Gravity-driven film flow down a uniformly heated, smoothly corrugated, rigid substrate - Supplementary Material #3

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Appendix F. RAM–θ_{para} - Linearised Periodic Coefficients

The periodic coefficients, $\varphi_k = \{\alpha_k, \beta_k, \gamma_k, \xi_k, \zeta_k, \eta_k, \mu_k, \nu_k\}$, read:

$$\alpha_0 = \epsilon Re + \epsilon^3 Re\left[\frac{107}{336}h_s\frac{\partial^2 h_s}{\partial x^4} + \frac{5}{12}h_s\frac{d^2 s}{dx^2} + \frac{29}{56}\left(\frac{\partial h_s}{\partial x}\right)^2 + \frac{11}{8}\frac{\partial h_s}{\partial x}\frac{ds}{dx}\left(\frac{ds}{dx}\right)^2\right], \quad \text{(F 1)}$$

$$\alpha_1 = -\epsilon^3 Re\frac{11h_s}{56}\frac{\partial h_s}{\partial x}, \quad \text{(F 2)}$$

$$\alpha_2 = -\epsilon^3 Re\frac{11h_s^2}{56}, \quad \text{(F 3)}$$

$$\begin{aligned}\beta_0 = &\frac{5}{2}\frac{\epsilon^5\left(1 - Ma\vartheta_s\right)\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2\frac{\partial f_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{5/2}} + \frac{5}{6}\frac{\epsilon^3\left(1 - Ma\vartheta_s\right)\frac{\partial^3 f_s}{\partial x^3}}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}} + \frac{5}{3}\frac{\epsilon^3 Ma\frac{\partial^2 f_s}{\partial x^2}\frac{\partial \vartheta_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}} \\ &+\frac{5}{6}\frac{\epsilon^3 Ma\frac{\partial^2 f}{\partial x}\frac{\partial^2 \vartheta_s}{\partial x^2}}{Ca\sqrt{1+\epsilon^2 g_s}} + \epsilon^4\left[\frac{53q_s}{2h_s^3}\left(\frac{\partial h_s}{\partial x}\frac{ds}{dx}\right)^2 - \frac{12q_s}{h_s^2}\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h_s}{\partial x}\frac{ds}{dx} + \frac{60q_s}{7h_s^3}\left(\frac{\partial h_s}{\partial x}\right)^4\right. \\ &-\frac{49q_s}{h_s^2}\frac{\partial h_s}{\partial x}\frac{d^2 s}{dx^2}\frac{ds}{dx} - \frac{107q_s}{336}\frac{\partial^4 h_s}{\partial x^4} + \frac{5q_s}{2h_s^3}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^3 - \frac{59q_s}{8h_s^2}\frac{\partial^2 h_s}{\partial x^2}\left(\frac{ds}{dx}\right)^2 \\ &+\frac{28q_s}{h_s^3}\left(\frac{\partial h_s}{\partial x}\right)^3\frac{ds}{dx} - \frac{5q_s}{h_s^3}\left(\frac{ds}{dx}\right)^4 - \frac{45q_s}{8h_s^2}\frac{d^2 s}{dx^2}\left(\frac{ds}{dx}\right)^2 - \frac{19q_s}{4h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{d^2 s}{dx^2} \\ &\left.-\frac{5q_s}{12}\frac{d^4 s}{dx^4} - \frac{113q_s}{28h_s^2}\frac{\partial^2 h_s}{\partial x^2}\left(\frac{\partial h_s}{\partial x}\right)^2\right] + \epsilon^3 Re\left[\frac{555q_s^2}{896h_s^2}\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h}{\partial x} - \frac{9q_s^2}{7h_s^2}\frac{d^2 s}{dx^2}\frac{ds}{dx}\right. \\ &+\frac{18q_s^2}{7h_s^3}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^2 + \frac{163q_s^2}{112h_s^3}\left(\frac{\partial h}{\partial x}\right)^2\frac{ds}{dx} + \frac{185q_s^2}{896h_s^2}\frac{\partial h_s}{\partial x}\frac{d^2 s}{dx^2} - \frac{163q_s^2}{448h_s^2}\frac{\partial^2 h_s}{\partial x^2}\frac{ds}{dx} \\ &\left.-\frac{115q_s^2}{336h_s^3}\left(\frac{\partial^3 h_s}{\partial x^3}\right)^2\right] - \frac{15q_s}{2h_s^3}\frac{\epsilon^2\frac{\partial f_s}{\partial x}\frac{\partial h_s}{\partial x}}{1-\epsilon^2 g_s} - \frac{15q_s}{8h_s^2}\frac{\epsilon^2\frac{\partial^2 f_s}{\partial x^2}}{1+\epsilon^2 g_s} + \epsilon^2\left[\frac{31q_s}{2h_s^3}\left(\frac{\partial h_s}{\partial x}\right)^2\right. \\ &\left.-\frac{33q_s}{8h_s^2}\frac{\partial^2 h_s}{\partial x^2} - \frac{15q_s}{8h_s^2}\frac{d^2 s}{dx^2} - \frac{10q_s}{h_s^3}\left(\frac{ds}{dx}\right)^2 + \frac{5q_s}{2h_s^3}\frac{\partial h_s}{\partial x}\frac{ds}{dx}\right] + \epsilon Re\frac{18q_s^2}{7h_s^3}\frac{\partial h_s}{\partial x} \\ &+\epsilon\frac{5}{3}\frac{\partial f_s}{\partial x}\cot\beta - \frac{5}{3} - \frac{5q_s}{h_s^3},\end{aligned} \quad \text{(F 4)}$$

