Supplementary material for "Membrane flutter in three-dimensional inviscid flow"

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1. Thin-membrane elasticity

We compute the stretching energy of the membrane using the position $\mathbf{r}(\alpha_1, \alpha_2, t)$. The membrane has thickness $h \ll L, W$, the lateral dimensions. We assume the stretching strain is constant through the thickness, accurate to leading order in h (Efrati *et al.* 2009).

We denote the flat prestrained configuration of the membrane central surface by $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, 0) = \mathbf{r}(\alpha_1, \alpha_2, 0)$. A small line of material connecting two material points $\boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\alpha}}$ in the flat prestrained configuration is $\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}$. We assume that the prestrained state is obtained from the zero-energy state by applying a uniform prestrain $\overline{e}$. Thus in the zero-energy state the small line of material is $\mathrm{d}\boldsymbol{\alpha}/(1+\overline{e})$, the prestrain having been removed by dividing by $(1+\overline{e})$.

One of the most common measures of deformation in nonlinear elasticity is the difference between the squared length of a material line in the deformed and zero-energy configurations (Landau & Lifshitz 1986; Tadmor *et al.* 2012):

$$\|\mathrm{d}\mathbf{r}\|^2 - \frac{1}{(1+\overline{e})^2}\|\mathrm{d}\boldsymbol{\alpha}\|^2 = 2\epsilon_{ij}\,\mathrm{d}\alpha_i \mathrm{d}\alpha_j\ , \quad \epsilon_{ij} = \frac{1}{2}\left(a_{ij} - \frac{1}{(1+\overline{e})^2}\delta_{ij}\right) \tag{S1.1}$$

where $\mathrm{d}\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2, 0) - \mathbf{r}(\tilde{\alpha}_1, \tilde{\alpha}_2, 0)$ and ϵ_{ij} is the (Green-Lagrange) strain tensor, written in terms of the metric tensor

$$a_{ij} = \partial_{\alpha_i}\mathbf{r} \cdot \partial_{\alpha_j}\mathbf{r} = \frac{\partial r_k}{\partial \alpha_i}\frac{\partial r_k}{\partial \alpha_j}, \quad i, j = 1, 2. \tag{S1.2}$$

and the identity tensor δ_{ij}.

For an isotropic membrane with Young's modulus E, Poisson ratio ν, and thickness h, the elastic energy per unit midsurface area (Alben *et al.* 2019) is

$$w_s = \frac{h}{2}\bar{A}^{mnop}\epsilon_{mn}\epsilon_{op}\ , \quad \bar{A}^{mnop} = \frac{E}{1+\nu}\left(\frac{\nu}{1-\nu}\delta_{mn}\delta_{op} + \delta_{mo}\delta_{np}\right) \tag{S1.3}$$

where $\bar{A}^{mnop}$ is the elasticity tensor for an isotropic material (Landau & Lifshitz 1986). For small-to-moderate prestrains, the focus of this work, the strain tensor in (S1.1) is approximately

$$\epsilon_{ij}(\alpha_1, \alpha_2, t) = \overline{e}\delta_{ij} + \frac{1}{2}\left(\partial_{\alpha_i}\mathbf{r} \cdot \partial_{\alpha_j}\mathbf{r} - \delta_{ij}\right), \quad i, j = 1, 2. \tag{S1.4}$$

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2. Derivation of pressure jump equation in 3D flow

In this section we derive an expression for the pressure jump $[p](\alpha_1, \alpha_2, t)$ across the membrane in terms of the membrane vortex sheet strength and other quantities, as a generalization of (Mavroyiakoumou & Alben 2020, appendix A).

