Navier–Stokes-based linear model for estimating large-scale fluctuations in wall turbulence: supplementary material

In this document, we present the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ introduced in Section 4 of the paper. Additionally, we also give a brief explanation on using the attached matlab codes for discretising the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and for calculation of covariance tensor $\langle \widehat{\boldsymbol{u}}\widehat{\boldsymbol{u}}^\dagger \rangle$ and the transfer function H_L.

I. EXPANSION OF THE LINEAR OPERATORS

The linearized models in the paper are created by first forming the nonlinear equations governing the evolution of velocity fluctuations by applying the Reynolds decomposition to the Navier–Stokes equations:

$$\frac{\partial u_i}{\partial t} = -U_k \frac{\partial u_i}{\partial x_k} - u_k \frac{\partial U_i}{\partial x_k} - \frac{\partial}{\partial x_k}\left(u_k u_i - \langle u_k u_i \rangle\right) - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}, \tag{1}$$

where U_i is the mean velocity and u_i and p are the velocity and pressure fluctuations, respectively. The subscripts take values $(1, 2, 3)$ and refer to coordinates (x, y, z) denoting the streamwise, spanwise and wall-normal directions, respectively. The mean and fluctuating velocity fields are represented as $(U, 0, 0)$ and (u, v, w), respectively. Because the flow is homogeneous in the streamwise and spanwise directions, we can decompose the velocity fluctuations into their Fourier coefficients characterized by the streamwise and spanwise wavenumbers (k_x, k_y). We then write the above equations in the spectral domain by applying the Fourier transformation in the horizontal directions as,

$$\frac{\partial \widehat{u}_i}{\partial t} = -U_k \frac{\partial \widehat{u}_i}{\partial x_k} - \widehat{u}_k \frac{\partial U_i}{\partial x_k} - \frac{\partial}{\partial x_k}\left(\widehat{u_k u_i}\right) - \frac{1}{\rho}\frac{\partial \widehat{p}}{\partial x_i} + \nu \frac{\partial^2 \widehat{u}_i}{\partial x_k \partial x_k}, \tag{2}$$

where $\widehat{\ }$ denotes a Fourier coefficient and the operators $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial^2/\partial x_1^2$ and $\partial^2/\partial x_2^2$ are equivalent to multiplication by ik_x, ik_y, $-k_x^2$ and $-k_y^2$, respectively. All the models in the present study are designed by replacing the nonlinear term (third term on the rhs) by a combination of an eddy dissipation term and a white-in-time spatially distributed body-forcing term. The resulting NS-based linear models are then represented as,

$$\frac{\partial \widehat{u}_i}{\partial t} = -U_k \frac{\partial \widehat{u}_i}{\partial x_k} - \widehat{u}_k \frac{\partial U_i}{\partial x_k} + \frac{\partial}{\partial x_k}\left(f_\nu \left(\frac{\partial \widehat{u}_i}{\partial x_k} + \frac{\partial \widehat{u}_k}{\partial x_i}\right)\right) + f_\sigma \widehat{d} - \frac{1}{\rho}\frac{\partial \widehat{p}}{\partial x_i} + \nu \frac{\partial^2 \widehat{u}_i}{\partial x_k \partial x_k}, \tag{3}$$

where f_ν and f_σ are, in general, functions of the wall-normal distance z and the wavenumbers k_x and k_y (see Table 1 in the paper).

