Dimples, jets and self-similarity in nonlinear, capillary waves: supplementary material

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1. Expressions for modal coefficients

With the shorthand notation representing the integrals

$$\mathcal{I}_{\nu_1-m_1,\nu_1-m_2,\nu_3-m_3,\ldots} \equiv \frac{1}{l_q^2}\int_0^{l_q} dr\; r \mathrm{J}_{\nu_1}(\alpha_{m_1,q}r)\mathrm{J}_{\nu_2}(\alpha_{m_2,q}r)\mathrm{J}_{\nu_3}(\alpha_{m_3,q}r)\ldots$$

With $\omega_{j,q}^2 \equiv \alpha_{j,q}^3$ and $\alpha_{j,q} \equiv \frac{l_j}{l_q}$, at $O(\epsilon^2)$ we obtain

$$\phi_2(r,z,\tau) = \frac{1}{2}\mathrm{J_0}^2(l_q)\sin(2\tau) + \sum_{j=1}^{\infty}\left[\xi_{j,q}^{(1)}\sin(\omega_{j,q}\tau) + \xi_{j,q}^{(2)}\sin(2\tau)\right]\exp(\alpha_{j,q}z)\mathrm{J_0}(\alpha_{j,q}r) \tag{1.1}$$

$$\text{and}\quad \eta_2(r,\tau) = \frac{1}{2}\sum_{j=1}^{\infty}\left[\zeta_{j,q}^{(1)}\cos(\omega_{j,q}\tau) + \zeta_{j,q}^{(2)}\cos(2\tau) + \zeta_{j,q}^{(3)}\right]\mathrm{J_0}(\alpha_{j,q}r) \tag{1.2}$$

where the expressions for the coefficients are

$$\xi_{j,q}^{(1)} \equiv \frac{2}{\mathrm{J_0}^2(l_j)(\omega_{j,q}^2-4)\omega_{j,q}}\left[(\alpha_{j,q}^3-\alpha_{j,q}^2-1)\mathcal{I}_{0-q,0-q,0-j} + (\alpha_{j,q}^2+1)\mathcal{I}_{1-q,1-q,0-j}\right]$$

$$\xi_{j,q}^{(2)} \equiv \frac{1}{\mathrm{J_0}^2(l_j)(\omega_{j,q}^2-4)}\left[(\alpha_{j,q}^2-3)\mathcal{I}_{0-q,0-q,0-j} - (\alpha_{j,q}^2+1)\mathcal{I}_{1-q,1-q,0-j}\right] \tag{1.3}$$

and

$$\zeta_{j,q}^{(1)} \equiv -\frac{4}{\alpha_{j,q}^2\mathrm{J_0}^2(l_j)(\omega_{j,q}^2-4)}\left[(\alpha_{j,q}^3-\alpha_{j,q}^2-1)\mathcal{I}_{0-q,0-q,0-j} + (\alpha_{j,q}^2+1)\mathcal{I}_{1-q,1-q,0-j}\right]$$

$$\zeta_{j,q}^{(2)} \equiv \frac{1}{\alpha_{j,q}^2\mathrm{J_0}^2(l_j)(\omega_{j,q}^2-4)}\left[(3\alpha_{j,q}^3-4\alpha_{j,q}^2)\mathcal{I}_{0-q,0-q,0-j} + (\alpha_{j,q}^3+4\alpha_{j,q}^2)\mathcal{I}_{1-q,1-q,0-j}\right]$$

$$\zeta_{j,q}^{(3)} \equiv \frac{1}{\alpha_{j,q}^2\mathrm{J_0}^2(l_j)}\left[\mathcal{I}_{0-q,0-q,0-j} - \mathcal{I}_{1-q,1-q,0-j}\right] \tag{1.4}$$

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At $O(\epsilon^3)$ we have with the expression for $\eta_3(r,\tau)$ viz.