The Euler momentum equation given by

$$\partial_t \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{u}(\boldsymbol{x}, t) = -\nabla p(\boldsymbol{x}, t) \tag{S2.1}$$

couples the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and the pressure $p(\boldsymbol{x}, t)$. We calculate the fluid pressure at a point that is adjacent to and follows a material point $\boldsymbol{r}(\alpha_1, \alpha_2, t)$ on the membrane. The rate of change of fluid velocity at such a point is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(\boldsymbol{r}(\alpha_1, \alpha_2, t), t) = \partial_t \boldsymbol{u}(\boldsymbol{x}, t)|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)} + (\partial_t \boldsymbol{r}(\alpha_1, \alpha_2, t) \cdot \nabla)\, \boldsymbol{u}(\boldsymbol{x}, t)|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)}. \tag{S2.2}$$

We use (S2.2) to replace the first term in (S2.1) and write the pressure gradient at a point that moves with $\boldsymbol{r}(\alpha_1, \alpha_2, t)$. To obtain the jump in fluid pressure at a material point on the membrane, we write (S2.1) (modified by (S2.2)) separately at points in the fluid that approach the membrane from either side:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(\boldsymbol{r}(\alpha_1, \alpha_2, t), t)^{\pm} + \left((\boldsymbol{u}(\boldsymbol{x}, t) - \partial_t \boldsymbol{r}) \cdot \nabla \boldsymbol{u}(\boldsymbol{x}, t)\Big|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)} \right)^{\pm} = -(\nabla p(\boldsymbol{x}, t)|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)})^{\pm}, \tag{S2.3}$$

using $+$ for the side toward which the membrane normal $\hat{\mathbf{n}}$ is directed and $-$ for the other side.

Next, we decompose the fluid velocity into components tangential and normal to the membrane. The normal component matches that of the membrane, ν_v. The tangential component of the fluid velocity may be written in terms of its jump across the membrane, using the vortex sheet strength components γ_1, γ_2 (Saffman 1992), and the average of the tangential components of the fluid velocity on the two sides of the membrane, denoted μ_1 and μ_2. The subscripts $\{1, 2\}$ denote the components in the $\widehat{\mathbf{s}}_1$ and $\widehat{\mathbf{s}}_2$ directions, respectively. The fluid velocity can be written

$$\boldsymbol{u}^{\pm} = \left(\mu_1 \pm \frac{\gamma_2}{2}\right)\widehat{\mathbf{s}}_1 + \left(\mu_2 \mp \frac{\gamma_1}{2}\right)\widehat{\mathbf{s}}_2 + \nu_v \hat{\mathbf{n}}. \tag{S2.4}$$

We take the difference of (S2.3) on the $+$ and $-$ sides:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{u}^{+}(\boldsymbol{r}(\alpha_1, \alpha_2, t), t) - \boldsymbol{u}^{-}(\boldsymbol{r}(\alpha_1, \alpha_2, t), t)\right) + \left((\boldsymbol{u}^{+}(\boldsymbol{x}, t) - \partial_t \boldsymbol{r}) \cdot \nabla \boldsymbol{u}^{+}(\boldsymbol{x}, t)\Big|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)} \right) \\ - \left((\boldsymbol{u}^{-}(\boldsymbol{x}, t) - \partial_t \boldsymbol{r}) \cdot \nabla \boldsymbol{u}^{-}(\boldsymbol{x}, t)\Big|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)} \right) = -(\nabla p(\boldsymbol{x}, t)^{+} - \nabla p(\boldsymbol{x}, t)^{-})|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)}.\end{aligned} \tag{S2.5}$$

We then take the $\widehat{\mathbf{s}}_1$ components of (S2.5), term by term. The $\widehat{\mathbf{s}}_1$ component of the right hand side of (S2.5) is $-\partial_{s_1}[p]^{+}_{-}$ which we will ultimately integrate to obtain $[p]^{+}_{-}$. $-\partial_{s_1}[p]^{+}_{-}$ is equal to the $\widehat{\mathbf{s}}_1$ components of the terms on the left hand side of (S2.5), which we now compute. Using (S2.4), the $\widehat{\mathbf{s}}_1$ component of the first term on the left-hand side of (S2.5) is