We now write the NS-based linear models (3) in the Orr-Sommerfeld-Squire form (see Jovanović and Bamieh [1]), which is more convenient for input-output analysis, as

$$\frac{\partial \widehat{\boldsymbol{q}}}{dt} = \mathbf{A}\widehat{\boldsymbol{q}} + \mathbf{B}\widehat{\boldsymbol{d}}, \tag{4a}$$

$$\widehat{\boldsymbol{u}} = \mathbf{C}\widehat{\boldsymbol{q}}, \tag{4b}$$

where $\widehat{\boldsymbol{q}} = (\widehat{w}, \widehat{\eta})$ comprises the wall-normal velocity and vorticity fluctuations. The linear operators $\mathbf{A}$ and $\mathbf{C}$ are similar to those in Hwang and Cossu [2] and operator $\mathbf{B}$ is similar to that in Ran *et al.* [3]. They are given as,

$$\mathbf{A} = \begin{bmatrix} \Delta^{-1}\mathcal{L}_{OS} & 0 \\ -ik_y \partial_z U & \mathcal{L}_{SQ} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -ik_x \Delta^{-1}\left(f_\sigma \mathcal{D} + \partial_z f_\sigma\right) & -ik_y \Delta^{-1}\left(f_\sigma \mathcal{D} + \partial_z f_\sigma\right) & -k^2 \Delta^{-1} \\ -ik_y f_\sigma & -ik_x f_\sigma & 0 \end{bmatrix}, \tag{5a}$$

$$\mathbf{C} = \frac{1}{k^2}\begin{bmatrix} ik_x \mathcal{D} & -ik_y \\ ik_y \mathcal{D} & ik_x \\ k^2 & 0 \end{bmatrix}, \tag{5b}$$

where $\Delta = \mathcal{D}^2 - k^2$, $k^2 = k_x^2 + k_y^2$ and $\mathcal{D}$ and ∂_z represent differentiation in the wall-normal direction. The operators $\mathcal{L}_{OS}$ and $\mathcal{L}_{SQ}$ are given as,

$$\mathcal{L}_{OS} = -ik_x U \Delta + ik_x \partial_z^2 U + \left(f_\nu + \nu\right)\Delta^2 + 2\partial_z f_\nu \mathcal{D}\Delta + \partial_z^2 f_\nu \left(\mathcal{D}^2 + k^2\right), \tag{6}$$

$$\mathcal{L}_{SQ} = ik_x U + \left(f_\nu + \nu\right)\Delta + \partial_z f_\nu \Delta. \tag{7}$$

Because the system is linearly stable. i.e. all the eigenvalues of $\mathbf{A}$ are stable, and the stochastic forcing is white-in-time, the system's response can be calculated in terms of the covariance tensor $\mathbf{X} = \langle \widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}^{\dagger} \rangle$ by solving the algebraic Lyapunov equation [2],

$$\mathbf{AX} + \mathbf{XA}^{\dagger} + \mathbf{BB}^{\dagger} = 0. \tag{8}$$

We calculate the covariance tensor $\langle \widehat{\boldsymbol{u}}\widehat{\boldsymbol{u}}^{\dagger} \rangle$ required for calculation of H_L as $\mathbf{CXC}^{\dagger}$.

II. MATLAB CODES

We attach the matlab codes for creating discretized form of the operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and calculating the transfer function H_L for a channel flow. The main code is named "Main.m", where the user can input the friction Reynolds number (Re_τ in the manuscript and Re in the code) and model name (B-model or W-model or L-model (L-model is for λ-model)).

The parameters for H_L are the wall-normal heights z_m (Z2 in the code) and z_p (Z1 in the code) and the wavenumbers (k_x, k_y) ((kx,ky) in the code).

The numerical parameter is method.N, which is the number of discretization points. To discretize the operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in the wall-normal direction (from $z = 0$ to 1), we use a Chebyshev-collocation method and impose the boundary conditions $\widehat{w}(0) = \widehat{\eta}(0) = \partial\widehat{w}(0)/\partial z = 0$. We divide the fluctuations into their symmetric and anti-symmetric components about the centreline ($z = 1$) and calculate their contributions separately.

[1] M. R. Jovanović and B. Bamieh, Journal of Fluid Mechanics **534**, 145 (2005).
[2] Y. Hwang and C. Cossu, J. Fluid Mech. **664**, 51 (2010).
[3] W. Ran, A. Zare, M. J. Hack, and M. R. Jovanović, Physical Review Fluids **4**, 1 (2019), arXiv:1807.07759.