# Algorithm and analytical formulae

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## Introduction

In this text, we concern ourselves with two issues. The first one is the algorithm for solving the system of nonlinear equations with respect to the parameters $a$, $b$, $c$, $d$ and $k$. Here we should remark that on the one hand, developing such an algorithm is a routine problem. Below we demonstrate that the system can be easily reduce to a system of five polynomial equations with five unknowns, to which one can apply, for example, the routine FindRoot (see Wolfram 2003) by choosing the zeroth approximations by a trial-and-error method. In such a manner, one can calculate several examples, but for systematical calculations this is an unpromising way. On the other hand, if one wants to solve the system almost instantly for any set of the parameters $\alpha$, $\beta$ and $\sigma$, the problem becomes challenging. One of the possible solutions to this problem is presented below. The second issue is the deduction of analytical formulae for hydrodynamic properties. The formulae are based on the analytical integral-free representation of the conformal mapping of the parametric domain onto the physical flow region and the generalization of the Blasius-Chaplygin formulae for the case of the re-entrant jet cavity model. For the flow over an oblique flat plate, the formulae include expressions for the lift, drag and moment coefficients, the length and width of the cavity. Such a deduction is again a challenging problem because something similar has never been deduced before for any cavity model. The solutions of both these problems allow very fast computations to be carried out.

## 1. Algorithm for finding the accessory parameters

Let us write down the system of equations needed to be solved:

$$\frac{[a+\mathrm{i}(b-k)][a-c+\mathrm{i}(b-d)][a+c+\mathrm{i}(b-d)]}{[a+\mathrm{i}(b+k)][a-c+\mathrm{i}(b+d)][a+c+\mathrm{i}(b+d)]} - R\,\mathrm{e}^{-\mathrm{i}\alpha} = 0, \tag{1.1}$$

$$\mathrm{i}\frac{a+\mathrm{i}b}{ab} - \frac{2(a+\mathrm{i}b)}{(a+\mathrm{i}b)^2-1} + \frac{4[a+\mathrm{i}(b+d)]}{[a+\mathrm{i}(b+d)]^2-c^2} + \frac{2}{a+\mathrm{i}(b+k)} = 0, \tag{1.2}$$

$$\frac{(1-\mathrm{i}k)(1-c-\mathrm{i}d)(1+c-\mathrm{i}d)}{(1+\mathrm{i}k)(1-c+\mathrm{i}d)(1+c+\mathrm{i}d)} - \mathrm{e}^{-\mathrm{i}(\alpha+\beta)} = 0, \tag{1.3}$$

were $\alpha$ is the angle of attack, $\beta$ is the inclination angle of the re-entrant jet, $R = 1/\sqrt{1+\sigma}$ and $\sigma$ is the cavity number. The unknowns in the system (1.1)–(1.3) are $a$, $b$, $c$, $d$ and $k$.

First, we reduce the system (1.1)–(1.3) to the system of polynomial equations. To do so, instead of $\alpha$ and $\beta$, we introduce the parameters

$$T = \tan\frac{\alpha}{2}, \quad P = \cot\frac{\beta}{2}. \tag{1.4$a$–$b$}$$

Now we insert in the right-hand sides of (1.1) and (1.3) the expressions

$$\mathrm{e}^{-\mathrm{i}\alpha} = \frac{1-\mathrm{i}T}{1+\mathrm{i}T}, \quad e^{-\mathrm{i}(\alpha+\beta)} = \frac{1-\mathrm{i}T}{1+\mathrm{i}T}\,\frac{P-\mathrm{i}}{P+\mathrm{i}}, \tag{1.5$a$–$b$}$$

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reduce each equation in the system (1.1)–(1.3) to a common denominator, calculate real and imaginary parts of the obtained three numerators, and equate these parts of numerators to zero. Carrying out these operations by means of the Mathematica package (see Wolfram 2003), we get

$$\begin{aligned} &b(R-1)\left[-3a^2-2aT(2d+k)+b^2+c^2+d^2+2dk\right] \\ &+(R+1)\left[a^3T-a^2(2d+k)-aT\left(3b^2+c^2+d^2+2dk\right)\right. \\ &\left.+b^2(2d+k)+k\left(c^2+d^2\right)\right]=0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} &(R-1)\left\{a^3+a^2T(2d+k)-a\left(3b^2+c^2+d^2+2dk\right)-T\left[b^2(2d+k)+k\left(c^2+d^2\right)\right]\right\} \\ &-b(R+1)\left[-3a^2T+2a(2d+k)+T\left(b^2+c^2+d^2+2dk\right)\right]=0, \end{aligned} \tag{1.7}$$

$$\begin{aligned} &a^4(2b+2d+k)+a^2\left\{4b^3-4b^2(2d+k)+b\left[-4\left(c^2+d^2\right)-8dk+2\right]\right. \\ &\left.-k\left(c^2+d^2+1\right)-2d\right\}+2b^5+3b^4(2d+k)+b^3\left[4\left(c^2+d^2\right)+8dk-2\right] \\ &-b^2\left[-5k\left(c^2+d^2\right)+2d+k\right]+k\left(c^2+d^2\right)=0, \end{aligned} \tag{1.8}$$

$$\begin{aligned} &a^6-a^4\left[-b^2+3b(2d+k)+c^2+d^2+2dk+1\right]-b\left(b^2+1\right)(b+k)\left[b(b+2d)+c^2+d^2\right] \\ &+a^2\left[-b^4+4b^3(2d+k)+\left(6b^2+5bk+1\right)\left(c^2+d^2\right)+6b^2(2dk-1)-b(2d+k)+2dk\right]=0, \end{aligned} \tag{1.9}$$

$$\left(c^2+d^2-1\right)(kU-V)-2d(kV+U)=0, \quad \text{where} \quad U=P-T, \quad V=1+PT. \tag{1.10$a$–$c$}$$

In the system (1.6)–(1.10), equations (1.6) and (1.7) are equivalent to (1.1), equations (1.8) and (1.9) are equivalent to (1.2) and equation (1.10) is equivalent to (1.3). So we have obtained the system of five polynomial equations with five unknowns.

Now, we exclude the parameters $c$ and $d$ from the system (1.6)–(1.10). With the help of (1.10), we express $c$ in terms of $k$ and $d$:

$$c=\sqrt{-d^2+\frac{2d(kV+U)}{kU-V}+1}, \quad \text{and, therefore,} \quad c^2+d^2=\frac{2d(kV+U)}{kU-V}+1. \tag{1.11$a$–$b$}$$