$$\widehat{\mathbf{s}}_1 \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^+(\boldsymbol{r}(\alpha_1,\alpha_2,t),t) - \boldsymbol{u}^-(\boldsymbol{r}(\alpha_1,\alpha_2,t),t)) = \widehat{\mathbf{s}}_1 \cdot \partial_t(-\gamma_1\widehat{\mathbf{s}}_2 + \gamma_2\widehat{\mathbf{s}}_1)$$
$$= \partial_t\gamma_2 - \partial_t\gamma_1(\widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2) - \gamma_1(\widehat{\mathbf{s}}_1 \cdot \partial_t\widehat{\mathbf{s}}_2), \tag{S2.6}$$

where we use $\widehat{\mathbf{s}}_i \cdot \widehat{\mathbf{s}}_i = 1$ and $\widehat{\mathbf{s}}_i \cdot \partial_t\widehat{\mathbf{s}}_i = 0$ for $i = 1, 2$.

For $\widehat{\mathbf{s}}_i \cdot [(\boldsymbol{u}^\pm(\boldsymbol{x},t) - \partial_t\boldsymbol{r}) \cdot \nabla\boldsymbol{u}^\pm(\boldsymbol{x},t)]$ where $i = 1, 2$, we first write $\boldsymbol{u}^\pm - \partial_t\boldsymbol{r}$ as $A\widehat{\mathbf{s}}_1 + B\widehat{\mathbf{s}}_2$. Then we compute the dot product between that and $\nabla\boldsymbol{u}^\pm$ and obtain $A\partial_{s_1}\boldsymbol{u}^\pm + B\partial_{s_2}\boldsymbol{u}^\pm$. If we substitute $\boldsymbol{u}^\pm$, we get an expression of the form $C\widehat{\mathbf{s}}_1 + D\widehat{\mathbf{s}}_2$, which we can finally dot with $\widehat{\mathbf{s}}_1$ and $\widehat{\mathbf{s}}_2$.

Using

$$\partial_t\boldsymbol{r} = \tau_1\widehat{\mathbf{s}}_1 + \tau_2\widehat{\mathbf{s}}_2 + \nu_v\hat{\mathbf{n}}, \tag{S2.7}$$

and (S2.4), we find that the $\widehat{\mathbf{s}}_1$ components of the second and third terms on the left-hand side of (S2.5) are

$$\begin{aligned}
&\widehat{\mathbf{s}}_1 \cdot \left[(\boldsymbol{u}^\pm(\boldsymbol{x},t) - \partial_t\boldsymbol{r}) \cdot \nabla\boldsymbol{u}^\pm(\boldsymbol{x},t)\right]\Big|_{\boldsymbol{x}=\boldsymbol{r}(\alpha_1,\alpha_2,t)} \\
&= \left(\mu_1 \pm \frac{\gamma_2}{2} - \tau_1\right)\left[\partial_{s_1}\left(\mu_1 \pm \frac{\gamma_2}{2}\right) + \left(\partial_{s_1}\left(\mu_2 \mp \frac{\gamma_1}{2}\right)\right)\widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2\right. \\
&\left.+ \left(\mu_2 \mp \frac{\gamma_1}{2}\right)\widehat{\mathbf{s}}_1 \cdot \partial_{s_1}\widehat{\mathbf{s}}_2 + \nu_v\widehat{\mathbf{s}}_1 \cdot \partial_{s_1}\hat{\mathbf{n}}\right] \\
&+ \left(\mu_2 \mp \frac{\gamma_1}{2} - \tau_2\right)\left[\partial_{s_2}\left(\mu_1 \pm \frac{\gamma_2}{2}\right) + \left(\partial_{s_2}\left(\mu_2 \mp \frac{\gamma_1}{2}\right)\right)\widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2\right. \\
&\left.+ \left(\mu_2 \mp \frac{\gamma_1}{2}\right)\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\widehat{\mathbf{s}}_2 + \nu_v\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\hat{\mathbf{n}} + \left(\mu_1 \pm \frac{\gamma_2}{2}\right)\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\widehat{\mathbf{s}}_1\right].
\end{aligned} \tag{S2.8}$$