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$$\beta_1 = -\frac{25}{2}\frac{\epsilon^7\left(1-Ma\vartheta_s\right)\left(\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}\right)^2}{\left[1+\epsilon^2 g_s\right]^{7/2}} + \frac{5h_s}{2}\frac{\epsilon^5\left(1-Ma\vartheta_s\right)\frac{\partial^3 f_s}{\partial x^3}\frac{\partial f_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{5/2}}$$
$$+\frac{5h_s}{2}\frac{\epsilon^5\left(1-Ma\vartheta_s\right)\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2}{Ca\left[1+\epsilon^2 g_s\right]^{5/2}} - \frac{5h_s\epsilon^5 Ma\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{5/2}} - \frac{5h_s}{6}\frac{\epsilon^5 Ma\left(\frac{\partial f_s}{\partial x}\right)^2\frac{\partial^2 \vartheta_s}{\partial x^2}}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}}$$
$$+\epsilon^4\left[\frac{49q_s}{4h_s}\frac{d^2 s}{dx^2}\frac{ds}{dx} + \frac{12q_s}{h_s}\frac{\partial^2 h_s}{\partial x^2}\frac{ds}{dx} - \frac{42q_s}{h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{ds}{dx} + \frac{3q_s}{7}\frac{\partial^3 h_s}{\partial x^3} - \frac{120q_s}{7h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^3\right.$$
$$\left.-\frac{q_s}{4}\frac{d^3 s}{dx^3} + \frac{53q_s}{2h_s^2}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^2 + \frac{113q_s}{14h_s}\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h_s}{\partial x} + \frac{19q_s}{2h_s}\frac{\partial h_s}{\partial x}\frac{d^2 s}{dx^2} - \frac{5q_s}{4h_s^2}\left(\frac{ds}{dx}\right)^3\right]$$
$$+\frac{5h_s}{6}\frac{\epsilon^3 Ma\frac{\partial^2 \vartheta_s}{\partial x^2}}{Ca\sqrt{1+\epsilon^2 g_s}} - \frac{5}{4}\frac{\epsilon^3\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}}{\left[1+\epsilon^2 g_s\right]^{3/2}\left(1-\epsilon^2 g_s\right)} + \frac{5}{2}\frac{\epsilon^3\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}}{\sqrt{1+\epsilon^2 g_s}\left(1-\epsilon^2 g_s\right)^2}$$
$$+\epsilon^3 Re\left[\frac{115q_s^2}{224h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2 - \frac{9q_s^2}{7h_s^2}\left(\frac{ds}{dx}\right)^2 - \frac{163q_s^2}{112h_s^2}\frac{\partial h_s}{\partial x}\frac{ds}{dx} - \frac{555q_s^2}{896h_s}\frac{\partial^2 h_s}{\partial x^2} - \frac{185q_s^2}{897h_s}\frac{d^2 s}{dx^2}\right]$$
$$-\epsilon^2\left[-\frac{31q_s}{2h_s^2}\frac{\partial h_s}{\partial x} + \frac{5q_s}{4h_s^2}\frac{ds}{dx}\right] - \frac{15q_s}{4h_s}\frac{\epsilon^4\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}}{\left(1+\epsilon^2 g_s\right)^2} + \frac{15q_s}{2h_s^2}\frac{\epsilon^4\left(\frac{\partial f_s}{\partial x}\right)^2\frac{\partial h_s}{\partial x}}{\left(1-\epsilon^2 g_s\right)^2}$$
$$+\frac{15q_s}{2h_s^2}\frac{\epsilon^2\frac{\partial f_s}{\partial x}}{1-\epsilon^2 g_s} - \epsilon Re\frac{9q_s^2}{7h_s^2} + \epsilon\frac{5h_s}{3}\cot\beta, \quad \text{(F 5)}$$