$$\eta_3(r,\tau) = \Bigg[\mu^{(q)}\cos(\tau) + \kappa^{(q)}\cos(3\tau) + \sum_{m=1,m\neq q}^{\infty}\left(\gamma_m^{(q)}\cos\left[\left(\omega_{m,q}+1\right)\tau\right] + \chi_m^{(q)}\cos\left[\left(\omega_{m,q}-1\right)\tau\right]\right)\Bigg]\mathrm{J}_0(r)$$
$$+ \sum_{j=1,j\neq q}^{\infty}\Bigg[\mu^{(j)}\cos(\tau) + \kappa^{(j)}\cos(3\tau) + \nu^{(j)}\cos\left(\omega_{j,q}\tau\right) +$$
$$\sum_{m=1,m\neq q}^{\infty}\left(\gamma_m^{(j)}\cos\left[\left(\omega_{m,q}+1\right)\tau\right] + \chi_m^{(j)}\cos\left[\left(\omega_{m,q}-1\right)\tau\right]\right)\Bigg]\mathrm{J}_0\left(\alpha_{j,q}r\right)$$

where

$$\begin{aligned}
\mu^{(j)} &\equiv \frac{1}{\alpha_{j,q}^2}\left(\mathcal{A}^{(j)} - \frac{\mathcal{T}^{(j)}}{(\omega_{j,q}^2-1)}\right)\\
\kappa^{(j)} &\equiv \frac{1}{\alpha_{j,q}^2}\left(\mathcal{B}^{(j)} - \frac{3\mathcal{U}^{(j)}}{(\omega_{j,q}^2-9)}\right)\\
\nu^{(j)} &\equiv -\frac{1}{\alpha_{j,q}^2}\omega_{j,q}\lambda^{(j)}\\
\gamma_m^{(j)} &\equiv \frac{1}{\alpha_{j,q}^2}\left(\mathcal{C}_m^{(j)} - \frac{(\omega_{m,q}+1)\mathcal{V}_m^{(j)}}{\omega_{j,q}^2-\left(\omega_{m,q}+1\right)^2}\right)\\
\chi_m^{(j)} &\equiv \frac{1}{\alpha_{j,q}^2}\left(\mathcal{D}_m^{(j)} - \frac{(\omega_{m,q}-1)\mathcal{W}_m^{(j)}}{\omega_{j,q}^2-\left(\omega_{m,q}-1\right)^2}\right)\\
\mu^{(q)} &\equiv \mathcal{A}^{(q)} - \lambda^{(q)}\\
\kappa^{(q)} &\equiv \mathcal{B}^{(q)} + \frac{3}{8}\mathcal{U}^{(q)}\\
\gamma_m^{(q)} &\equiv \mathcal{C}_m^{(q)} + \frac{(\omega_{m,q}+1)}{\omega_{m,q}(\omega_{m,q}+2)}\mathcal{V}_m^{(q)}\\
\chi_m^{(q)} &\equiv \mathcal{D}_m^{(q)} + \frac{(\omega_{m,q}-1)}{\omega_{m,q}(\omega_{m,q}-2)}\mathcal{W}_m^{(q)}
\end{aligned}$$

and

$$\mathcal{A}^{(j)} \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[\sum_{m=1}^{\infty}\left\{\left(\frac{1}{2}\zeta^{(2)}_{m,q}+\zeta^{(3)}_{m,q}-\alpha_{m,q}\xi^{(2)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}+\alpha_{m,q}\xi^{(2)}_{m,q}\mathcal{I}_{1-q,1-m,0-j}\right\}\right.$$
$$\left.+\frac{1}{4}\mathcal{I}_{0-q,0-q,0-q,0-j}+\mathcal{I}_{0-q,1-q,1-q,0-j}-\frac{3}{4}\mathcal{I}_{2-q,1-q,1-q,0-j}\right]+\delta_{jq}\Omega_2$$
$$\mathcal{B}^{(j)} \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[\sum_{m=1}^{\infty}\left\{\left(\frac{1}{2}\zeta^{(2)}_{m,q}-3\alpha_{m,q}\xi^{(2)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}-\alpha_{m,q}\xi^{(2)}_{m,q}\mathcal{I}_{1-q,1-m,0-j}\right\}\right.$$
$$\left.+\frac{3}{4}\mathcal{I}_{0-q,0-q,0-q,0-j}+\mathcal{I}_{0-q,1-q,1-q,0-j}-\frac{1}{4}\mathcal{I}_{2-q,1-q,1-q,0-j}\right]$$
$$\mathcal{C}^{(j)}_m \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[\left(\frac{1}{2}\zeta^{(1)}_{m,q}-\alpha_{m,q}\left(\omega_{m,q}+1\right)\xi^{(1)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}-\alpha_{m,q}\xi^{(1)}_{m,q}\mathcal{I}_{1-q,1-m,0-j}\right]$$
$$\mathcal{D}^{(j)}_m \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[\left(\frac{1}{2}\zeta^{(1)}_{m,q}-\alpha_{m,q}\left(\omega_{m,q}-1\right)\xi^{(1)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}+\alpha_{m,q}\xi^{(1)}_{m,q}\mathcal{I}_{1-q,1-m,0-j}\right]$$