Equations (1.6)–(1.9) are quadratic with respect to the unknowns $c$ and $d$ and do not contain $c$ being raised to the first power. Moreover, if somewhere in the equations $c^2$ is presented, then $d^2$ is also presented forming the sum $c^2+d^2$. This means that inserting equation (1.11$b$) into (1.6)–(1.9) gives rise to four equations which will not contain $c$ and will be linear with respect to $d$. Solving these linear equations, we obtain

$$d=\frac{1}{2}(V-kU)\frac{E_{11}}{E_{21}}, \quad d=\frac{1}{2}(V-kU)\frac{E_{12}}{E_{22}}, \quad d=\frac{1}{2}(V-kU)\frac{E_{13}}{E_{23}}, \quad d=\frac{1}{2}(V-kU)\frac{E_{14}}{E_{24}}, \tag{1.12$a$–$d$}$$

where

$$E_{11}=(R+1)\left[k\left(-a^2+b^2+1\right)+aT\left(a^2-3b^2-1\right)\right]+b(R-1)\left(-3a^2-2akT+b^2+1\right), \tag{1.13}$$

$$\begin{aligned} E_{21}=U&\left\{(R+1)\left[k\left(-a^2+b^2+1\right)-ak^2T-aT\right]+b(R-1)\left(-2akT+k^2+1\right)\right\} \\ &+V\left[(R+1)\left(a^2-b^2+k^2\right)+2ab(R-1)T\right], \end{aligned} \tag{1.14}$$

$$E_{12}=b(R+1)\left(-3a^2T+2ak+b^2T+T\right)-(R-1)\left[a^3+a^2kT-a\left(3b^2+1\right)-\left(b^2+1\right)kT\right], \tag{1.15}$$

$$E_{22} = U\left\{b(R+1)\left(2ak+k^2T+T\right)-(R-1)\left[a^2kT-a\left(k^2+1\right)-\left(b^2+1\right)kT\right]\right\} \\ +V\left[(R-1)T\left(a^2-b^2+k^2\right)-2ab(R+1)\right], \tag{1.16}$$

$$E_{13} = k\left(a^2-3b^2-1\right)\left(a^2-b^2-1\right)+2b\left[a^4+a^2\left(2b^2-1\right)+b^4+b^2\right], \tag{1.17}$$

$$E_{23} = U\left\{a^4k-2a^2[2(b^2k+bk^2+b)+k]+b^2\left[b\left(3bk+4k^2+4\right)+4k\right]+k\right\} \\ +V\left[-a^4+a^2\left(4b^2-k^2+1\right)-3b^4+b^2\left(5k^2+1\right)+k^2\right], \tag{1.18}$$

$$E_{14} = -\left(a^2-b^2-1\right)\left[a^4+a^2\left(2b^2-3bk-1\right)+b\left(b^2+1\right)(b+k)\right], \tag{1.19}$$

$$E_{24} = U\left\{a^4\left(3bk+k^2+1\right)-a^2\left[4b^3k+6b^2\left(k^2+1\right)+4bk+k^2+1\right] \\ +b\left(b^2+1\right)(b+k)(bk+1)\right\}+Vb\left[-3a^4+a^2\left(4b^2-5k^2-1\right)+\left(b^2+1\right)\left(k^2-b^2\right)\right]. \tag{1.20}$$

With the help of the first equation in the system (1.12), we eliminate $d$ from the three remaining equations to obtain

$$E_{11}E_{22}-E_{12}E_{12}=0, \quad E_{11}E_{23}-E_{12}E_{13}=0, \quad E_{11}E_{24}-E_{12}E_{14}=0. \tag{1.21a–c}$$

The system (1.21) is a rather cumbersome system of three polynomial equations with respect to the unknowns $a$, $b$ and $k$ that must satisfy the complementary conditions

$$a>0, \quad b>0, \quad k>0. \tag{1.22a–c}$$

The most surprising fact is that it is possible to exclude $k$ from this system. Indeed, note that each equation of the system is cubic with respect to $k$. Therefore, the system can be rewritten as follows

$$\Lambda_{11}k^3+\Lambda_{12}k^2+\Lambda_{13}k+\Lambda_{14}=0, \tag{1.23}$$
$$\Lambda_{21}k^3+\Lambda_{22}k^2+\Lambda_{23}k+\Lambda_{24}=0, \tag{1.24}$$
$$\Lambda_{31}k^3+\Lambda_{32}k^2+\Lambda_{33}k+\Lambda_{34}=0. \tag{1.25}$$

Here, the coefficients $\Lambda_{ij}$, $(i=\overline{1,3},\ j=\overline{1,4})$ depend on the unknowns $a$ and $b$ and the known parameters $R, T, U$ and $V$. The values of these twelve coefficient are presented in Appendix to this text. With the help of equation (1.23), we eliminate $k^3$ from equations (1.24) and (1.25). Namely, we insert

$$k^3=-\frac{1}{\Lambda_{11}}(\Lambda_{12}k^2+\Lambda_{13}k+\Lambda_{14})$$

into the first terms of equations (1.24) and (1.25). The result of these simple algebraic operations turns out to be rather unexpected, namely, equations (1.24) and (1.25) are equivalent to the following ones:

$$4bM_1(a,b)E_{21}(a,b,k)=0, \quad 4abM_2(a,b)E_{21}(a,b,k)=0, \tag{1.26a–b}$$

where

$$M_1(a,b)=\Omega_{11}U+\Omega_{12}V, \quad M_2(a,b)=\Omega_{21}U+\Omega_{22}V, \tag{1.27a–b}$$

and $\Omega_{ij}$ are polynomials of the unknowns $a$ and $b$ with coefficients that depend on the known parameters $R$ and $T$. The parameters $U$ and $V$ are also known. So the notations $M_1(a,b)$ and $M_1(a,b)$ emphasize that $M_1$ and $M_2$ are polynomials of unknowns $a$ and $b$ with known coefficients. The analytical formulae for the polynomials $\Omega_{ij}$ $(i,j=1,2)$ are presented in Appendix to this text.

Since $E_{21}(a,b,k)$ is the denominator of the first equation in (1.12), we must assume that

$E_{21}(a,b,k) \neq 0$. Then by virtue of (1.22), we infer that equations (1.26) are equivalent to the pair of polynomial equations with only two unknowns, $a$ and $b$:

$$M_1(a,b) = 0, \quad M_2(a,b) = 0. \tag{1.28$a$–$b$}$$

Taking into account the awkwardness of the coefficients $\Lambda_{ij}$ (see Appendix), the passage from equations (1.23)–(1.25) to the system (1.28) can be considered as a small wonder.

One can easily see that if $a$ and $b$ satisfy (1.28), then all equations in the system (1.23)–(1.25) will be proportional to each other. So, we have reduced the system of five equations (1.6)–(1.10) with five unknown accessory parameters $a$, $b$, $c$, $d$ and $k$, to the system (1.28) with only two unknowns $a$ and $b$. After finding $a$ and $b$ from (1.28), we determine $k$ from the cubic equation (1.23) and the parameters $d$ and $c$ by means of formulae (1.12$a$) and (1.11$a$).

The system (1.28) is a system of nonlinear polynomial equations with respect to $a$ and $b$. To solve such systems, there are two routines in the Mathematica package (see Wolfram 2003): NSolve and FindRoot. Comparing these two routines, we should remark that NSolve is especially intended for solving polynomial systems and is able to find all roots. The latter is the reason why NSolve is rather time-consuming. FindRoot uses the Newton method and works thousand times faster than NSolve, but for its application one needs to know a good zeroth approximations for unknowns. So, we shall use FindRoot to solve the system (1.28). As to the zeroth approximations, we construct three types of them.