$$\beta_2 = \frac{5\epsilon^5\left(1-Ma\vartheta_s\right)h_s\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{5/2}} + \frac{5}{3}\frac{\epsilon^3 h_s Ma\frac{\partial \vartheta_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}}$$
$$+\epsilon^4\left[\frac{113q_s}{28h_s}\left(\frac{\partial h_s}{\partial x}\right)^2 + \frac{59q_s}{8h_s}\left(\frac{ds}{dx}\right)^2 + \frac{53q_s}{28}\frac{\partial^2 h_s}{\partial x^2} + \frac{7q_s}{8}\frac{d^2 s}{dx^2}\frac{12q_s}{h_s}\frac{\partial h_s}{\partial x}\frac{ds}{dx}\right]$$
$$+\epsilon^3 Re\left[\frac{163q_s^2}{448h_s}\frac{ds}{dx} - \frac{555q_s^2}{896h_s}\frac{\partial h_s}{\partial x}\right] + \epsilon^2\left[\frac{33q_s}{8h_s} + \frac{15q_s}{8h_s}\frac{1}{1+\epsilon^2 g_s}\right], \quad \text{(F 6)}$$

$$\beta_3 = \epsilon^3 Re\frac{1205q_s^2}{2688} - \frac{5h_s}{6}\frac{\epsilon^3\left(1-Ma\vartheta_s\right)}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}} + \epsilon^4\left[\frac{3q_s}{7}\frac{\partial h_s}{\partial x} - \frac{q_s}{4}\frac{ds}{dx}\right], \quad \text{(F 7)}$$

$$\beta_4 = -\epsilon^4\frac{107h_s q_s}{336}, \quad \text{(F 8)}$$

$$\gamma_0 = \epsilon^4\left[\frac{3}{7}\frac{\partial^3 h_s}{\partial h_s}\partial x - \frac{5h_s}{12}\frac{d^4 s}{dx^4} - \frac{107h_s}{336}\frac{\partial^4 h_s}{\partial x^4} - \frac{30}{7h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^4 + \frac{12}{h_s}\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h_s}{\partial x}\frac{ds}{dx}\right.$$
$$-\frac{5}{4h_s^2}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^3 - \frac{1}{2}\left(\frac{d^2 s}{dx^2}\right)^2 + \frac{53}{56}\left(\frac{\partial^2 h_s}{\partial x^2}\right)^2 + \frac{59}{8h_s}\frac{\partial^2 h_s}{\partial x^2}\left(\frac{ds}{dx}\right)^2$$
$$-\frac{1}{4}\frac{\partial^3 h_s}{\partial x^3}\frac{ds}{dx} + \frac{5}{2h_s^2}\left(\frac{ds}{dx}\right)^4 - \frac{3}{2}\frac{d^3 s}{dx^3}\frac{ds}{dx} + \frac{7}{8}\frac{\partial^2 h_s}{\partial x^2}\frac{d^2 s}{dx^2} - \frac{1}{4}\frac{\partial h_s}{\partial x}\frac{d^3 s}{dx^3}$$
$$+\frac{113}{28h_s}\frac{\partial^2 h_s}{\partial x^2}\left(\frac{\partial h_s}{\partial x}\right)^2 - \frac{14}{h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^3\frac{ds}{dx} - \frac{53}{4h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2\left(\frac{ds}{dx}\right)^2$$