The difference of the + and − terms on the right-hand side of (S2.8) is

$$\begin{aligned}
&(\widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2)(-\mu_1\partial_{s_1}\gamma_1 + \gamma_2\partial_{s_1}\mu_2 + \tau_1\partial_{s_1}\gamma_1 - \mu_2\partial_{s_2}\gamma_1 - \gamma_1\partial_{s_2}\mu_2 + \tau_2\partial_{s_2}\gamma_1) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \partial_{s_1}\widehat{\mathbf{s}}_2)(-\mu_1\gamma_1 + \gamma_2\mu_2 + \tau_1\gamma_1) \\
&+ (\mu_1\partial_{s_1}\gamma_2 + \gamma_2\partial_{s_1}\mu_1 - \tau_1\partial_{s_1}\gamma_2 + \mu_2\partial_{s_2}\gamma_2 - \gamma_1\partial_{s_2}\mu_1 - \tau_2\partial_{s_2}\gamma_2) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\widehat{\mathbf{s}}_1)(\mu_2\gamma_2 - \gamma_1\mu_1 - \tau_2\gamma_2) + (\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\widehat{\mathbf{s}}_2)(-2\mu_2\gamma_1 + \tau_2\gamma_1) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \partial_{s_1}\hat{\mathbf{n}})\gamma_2\nu_v - (\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\hat{\mathbf{n}})\gamma_1\nu_v,
\end{aligned} \tag{S2.9}$$

using $\widehat{\mathbf{s}}_i \cdot \partial_{s_i}\widehat{\mathbf{s}}_i = 0$ and $\hat{\mathbf{n}} \cdot \widehat{\mathbf{s}}_i = 0$ for $i = 1, 2$.

Combining (S2.6) and (S2.9), the $\widehat{\mathbf{s}}_1$ component of (S2.5) is

$$\begin{aligned}
&\partial_t\gamma_2 - \partial_t\gamma_1(\widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2) - \gamma_1(\widehat{\mathbf{s}}_1 \cdot \partial_t\widehat{\mathbf{s}}_2) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2)(-\mu_1\partial_{s_1}\gamma_1 + \gamma_2\partial_{s_1}\mu_2 + \tau_1\partial_{s_1}\gamma_1 - \mu_2\partial_{s_2}\gamma_1 - \gamma_1\partial_{s_2}\mu_2 + \tau_2\partial_{s_2}\gamma_1) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \partial_{s_1}\widehat{\mathbf{s}}_2)(-\mu_1\gamma_1 + \gamma_2\mu_2 + \tau_1\gamma_1) \\
&+ (\mu_1\partial_{s_1}\gamma_2 + \gamma_2\partial_{s_1}\mu_1 - \tau_1\partial_{s_1}\gamma_2 + \mu_2\partial_{s_2}\gamma_2 - \gamma_1\partial_{s_2}\mu_1 - \tau_2\partial_{s_2}\gamma_2) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\widehat{\mathbf{s}}_1)(\mu_2\gamma_2 - \gamma_1\mu_1 - \tau_2\gamma_2) + (\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\widehat{\mathbf{s}}_2)(-2\mu_2\gamma_1 + \tau_2\gamma_1) \\
&+ (\widehat{\mathbf{s}}_1 \cdot \partial_{s_1}\hat{\mathbf{n}})\gamma_2\nu_v - (\widehat{\mathbf{s}}_1 \cdot \partial_{s_2}\hat{\mathbf{n}})\gamma_1\nu_v \\
&= -\partial_{s_1}[p]^+_-.
\end{aligned} \tag{S2.10}$$

We multiply (S2.10) through by $\partial_{\alpha_1}s_1$ which converts $\partial_{s_1}[p]^+_-$ to $\partial_{\alpha_1}[p]^+_-$. We integrate with respect to α_1 from the trailing edge, applying $[p]^+_- = 0$ at the trailing edge, to obtain $[p]^+_-$ at all points on the membrane.