**Type 1.** The parameter $\varepsilon = 1 - R$, i.e. the cavity number $\sigma$, is small, whereas the parameters $T$ and $P$, i.e. the angle of attack $\alpha$ and the re-entrant jet inclination angle $\beta$, are finite:

$$\begin{aligned} a &= 1 + \varepsilon^2 \frac{T^2+1}{32T^2}\left[T^2 - 4PT - 3 - \frac{P^2\left(T^2-3\right) + 24PT - 7T^2 + 13}{4T^2}\varepsilon\right], \\ b &= \varepsilon\left[\frac{1}{4}\left(T + \frac{1}{T}\right) - \frac{\left(P^2-3\right)\left(T^2+1\right)}{32T}\varepsilon\right]. \end{aligned} \tag{1.29}$$

**Type 2.** The parameter $T$ is small, whereas the parameters $R$ and $P$ are finite:

$$\begin{aligned} a &= \frac{\sqrt[4]{R}\sqrt{R+1}\sqrt{T}}{\sqrt{1-R}\sqrt{(R-1)P^2+R+1}} + \frac{(1-3R)PT}{2\left[(R-1)P^2+R+1\right]}, \\ b &= \frac{R^{3/4}\sqrt{R+1}\sqrt{T}}{\sqrt{1-R}\sqrt{(R-1)P^2+R+1}} - \frac{\sqrt{R}(5R+1)PT}{2\left[(R-1)P^2+R+1\right]}. \end{aligned} \tag{1.30}$$

**Type 3.** The parameters $T$ and $\varepsilon$ are both small, but the ratio $T/\varepsilon = A$ as well as the parameter $P$ are finite:

$$a = A_1 + \varepsilon\frac{A_2}{A_3}, \quad b = B_1 + \varepsilon\frac{B_2}{B_3}, \tag{1.31$a$–$b$}$$

where

$$A_1 = \sqrt{\frac{A\left(\sqrt{4A^2+1}+2A\right)}{4A^2+1}}, \quad B_1 = \sqrt{\frac{A\left(\sqrt{4A^2+1}-2A\right)}{4A^2+1}}, \tag{1.32$a$–$b$}$$

$$\begin{aligned} A_2 = & -32A^5\left(4A_1^5 + 5A_1^4 - 4A_1^3 - 4A_1^2 - 1\right)A_1P \\ & +4A^4\left(A_1^2-1\right)\left[\left(8A_1^4 - 4A_1^2 - 1\right)P^2 + 20A_1^4 + 2A_1^2 + 1\right] \\ & +8A^3\left(-8A_1^5 - 10A_1^4 + 4A_1^3 + 4A_1^2 + A_1 + 1\right)A_1P \\ & +A^2A_1^2\left[\left(16A_1^4 - 12A_1^2 - 1\right)P^2 + 32A_1^4 - 14A_1^2 - 4\right] \\ & +A_1^6\left(2P^2+3\right) - 2A\left(4A_1+5\right)A_1^5P, \end{aligned} \tag{1.33}$$

$$A_3 = 4\left[8A^4\left(4A_1^2-3\right)A_1^3 + A^2\left(16A_1^4-6A_1^2-1\right)A_1 + 2A_1^5\right], \tag{1.34}$$

$$\begin{aligned} B_2 \;&= 18A^2\left(4A^2+1\right)B_1^3P - 6A^2B_1P + A^2\left(8AP-P^2-7\right) \\ &-2\left(4A^2+1\right)B_1^6\left[16A^3P-4A^2\left(P^2+3\right)+4AP-P^2-2\right] \\ &-A^2B_1^2\left(128A^3P-4A^2\left(3P^2+31\right)+24AP+P^2-23\right)+2B_1^4 \\ &\times\left[-128A^5P+A^4\left(40P^2+92\right)-48A^3P+A^2\left(14P^2+29\right)-4AP+P^2+2\right], \end{aligned} \tag{1.35}$$

$$B_3 = 4B_1\left[8A^4\left(4B_1^4+5B_1^2+1\right)+A^2\left(16B_1^4+14B_1^2+1\right)+2B_1^4+B_1^2\right]. \tag{1.36}$$

The zeroth approximations (1.29)–(1.31) have been found by the following method. With the help of the routine Resultant of the Mathematica package (see Wolfram 2003), we compute the resultants (see Kurosh 1980) of polynomials $M_1(a,b)$ and $M_2(a,b)$ with respect to the variables $b$ (at a fixed $a$) and $a$ (at a fixed $b$). This operation leads to two separated polynomials, one of the variable $a$ and another of the variable $b$, both of degree 41. We factorize these polynomials and remove the factors that do not satisfy the following conditions:

- for the polynomial of $a$, the factor vanishes at $R = 1$ and $a = 1$;
- for the polynomial of $b$, the factor vanishes at $R = 1$ and $b = 0$.

These conditions follow from the following reasoning. The re-entrant jet cavity flow must tend to the Helmholtz-Kirchhoff flow with infinite cavity when the cavity number $\sigma \to 0$ ($R \to 1$). Since for the Helmholtz-Kirchhoff flow, the infinity $I$ of the re-entrant jet and infinity $D$ of the main stream coincide, we must have

$$R \to 1 \quad\Longrightarrow\quad a \to 1 \;\text{ and }\; b \to 0. \tag{1.37}$$

The procedure of removing factors from the initial resultants allows us to exclude from consideration a significant number of spurious roots and leads to two polynomials $P_a(a)$ and $P_b(b)$, both of twelfth degree, with coefficients depending on $R$, $T$, and $P$. So, we have succeeded in reducing the system of five equations (1.6)–(1.10) to two polynomial equations

$$P_a(a) = 0, \quad P_b(b) = 0, \tag{1.38a–b}$$

and each of the equations has only one unknown. The polynomials $F_a(a)$ and $F_b(b)$ are stored in the Supplementary Matherials in the form of expressions of the Mathematica package (see Wolfram 2003).

Further, we apply a standard perturbation technique (see Murdock 1991) to the equations $P_a(a) = 0$ and $P_b(b) = 0$ to construct the two-term asymptotic expansions (1.29)–(1.31). Here, the routine Series of the Mathematica package turns out to be very helpful.

Computations have shown that for any angle of attack $0 < \alpha \le \pi/2$, any re-entrant jet inclination angle $\pi/2 \le \beta \le 3\pi/2$ and any cavity number $0 < \sigma \le 5$, at least one of the zeroth approximations (1.29)–(1.31) leads to the convergence of the routine FindRoot. The question is what is the priority of using these zeroth approximations? Here, the following approach has been applied. If $\alpha \ge \pi/4$, we always use the zeroth approximation of type 1. In the opposite case, we organize a competition between the zeroth approximations of types 1–3. Namely, at the values of $a$ and $b$, calculated by formulae (1.29)–(1.31), we compute the values of the norms

$$\sqrt{M_1^2(a,b) + M_2^2(a,b)}, \tag{1.39}$$

locate these norms in the increasing order and sequentially apply FindRoot with the zero approximations corresponding to this order.

Applying this algorithm, we have used the floating point arithmetics with 56 decimal places, so the obtained values of $a$, $b$, $c$, $d$ and $k$ are very accurate. At these values, the maximum order of the right-hand sides of the initial equations (9), (12) and (13) is $10^{-55}$ . The algorithm is also fast: for one set of the parameters $\alpha$, $\beta$ and $\sigma$, the average time of computing the unknown parameters is approximately 0.01 second on the notebook HP G1 EliteBook Folio 1040. The algorithm is realized in the routine fnum[$\alpha, \beta, \sigma$] of the package RerntrantJet.m.