$$+\frac{45}{8h_s}\frac{d^2s}{dx^2}\left(\frac{ds}{dx}\right)^2+\frac{19}{4h_s}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{d^2s}{dx^2}+\frac{49}{4h_s}\frac{\partial h_s}{\partial x}\frac{d^2s}{dx^2}\frac{ds}{dx}\Bigg]+\frac{5}{2h_s^2}$$

$$+\epsilon^3 Re\left[\frac{115q_s}{336h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^3+\frac{18q_s}{7h_s}\frac{d^2s}{dx^2}\frac{ds}{dx}-\frac{555q_s}{448h_s}\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h_s}{\partial x}-\frac{185q_s}{448h_s}\frac{\partial h_s}{\partial x}\frac{d^2s}{dx^2}\right.$$

$$+\frac{1205q_s}{1344}\frac{\partial^3 h_s}{\partial x^3}+\frac{73q_s}{64}\frac{d^3s}{dx^3}-\frac{18q_s}{7h_s^2}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^2-\frac{163q_s}{112h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{ds}{dx}$$

$$\left.+\frac{163q_s}{224h_s}\frac{\partial^2 h_s}{\partial x^2}\frac{ds}{dx}\right]+\frac{15}{4h_s^2}\frac{\epsilon^2\frac{\partial f_s}{\partial x}\frac{\partial h_s}{\partial x}}{1-\epsilon^2 g_s}+\frac{15}{8h_s}\frac{\epsilon^2\frac{\partial^2 f_s}{\partial x^2}}{1+\epsilon^2 g_s}-\epsilon Re\frac{18q_s}{7h_s^2}\frac{\partial h_s}{\partial x}$$

$$+\epsilon^2\left[\frac{5}{h_s^2}\left(\frac{ds}{dx}\right)^2-\frac{31}{4h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2+\frac{33}{8h_s}\frac{\partial^2 h_s}{\partial x^2}+\frac{15}{8h_s}\frac{d^2s}{dx^2}-\frac{5}{4h_s^2}\frac{\partial h_s}{\partial x}\frac{ds}{dx}\right], \tag{F 9}$$

$$\gamma_1=\epsilon^4\left[\frac{29}{2h_s}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{ds}{dx}+\frac{59}{4h_s}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^2-\frac{107h_s}{84}\frac{\partial^3 h_s}{\partial x^3}-\frac{5h_s}{3}\frac{d^3s}{dx^3}+\frac{15}{4h_s}\left(\frac{ds}{dx}\right)^3\right.$$

$$\left.-\frac{17}{14}\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h_s}{\partial x}+\frac{61}{14h_s}\left(\frac{\partial h_s}{\partial x}\right)^3-\frac{13}{4}\frac{\partial h_s}{\partial x}\frac{d^2s}{dx^2}-\frac{13}{4}\frac{\partial^2 h_s}{\partial x^2}\frac{ds}{dx}-7\frac{d^2s}{dx^2}\frac{ds}{dx}\right]$$

$$+\epsilon^3 Re\left[\frac{277q_s}{128}\frac{d^2s}{dx^2}+\frac{4633q_s}{2688}\frac{\partial^2 h_s}{\partial x^2}+\frac{297q_s}{224h_s}\frac{\partial h_s}{\partial x}\frac{ds}{dx}+\frac{17q_s}{7h_s}\left(\frac{ds}{dx}\right)^2-\frac{253q_s}{672h_s}\left(\frac{\partial h_s}{\partial x}\right)^2\right]$$

$$+\epsilon^2\left[\frac{33}{4h_s}\frac{\partial h_s}{\partial x}+\frac{15}{4h_s}\frac{ds}{dx}\right]-\frac{15}{4h_s}\frac{\epsilon^2\frac{\partial f_s}{\partial x}}{1-\epsilon^2 g_s}+\epsilon Re\frac{17q_s}{7h_s}, \tag{F 10}$$