If we directly integrate $\partial_{\alpha_1}[p]^+_-$ numerically (e.g. using the trapezoidal rule), the results disagree significantly with our 2D benchmark results (Mavroyiakoumou & Alben 2020). Using a different formulation that agrees with the 2D results in the small-amplitude regime, we have found empirically that the following method agrees well with the 2D results in both the small- and large-amplitude regimes. We first write $\partial_{\alpha_1}[p]^+_-$ as a sum of two terms:

$$\partial_{\alpha_1}[p]^+_- = (\partial_{\alpha_1}[p]^+_- + \partial_{\alpha_1}s_1\partial_t\gamma_2 + \partial_{\alpha_1}\gamma_2) + (-\partial_{\alpha_1}s_1\partial_t\gamma_2 - \partial_{\alpha_1}\gamma_2). \tag{S2.11}$$

We have added and subtracted $\partial_{\alpha_1}s_1\partial_t\gamma_2 + \partial_{\alpha_1}\gamma_2$. Its integral with respect to α_1 can be written $\partial_t\Gamma + \partial_{s_1}\Gamma$, where Γ is the integrated vortex sheet strength (Saffman 1992). It is conserved at points of a free vortex sheet that move at the average of the tangential flow velocities on the two sides of the sheet, equal to 1 in this case. Hence $\partial_t\Gamma + \partial_{s_1}\Gamma = 0$ at the trailing edge, as does $[p]^+_-$ by the unsteady Kutta condition (Saffman 1992; Katz & Plotkin 2001).

Therefore we integrate the first term in (S2.11) (in parentheses) with respect to α_1 using the trapezoidal rule, with the boundary condition that its integral is 0 at the trailing edge. The integral of the remaining terms in (S2.11) is $-\partial_t\Gamma - \partial_{s_1}\Gamma$ evaluated at α_1 minus its value at the trailing edge, which is zero as we have just discussed. To evaluate $-\partial_t\Gamma - \partial_{s_1}\Gamma$ on the (α_1,α_2) grid, we approximate Γ at the center point of a membrane vortex panel by the circulation of the vortex ring at that panel. Derivatives of Γ with respect to t and α_1 are obtained by the usual finite-difference formulas and extrapolation to the (α_1,α_2) grid points—i.e. the corner points of the panels.

3. Residual membrane equations in Broyden's method

In this section we write down the discretized system of nonlinear membrane equations that we solve using Broyden's method. Having computed each of the quantities in the membrane equation ((2.16) in the main manuscript) we keep iterating until $\mathbf{f}(\mathbf{x})$ drops below a certain tolerance that we set to 10^{-5}. Here $\mathbf{f}(\mathbf{x})$ is given by:

$$\begin{aligned} f_j(\mathbf{x}) = R_1\partial_{tt}x_j^k - K_s\Big\{&(D^2_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_1}\mathbf{r}^k_j)D^1_{\alpha_1}x^k_j + \epsilon_{11}D^2_{\alpha_1}x^k_j \\ &+ \nu\left((D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j)D^1_{\alpha_1}x^k_j + \epsilon_{22}D^2_{\alpha_1}x^k_j\right) \\ &+ (1-\nu)\left[((D^2_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j + D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j)/2)D^1_{\alpha_2}x^k_j + \epsilon_{12}D^2_{\alpha_1\alpha_2}x^k_j\right] \\ &+ (D^2_{\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j)D^1_{\alpha_2}x^k_j + \epsilon_{22}D^2_{\alpha_2}x^k_j + \nu((D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_1}\mathbf{r}^k_j)D^1_{\alpha_2}x^k_j + \epsilon_{11}D^2_{\alpha_2}x^k_j) \\ &+ (1-\nu)\left[((D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j + D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^2_{\alpha_2}\mathbf{r}^k_j)/2)D^1_{\alpha_1}x^k_j + \epsilon_{12}D^2_{\alpha_1\alpha_2}x^k_j\right]\Big\} \\ &+ [p]^k_j\hat{\mathbf{n}}^k_{x,j}\sqrt{(D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_1}\mathbf{r}^k_j)(D^1_{\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j) - (D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j)^2}, \end{aligned} \tag{S3.1}$$