## 2. Analytical formulae for hydrodynamic properties

### 2.1. *Formulae from the main text of the paper*

Below we present the formulae from the main text of the paper to make easier the reading of the further reasonings. Equations for the lift and drag forces $L$ and $D$ have been deduced by Gilbarg & Serrin (1950):

$$L = -Q\rho v_0 \sin\beta - \rho v_\infty \Gamma, \quad D = -Q\rho v_0 \cos\beta + \rho v_\infty Q. \tag{2.1$a,b$}$$

Hear $\rho$ is the density of the fluid, $v_\infty$ is the incident velocity, $v_0$ is the constant fluid velocity along the cavity boundaries, $\Gamma$ is the circulation around the body-cavity system, $Q$ is the flow flux in the re-entrant jet and $\beta$ is the angle of inclination of the re-entrant jet with respect the incident flow direction.

The formulae connecting the variables $w = \varphi + \mathrm{i}\psi$, $z = x + \mathrm{i}y$ and $u = \xi + \mathrm{i}\eta$ are as follows

$$\frac{\mathrm{d}w}{\mathrm{d}u} = l_0 v_0 f(u), \quad f(u) = \frac{u(u^2+k^2)(u^2-u_0^2)(u^2-\overline{u_0}^2)}{(1-u^2)(u^2-u_\infty^2)^2(u^2-\overline{u_\infty}^2)^2}, \tag{2.2$a,b$}$$

$$\frac{\mathrm{d}w}{v_0 \mathrm{d}z} = \mathrm{e}^{\mathrm{i}\alpha} F(u), \quad F(u) = \frac{(u-\mathrm{i}k)(u-u_0)(u+\overline{u_0})}{(u+\mathrm{i}k)(u-\overline{u_0})(u+u_0)}, \tag{2.3$a,b$}$$

$$\frac{\mathrm{d}z}{\mathrm{d}u} = l_0 \mathrm{e}^{-\mathrm{i}\alpha} G(u), \quad G(u) = \frac{f(u)}{F(u)} = \frac{u(u+\mathrm{i}k)^2(u+u_0)^2(u-\overline{u_0})^2}{(1-u^2)(u^2-u_\infty^2)^2(u^2-\overline{u_\infty}^2)^2}. \tag{2.4$a,b$}$$

where

$$u_\infty = a + \mathrm{i}b, \quad u_0 = c + \mathrm{i}d, \tag{2.5$a,b$}$$

the overbars mean the complex conjugate values, and $l_0$ is an unknown positive constant, which is related the the length $l$ of the plate by the relationships

$$l = l_0 J(a, b, c, d, k), \tag{2.6}$$

where

$$J = \int_0^\infty G_1(\eta)\mathrm{d}\eta, \quad G_1(\eta) = \frac{\eta(\eta+k)^2[(\eta+d)^2+c^2]^2}{(\eta^2+1)[\eta^4+2(a^2-b^2)\eta^2+(a^2+b^2)^2]^2}. \tag{2.7$a,b$}$$

### 2.2. *Deduction of analytical formulae*

Assume that the system of equations (1.1), (1.2) and (1.3) has been solved and the accessory parameters $a$, $b$, $c$, $d$ and $k$ have been found.

It is clear that

$$\oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{d}z = \Gamma - \mathrm{i}Q, \tag{2.8}$$

where $\Gamma$ is the circulation, $Q$ is the flux in the re-entrant jet, $C$ is a closed contour surrounding the plate-cavity system, and the integration along $C$ is in the anticlockwise direction. In the parametric $u$-plane, the contour $C$ transforms to a contour surrounding the point $u_\infty$, but the integration will be in the clockwise direction, therefore,

$$Q + \mathrm{i}\Gamma = 2\pi \operatorname*{res}_{u=u_\infty} \frac{\mathrm{d}w}{\mathrm{d}u} = 2\pi l_0 v_0 \operatorname*{res}_{u=u_\infty} f(u). \tag{2.9}$$

Calculating the residue, we obtain

$$Q = l_0 v_0 q, \quad q = \frac{\pi\left(k^2+1\right)\left[c^4 + 2c^2\left(d^2-1\right) + \left(d^2+1\right)^2\right]}{2\left[a^4 + 2a^2\left(b^2-1\right) + \left(b^2+1\right)^2\right]^2}. \tag{2.10$a$–$b$}$$

The flux in the re-entrant jet $Q = v_0\delta$, where $\delta$ is the width of the re-entrant jet. Thus, the dimensionless width of the re-entrant jet is $\delta/l = q/J$.

Computing the imaginary part of the residue in (2.9) leads to an expression for the circulation $\Gamma$, but the formula turns out to be very cumbersome. Another way of determining $\Gamma$ is to express the circulation in terms of $q$. Indeed, for the flat plate the lift-to-drag ratio $L/D = \cot\alpha$. Taking into account (2.1), we get

$$\Gamma = l_0 v_0 \gamma, \quad \gamma = \frac{q}{R}\left[(\cos\beta - R)\cot\alpha - \sin\beta\right], \tag{2.11$a$–$b$}$$

where

$$R = \frac{v_\infty}{v_0} = \frac{1}{\sqrt{1+\sigma}}. \tag{2.12}$$

Let $N$ be the normal force acting on the plate and $M$ be the moment about the trailing edge $B$. The positive direction of the moment $M$ is anticlockwise. We introduce the following hydrodynamic coefficients:

$$C_N = \frac{2N}{\rho v_\infty^2 l}, \quad C_L = \frac{2L}{\rho v_\infty^2 l}, \quad C_D = \frac{2D}{\rho v_\infty^2 l}, \quad C_M = \frac{2M}{\rho v_\infty^2 l^2}. \tag{2.13$a$–$d$}$$

It is evident that the lift and drag coefficients $C_L$ and $C_D$ can be expressed in terms of $C_N$:

$$C_L = C_N\cos\alpha, \quad C_D = C_N \sin\alpha, \tag{2.14$a$,$b$}$$

and with allowance for (2.1) we infer that

$$C_N = 2\csc\alpha(1+\sigma)(R-\cos\beta)\,\delta/l, \quad \delta/l = q/J. \tag{2.15$a$–$b$}$$

Now we find explicit analytical expressions for the conformal mapping $z(u)$ and the parameter $J$. It follows from (2.4) that

$$z(u) = l_0 e^{-\mathrm{i}\alpha}\zeta(u), \quad \zeta(u) = \int_\infty^u G(u_1)\mathrm{d}u_1, \tag{2.16$a$–$b$}$$

where we have taken into account that the origin of the Cartesian coordinate system is located at the point $B$. The integrand in (2.16) is a rational function, hence the integral can be calculated analytically. Nevertheless, the direct application of the routine Integrate of the Mathematica package (see Wolfram 2003) leads to an expression which is too long to be printed. Yet, some simplifications are possible. Let us introduce the coefficient

$$\mu = \lim_{u\to u_\infty} f(u)(u-u_\infty)^2. \tag{2.17}$$

Calculating the limit, we obtain

$$\mu = \frac{\left[k^2 + (a+\mathrm{i}b)^2\right]\left\{d^4 + 2d^2\left[(a+\mathrm{i}b)^2 + c^2\right] + \left[(a+\mathrm{i}b)^2 - c^2\right]^2\right\}}{64a^2b^2\left[(a+\mathrm{i}b)^2 - 1\right](a+\mathrm{i}b)}. \tag{2.18}$$

