression 0.7 contains $\frac{\mathrm{d}\hat{k}_0(\hat{\omega})}{\mathrm{d}\hat{\omega}}$ terms, so to eliminate them substitute 0.6 to give,

$$\begin{aligned}\frac{\mathrm{d}^2\hat{k}_0\left(\hat{\omega}\right)}{\mathrm{d}\hat{\omega}^2} = &\frac{8\hat{\omega}^2\left(\hat{k}_0\left(\hat{\omega}\right)\tanh^3 \hat{k}_0\left(\hat{\omega}\right)\right) - \tanh^2 \hat{k}_0\left(\hat{\omega}\right) - \hat{k}_0\left(\hat{\omega}\right)\tanh \hat{k}_0\left(\hat{\omega}\right) + 1}{\left(\hat{k}_0\left(\hat{\omega}\right)\tanh^2 \hat{k}_0\left(\hat{\omega}\right) - \tanh \hat{k}_0\left(\hat{\omega}\right) - \hat{k}_0\left(\hat{\omega}\right)\right)^3} \\ &- \frac{2}{\hat{k}_0\left(\hat{\omega}\right)\tanh^2 \hat{k}_0\left(\hat{\omega}\right) - \tanh \hat{k}_0\left(\hat{\omega}\right) - \hat{k}_0\left(\hat{\omega}\right)}.\end{aligned} \tag{0.8}$$

Further remove the tanh terms by substituting from the dispersion relation $\tanh \hat{k}_0\left(\hat{\omega}\right) = \frac{\hat{\omega}^2}{\hat{k}_0(\hat{\omega})}$ to arrive at,

$$\begin{aligned}\frac{\mathrm{d}^2\hat{k}_0\left(\hat{\omega}\right)}{\mathrm{d}\hat{\omega}^2} = &\frac{8\hat{\omega}^2\hat{k}_0\left(\hat{\omega}\right)\left(-\hat{\omega}^6 + \hat{\omega}^2\hat{k}_0\left(\hat{\omega}\right)^2 + \hat{\omega}^4 - \hat{k}_0\left(\hat{\omega}\right)^2\right)}{\left(-\hat{\omega}^4 + \hat{k}_0\left(\hat{\omega}\right)^2 + \hat{\omega}^2\right)^3} \\ &+ \frac{2\hat{k}_0\left(\hat{\omega}\right)}{-\hat{\omega}^4 + \hat{k}_0\left(\hat{\omega}\right)^2 + \hat{\omega}^2}.\end{aligned} \tag{0.9}$$

Finally making use of the shallow water approximation $\hat{k}_0\left(\hat{\omega}\right) = \hat{\omega}$ [Stiassnie(2010)] reduces the expression for the second derivative to,

$$\frac{\mathrm{d}^2\hat{k}_0\left(\hat{\omega}\right)}{\mathrm{d}\hat{\omega}^2} = \frac{6\hat{\omega}^3 - 8\hat{\omega}}{\left(\hat{\omega}^2-2\right)^3} = \hat{\omega} + \frac{3}{4}\hat{\omega}^3 + \frac{3}{8}\hat{\omega}^5 + O(\hat{\omega}^7). \tag{0.10}$$

Which to leading order becomes,

$$\frac{\mathrm{d}^2\hat{k}_0\left(\hat{\omega}\right)}{\mathrm{d}\hat{\omega}^2} = \hat{\omega} = \hat{\Omega}_0 \quad \text{at point of stationary phase.} \tag{0.11}$$

Therefore,

$$\frac{\partial^2 g_0(\hat{\omega})}{\partial \hat{\omega}^2} = \hat{\Omega}_0 \frac{\hat{x}}{\hat{t}}$$

Stationary phase approximation now gives,

$$\frac{8\rho g\tilde{\zeta}_0}{\pi\hat{\tau}} \frac{\sin\left(\hat{\Omega}_0\hat{\tau}/2\right)\sin\left(\hat{K}_0\hat{b}\right)}{\hat{K}_0\left[2\hat{K}_0+\sinh\left(2\hat{K}_0\right)\right]} \sqrt{\frac{2\pi}{\hat{t}}} \frac{1}{\left[\hat{\Omega}_0\frac{\hat{x}}{\hat{t}}\right]^{\frac{1}{2}}} \cos\left(\hat{K}_0\hat{x}-\hat{\Omega}_0\hat{t}+\frac{\pi}{4}\right)$$

$$\frac{8\rho g\tilde{\zeta}_0}{\sqrt{\pi}\hat{\tau}\hat{x}^{\frac{1}{2}}} \frac{2^{\frac{1}{2}}\sin\left(\hat{\Omega}_0\hat{\tau}/2\right)\sin\left(\hat{K}_0\hat{b}\right)}{\hat{K}_0\hat{\Omega}_0^{\frac{1}{2}}\left[2\hat{K}_0+\sinh\left(2\hat{K}_0\right)\right]} \cos\left(\hat{K}_0\hat{x}-\hat{\Omega}_0\hat{t}+\frac{\pi}{4}\right) \tag{0.12}$$

