Supplemental material accompanying 'Variational methods for finding periodic orbits in the incompressible Navier-Stokes equations'

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Here we give the full gradients of the two objective functionals considered for relative periodic orbits, namely

$$J_{PV}[u,v,p,c,T] = \frac{1}{2}\int \left\{ \left(I^{(1)}_{PV}\right)^2 + \left(I^{(2)}_{PV}\right)^2 + \left(I^{(3)}_{PV}\right)^2 \right\} \mathrm{d}V, \tag{1}$$

and

$$J_{SV}[\psi,\omega,c,T] = \frac{1}{2}\int \left\{ \left(I^{(1)}_{SV}\right)^2 + \left(I^{(2)}_{SV}\right)^2 \right\} \mathrm{d}V, \tag{2}$$

where we have defined

$$I^{(1)}_{PV} = \frac{2\pi}{T}\partial_s u - c\partial_x u + u\partial_x u + v\partial_y u + \partial_x p - \frac{1}{Re}\Delta u - \sin 4y, \tag{3}$$

$$I^{(2)}_{PV} = \frac{2\pi}{T}\partial_s v - c\partial_x v + u\partial_x v + v\partial_y v + \partial_y p - \frac{1}{Re}\Delta v, \tag{4}$$

$$I^{(3)}_{PV} = \partial_x u + \partial_y v, \tag{5}$$

$$I^{(1)}_{SV} = \frac{2\pi}{T}\partial_s \omega - c\partial_x \omega + \partial_y\psi\partial_x\omega - \partial_x\psi\partial_y\omega - \frac{1}{Re}\Delta\omega - 4\cos 4y, \tag{6}$$

$$I^{(2)}_{SV} = \omega + \Delta\psi. \tag{7}$$

Then we derive the partial derivatives

$$\frac{\partial J_{PV}}{\partial c} = -\int \left(I^{(1)}_{PV}\partial_x u + I^{(2)}_{PV}\partial_x v\right)\mathrm{d}V, \tag{8}$$

$$\frac{\partial J_{PV}}{\partial T} = -\int \frac{2\pi}{T^2}\left(I^{(1)}_{PV}\partial_s u + I^{(2)}_{PV}\partial_s v\right)\mathrm{d}V, \tag{9}$$

$$\frac{\partial J_{SV}}{\partial c} = -\int I^{(1)}_{SV}\partial_x\omega \,\mathrm{d}V, \tag{10}$$

$$\frac{\partial J_{SV}}{\partial T} = -\int \frac{2\pi}{T^2} I^{(1)}_{SV}\partial_s\omega \,\mathrm{d}V, \tag{11}$$

and the functional derivatives

$$\frac{\delta J_{PV}}{\delta u} = -\frac{2\pi}{T}\partial_s I_{PV}^{(1)} - c\partial_x I_{PV}^{(1)} + I_{PV}^{(1)}\partial_x u + I_{PV}^{(2)}\partial_x v - \partial_x\left(uI_{PV}^{(1)}\right) - \partial_y\left(vI_{PV}^{(1)}\right) - \frac{1}{Re}\Delta I_{PV}^{(1)} - \partial_x I_{PV}^{(3)}, \quad (12)$$

$$\frac{\delta J_{PV}}{\delta v} = -\frac{2\pi}{T}\partial_s I_{PV}^{(2)} - c\partial_x I_{PV}^{(2)} + I_{PV}^{(1)}\partial_y u + I_{PV}^{(2)}\partial_y v - \partial_x\left(uI_{PV}^{(2)}\right) - \partial_y\left(vI_{PV}^{(2)}\right) - \frac{1}{Re}\Delta I_{PV}^{(2)} - \partial_y I_{PV}^{(3)}, \quad (13)$$

$$\frac{\delta J_{PV}}{\delta p} = -\left(\partial_x I_{PV}^{(1)} + \partial_y I_{PV}^{(2)}\right), \quad (14)$$

$$\frac{\delta J_{SV}}{\delta \psi} = -\partial_y I_{SV}^{(1)}\partial_x\omega + \partial_x I_{SV}^{(1)}\partial_y\omega + \Delta I_{SV}^{(2)}, \quad (15)$$

$$\frac{\delta J_{SV}}{\delta \omega} = -\frac{2\pi}{T}\partial_s I_{SV}^{(1)} - c\partial_x I_{SV}^{(1)} + \partial_y I_{SV}^{(1)}\partial_x\psi - \partial_x I_{SV}^{(1)}\partial_y\psi - \frac{1}{Re}\Delta I_{SV}^{(1)} + I_{SV}^{(2)}. \quad (16)$$

Slightly different but analytically equivalent formulations of the nonlinear terms are possible. The forms given here are those that were implemented in the code.