It can be easily demonstrated that the functions $f(u)$ and $F(u)$ possess the following properties:

$$f(u)=\overline{f(\bar{u})},\quad f(u)=-\overline{f(-\bar{u})},\quad F(u)=\overline{F(-\bar{u})},\quad F(u)=\frac{1}{\overline{F(\bar{u})}}. \tag{2.19a–d}$$

Besides, we take into account that the system of equations (1.1)–(1.3) has been deduced from the following relations:

$$\mathrm{e}^{\mathrm{i}\alpha}F(u_\infty)=R,\quad \operatorname*{res}_{u=u_\infty}G(u)=0,\quad \mathrm{e}^{\mathrm{i}\alpha}F(1)=\mathrm{e}^{-\mathrm{i}\beta} \tag{2.20a–c}$$

The properties (2.19) and the relations (2.20) allow us to determine the finite principle parts of Laurent series of the function $G(u)=f(u)/F(u)$ in the vicinity of the points $u=\pm u_\infty$, $u=\pm\overline{u_\infty}$ and $u=\pm 1$. With the use of these expansions, we decompose $G(u)$ into a sum of partial fractions which can be easily integrated. The final result looks as follows:

$$\zeta(u)=\zeta_1(u)+\zeta_2(u)+\zeta_3(u), \tag{2.21}$$

where

$$\zeta_1(u)=-\frac{\mathrm{e}^{\mathrm{i}\alpha}}{R}\frac{\mu}{u-u_\infty},\quad \zeta_2(u)=-\mathrm{e}^{\mathrm{i}(\alpha+\beta)}\frac{q}{\pi}\log\frac{u-1}{u+1},\quad \zeta_3(u)=\frac{\mathrm{e}^{-\mathrm{i}\alpha}}{R}\frac{\bar{\mu}}{u+\overline{u_\infty}}$$
$$-R\,\mathrm{e}^{\mathrm{i}\alpha}\frac{\bar{\mu}}{u-\overline{u_\infty}}+R\,\mathrm{e}^{-\mathrm{i}\alpha}\frac{\mu}{u+u_\infty}+R\,\mathrm{e}^{\mathrm{i}\alpha}\frac{q-\mathrm{i}\gamma}{\pi}\log\frac{u-\overline{u_\infty}}{u+1}+R\,\mathrm{e}^{-\mathrm{i}\alpha}\frac{q+\mathrm{i}\gamma}{\pi}\log\frac{u+u_\infty}{u+1}, \tag{2.22}$$

and the standard branch of the logarithm with the cut along the negative part of the $\xi$-axis is chosen, i.e.

$$\log u=\log|u|+\mathrm{i}\arg u,\quad -\pi<\arg u\le\pi. \tag{2.23a,b}$$

In the $z$-plane, the complex coordinate of the point $A$ has an argument of $\pi-\alpha$. This means that the length of the plate $l=-l_0\zeta(0)$, and, as follows from (2.6), $J=-\zeta(0)$. An explicit expression for $J$ is as follows

$$\begin{aligned} J=&\frac{2\left(R^2-1\right)\sin\alpha(b\mu_1-a\mu_2)-2\left(R^2+1\right)\cos\alpha(a\mu_1+b\mu_2)}{\sigma\left(a^2+b^2\right)}\\ &+\frac{R}{\pi}\left[2\tan^{-1}\left(\frac{b}{a}\right)(\gamma\cos\alpha-q\sin\alpha)-\log\left(a^2+b^2\right)(\gamma\sin\alpha+q\cos\alpha)\right]\\ &-q\sin(\alpha+\beta)-\gamma R\cos\alpha+qR\sin\alpha, \end{aligned} \tag{2.24}$$

where $\mu_1=\operatorname{Re}\mu$ and $\mu_2=\operatorname{Im}\mu$.

The normal force $N$ and the moment $M$ acting on the plate are expressed by the integrals

$$N=\frac{\rho v_\infty^2}{2}\int_0^l C_p(s)\mathrm{d}s,\quad M=-\frac{\rho v_\infty^2}{2}\int_0^l C_p(s)s\mathrm{d}s, \tag{2.25a,b}$$

where $C_p$ is the coefficient of the pressure $p$ calculated with respect to the pressure $p_0$ inside the cavity:

$$C_p=\frac{p-p_0}{\rho v_\infty^2/2}=(1+\sigma)\left(1-\frac{v^2}{v_0^2}\right), \tag{2.26}$$

$v$ is the velocity on the plate, and $s$ is the distance between a point on the plate and the trailing edge $B$,

Passing in (2.25) to the integration in the parametric $u$-plane along the imaginary axis and taking into account (2.3), (2.6) and (2.16), we infer that

$$C_N = \frac{1+\sigma}{J}\int_0^\infty \left(1-|F(\mathrm{i}\eta)|^2\right) G_1(\eta)\mathrm{d}\eta, \quad C_M = \frac{1+\sigma}{J^2}\int_0^\infty \left(1-|F(\mathrm{i}\eta)|^2\right)|\zeta(\mathrm{i}\eta)|G_1(\eta)\mathrm{d}\eta, \tag{2.27a,b}$$

where $G_1(\eta)$ is determined by (2.7*b*).

By making use of (2.1), we have already derived formula (2.15) that, opposite to (2.27*a*), expresses the coefficient $C_N$ of the normal force $N$ in the integral-free form. Let us deduce now an integral-free expression for $C_M$. First, we present two formulae that generalize the Blasius-Chaplygin theorem to the case of the re-entrant jet cavity flows:

$$D - \mathrm{i}L = \frac{\mathrm{i}\rho}{2}\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z - \rho v_0^2 \mathrm{e}^{-\mathrm{i}\beta}\delta, \quad M = -\frac{\rho}{2}\,\mathrm{Re}\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 z\mathrm{d}z - \rho v_0^2\delta h, \tag{2.28a,b}$$

where $C$ is a closed contour surrounding the plate-cavity system, $h$ is the distance from the origin to the mean line of the re-entrant jet at the infinity $I$, the integration along $C$ being in the anticlockwise direction. Equations (2.28) are correct for a curved plate of arbitrary shape, and their derivation can be performed by the standard approach, described, for example, in Milne-Thomson (1968, §6·41). It is worthwhile to note that the vector $\rho v_0^2\delta \mathrm{e}^{\mathrm{i}\beta}$ is that of the mean algebraic momentum of the remote part of the re-entrant jet, thus, $\rho v_0^2\delta h$ is the moment of the momentum $\rho v_0^2\delta \mathrm{e}^{\mathrm{i}\beta}$ with the lever $h$. Therefore, $h$ is positive if the momentum acts in the counterclockwise direction with respect to the origin.

