Thin-film Rayleigh-Taylor instability in the presence of a deep periodic corrugated wall

Supplementary Material

S1. Two-phase WRIBL derivation

We consider hydrodynamically active immiscible Newtonian fluids 1 and 2, with constant properties. A heavy fluid 1 is underneath a wavy wall and a light fluid 2 is beneath fluid 1 as shown in figure S 1. The density, viscosity and the interfacial tension of the fluids are denoted by ρ_1, ρ_2, μ_1, μ_2 and γ. The horizontal and vertical components of the velocity vector ($\mathbf{v}$) are denoted by u and w. The bottom wavy wall is denoted by the function $f(x)$. This system is subjected to gravity along the z-coordinate. The long-wave approximation is used, i.e. $H \ll \lambda$, where λ is the characteristic horizontal length scale arising from the instability and H is the thickness of the bilayer thin film.

S1.1. *Governing equations*

Fluids 1 and 2 are hydrodynamically active and satisfy the continuity and Navier-Stokes equations. These equations are given by

$$\nabla \cdot \mathbf{v_j} = 0, \rho_j(\frac{\partial \mathbf{v_j}}{\partial t} + \mathbf{v_j} \cdot \nabla \mathbf{v_j}) = \nabla \cdot \boldsymbol{T_j} + \rho_j g \boldsymbol{i}_z \tag{S 1.1}$$

Here, $\boldsymbol{T}_j = -p_j \boldsymbol{I} + \mu_j(\nabla \mathbf{v_j} + \nabla \mathbf{v_j}^T)$, $\boldsymbol{i}_z$ is the unit vector along the positive z direction and I is the identity tensor. The subscripts $j = 1, 2$ represent fluids 1 and 2 respectively.

The interface speed $\mathcal{U}$, the unit normal vector ($\mathbf{n}$) and the unit tangent vector ($\mathbf{t}$) are given by

$$\mathcal{U} = \frac{\frac{\partial h}{\partial t}}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{1/2}}, \ \boldsymbol{n} = \frac{-\frac{\partial h}{\partial x} i_x + i_z}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{1/2}} \text{ and } \boldsymbol{t} = \frac{i_x + \frac{\partial h}{\partial x} i_z}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{1/2}} \tag{S 1.2}$$

At the flat and the wavy walls, no slip and no penetration are satisfied. At the fluid-fluid interface i.e., $z = h(x, t)$, $\mathbf{v}_1 \cdot \boldsymbol{n} - \mathcal{U} = 0$ must hold since it is a material surface and $\mathbf{v}_1 = \mathbf{v}_2$ also holds, while the interfacial force balances are given by

$$\mathbf{n} \cdot \mathbf{T}_1 \cdot \mathbf{t} - \mathbf{n} \cdot \mathbf{T}_2 \cdot \mathbf{t} = 0 \text{ and } \mathbf{n} \cdot \mathbf{T}_1 \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{T}_2 \cdot \mathbf{n} = -\gamma \nabla \cdot \mathbf{n} \tag{S 1.3}$$

To analyse the effect of the wavy wall, the governing equations are investigated in the long wave limit.

S1.2. *Long-wave model and boundary layer equations*

The long-wave model is obtained using separation of length scales, where the governing equations are made dimensionless by using the following scales denoted by the subscript 'c':

$$x_c = \lambda, \ z_c = H, \ u_c = U, \ w_c = \epsilon U, \ t_c = \frac{\lambda}{U}, \ p_{jc} = \rho_j U^2 \tag{S 1.4}$$

Here ϵ is designated as film or long-wave parameter and is defined as $\epsilon = H/\lambda$ and U is a characteristic velocity scale. Using the above scales the nondimensional model is obtained. The wavy wall is represented by the $f(x) = \mathcal{A} cos(2k_w x)$, where the amplitude, $\mathcal{A}$, of the wavy wall is scaled with the thickness, H, of the thin film. The boundary layer

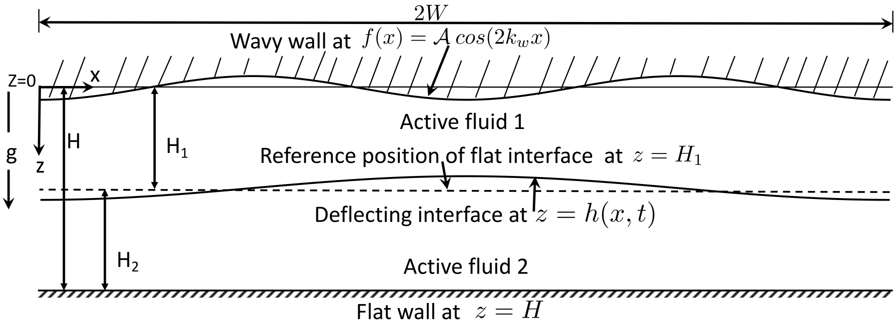

FIGURE S 1. Schematic of a heavy fluid overlying on a light fluid under gravity on a corrugated surface is shown here. Here $k_w = \frac{\pi}{W}$, where the horizontal width of the wall is $2W$.