$$\begin{aligned} f_{j+(M-1)(N-1)}(\mathbf{x}) = R_1\partial_{tt}y_j^k - K_s\Big\{&(D^2_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_1}\mathbf{r}^k_j)D^1_{\alpha_1}y^k_j + \epsilon_{11}D^2_{\alpha_1}y^k_j \\ &+ \nu\left((D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j)D^1_{\alpha_1}y^k_j + \epsilon_{22}D^2_{\alpha_1}y^k_j\right) \\ &+ (1-\nu)\left[((D^2_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j + D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j)/2)D^1_{\alpha_2}y^k_j + \epsilon_{12}D^2_{\alpha_1\alpha_2}y^k_j\right] \\ &+ (D^2_{\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j)D^1_{\alpha_2}y^k_j + \epsilon_{22}D^2_{\alpha_2}y^k_j + \nu((D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_1}\mathbf{r}^k_j)D^1_{\alpha_2}y^k_j + \epsilon_{11}D^2_{\alpha_2}y^k_j) \\ &+ (1-\nu)\left[((D^2_{\alpha_1\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j + D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^2_{\alpha_2}\mathbf{r}^k_j)/2)D^1_{\alpha_1}y^k_j + \epsilon_{12}D^2_{\alpha_1\alpha_2}y^k_j\right]\Big\} \\ &+ [p]^k_j\hat{\mathbf{n}}^k_{y,j}\sqrt{(D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_1}\mathbf{r}^k_j)(D^1_{\alpha_2}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j) - (D^1_{\alpha_1}\mathbf{r}^k_j\cdot D^1_{\alpha_2}\mathbf{r}^k_j)^2}, \end{aligned} \tag{S3.2}$$

$$
\begin{aligned}
f_{j+2(M-1)(N-1)}(\mathbf{x}) &= R_1\partial_{tt}z_j^k - K_s\left\{(D^2_{\alpha_1}\mathbf{r}_j^k\cdot D^1_{\alpha_1}\mathbf{r}_j^k)D^1_{\alpha_1}z_j^k + \epsilon_{11}D^2_{\alpha_1}z_j^k\right.\\
&+\nu\left((D^2_{\alpha_1\alpha_2}\mathbf{r}_j^k\cdot D^1_{\alpha_2}\mathbf{r}_j^k)D^1_{\alpha_1}z_j^k + \epsilon_{22}D^2_{\alpha_1}z_j^k\right)\\
&+(1-\nu)\left[((D^2_{\alpha_1}\mathbf{r}_j^k\cdot D^1_{\alpha_2}\mathbf{r}_j^k + D^1_{\alpha_1}\mathbf{r}_j^k\cdot D^2_{\alpha_1\alpha_2}\mathbf{r}_j^k)/2)D^1_{\alpha_2}z_j^k + \epsilon_{12}D^2_{\alpha_1\alpha_2}z_j^k\right]\\
&+(D^2_{\alpha_2}\mathbf{r}_j^k\cdot D^1_{\alpha_2}\mathbf{r}_j^k)D^1_{\alpha_2}z_j^k + \epsilon_{22}D^2_{\alpha_2}z_j^k + \nu((D^2_{\alpha_1\alpha_2}\mathbf{r}_j^k\cdot D^1_{\alpha_1}\mathbf{r}_j^k)D^1_{\alpha_2}z_j^k + \epsilon_{11}D^2_{\alpha_2}z_j^k)\\
&+(1-\nu)\left.\left[((D^2_{\alpha_1\alpha_2}\mathbf{r}_j^k\cdot D^1_{\alpha_2}\mathbf{r}_j^k + D^1_{\alpha_1}\mathbf{r}_j^k\cdot D^2_{\alpha_2}\mathbf{r}_j^k)/2)D^1_{\alpha_1}z_j^k + \epsilon_{12}D^2_{\alpha_1\alpha_2}z_j^k\right]\right\}\\
&+[p]_j^k\hat{\mathbf{n}}_{z,j}^k\sqrt{(D^1_{\alpha_1}\mathbf{r}_j^k\cdot D^1_{\alpha_1}\mathbf{r}_j^k)(D^1_{\alpha_2}\mathbf{r}_j^k\cdot D^1_{\alpha_2}\mathbf{r}_j^k) - (D^1_{\alpha_1}\mathbf{r}_j^k\cdot D^1_{\alpha_2}\mathbf{r}_j^k)^2},
\end{aligned}
\tag{S3.3}
$$