In the neighborhood of the infinity $D$, we have

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_\infty + \frac{\Gamma - \mathrm{i}Q}{2\pi \mathrm{i}z} + \frac{\omega_1 + \mathrm{i}\omega_2}{z^2} + O(1/|z|^3), \tag{2.29}$$

where $\omega_1$ and $\omega_2$ are real constants. With the help of the first equation in (2.28) and (2.29), one can easily derive equations (2.1) for the lift and drag forces $L$ and $D$. For the moment $M$, formula (2.29) and the second equation in (2.28) yield

$$M = \rho\left[2\pi\omega_2 v_\infty + \frac{\Gamma Q}{2\pi} - v_0^2\delta h\right]. \tag{2.30}$$

So, to calculate the moment $M$, one needs to determine the constants $\omega_2$ and $h$. Since the mean line of the re-entrant jet is rotated with respect to the $x$-axis by the angle $\pi - \beta$, it is evident that

$$h = \frac{\delta}{2} + \lim_{\xi\to 1+0} \mathrm{Im}\left[\mathrm{e}^{(\pi-\beta)\mathrm{i}} z(\xi)\right]. \tag{2.31}$$

Taking into account equation (2.16) and the equalities $l_0 = l/J$ and $\delta/l = q/J$, we obtain

$$\frac{h}{l} = \frac{q}{2J} - \frac{1}{J}\,\mathrm{Im}\left[\mathrm{e}^{-(\alpha+\beta)\mathrm{i}} p_1\right], \quad \text{where} \quad p_1 = \zeta_1(1) + \zeta_3(1). \tag{2.32a,b}$$

Thus, to find the constant $p_1$, one needs to remove from the expression (2.22) the second term and to calculate the remainder sum at $u = 1$. To determine $\omega_2$, we write

$$\omega_1 + \mathrm{i}\omega_2 = \frac{1}{2\pi i}\oint_C \frac{\mathrm{d}w}{\mathrm{d}z} z\,\mathrm{d}z = -\frac{l_0^2 v_0}{2\pi \mathrm{i}}\mathrm{e}^{-\mathrm{i}\alpha}\oint_{u_\infty} F(u)\zeta(u)\frac{\mathrm{d}\zeta}{\mathrm{d}u}\mathrm{d}u = \frac{l_0^2 v_0}{4\pi \mathrm{i}}\mathrm{e}^{-\mathrm{i}\alpha}\oint_{u_\infty} F'(u)\zeta^2(u)\mathrm{d}u, \tag{2.33}$$

where $F(u)$ is defined in (2.3), the symbol $\oint_{u_\infty}$ means that the integration is in the anticlockwise direction along a closed contour surrounding the point $u_\infty$ in the parametric $u$-plane, and the second equality has been obtained by making use of integration by parts. Therefore,

$$\omega_2 = \frac{l_0^2 v_0}{2}\,\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\alpha} \operatorname*{res}_{u=u_\infty}\left[F'(u)\zeta^2(u)\right]\right\}. \tag{2.34}$$

Now we introduce the following complex constants:

$$p_2 = \zeta_2(u_\infty) + \zeta_3(u_\infty), \quad p_3 = \frac{F'(u_\infty)}{F(u_\infty)} = \frac{2ik}{u_\infty^2 + k^2} + \frac{2u_0}{u_\infty^2 - u_0^2} - \frac{2\overline{u_0}}{u_\infty^2 - \overline{u_0}^2}, \tag{2.35a,b}$$

$$p_4 = \frac{\mathrm{d}}{du}\frac{F'(u)}{F(u)}\bigg|_{u=u_\infty} = -\frac{4iku_\infty}{(u_\infty^2 + k^2)^2} - \frac{4u_0 u_\infty}{(u_\infty^2 - u_0^2)^2} + \frac{4\overline{u_0}u_\infty}{(u_\infty^2 - \overline{u_0}^2)^2}. \tag{2.36}$$

Taking into account that $F(u_\infty) = R\mathrm{e}^{-\mathrm{i}\alpha}$, we write

$$F'(u_\infty) = F(u_\infty)\frac{F'(u_\infty)}{F(u_\infty)} = R\mathrm{e}^{-\mathrm{i}\alpha}p_3, \quad F''(u_\infty) = \frac{\mathrm{d}}{\mathrm{d}u}\left[F(u)\frac{F'(u)}{F(u)}\right]\bigg|_{u=u_\infty} = R\mathrm{e}^{-\mathrm{i}\alpha}(p_3^2 + p_4). \tag{2.37a,b}$$

Thus, in the neighborhood of the point $u = u_\infty$, we have

$$\zeta(u) = -\frac{\mathrm{e}^{\mathrm{i}\alpha}}{R}\frac{\mu}{u - u_\infty} + p_2 + O(|u - u_\infty|), \quad F'(u) = R\mathrm{e}^{-\mathrm{i}\alpha}\left[p_3 + (p_3^2 + p_4)(u - u_\infty)\right] + O\left(|u - u_\infty|^2\right). \tag{2.38a,b}$$

The last two formulae allows us to calculate the residue in (2.34) and with allowance for (2.30) to write finally

$$C_M = \frac{2(\sigma+1)}{J^2}\left\{\mathrm{Im}\left[-2\pi e^{-i\alpha}R\mu p_2 p_3 + \pi\mu^2\left(p_3^2 + p_4\right) + p_1 q e^{-i(\alpha+\beta)}\right] - \frac{q^2}{2} + \frac{\gamma q}{2\pi}\right\}. \tag{2.39}$$

Thus, we have calculated analytically the second integral in (2.27). Let $r$ be the distance between the center of pressure on the plate and the trailing edge $B$. Since we calculate the moment about the trailing edge, the formula for $r$ takes the form $r/l = -C_M/C_N$.

We denote by $L_c$ the length of the cavity defined as the distance between the extreme left and right vertical lines that touch the cavity surface. Analogously, $H_c$ is the width of the cavity defined as the distance between the extreme above and below horizonal lines that touch the same surface. Thus, $L_c$ and $H_c$ determine the minimal rectangle in which it is possible to inscribe the plate-cavity system, located on the main sheet of the flow region. The simplest way of finding $L_c$ and $H_c$ is to determine the positive roots of the equations

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = \mathrm{Re}\,\frac{\mathrm{d}z}{\mathrm{d}\xi} = l_0\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\alpha}G(\xi)\right] = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \mathrm{Im}\,\frac{\mathrm{d}z}{\mathrm{d}\xi} = l_0\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\alpha}G(\xi)\right] = 0. \tag{2.40a,b}$$

These roots correspond to the images of the contact points of the horizontal (first equation) and vertical (second equation) tangent lines to the surface of the cavity. Fortunately, the left-hand sides of the equations can be factorized:

$$-l_0\xi\frac{\Delta_1(\xi)\Delta_2(\xi)}{\Delta(\xi)} = 0, \quad 2l_0\xi\frac{\Delta_3(\xi)\Delta_4(\xi)}{\Delta(\xi)} = 0, \tag{2.41a,b}$$

where

$$\Delta_1(\xi) = -k(T-1)\left(c^2 + d^2\right) - \xi(T+1)\left[c^2 + d(d+2k)\right] + \xi^2(T-1)(2d+k) + \xi^3(T+1), \tag{2.42}$$

$$\Delta_2(\xi) = -k(T+1)\left(c^2 + d^2\right) + \xi(T-1)\left(c^2 + d(d+2k)\right) + \xi^2(T+1)(2d+k) + \xi^3(1-T), \tag{2.43}$$

$$\Delta_3(\xi) = -kT\left(c^2 + d^2\right) + \xi\left(-c^2 - d(d+2k)\right) + \xi^2 T(2d+k) + \xi^3, \tag{2.44}$$

$$\Delta_4(\xi) = k\left(c^2 + d^2\right) - \xi T\left(c^2 + d(d+2k)\right) + \xi^2(-2d-k) + \xi^3 T, \tag{2.45}$$

$$\Delta(\xi) = (\xi^2 - 1)\left(T^2 + 1\right)\left(a^2 - 2a\xi + b^2 + \xi^2\right)^2\left(a^2 + 2a\xi + b^2 + \xi^2\right)^2, \tag{2.46}$$

where $T = \tan(\alpha/2)$.