$$\gamma_2=-\epsilon^4\left[\frac{13}{7}\left(\frac{\partial h_s}{\partial x}\right)^2+\frac{19}{4}\left(\frac{ds}{dx}\right)^2+\frac{181h_s}{168}\frac{\partial^2 h_s}{\partial x^2}+\frac{5h_s}{3}\frac{d^2s}{dx^2}+\frac{23}{4}\frac{\partial h_s}{\partial x}\frac{ds}{dx}\right]$$

$$-\epsilon^3 Re\left[\frac{15q_s}{64}\frac{ds}{dx}+\frac{187q_s}{1344}\frac{\partial h_s}{\partial x}\right]-\epsilon^2\left[\frac{13}{4}+\frac{5}{4}\frac{1}{1+\epsilon^2 g_s}\right], \tag{F 11}$$

$$\gamma_3=-\epsilon^3 Re\frac{601h_s q_s}{1008}+\epsilon^4\frac{11h_s}{28}\frac{\partial h_s}{\partial x}, \tag{F 12}$$

$$\gamma_4=\epsilon^4\frac{11h_s^2}{56}, \tag{F 13}$$

$$\xi_0=-\frac{5h_s}{2}\frac{\epsilon^5 Ma\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2\frac{\partial f_s}{\partial x}}{Ca\left[1+\epsilon^2 g_s\right]^{5/2}}+\frac{5h_s}{6}\frac{\epsilon^3 Ma\frac{\partial^3 f_s}{\partial x^3}}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}}, \tag{F 14}$$

$$\xi_1=\frac{5h_s}{3}\frac{\epsilon^3 Ma\frac{\partial^2 f_s}{\partial x^2}}{Ca\left[1+\epsilon^2 g_s\right]^{3/2}}+\frac{5}{4}\frac{\epsilon Ma}{Ca\sqrt{1+\epsilon^2 g_s}\left(1-\epsilon^2 g_s\right)}, \tag{F 15}$$

$$\xi_2=\frac{5h_s}{6}\frac{\epsilon^3 Ma\frac{\partial f_s}{\partial x}}{Ca\sqrt{1+\epsilon^2 g_s}}, \tag{F 16}$$

$$\zeta_0=\epsilon RePr\left[1+\frac{Bih_s}{5\sqrt{1+\epsilon^2 g_s}}\right], \tag{F 17}$$

$$\eta_0 = -\frac{2h_s}{5}\frac{\epsilon^6 Bi\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2\left(\frac{\partial f_s}{\partial x}\right)^2}{\left[1+\epsilon^2 g_s\right]^{5/2}} + \frac{h_s}{5}\frac{\epsilon^4 Bi\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2}{\left[1+\epsilon^2 g_s\right]^{5/2}} + \frac{1}{5}\frac{\epsilon^4 Bi\frac{\partial^3 f_s}{\partial x^3}\frac{\partial f_s}{\partial x}}{\left[1+\epsilon^2 g_s\right]^{3/2}}$$
$$+\frac{\epsilon^4 Bi\frac{\partial^2 f_s}{\partial x^2}}{\left[1+\epsilon^2 g_s\right]^{3/2}}\left[\frac{6}{5}\left(\frac{\partial h_s}{\partial x}\right)^2 + 2\frac{\partial h_s}{\partial x}\frac{ds}{dx} + \frac{4}{5}\left(\frac{ds}{dx}\right)^2\right] - \frac{6q_s}{25}\frac{\epsilon^3 RePrBi\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}}{\left[1+\epsilon^2 g_s\right]^{3/2}}$$
$$+\frac{\epsilon^2 Bi}{\sqrt{1+\epsilon^2 g_s}}\left[\frac{5}{6h_s}\left(\frac{\partial h_s}{\partial x}\right)^2 + \frac{16}{5h_s}\frac{\partial h_s}{\partial x}\frac{ds}{dx} + \frac{12}{5h_s}\left(\frac{ds}{dx}\right)^2 - \frac{3}{5}\frac{\partial^2 h_s}{\partial x^2} - \frac{2}{5}\frac{d^2 s}{dx^2}\right]$$
$$+\epsilon^2\left[\frac{8}{5h_s^2}\frac{\partial h_s}{\partial x}\frac{ds}{dx} + \frac{4}{5h_s}\frac{d^2 s}{dx^2} + \frac{2}{5h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2 + \frac{2}{5h_s}\frac{\partial^2 h_s}{\partial x^2} + \frac{12}{5h_s^2}\left(\frac{ds}{dx}\right)^2\right]$$
$$+\frac{6q_s}{25h_s}\frac{\epsilon RePrBi}{\sqrt{1+\epsilon^2 g_s}}\frac{\partial h_s}{\partial x} + \frac{12}{5h_s^2}\left[1+\frac{Bi h_s}{\sqrt{1+\epsilon^2 g_s}}\right], \quad \text{(F 18)}$$