$$\mathcal{P}^{(j)} \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[\sum_{m=1}^{\infty}\left\{-\left(\frac{1}{2}\zeta^{(2)}_{m,q}-\zeta^{(3)}_{m,q}+\alpha^2_{m,q}\xi^{(2)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}\right.\right.$$
$$\left.+\alpha_{m,q}\left(\frac{1}{2}\zeta^{(2)}_{m,q}-\zeta^{(3)}_{m,q}+\xi^{(2)}_{m,q}\right)\mathcal{I}_{1-q,1-m,0-j}\right\}$$
$$\left.+\frac{1}{4}\mathcal{I}_{0-q,0-q,0-q,0-j}-\frac{1}{2}\mathcal{I}_{0-q,1-q,1-q,0-j}\right]-\delta_{jq}\Omega_2$$
$$\mathcal{Q}^{(j)} \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[\sum_{m=1}^{\infty}\left\{-\left(-\frac{1}{2}\zeta^{(2)}_{m,q}+\alpha^2_{m,q}\xi^{(2)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}\right.\right.$$
$$\left.+\alpha_{m,q}\left(-\frac{1}{2}\zeta^{(2)}_{m,q}+\xi^{(2)}_{m,q}\right)\mathcal{I}_{1-q,1-m,0-j}\right\}$$
$$\left.+\frac{1}{4}\mathcal{I}_{0-q,0-q,0-q,0-j}-\frac{1}{2}\mathcal{I}_{0-q,1-q,1-q,0-j}\right]$$
$$\mathcal{R}^{(j)}_m \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[-\left(-\frac{1}{2}\zeta^{(1)}_{m,q}+\alpha^2_{m,q}\xi^{(1)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}+\alpha_{m,q}\left(-\frac{1}{2}\zeta^{(1)}_{m,q}+\xi^{(1)}_{m,q}\right)\mathcal{I}_{1-q,1-m,0-j}\right]$$
$$\mathcal{S}^{(j)}_m \equiv \frac{1}{\mathrm{J_0}^2(l_j)}\left[-\left(\frac{1}{2}\zeta^{(1)}_{m,q}+\alpha^2_{m,q}\xi^{(1)}_{m,q}\right)\mathcal{I}_{0-q,0-m,0-j}+\alpha_{m,q}\left(\frac{1}{2}\zeta^{(1)}_{m,q}+\xi^{(1)}_{m,q}\right)\mathcal{I}_{1-q,1-m,0-j}\right]$$

and

$$\mathcal{T}^{(j)} = \alpha^2_{j,q}\mathcal{P}^{(j)}-\mathcal{A}^{(j)}$$
$$\mathcal{U}^{(j)} = \alpha^2_{j,q}\mathcal{Q}^{(j)}-3\mathcal{B}^{(j)}$$
$$\mathcal{V}^{(j)}_m = \alpha^2_{j,q}\mathcal{R}^{(j)}_m-\left(\omega_{m,q}+1\right)\mathcal{C}^{(j)}_m$$
$$\mathcal{W}^{(j)}_m = \alpha^2_{j,q}\mathcal{S}^{(j)}_m-\left(\omega_{m,q}-1\right)\mathcal{D}^{(j)}_m$$