where $j = 1, \dots, (M-1)(N-1)$ and $[p]$ is the result of integrating (S2.11). Here $D^1_{\alpha_i} \approx \partial_{\alpha_i}$, $D^2_{\alpha_i} \approx \partial_{\alpha_i\alpha_i}$ for $i = 1, 2$ and $D^2_{\alpha_1\alpha_2} \approx \partial_{\alpha_1\alpha_2}$ are second-order accurate finite-difference matrix approximations to first and second-order derivatives (order listed in the superscript) on uniform grid nodes, one-sided at boundaries. $\partial_{tt}\{x_j^k, y_j^k, z_j^k\}$ are the second-order accurate finite difference formulas (backward differentiation formulas) at time step k based on the values of $\{x_j, y_j, z_j\}$ at times steps $k-3, \dots, k$.

4. Classification of instabilities in the linear growth regime

Here figure S1 marks (R_1, T_0) pairs where the small-amplitude instability corresponds to divergence or flutter with divergence, for each of the 12 boundary conditions and membrane aspect ratio 1. Divergence (without flutter) is most common for group 1 (fixed–fixed), while flutter with divergence is more common for the other three groups.

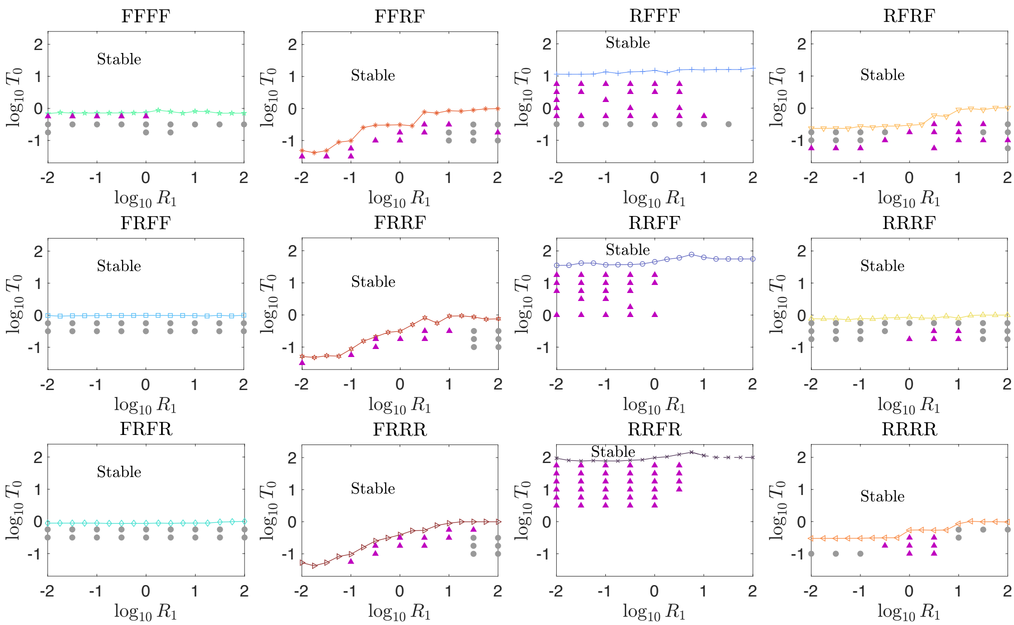

FIGURE S1. Classification of instability types for the 12 boundary conditions. Columns 1–4 correspond to groups 1–4 in figures 1 and 9 of the main manuscript. The colored lines separate the regions where the membranes are stable and unstable. The purple triangles mark where membranes become unstable through flutter and divergence and the gray dots mark where membranes lose stability by divergence.

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