assumption is invoked i.e., $\epsilon < 1$ and upon retaining terms of $\mathcal{O}(\epsilon)$, the nondimensional model is given by

$$\frac{\partial u_j}{\partial x} + \frac{\partial w_j}{\partial z} = 0 \tag{S 1.5}$$

$$\epsilon\left(\frac{\partial u_j}{\partial t} + u_j\frac{\partial u_j}{\partial x} + w_j\frac{\partial u_j}{\partial z}\right) = -\epsilon\,\frac{\partial p_j}{\partial x} + \frac{1}{Re_j}\frac{\partial^2 u_j}{\partial z^2}, \quad \text{where } Re_j = \frac{\rho_j U H}{\mu_j} \tag{S 1.6}$$

and

$$-\epsilon\,\frac{\partial p_j}{\partial z} + \frac{\epsilon\,G_j}{Re_j} = 0, \quad \text{where } G_j = \frac{\rho_j g H^2}{\mu_j U} \tag{S 1.7}$$

The above equations are obtained by considering that Re_j and G_j to be of at least $\mathcal{O}(1)$. At the liquid-liquid interface $z = h(x,t)$, continuity of velocities, kinematic, tangential and normal force balance conditions are imposed i.e.,

$$u_1 = u_2 \text{ and } w_1 = w_2 \tag{S 1.8}$$

$$\epsilon\,\frac{\partial h}{\partial t} = -\epsilon\,u_1\frac{\partial h}{\partial x} + \epsilon\,w_1 \tag{S 1.9}$$

$$\frac{\partial u_1}{\partial z} = \mu_{21}\frac{\partial u_2}{\partial z}, \quad \text{where } \mu_{21} = \frac{\mu_2}{\mu_1} \tag{S 1.10}$$

and

$$-\epsilon\,p_1 + \epsilon\,\rho_{21}p_2 = \frac{\epsilon^2}{Ca\,Re_1}\frac{\partial^2 h}{\partial x^2}, \quad \text{where } Ca = \frac{\mu_1 U}{\gamma} \text{ and } \rho_{21} = \frac{\rho_2}{\rho_1} \tag{S 1.11}$$

The equations S 1.5-S 1.11 are referred to as the $1+\epsilon$ model. In the above equation, Ca is taken to be at most $\mathcal{O}(\epsilon^3)$, this allows us to include the effect of surface curvature in the $1+\epsilon$ model.

Integrating the vertical components of the momentum balances corresponding to each fluid layer from the bulk of the fluids to the interface and substituting the resulting equations into the x-momentum equations gives

$$\epsilon\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + w_1\frac{\partial u_1}{\partial z}\right) = -\epsilon\,\frac{\partial p_1}{\partial x}\bigg|_h + \frac{1}{Re_1}\frac{\partial^2 u_1}{\partial z^2} + \frac{\epsilon\,G_1}{Re_1}\frac{\partial h}{\partial x} \tag{S 1.12}$$

and

$$\epsilon \left(\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} \right) = -\epsilon \left. \frac{\partial p_2}{\partial x} \right|_h + \frac{1}{Re_2} \frac{\partial^2 u_2}{\partial z^2} + \frac{\epsilon\, G_2}{Re_2} \frac{\partial h}{\partial x} \tag{S 1.13}$$

The normal force balance condition at $z = h(x,t)$, i.e., equation S 1.11 is differentiated in the horizontal direction and is used to express the pressure, p_1, at the free surface in the x-momentum equation in terms of pressure, p_2 and the surface curvature term. This gives

$$\epsilon \left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} \right) = -\epsilon \rho_{21} \left. \frac{\partial p_2}{\partial x} \right|_h + \frac{1}{Re_1} \frac{\partial^2 u_1}{\partial z^2} + \frac{\epsilon\, G_1}{Re_1} \frac{\partial h}{\partial x} + \frac{\epsilon^3}{CaRe_1} \frac{\partial^3 h}{\partial x^3} \tag{S 1.14}$$

