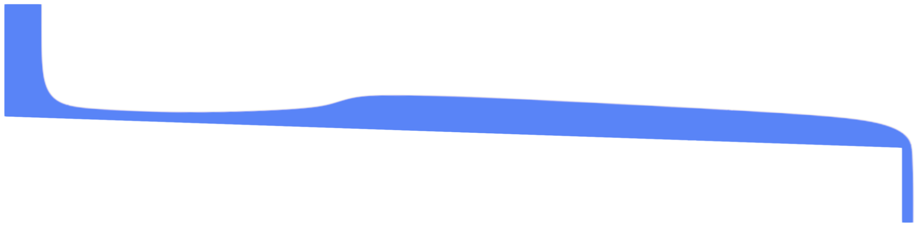

FIGURE 1. Steady flow for $Q_1 = 0$, $\beta = 92°$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$.

Supplementary Material for the Paper

Hydraulic Jump on the Surface of a Cone

by Guangzhao Zhou and Andrea Prosperetti

1. Description of the movies

The movies show the time dependent film flow when the jet flow rate executes periodic oscillations according to

$$Q(t) = Q_0 + Q_1 \sin(2\pi f t)\,.$$

The period is $T = 1/f$. Movie 1 shows results from the time-dependent reduced-order model; Movies 2 to 5 show DNS results.

Movie 1. Sequence generated by the time-dependent reduced-order model for the pulsating jet of section 10 of the paper; figure 16 is taken from this sequence. The inclination angle of the cone surface is $\beta = 120°$, which is too large for a jump to form in steady conditions. Here $Q_1/Q_0 = 0.05$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$ and $a^2 f/\nu = 2.5$. These values can be realized, for example, with $Q = 30 \times 10^{-6}$ m^3/s, $a = 2.5$ mm, $\nu = 20 \times 10^{-6}$ m^2/s and $f = 8$ Hz.

Movie 2. $\beta = 92°$, $Q_1/Q_0 = 0.1$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$, $a^2 f/\nu = a^2/(\nu T) = 2.5$.

Movie 3. $\beta = 92°$, $Q_1/Q_0 = 0.2$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$, $a^2 f/\nu = a^2/(\nu T) = 2.5$.

Movie 4. $\beta = 92°$, $Q_1/Q_0 = 0.1$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$, $a^2 f/\nu = a^2/(\nu T) = 5$.

Movie 5. $\beta = 92°$, $Q_1/Q_0 = 0.2$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$, $a^2 f/\nu = a^2/(\nu T) = 5$.

Figure 1 shows the steady flow when $Q_1 = 0$, $\beta = 92°$, $Re_{j0} = 2Q_0/(\pi a\nu) = 382$, $N = 0.0419$. Note the hydraulic jump.

TABLE 1. Comparison of the jump radius between the present DNS results and Bhagat et al.'s Eq. (3.6)

Case	**DNS**	**Bhagat et al.**	**Error**	**Experiment**	**Note**
figure 2	3.79	4.51	19%	3.49	–
figure 4(a)	2.93	3.37	15%	–	$\sigma = 0$
figure 12(d)	2.80	3.13	12%	–	–
figure C1(a)	13.13	12.45	-5%	–	–
figure C1(b)	14.02	14.82	6%	–	$\sigma = 0$

2. Comparison between DNS results and Bhagat et al.'s theory

As requested by a referee, we compare the predictions of Bhagat *et al.* (2020) for the jump radius with our DNS results for the cases with $\beta = 90°$ in the paper. The comparison is shown in Table 1. The values shown are normalized by the jet diameter and the relative error is defined as

$$\text{Error} = \frac{r_{\text{Bhagat}} - r_{\text{DNS}}}{r_{\text{DNS}}} \times 100\%. \tag{2.1}$$

Here, as in the paper, the jump radius is determined as the position with the maximum free-surface curvature (illustrated in figure 3 of the paper). It is not clear from Bhagat *et al.* (2020) how precisely the authors define the jump radius.

The entries in the table show that our DNS results are compatible with the predictions of Bhagat *et al.* (2020). For the only case in which data is available, our result is significantly more accurate.