and

$$\Omega_2 = \frac{1}{2\mathrm{J_0}^2(l_j)}\left[\sum_{m=1}^{\infty}\left\{\left(-\zeta^{(2)}_{m,q} - (\alpha^2_{m,q} - \alpha_{m,q})\xi^{(2)}_{m,q}\right)\mathcal{I}_{0-q,0-q,0-m}\right.\right.$$
$$\left.\left. +\alpha_{m,q}\left(\frac{1}{2}\zeta^{(2)}_{m,q} - \zeta^{(3)}_{m,q}\right)\mathcal{I}_{0-q,1-q,1-m}\right\} - \frac{3}{2}\mathcal{I}_{0-q,0-q,1-q,1-q} + \frac{3}{4}\mathcal{I}_{0-q,1-q,1-q,2-q}\right]$$

and

$$\lambda^{(q)} \equiv \mathcal{A}^{(q)} + \mathcal{B}^{(q)} + \frac{3}{8}\mathcal{U}^{(q)} + \sum_{\substack{m=1\\ m\neq q}}^{\infty}\left\{\mathcal{C}^{(q)}_m + \frac{(\omega_{m,q}+1)}{\omega_{m,q}(\omega_{m,q}+2)}\mathcal{V}^{(q)}_m + \mathcal{D}^{(q)}_m + \frac{(\omega_{m,q}-1)}{\omega_{m,q}(\omega_{m,q}-2)}\mathcal{W}^{(q)}_m\right\}$$

$$\lambda^{(j)} \equiv \frac{1}{\omega_{j,q}}\left[\left\{\mathcal{A}^{(j)} - \frac{\mathcal{T}^{(j)}}{(\omega^2_{j,q}-1)}\right\} + \left\{\mathcal{B}^{(j)} - \frac{3\mathcal{U}^{(j)}}{(\omega^2_{j,q}-9)}\right\}\right.$$
$$\left. + \sum_{\substack{m=1\\ m\neq q}}^{\infty}\left\{\mathcal{C}^{(j)}_m - \frac{\mathcal{V}^{(j)}_m(\omega_{m,q}+1)}{\left[\omega^2_{j,q} - (\omega_{m,q}+1)^2\right]} + \mathcal{D}^{(q)}_m - \frac{\mathcal{W}^{(j)}_m(\omega_{m,q}-1)}{\left[\omega^2_{j,q} - (\omega_{m,q}-1)^2\right]}\right\}\right]$$

2. Similarity solutions

2.1. *Delta function*

The algebra below obtains the axisymmetric Cauchy-Poisson solution for two different intial conditions. These conditions are discussed in Debnath (1994) for pure gravity waves and are re-derived for pure capillary waves, of interest to us here. With the initial condition

$$\hat{\eta}(\hat{r},0) = \frac{\hat{V}_0}{2\pi\hat{r}}\delta(\hat{r}) \tag{2.1}$$

The zeroth order Hankel transformation of $\hat{\eta}(\hat{r},0)$ viz. $\tilde{\eta}_0(k)$ is Debnath (1994)

$$\tilde{\eta}_0(k) = \frac{\hat{V}_0}{2\pi} \tag{2.2}$$

From the linearised Cauchy-Poisson solution for evolution of the interface $\eta(r,t)$, we obtain Debnath (1994)

$$\hat{\eta}(\hat{r},\hat{t}) = \int_0^\infty k\mathrm{J}_0(k\hat{r})\tilde{\eta}_0(k)\cos(\omega\hat{t})dk, \quad \omega^2 = \frac{Tk^3}{\rho} \tag{2.3}$$

When $k\hat{r} >> 1$, $\mathrm{J}_0(k\hat{r})$ can be approximated as

$$\mathrm{J}_0(k\hat{r}) \approx \left(\frac{2}{\pi k\hat{r}}\right)^{\frac{1}{2}}\cos\left(k\hat{r} - \frac{\pi}{4}\right) \tag{2.4}$$