The equations S 1.13 and S 1.14 are integrated with respect to weight functions F_1 and F_2, which are defined later. The resulting equations are then added to obtain

$$\begin{aligned} \int_f^h \epsilon \left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} \right) F_1 \, dz + \int_h^1 \epsilon \rho_{21} \left(\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} \right) F_2 \, dz = \\ -\epsilon \rho_{21} \left. \frac{\partial p_2}{\partial x} \right|_h \left(\int_f^h F_1 \, dz + \int_h^1 F_2 \, dz \right) + \int_f^h \frac{1}{Re_1} \left[\frac{\partial^2 u_1}{\partial z^2} + \epsilon G_1 \frac{\partial h}{\partial x} \right] F_1 \, dz \\ + \int_h^1 \frac{\rho_{21}}{Re_2} \left[\frac{\partial^2 u_2}{\partial z^2} + \epsilon G_2 \frac{\partial h}{\partial x} \right] F_2 \, dz + \int_f^h \left(\frac{\epsilon^3}{Ca\, Re_1} \frac{\partial^3 h}{\partial x^3} \right) F_1 \end{aligned} \tag{S 1.15}$$

To obtain the final evolution equations, we apply the Weighted Residual Integral Boundary Layer (WRIBL) method (Kalliadasis *et al.* 2011). To perform the integration of the boundary layer equations, we decompose the horizontal component of the velocity into an $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ part i.e.,

$$u_j(x,z,t) = \underbrace{\widehat{u}_j(x,z,t)}_{\mathcal{O}(1)} + \underbrace{\widetilde{u}_j(x,z,t)}_{\mathcal{O}(\epsilon)} \tag{S 1.16}$$

The leading order velocity $\widehat{u}_j(x,z,t)$ is chosen to be parabolic along the horizontal direction and is determined such that the following conditions are satisfied, i.e.,

$$\begin{aligned} \widehat{u}_1\big|_f = 0, \;\; \widehat{u}_2\big|_1 = 0, \;\; \widehat{u}_1\big|_{h(x,t)} = \widehat{u}_2\big|_{h(x,t)}, \;\; \text{and} \;\; \left. \frac{\partial \widehat{u}_1}{\partial z} \right|_{h(x,t)} = \mu_{21} \left. \frac{\partial \widehat{u}_2}{\partial z} \right|_{h(x,t)} \\ \frac{\partial^2 \widehat{u}_1}{\partial z^2} = K_1, \quad \frac{\partial^2 \widehat{u}_2}{\partial z^2} = K_2, \quad \int_f^{h(x,t)} \widehat{u}_1 \, dz = q_1 \quad \text{and} \quad \int_{h(x,t)}^1 \widehat{u}_2 \, dz = q_2 \end{aligned} \tag{S 1.17}$$

Here K_1 and K_2 are obtained in terms of the flow rates, q_1 and q_2 using the integral conditions. We use the Galerkin method, where the weight functions have the same functional from as the leading order velocities. The weight functions F_1 and F_2, are defined as follows

$$\begin{aligned} F_1\big|_f = 0, \;\; F_2\big|_1 = 0, \;\; F_1\big|_{h(x,t)} = F_2\big|_{h(x,t)}, \;\; \text{and} \;\; \left. \frac{\partial F_1}{\partial z} \right|_{h(x,t)} = \mu_{21} \left. \frac{\partial F_2}{\partial z} \right|_{h(x,t)} \\ \frac{\partial^2 F_1}{\partial z^2} = C_1, \quad \frac{\partial^2 F_2}{\partial z^2} = C_2 \end{aligned} \tag{S 1.18}$$

In order to eliminate pressure, p_2, from equation S 1.15, we now impose the following condition

$$\int_f^h F_1\,dz = -\int_h^1 F_2\,dz \quad \text{and} \quad C_1 = 1 \tag{S 1.19}$$

The equation S 1.18 along with equation S 1.19 are used in determining the complete solution of the weight functions F_1 and F_2. The transverse components of velocities are obtained from the continuity equation.

By integrating the continuity equation across respective fluid layers, we get

$$\frac{\partial h}{\partial t} + \frac{\partial q_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \delta}{\partial t} + \frac{\partial q_2}{\partial x} = 0 \tag{S 1.20}$$

where Leibniz's integration rule and the kinematic condition are used.