$$\eta_1 = \frac{2h_s}{5}\frac{\epsilon^4 Bi\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}}{\left[1+\epsilon^2 g_s\right]^{3/2}} + \epsilon^2\left[\frac{4}{5h_s}\frac{\partial h_s}{\partial x} + \frac{8}{5h_s}\frac{ds}{dx}\right] - \frac{\epsilon^2 Bi}{\sqrt{1+\epsilon^2 g_s}}\left[\frac{6}{5}\frac{\partial h_s}{\partial x} + \frac{4}{5}\frac{ds}{dx}\right]$$
$$-\frac{12}{5h_s}\frac{\epsilon^2\frac{\partial f_s}{\partial x}}{1+\epsilon^2 g_s} + \epsilon RePr\left[\frac{33q_s}{25h_s} + \frac{6q_s}{25}\frac{Bi}{\sqrt{1+\epsilon^2 g_s}}\right], \quad \text{(F 19)}$$

$$\eta_2 = -\epsilon^2\left[1+\frac{Bi h_s}{5\sqrt{1+\epsilon^2 g_s}}\right], \quad \text{(F 20)}$$

$$\mu_0 = -\frac{2\vartheta_s}{5}\frac{\epsilon^6 Bi\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2\left(\frac{\partial f_s}{\partial x}\right)^2}{\left[1+\epsilon^2 g_s\right]^{5/2}} + \frac{\vartheta_s}{5}\frac{\epsilon^4 Bi\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2}{\left[1+\epsilon^2 g_s\right]^{5/2}} + \frac{\vartheta_s}{5}\frac{\epsilon^4 Bi\frac{\partial^3 f_s}{\partial x^3}\frac{\partial f_s}{\partial x}}{\left[1+\epsilon^2 g_s\right]^{3/2}}$$
$$-\frac{\epsilon^2 Bi}{\sqrt{1+\epsilon^2 g_s}}\left[\frac{1}{5}\frac{\partial^2 \vartheta_s}{\partial x^2} + \frac{6\vartheta_s}{5h_s^2}\left(\frac{\partial h_s}{\partial x}\right)^2 + \frac{16\vartheta_s}{5h_s^2}\frac{\partial h_s}{\partial x}\frac{ds}{dx} + \frac{12\vartheta_s}{5h_s^2}\left(\frac{ds}{dx}\right)^2\right]$$
$$+\frac{2}{5}\frac{\epsilon^4 Bi\frac{\partial^2 f_s}{\partial x^2}\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}}{\left[1+\epsilon^2 g_s\right]^{3/2}} - \epsilon^2\left[\frac{2\left(\vartheta_s-1\right)}{5h_s^2}\frac{\partial^2 h_s}{\partial x^2} + \frac{4}{5h_s^2}\frac{\partial h_s}{\partial x}\frac{\partial \vartheta_s}{\partial x} + \frac{8}{5h_s^2}\frac{ds}{dx}\frac{\partial \vartheta_s}{\partial x}\right.$$
$$+\frac{4\left(\vartheta_s-1\right)}{5h_s^2}\frac{d^2 s}{dx^2} + \frac{16\left(\vartheta_s-1\right)}{5h_s^3}\frac{\partial h_s}{\partial x}\frac{ds}{dx} + \frac{4\left(\vartheta_s-1\right)}{5h_s^3}\left(\frac{\partial h_s}{\partial x}\right)^2$$
$$\left.+\frac{24\left(\vartheta_s-1\right)}{5h_s^3}\left(\frac{ds}{dx}\right)^2\right] - \frac{24\left(\vartheta_s-1\right)}{5h_s^3} - \frac{12Bi\vartheta_s}{5h_s^2\sqrt{1+\epsilon^2 g_s}} + \frac{12}{5h_s^2}\frac{\epsilon^2\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}}{1+\epsilon^2 g_s}$$
$$-\epsilon RePr\left[\frac{33q_s}{25h_s^2}\frac{\partial \vartheta_s}{\partial x} + \frac{6q_s\vartheta_s}{25h_s^2}\frac{Bi\frac{\partial h_s}{\partial x}}{\sqrt{1+\epsilon^2 g_s}}\right], \quad \text{(F 21)}$$