The functions $\Delta_i(\xi)$, $i = \overline{1,4}$, are cubic polynomials, therefore, their roots can be found analytically by means, for example, of the trigonometric method. Let $\{\xi_1, \xi_2, \ldots, \xi_N\}$ be the positive roots of the polynomials $\Delta_i(\xi)$, $i = \overline{1,4}$. Then

$$\frac{L_c}{l} = \frac{1}{J}\left\{\max_i\,[\mathrm{Re}\,\zeta(\xi_i)] - \min_i\,[\mathrm{Re}\,\zeta(\xi_i)]\right\}, \quad \frac{H_c}{l} = \frac{1}{J}\left\{\max_i\,[\mathrm{Im}\,\zeta(\xi_i)] - \min_i\,[\mathrm{Im}\,\zeta(\xi_i)]\right\}. \tag{2.47a,b}$$

Thus, if the accessory parameters $a$, $b$, $c$, $d$ , $k$ and $J$ are found, all desired features of the flow can be determined from explicit analytical integral-free formulae. On the notebook HP G1 EliteBook Folio 1040, the total time of computing the accessory parameters and all hydrodynamic properties mentioned above is not more than 0.02 second. Analytical formulae of this section are realized in the routine param$[\alpha, \beta, \sigma]$ of the package ReentrantJet.m stored in the Supplementary Materials.

## Appendix

Below, the formulae for the twelve coefficients $\Lambda_{ij}$ ($i = \overline{1,3}$, $j = \overline{1,4}$) of equations (1.23)–(1.25) are written down. The formulae have been generated and printed out by computer, so they cannot contain any typos.

$$\begin{aligned}\Lambda_{11} = \;& U\left[-a\left(R^2-1\right)(T^2+1)\left(a^2+b^2-1\right)+4bRT\left(a^2+b^2+1\right)\right]\\ &-2abV\left[(R-1)^2T^2+(R+1)^2\right],\end{aligned}$$

$$\begin{aligned}\Lambda_{12} = &-2abU\left(a^2+b^2\right)\left[(R+1)^2T^2+(R-1)^2\right]\\ &+V\left[a\left(R^2-1\right)\left(T^2+1\right)(a^2-3b^2-1)+12a^2bRT-4b\left(b^2+1\right)RT\right],\end{aligned}$$

$$\begin{aligned}\Lambda_{13} = \;& U\left\{-a\left(R^2-1\right)\left(T^2+1\right)\left[a^4+a^2\left(2b^2-1\right)+b^2\left(b^2+3\right)\right]\right.\\ &\left.-4bRT\left[a^4+a^2\left(2b^2-3\right)+b^4+b^2\right]\right\}\\ &-2abV\left[(R^2+1)\left(T^2+1\right)-2R\left(T^2-1\right)\right],\end{aligned}$$

$$\begin{aligned}\Lambda_{14} = &-2abU\left(a^2+b^2\right)\left[(R+1)^2T^2+(R-1)^2\right]\\ &+V\left(a^2+b^2\right)\left[a\left(R^2-1\right)(T^2+1)\left(a^2+b^2-1\right)+4bRT\left(a^2+b^2+1\right)\right],\end{aligned}$$

$$\begin{aligned}\Lambda_{21} = \;& (R+1)\left(a^2-b^2-1\right)\left[a^3TU+4a^2bU-a\left(3b^2+1\right)TU-2b^2(2bU+V)\right]\\ &-b(R-1)\left\{U\left[a^4-8a^3bT-2a^2\left(2b^2+1\right)+8ab^3T+3b^4+4b^2+1\right]\right.\\ &\left.-2aTV\left(a^2-5b^2-1\right)\right\},\end{aligned}$$

$$
\begin{aligned}
\Lambda_{22} = \quad & U\left\{2b^2(R-1)\left[5a^4-a^2\left(10b^2+1\right)+b^4+b^2\right]\right.\\
& \left.-2ab(R+1)T\left[a^4-a^2\left(10b^2+1\right)+5b^4+b^2\right]\right\}\\
& +V\left\{b(R-1)\left[3a^4-4a^2\left(4b^2+1\right)+5b^4+6b^2+1\right]\right.\\
& +\,(R+1)\left[-Ta^5-2a^4b+2a^3\left(4b^2T+T\right)+a^2\left(2b-4b^3\right)\right.\\
& \left.\left.-a\left(15b^4+8b^2+1\right)T-2\left(b^5+b^3\right)\right]\right\},
\end{aligned}
$$

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$$
\begin{aligned}
\Lambda_{23} = \quad & (R+1)\left(a^2-b^2-1\right)\left\{a^5TU+2a^4bU-a^3\left(6b^2+1\right)TU\right.\\
& \left.+2a^2\left(2b^3+b\right)U+3ab^2\left(3b^2+1\right)TU+2b^2\left[b\left(b^2-1\right)U-V\right]\right\}\\
& -\,b(R-1)\left[3a^6U-4a^5bTU-a^4\left(13b^2+6\right)U-2a^3T\left(4b^3U+2bU+V\right)\right.\\
& +a^2\left(13b^4+14b^2+3\right)U+2aT\left(-2b^5U+2b^3U+5b^2V+V\right)\\
& \left.-b^2\left(3b^4+4b^2+1\right)U\right],
\end{aligned}
$$

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$$
\begin{aligned}
\Lambda_{24} = \quad & 2bU\left\{b(R-1)\left[5a^4-a^2\left(10b^2+1\right)+b^4+b^2\right]\right.\\
& \left.-a(R+1)T\left[a^4-a^2\left(10b^2+1\right)+5b^4+b^2\right]\right\}\\
& +V\left\{-a^7(R+1)T+a^6b(R-5)+a^5T\left[b^2(3R+11)+2(R+1)\right]\right.\\
& +a^4b\left[b^2(11-15R)-2R+6\right]-a^3T\left[b^4(23R+7)+2b^2(R+5)+R+1\right]\\
& +a^2b\left[b^4(15R-11)-2b^2(R+3)+R-1\right]\\
& \left.+ab^2T\left[b^4(5R+13)-4b^2(R-1)-R-1\right]+b^3\left[-b^4(R-5)+4b^2+R-1\right]\right\},
\end{aligned}
$$