The final model consists of the following dimensionless evolution equations

$$\begin{aligned}\int_f^h \epsilon\left(\frac{\partial \widehat{u}_1}{\partial t} + \widehat{u}_1\frac{\partial \widehat{u}_1}{\partial x} + \widehat{w}_1\frac{\partial \widehat{u}_1}{\partial z}\right)F_1\,dz + \int_h^1 \epsilon\rho_{21}\left(\frac{\partial \widehat{u}_2}{\partial t} + \widehat{u}_2\frac{\partial \widehat{u}_2}{\partial x} + \widehat{w}_2\frac{\partial \widehat{u}_2}{\partial z}\right)F_2\,dz = \\ \int_f^h \frac{1}{Re_1}\left[\frac{\partial^2 \widehat{u}_1}{\partial z^2} + \epsilon G_1\frac{\partial h}{\partial x}\right]F_1\,dz + \int_h^1 \frac{\rho_{21}}{Re_2}\left[\frac{\partial^2 \widehat{u}_2}{\partial z^2} + \epsilon G_2\frac{\partial h}{\partial x}\right]F_2\,dz \\ + \int_f^h \left(\frac{\epsilon^3}{Ca\,Re_1}\frac{\partial^3 h}{\partial x^3}\right)F_1\,dz\end{aligned} \tag{S 1.21}$$

$$\frac{\partial h}{\partial t} + \frac{\partial q_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \delta}{\partial t} + \frac{\partial q_2}{\partial x} = 0 \tag{S 1.22}$$

The expressions for the leading order velocities and weight functions i.e., $\widehat{u}_1$, $\widehat{u}_2$, F_1 and F_2 are provided in the next section. These expressions are substituted in the above equations to obtain the final evolution equations in terms of h, q_1 and q_2 alone.

S2. Calculation of leading order velocities, i.e., $\widehat{u}_1$ and $\widehat{u}_2$ and weight functions, i.e., F_1 and F_2

The leading order velocities are obtained from the equations S 1.17 i.e.,

$$\begin{aligned}\widehat{u}_1 = 3(z-f)(q_1(\delta-1)(f(\delta+4\mu_{21}(z-\delta)-1)+\delta \\ (-2\delta+4\mu_{21}(\delta-z)+z+2)-z)-\mu_{21}q_2(f \\ -\delta)^2(-2\delta-f+3z))/2(\delta-1)(f-\delta)^3(\mu_{21}(f-\delta)+\delta-1)\end{aligned} \tag{S 2.1}$$

$$\begin{aligned}\widehat{u}_2 = 3(z-1)(q_2(f-\delta)(\mu_{21}f(-2\delta+z+1)+2(\mu_{21}-2) \\ \delta^2-(\mu_{21}-4)(z+1)\delta-4z)-q_1(\delta-1)^2 \\ (-2\delta+3z-1))/2(\delta-1)^3(f-\delta)(\mu_{21}(f-\delta)+\delta-1)\end{aligned} \tag{S 2.2}$$

The weight functions are obtained by solving the governing equations S 1.18-S 1.19, i.e.,

$$\begin{aligned}F_1 = \frac{f(f(\delta(-\mu_{21}(\delta-4)+\delta-2)+1)+2\delta((\mu_{21}-1)(\delta-2)\delta-1)-\mu_{21}f^3)}{2\delta^2(2\mu_{21}(f-2)\delta+(\mu_{21}-1)\delta^2+2\delta+\mu_{21}f(4-3f)-1)} \\ +\frac{z(-\mu_{21}f^2(\delta+2)-(\mu_{21}-1)(\delta-2)\delta^2+\delta+2\mu_{21}f^3)}{\delta^2(2\mu_{21}(f-2)\delta+(\mu_{21}-1)\delta^2+2\delta+\mu_{21}f(4-3f)-1)}+\frac{z^2}{2\delta^2}\end{aligned} \tag{S 2.3}$$

$$F_2 = \frac{(1-z)^2(f-\delta)^2(-2f((\mu_{21}-2)\delta+2)+\delta((\mu_{21}-1)\delta-2)+\mu_{21}f^2+3)}{2(\delta-1)^2\delta^2(2\mu_{21}(f-2)\delta+(\mu_{21}-1)\delta^2+2\delta+\mu_{21}f(4-3f)-1)} + \frac{(1-z)(f-\delta)^2(-2f((\mu_{21}-1)\delta+1)+(\mu_{21}-1)\delta^2+\mu_{21}f^2+1)}{(\delta-1)\delta^2(2\mu_{21}(f-2)\delta+(\mu_{21}-1)\delta^2+2\delta+\mu_{21}f(4-3f)-1)} \tag{S 2.4}$$

REFERENCES

Kalliadasis, S., Ruyer-Quil, C., Scheid, B. & Velarde, M. G. 2011 *Falling liquid films*. Springer Science & Business Media.