In order to obtain from the theory of Bhagat *et al.* (2020) the predictions shown in the table for $\sigma = 0$ we start from their relation

$$R = 0.2705 R_{ST} \left[\sqrt{Q^{*2} + 2Q^*} - Q^* \right]^{1/4}, \tag{2.2}$$

where

$$R_{ST} = \frac{Q^{3/4} \rho^{1/4}}{\nu^{1/4} \sigma^{1/4}}, \quad Q^* = 5.78 \frac{\sigma^2}{\nu \rho^2 g Q}. \tag{2.3}$$

Although not shown in their preprint, it is easy to calculate that, in the small surface tension limit,

$$\lim_{\sigma \to 0} R = \lim_{\sigma \to 0} 0.2705 R_{ST} \left(2Q^*\right)^{1/8} \approx 1.16 \frac{q^{5/8}}{\nu^{3/8} g^{1/8}}, \tag{2.4}$$

where $q = Q/2\pi$. This scaling is the same as that proposed in Bohr *et al.* (1993), with the coefficient 0.73 in place of 1.16.

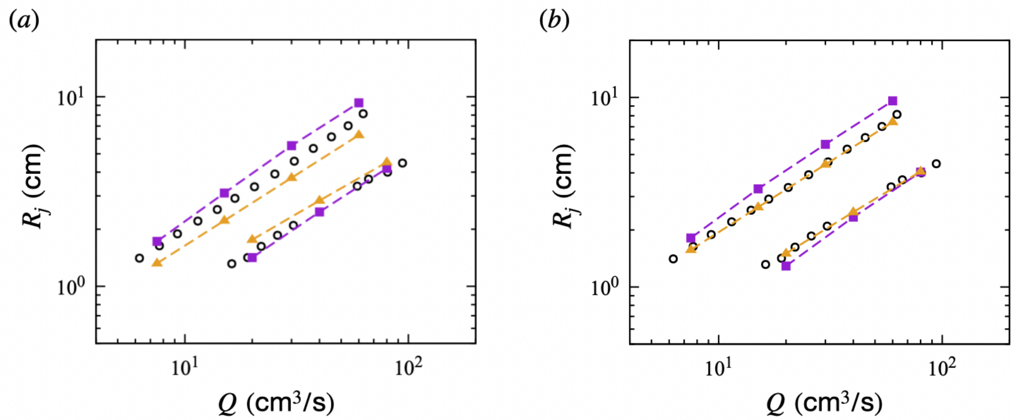

FIGURE 2. Comparison of the DNS result and Bhagat et al.'s theory. The circles are experimental data adapted from Hansen *et al.* (1997). The purple squares are the DNS results. The yellow triangles are the prediction of Bhagat et al.'s theory. The top group of data in each panel is for water, whereas the bottom group is for oil. The predicted data in the left panel are obtained with the parameters reported in Hansen et al. (1997) (water) and Rojas et al. (2010) (oil), whereas those in the right panel are obtained with the parameters reported in Bhagat et al. (2020).

3. Comparison of our DNS results, Bhagat *et al.* (2020) and the data of Hansen *et al.* (1997)

The comparison of our DNS results with the data of Hansen *et al.* (1997) in figure 3 of our paper is not straightforward because of the uncertainty affecting the liquid density ρ and the surface tension coefficient σ which appear in the combination σ/ρ in the computation. The comparison is for two sets of experimental data presented in figure 3 of Hansen *et al.* (1997), for water and oil, respectively. The authors state that "for water, [the surface tension coefficient] $\gamma = 74$ dyn/cm..." However, the value for the same quantity is given in table 1 of Bhagat *et al.* (2020) as 0.037 kg/s^2, i.e. half of that of the original reference which is the standard value for water. The liquid used by Hansen et al. is shown in the same table of Bhagat *et al.* (2020) as "water+surfactant/water" in spite of the fact Hansen et al.'s paper makes no mention of the contamination of their water by surfactants. To be on the safe side, in figure 2 we show the DNS results obtained with both values of σ.

A similar problem arises for Hansen et al's oil data in figure 3 of our paper. Hansen et al. did not provide the value of the density and surface tension coefficient of their oil. For our DNS we used the values given by Rojas et al. (2010), where the authors write "The density and surface tension of both oils are not given in Hansen et al. (1997). We use the standard values 9.5×10^2 kg/m^3 and $\gamma = 2.0 \times 10^{-2}$ N/m in our fit." These values give $\sigma/\rho \approx 2.11 \times 10^{-5}$ m^3/s^2. However, for the same experiment, Bhagat *et al.* (2020) used $\rho = 875$ kg/m^3 and $\gamma = 0.045$ N/m, resulting in $\sigma/\rho \approx 5.14 \times 10^{-5}$ m^3/s^2, which is about 2.5 times the value used by Rojas et al. (2010). Again, to be on the safe side, we show in figure 2 the DNS results obtained with both values of σ/ρ.