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$$
\begin{aligned}
\Lambda_{31} = \quad & 2abV\left[a(R+1)\left(a^2-b^2-1\right)-b(R-1)T\left(-5a^2+b^2+1\right)\right]\\
& +U\left\{-a^6(R+1)+a^5b(R+5)T+2a^4\left[b^2(2R+5)+R+1\right]\right.\\
& +2a^3bT\left[4b^2(R-2)-R-3\right]-a^2\left(b^2+1\right)\left[b^2(3R+11)+R+1\right]\\
& \left.+abT\left[-b^4(R-3)+4b^2+R+1\right]+2\left(b^3+b\right)^2\right\},
\end{aligned}
$$

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$$
\begin{aligned}
\Lambda_{32} = \;& 2abU\Big\{-b(R+1)T\left(5a^4-5a^2\left(2b^2+1\right)+b^4+b^2\right)\\
&-a(R-1)\left[a^4-a^2\left(10b^2+1\right)+5\left(b^4+b^2\right)\right]\Big\}\\
&+V\Big\{a^6(R+1)-5a^5b(R+1)T+2a^4\left[b^2(8R-7)-R-1\right]\\
&+2a^3b\left(8b^2+3\right)(R+1)T-a^2\left(b^2+1\right)\left[b^2(9R-7)-R-1\right]\\
&-ab\left(3b^4+4b^2+1\right)(R+1)T-2\left(b^3+b\right)^2\Big\},
\end{aligned}
$$

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$$
\begin{aligned}
\Lambda_{33} = \;& 2abV\left[a(R+1)\left(a^2-b^2-1\right)-b(R-1)T\left(-5a^2+b^2+1\right)\right]\\
&+U\Big\{-a^8(R+1)+a^7b(R+5)T+a^6\left[2(R+1)-9b^2(R-1)\right]\\
&-a^5bT\left[b^2(15R+11)+2(R+3)\right]+a^4\Big[b^4(17R-13)\\
&+6b^2(3R-1)-R-1\Big]+a^3bT\left[b^4(15R+11)+b^2(26R+2)+R+1\right]\\
&-a^2b^2\left(b^2+1\right)\left[7b^2(R-1)+9R+3\right]-ab^3\left(b^2+1\right)T\left[b^2(R+5)+3(R+1)\right]\\
&-2b^4\left(b^2+1\right)^2\Big\},
\end{aligned}
$$

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$$
\begin{aligned}
\Lambda_{34} = \;& 2abU\Big\{-b(R+1)T\left[5a^4-5a^2\left(2b^2+1\right)+b^4+b^2\right]\\
&-a(R-1)\left[a^4-a^2\left(10b^2+1\right)+5\left(b^4+b^2\right)\right]\Big\}\\
&+V\Big\{a^8(R+1)-a^7b(R+5)T+a^6\left[9b^2(R-1)-2(R+1)\right]\\
&+a^5bT\left[b^2(15R+11)-2R+6\right]+a^4\left[b^4(13-17R)+2b^2(R+1)+R+1\right]\\
&-a^3bT\left[b^4(15R+11)+2b^2(R+1)-3R+1\right]\\
&+a^2b^2\left(b^2+1\right)\left[7b^2(R-1)-3R-1\right]+ab^3T\left[b^4(R+5)+8b^2-R+3\right]\\
&+2b^4\left(b^2+1\right)^2\Big\}.
\end{aligned}
$$

---

Now we print out the formulae for the four coefficients $\Omega_{ij}$ ($i = 1,2;\ j = 1,2$) that defines the polynomials $M_1(a,b)$ and $M_2(a,b)$ in equations (1.27) and (1.28). As well as formulae for $\Lambda_{ij}$, these ones have bee generated by computer too.

---

$$
\begin{aligned}
\Omega_{11} = &- a^3\left(R^2-1\right)\left(T^2+1\right)\left[a^4-2a^2\left(b^2+1\right)-3b^4+6b^2+1\right] \\
&- (R-1)\Big[-a^7+3a^6bT+a^5\left(3b^2+2\right)+a^4b\left(5b^2-4\right)T \\
&+a^3\left(b^4-2b^2-1\right)+a^2b\left(b^4+10b^2+1\right)T \\
&-a\left(3b^6+4b^4+b^2\right)-b^3\left(b^2+1\right)^2T\Big] \\
&- (R+1)T\Big[a^7T+3a^6b-a^5\left(3b^2+2\right)T+a^4b\left(5b^2-4\right) \\
&+a^3\left(-b^4+2b^2+1\right)T+a^2\left(b^5+10b^3+b\right) \\
&+ab^2\left(3b^4+4b^2+1\right)T-b^3\left(b^2+1\right)^2\Big],
\end{aligned}
$$

---

$$
\begin{aligned}
\Omega_{12} = &8a^3b^3\left(R^2-1\right)\left(T^2+1\right)-(R-1)T\Big[-a^6+2a^5bT+a^4\left(5b^2+2\right) \\
&+2a^3b\left(2b^2-1\right)T-a^2\left(11b^4+4b^2+1\right)+2ab^3\left(b^2+1\right)T-\left(b^3+b\right)^2\Big] \\
&+(R+1)\Big[a^6T+2a^5b-a^4\left(5b^2+2\right)T \\
&+a^3\left(4b^3-2b\right)+a^2\left(11b^4+4b^2+1\right)T+2a\left(b^5+b^3\right)+\left(b^3+b\right)^2T\Big],
\end{aligned}
$$

---

$$
\begin{aligned}
\Omega_{21} = &- a^2b\left(R^2-1\right)\left(T^2+1\right)\left[3a^4+2a^2\left(b^2-3\right)-b^4+2b^2+3\right] \\
&+(R+1)T\Big[a^7-3a^6bT-a^5\left(b^2+2\right)+a^4b\left(b^2+4\right)T+a^3\left(-5b^4+6b^2+1\right) \\
&+a^2b\left(3b^4+2b^2-1\right)T-ab^2\left(3b^4+8b^2+5\right)-b^3\left(b^2+1\right)^2T\Big] \\
&-(R-1)\Big[-a^7T-3a^6b+a^5\left(b^2+2\right)T+a^4b\left(b^2+4\right)+a^3\left(5b^4-6b^2-1\right)T \\
&+a^2b\left(3b^4+2b^2-1\right)+ab^2\left(3b^4+8b^2+5\right)T-b^3\left(b^2+1\right)^2\Big],
\end{aligned}
$$

---

$$
\begin{aligned}
\Omega_{22} = & -a^2\left(R^2-1\right)\left(T^2+1\right)\left[a^4+a^2\left(6b^2-2\right)-3b^4-2b^2+1\right] \\
& +(R+1)\left[-a^6+2a^5bT-a^4\left(b^2-2\right)\right. \\
& \left.-2a^3\left(6b^3+b\right)T+a^2\left(b^4-1\right)+2ab^3\left(b^2+1\right)T+\left(b^3+b\right)^2\right] \\
& +(R-1)T\left[a^6T+2a^5b+a^4\left(b^2-2\right)T-2a^3\left(6b^3+b\right)\right. \\
& \left.+a^2\left(T-b^4T\right)+2a\left(b^5+b^3\right)-\left(b^3+b\right)^2T\right].
\end{aligned}
$$

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