Note that,

$$\hat{K}_0\hat{\Omega}_0^{\frac{1}{2}} = \sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\left[\sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} = 2^{\frac{1}{2}}2^{\frac{1}{4}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{4}}$$

substituting and re-arranging 0.12 gives,

$$\frac{8.2^{\frac{1}{2}}\rho g\tilde{\zeta}_0}{2^{\frac{1}{2}}2^{\frac{1}{4}}\sqrt{\pi}\hat{\tau}\hat{x}^{\frac{1}{2}}} \frac{\sin\left(\hat{\Omega}_0\hat{\tau}/2\right)\sin\left(\hat{K}_0\hat{b}\right)}{\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{4}}\left[2\hat{K}_0+\sinh\left(2\hat{K}_0\right)\right]} \cos\left(\hat{K}_0\hat{x}-\hat{\Omega}_0\hat{t}+\frac{\pi}{4}\right)$$

$$\frac{2^{\frac{11}{4}}\rho g\tilde{\zeta}_0}{\sqrt{\pi}\hat{\tau}\hat{x}^{\frac{1}{2}}} \frac{\sin\left(\hat{\Omega}_0\hat{\tau}/2\right)\sin\left(\hat{K}_0\hat{b}\right)}{\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{4}}\left[2\hat{K}_0+\sinh\left(2\hat{K}_0\right)\right]} \cos\left(\hat{K}_0\hat{x}-\hat{\Omega}_0\hat{t}+\frac{\pi}{4}\right) \tag{0.13}$$

Since we also have,

$$\sin\left(\hat{\Omega}_0\hat{\tau}/2\right) = \sin\left[\sqrt{2}\hat{\tau}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}/2\right]$$

$$\sin\left(\hat{K}_0\hat{b}\right) = \sin\left[\sqrt{2}\hat{b}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\right]$$

$$2\hat{K}_0 = 2\sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}} = 2^{\frac{3}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}$$

Using $\cos(-\theta) = \cos(\theta)$,

$$\cos\left[\hat{K}_0\hat{x}-\hat{\Omega}_0\hat{t}+\frac{\pi}{4}\right] = \cos\left[-\left(\hat{K}_0\hat{x}-\hat{\Omega}_0\hat{t}+\frac{\pi}{4}\right)\right]$$

$$\cos\left[\hat{\Omega}_0\hat{t}-\hat{K}_0\hat{x}-\frac{\pi}{4}\right] = \cos\left[\sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\hat{t}-\sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\hat{x}-\frac{\pi}{4}\right]$$

$$= \cos\left[\sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\left(\hat{t}-\hat{x}\right)-\frac{\pi}{4}\right]$$

but,

$$\left(\hat{t}-\hat{x}\right) = \hat{x}\left(\frac{\hat{t}}{\hat{x}}-1\right)$$

so that,

$$\cos\left[\sqrt{2}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}(\hat{t}-\hat{x})-\frac{\pi}{4}\right]=\cos\left[\sqrt{2}\hat{x}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{2}}-\frac{\pi}{4}\right]$$

so the final result is identical with the first line of equation (4.2) in [Stiassnie(2010)] with the exception of an extra factor of 2 in the numerator.

$$2^{\frac{7}{4}}.2=2^{\frac{7}{4}}.2^{\frac{4}{4}}=2^{\frac{11}{4}}$$

In summary the first line of equation (4.2) in [Stiassnie(2010)] is given as,

$$\frac{2^{\frac{7}{4}}\rho g\tilde{\zeta}_0\sin\left[\sqrt{2}\hat{\tau}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}/2\right]\sin\left[\sqrt{2}\hat{b}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\right]\cos\left[\sqrt{2}\hat{x}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{2}}-\frac{\pi}{4}\right]}{\sqrt{\pi}\hat{\tau}\hat{x}^{\frac{1}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{4}}\left[2^{\frac{3}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}+\sinh\left(2^{\frac{3}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\right)\right]}, \tag{0.14}$$

but should read,

$$\frac{2^{\frac{11}{4}}\rho g\tilde{\zeta}_0\sin\left[\sqrt{2}\hat{\tau}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}/2\right]\sin\left[\sqrt{2}\hat{b}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\right]\cos\left[\sqrt{2}\hat{x}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{2}}-\frac{\pi}{4}\right]}{\sqrt{\pi}\hat{\tau}\hat{x}^{\frac{1}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{3}{4}}\left[2^{\frac{3}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}+\sinh\left(2^{\frac{3}{2}}\left(\frac{\hat{t}}{\hat{x}}-1\right)^{\frac{1}{2}}\right)\right]}. \tag{0.15}$$

A similar derivation applies to the first term in equation (4.1) of [Stiassnie(2010)] relating to the surface elevation.

References

[Stiassnie(2010)] Stiassnie, M. 2010 Tsunamis and acoustic-gravity waves from underwater earthquakes. *Journal of Engineering Mathematics* **67** (1-2), 23–32.