The left panel of figure 2 shows the result with the parameters from Hansen *et al.* (1997) and Rojas *et al.* (2010), i.e., $\sigma/\rho = 7.4 \times 10^{-5}$ m^3/s^2 for water, and 2.11×10^{-5} m^3/s^2 for oil, while the right panel shows the results with the parameters from Bhagat et al. (2020), i.e., $\sigma/\rho = 3.7 \times 10^{-5}$ m^3/s^2 for water, and 5.14×10^{-5} m^3/s^2 for oil. Note that only σ/ρ is different between the two panels.

It is seen from the left panel of figure 2 that, with the parameters given by Hansen et

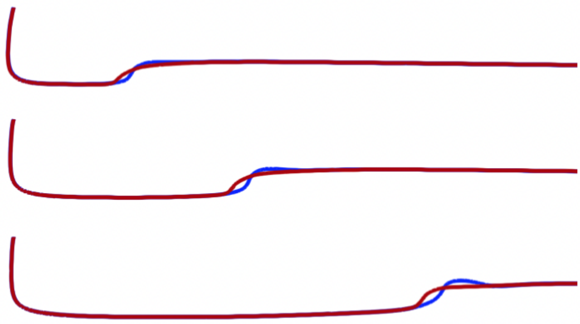

Figure 3. Comparison of the shape of the free surface for oil obtained from DNS; $\sigma/\rho = 2.11 \times 10^{-5}$ m^3/s^2 for the blue line; $\sigma/\rho = 5.14 \times 10^{-5}$ m^3/s^2 for the red line. From top to bottom the flow rates are $Q = 20$, 40, and 80 cm^3/s.

al. (1997), the predicted water jump radius from DNS is larger than the experimental data, whereas that from Bhagat et al.'s theory is smaller with a similar magnitude of discrepancy. (As noted in the paper, Hansen et al. mention some unsteadiness of the position of the hydraulic jump for water which may have affected the accuracy of the reported value.) For oil, the DNS result with the parameters given by Rojas et al. (2010) has a better agreement with the experimental data than Bhagat et al.'s theory. When the parameters from Bhagat et al. (2020) are used (right panel), the predictions of their theory improve significantly, while the DNS results differ little from those shown in the left panel. Since only the parameter σ/ρ is different between the left and right panels, the small difference of the DNS results implies again that surface tension does not have a big influence on the jump radius.

This conclusion can be demonstrated more effectively by comparing the shape of the free surface obtained from the DNS with different surface tensions. Figure 3 shows this comparison for oil. The blue line is the free surface simulated with σ/ρ from Rojas et al. (2010). The red line is the free surface simulated with σ/ρ from Bhagat et al. (2020). It is seen here that the small change of the jump radius is largely related to the reshaping effect of the surface tension in the jump region.

A similar comparison for water is shown in figure 4. Even with the vertical scale enlarged 10 times to magnify the difference, the two lines are very nearly on top of one another. Actually, as the inset shows, when plotted using the same scale in both directions, the jump is so mild that it can hardly be distinguished. With such a small curvature in the jump region, the influence of surface tension would be expected to be small.

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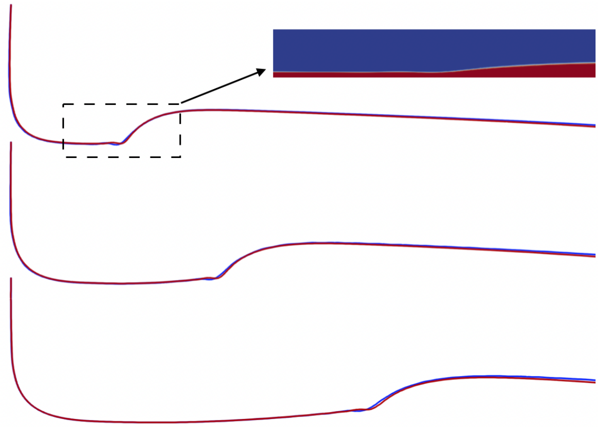

Figure 4. Comparison of the shape of the free surface for water obtained from DNS; $\sigma/\rho = 7.4 \times 10^{-5}$ $\mathrm{m}^3/\mathrm{s}^2$ for the blue line; $\sigma/\rho = 3.7 \times 10^{-5}$ $\mathrm{m}^3/\mathrm{s}^2$ for the red line. From top to bottom the flow rates are $Q = 15$, 30, and 60 cm^3/s. Note that the vertical scale is enlarged to 10 times that of the horizontal one. The colored inset shows the jump region with both scales equal.