$$\mu_1 = \frac{2\epsilon^8 Bi h_s\vartheta_s\left(\frac{\partial f_s}{\partial x}\right)^3\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2}{\left[1+\epsilon^2 g_s\right]^{7/2}} - \frac{\epsilon^6 Bi h_s\left(\frac{\partial f_s}{\partial x}\right)^2\left[\frac{6}{5}\frac{\partial^2 f_s}{\partial x^2}\frac{\partial \vartheta_s}{\partial x} + \frac{3}{5}\vartheta_s\frac{\partial^3 f_s}{\partial x^3}\right]}{\left[1+\epsilon^2 g_s\right]^{5/2}}$$
$$-\frac{\epsilon^6 Bi\vartheta_s\frac{\partial^2 f_s}{\partial x^2}}{\left[1+\epsilon^2 g_s\right]^{5/2}}\left[\frac{18}{5}\left(\frac{\partial h_s}{\partial x}\right)^3 + \frac{12}{5}\left(\frac{ds}{dx}\right)^3 + \frac{42}{5}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^2 + \frac{48}{5}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{ds}{dx}\right]$$
$$+\frac{\epsilon^4 Bi\vartheta_s}{\left[1+\epsilon^2 g_s\right]^{3/2}}\left[3\frac{\partial^2 h_s}{\partial x^2}\frac{\partial h_s}{\partial x} + \frac{13}{5}\frac{\partial^2 h_s}{\partial x^2}\frac{ds}{dx} + \frac{14}{5}\frac{\partial h_s}{\partial x}\frac{d^2 s}{dx^2} + \frac{12}{5}\frac{d^2 s}{dx^2}\frac{ds}{dx}\right.$$

$$-\frac{6}{5h_s}\left(\frac{\partial h_s}{\partial x}\right)^3-\frac{12}{5h_s}\left(\frac{ds}{dx}\right)^3-\frac{22}{5h_s}\left(\frac{\partial h_s}{\partial x}\right)^2\frac{ds}{dx}-\frac{28}{5h_s}\frac{\partial h_s}{\partial x}\left(\frac{ds}{dx}\right)^2\Bigg]$$

$$+\frac{\epsilon^4 Bi\frac{\partial \vartheta_s}{\partial x}}{[1+\epsilon^2 g_s]^{3/2}}\left[\frac{2}{5}\frac{\partial^2 f_s}{\partial x^2}+\frac{6}{5}\left(\frac{\partial h_s}{\partial x}\right)^2+2\frac{\partial h_s}{\partial x}\frac{ds}{dx}+\frac{4}{5}\left(\frac{ds}{dx}\right)^2\right]$$

$$+\frac{24}{5h_s}\frac{\epsilon^4\left(\frac{\partial f_s}{\partial x}\right)^2\frac{\partial \vartheta_s}{\partial x}}{(1+\epsilon^2 g_s)^2}+\frac{\epsilon^4 Bi\left[\frac{1}{5}\frac{\partial f_s}{\partial x}\frac{\partial^2 \vartheta_s}{\partial x^2}+h_s\vartheta_s\frac{\partial^3 f_s}{\partial x^3}\right]}{[1+\epsilon^2 g_s]^{3/2}}-\frac{4}{5}\frac{\epsilon^6 Bi h_s\vartheta_s\left(\frac{\partial^2 f_s}{\partial x^2}\right)^2\frac{\partial f_s}{\partial x}}{[1+\epsilon^2 g_s]^{5/2}}$$