Using this approximation $\hat{\eta}(\hat{r},\hat{t})$ can be written as follows

$$\hat{\eta}(\hat{r},\hat{t}) \sim \left(\frac{2}{\pi\hat{r}}\right)^{\frac{1}{2}} \times \frac{\hat{V}_0}{2\pi}\int_0^\infty k^{\frac{1}{2}}\cos\left(k\hat{r} - \frac{\pi}{4}\right)\cos(\omega\hat{t})dk \tag{2.5}$$

Using the formulae

$$\cos\left(k\hat{r} - \frac{\pi}{4}\right)\cos(\omega\hat{t}) = \frac{1}{4}\left[\exp\left[I\left(\omega\hat{t} + k\hat{r} - \frac{\pi}{4}\right)\right] + \mathrm{cc}_1 + \exp\left[I\left(\omega\hat{t} - k\hat{r} + \frac{\pi}{4}\right)\right] + \mathrm{cc}_2\right] \tag{2.6}$$

Here cc_1 and cc_2 are complex conjugates of terms on their left in equation 2.6. Considering outward travelling waves (term 3 and cc_2) one can obtain

$$\hat{\eta}(\hat{r},\hat{t}) \sim \frac{1}{4}\left(\frac{2}{\pi\hat{r}}\right)^{\frac{1}{2}} \frac{\hat{V}_0}{2\pi}\int_0^\infty k^{\frac{1}{2}} \exp\left[I\left(\omega\hat{t} - k\hat{r} + \frac{\pi}{4}\right)\right] dk + \mathrm{cc} \tag{2.7}$$

cc is the corresponding complex conjugate. Eqn. 2.7 can be asymptotically solved by method of stationary phase ($\hat{t} \longrightarrow \infty$) Rozman (2017) leading to,

$$\hat{\eta}(\hat{r},\hat{t}) \sim \frac{\hat{V}_0}{4\pi}\left(\frac{k_0}{\omega_0''\hat{r}\hat{t}}\right)^{\frac{1}{2}} \exp\left[I\left(\omega_0\hat{t} - k_0\hat{r} + \frac{\pi}{2}\right)\right] + \mathrm{cc} \tag{2.8}$$

Where k_0 is the stationary point of $g(k) = \omega\hat{t} - k\hat{r} + \frac{\pi}{2}$ (i.e $g'(k_0) = 0$) and $\omega_0 = \omega(k_0)$. For pure capillary waves ($\omega = (T'k^3)^{\frac{1}{2}}$, with $T' = \frac{T}{\rho}$) one can obtain

$$\boxed{k_0 = \frac{4\hat{r}^2}{9T'\hat{t}^2}} \tag{2.9}$$

Using the value of k_0 and ω_0 we find

$$\left(\frac{k_0}{\omega_0''\hat{r}\hat{t}}\right)^{\frac{1}{2}} = \frac{4\sqrt{2}}{9} \times \frac{\hat{r}}{\hat{t}^2 T'} \tag{2.10}$$

$$\omega_0\hat{t} - k_0\hat{r} = -\frac{4}{27} \times \frac{\hat{r}^3}{T'\hat{t}^2} \tag{2.11}$$

Substituting the expressions from 2.10 and 2.11 to 2.8 we obtain

$$\boxed{\hat{\eta}(\hat{r},\hat{t}) \sim \frac{2\sqrt{2}}{9\pi}\frac{\hat{V}_0\hat{r}}{T'\hat{t}^2} \sin\left(\frac{4}{27}\frac{\hat{r}^3}{T'\hat{t}^2}\right)} \tag{2.12}$$

From the asymptotic expression of $\hat{\eta}(\hat{r},\hat{t})$ (2.12) only two non dimensional parameters π_1, π_2 can be obtained. Where $\pi_1 = \frac{\hat{\eta}T'\hat{t}^2}{\hat{V}_0\hat{r}}$ and $\pi_2 = \frac{\hat{r}}{(T')^{1/3}\hat{t}^{2/3}}$. Here π_2 is the Keller & Miksis (1983) scale and π_1 and be written as a function of π_2 only ($\pi_1 = f(\pi_2)$), which makes the system self similar.

2.2. *An initial cavity*

In contrast to the delta function initial condition earlier which had only one length scale, we now choose an initial condition with two length-scales. This is Debnath (1994)

$$\hat{\eta}_0(\hat{r}) = \hat{d}\left(1 - \frac{\hat{r}^2}{\hat{a}^2}\right)\exp\left(-\frac{\hat{r}^2}{\hat{a}^2}\right) \tag{2.13}$$

representing a cavity or a hump for $\hat{d} \lessgtr 0$. The Hankel transform Debnath (1994) of this is

$$\tilde{\eta}_0(k) = \frac{\hat{d}\hat{a}^4}{8}k^2\exp\left(-\frac{1}{4}k^2\hat{a}^2\right) \tag{2.14}$$

Like earlier we can obtain an asymptotic expression like 2.7 using method of stationary phase

$$\hat{\eta}(\hat{r},\hat{t}) \sim \frac{1}{4}\left(\frac{2}{\pi\hat{r}}\right)^{\frac{1}{2}} \frac{\hat{d}\hat{a}^4}{8}\int_0^\infty k^{\frac{5}{2}}\exp\left(-\frac{1}{4}k^2\hat{a}^2\right)\exp\left[I\left(\omega\hat{t} - k\hat{r} + \frac{\pi}{4}\right)\right] dk + \mathrm{c.c.} \tag{2.15}$$

Using the method of stationary phase like before we get

$$\hat{\eta}(\hat{r},\hat{t}) \sim \frac{8\sqrt{2}}{729}\frac{\hat{d}\hat{a}^4\hat{r}^5}{(T')^3\hat{t}^6}\exp\left(-\frac{4\hat{a}^2\hat{r}^4}{81(T')^2\hat{t}^4}\right)\sin\left(\frac{4}{27}\frac{\hat{r}^3}{T'\hat{t}^2}\right) \tag{2.16}$$

In this case, along with two non dimensional parameters involving $\hat{\eta}$ and Keller & Miksis (1983) scale respectively viz. $\pi_1 = \frac{\hat{\eta}(T')^3\hat{t}^6}{\hat{d}\hat{a}^4\hat{r}^5}$ and $\pi_2 = \frac{\hat{r}}{(T')^{1/3}\hat{t}^{2/3}}$, another non dimensional group $\pi_3 = \frac{\hat{a}^2\hat{r}^4}{(T')^2\hat{t}^4}$ is present. In this case we get $\pi_1 = g(\pi_2,\pi_3)$, and the behaviour is not self similar.

3. Numerical simulations

The simulation geometry is shown in fig. 1 and uses the same boundary conditions as Basak *et al.* (2021).

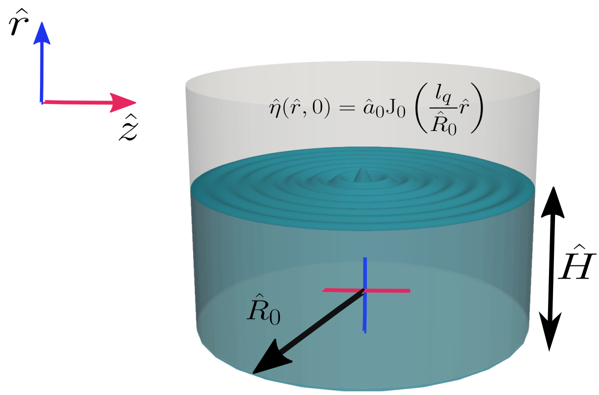

Figure 1: Simulation geometry

The simulation parameters are provided in table 1

Case	ϵ	l_q	$\hat{R}_0$	$\hat{a}_0$	$\hat{H}$
1	0.8	47.9	0.4	0.0059	0.5
2	1.8	47.90	0.4	0.015	0.5
3	2.3	47.90	0.4	0.0192	0.5
4	2.3	74.18	0.4	0.0124	0.5

Table 1: Simulations parameters (CGS units)

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