$$-\frac{6}{25}\frac{\epsilon^3 RePrBi}{[1+\epsilon^2 g_s]^{3/2}}\left[\frac{q_s\vartheta_s}{h_s}\frac{\partial f_s}{\partial x}\frac{\partial h_s}{\partial x}+q_s\vartheta_s\frac{\partial^2 f_s}{\partial x^2}+q_s\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}\right]$$

$$+\epsilon^2\left[\frac{4(\vartheta_s-1)}{5h_s^2}+\frac{8(\vartheta_s-1)}{5h_s^2}+\frac{4}{5h_s}\frac{\partial \vartheta_s}{\partial x}-\frac{12}{5h_s}\frac{\frac{\partial \vartheta_s}{\partial x}}{1+\epsilon^2 g_s}+\frac{6q_s\vartheta_s}{25h_s}\frac{\epsilon RePrBi}{\sqrt{1+\epsilon^2 g_s}},\right]$$

$$-\frac{12\vartheta_s}{5h_s}\frac{\epsilon^2\frac{\partial f_s}{\partial x}}{[1+\epsilon^2 g_s]^{3/2}}+\frac{\epsilon^2\left[\frac{12\vartheta_s}{5h_s}\frac{\partial h_s}{\partial x}+\frac{16\vartheta_s}{5h_s}\frac{ds}{dx}-\frac{6}{5}\frac{\partial \vartheta_s}{\partial x}\right]}{\sqrt{1+\epsilon^2 g_s}}, \tag{F 22}$$

$$\mu_2=-\frac{\epsilon^6 Bih\vartheta_s\left(\frac{\partial f_s}{\partial x}\right)^2\frac{\partial^2 f_s}{\partial x^2}}{[1+\epsilon^2 g_s]^{5/2}}+\frac{\epsilon^4 Bi\left[\frac{2h_s}{5}\frac{\partial f_s}{\partial x}\frac{\partial \vartheta_s}{\partial x}+\vartheta_s\left(\frac{6}{5}\left(\frac{\partial h_s}{\partial x}\right)^2+2\frac{\partial h_s}{\partial x}\frac{ds}{dx}+\frac{4}{5}\left(\frac{ds}{dx}\right)^2\right)\right]}{[1+\epsilon^2 g_s]^{3/2}}$$

$$+\frac{2}{5}\frac{\epsilon^4 Bih_s\vartheta_s\frac{\partial^2 f_s}{\partial x^2}}{[1+\epsilon^2 g_s]^{5/2}}-\frac{6q_s}{25}\frac{\epsilon^3 RePrBi\vartheta_s\frac{\partial f_s}{\partial x}}{[1+\epsilon^2 g_s]^{3/2}}+\epsilon^2\left[\frac{2}{5}\frac{(\vartheta_s-1)}{h_s}-\frac{3}{5}\frac{Bi\vartheta_s}{\sqrt{1+\epsilon^2 g}}\right], \tag{F 23}$$

$$\mu_3=\frac{1}{5}\frac{\epsilon^4 Bih_s\vartheta_s}{[1+\epsilon^2 g_s]^{3/2}}\frac{\partial f_s}{\partial x}, \tag{F 24}$$

$$\nu_0=\epsilon RePr\left[\frac{33}{25h_s}\frac{\partial \vartheta_s}{\partial x}+\frac{6}{25}\frac{Bi\left(\frac{\vartheta_s}{h_s}\frac{\partial h_s}{\partial x}+\frac{\partial \vartheta_s}{\partial x}\right)}{\sqrt{1+\epsilon^2 g_s}}\right]-\frac{6}{25}\frac{\epsilon^3 RePrBi\vartheta_s\frac{\partial f_s}{\partial x}\frac{\partial^2 f_s}{\partial x^2}}{\sqrt{1+\epsilon^2 g_s}}, \tag{F 25}$$

$$\nu_1=\epsilon RePr\left[\frac{3}{25}\frac{(\vartheta_s-1)}{h}-\frac{11}{50}\frac{Bi\vartheta_s}{\sqrt{1+\epsilon^2 g_s}}\right], \tag{F 26}$$

where $(q_s, h_s, \vartheta_s, f_s, g_s)$ are the steady-state quantities for the flow rate, film thickness, free-surface temperature, free-surface position and curvature pre-factor, respectively.