Supplementary Material

1. Derivation of F_l

To begin, each force defined listed in (2.4) is written as a combination of scalar potentials $(\mathcal{R}, \mathcal{S}, \mathcal{T})$ such that

$$\boldsymbol{F} = \mathcal{R}\hat{\boldsymbol{r}} + r\nabla\mathcal{S} + \boldsymbol{r} \times \nabla\mathcal{T}, \tag{S.1}$$

where $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ represent the radial, spheroidal, and toroidal fields, respectively. These scalar potentials are then expanded in terms of spherical harmonics (i.e. $\{\mathcal{R}, \mathcal{S}, \mathcal{T}\} = \{\mathcal{R}_l^m(r), \mathcal{S}_l^m(r), \mathcal{T}_l^m(r)\}\, Y_l^m(\theta, \phi)$) whence

$$\boldsymbol{F} = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \mathcal{R}_l^m Y_l^m \hat{\boldsymbol{r}} + r\mathcal{S}_l^m \nabla Y_l^m + \mathcal{T}_l^m \boldsymbol{r} \times \nabla Y_l^m \tag{S.2}$$

$$= \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \mathcal{R}_l^m \boldsymbol{Y}_l^m + \mathcal{S}_l^m \boldsymbol{\Psi}_l^m + \mathcal{T}_l^m \boldsymbol{\Phi}_l^m, \tag{S.3}$$

where $\boldsymbol{Y}_l^m = Y_l^m \hat{\boldsymbol{r}}$, $\boldsymbol{\Psi}_l^m = r\nabla Y_l^m$, and $\boldsymbol{\Phi}_l^m = \boldsymbol{r} \times \nabla Y_l^m$ are the vector spherical harmonics. and $l_{\max}$ is the truncation of the spherical harmonic degree for any given simulation. Using the orthonormal basis properties of the spherical harmonics, the energy of the force vector is

$$F^2 = \int \boldsymbol{F}^2 \mathrm{d}V \tag{S.4}$$

$$= \sum_{l=0}^{l_{\max}} \int_{r_\mathrm{i}+b}^{r_\mathrm{o}-b} \left[|\mathcal{R}_l^0|^2 \;+\; l(l+1) \left(|\mathcal{S}_l^0|^2 + |\mathcal{T}_l^0|^2 \right) \right] r^2 \mathrm{d}r \tag{S.5}$$

$$+\; 2 \sum_{l=0}^{l_{\max}} \sum_{m=1}^{l} \int_{r_\mathrm{i}+b}^{r_\mathrm{o}-b} \left[|\mathcal{R}_l^m|^2 \;+\; l(l+1) \left(|\mathcal{S}_l^m|^2 + |\mathcal{T}_l^m|^2 \right) \right] r^2 \mathrm{d}r \tag{S.6}$$

$$= 2 \sum_{l=0}^{l_{\max}} {\sum_{m=0}^{l}}' \int_{r_\mathrm{i}+b}^{r_\mathrm{o}-b} \left[|\mathcal{R}_l^m|^2 \;+\; l(l+1) \left(|\mathcal{S}_l^m|^2 + |\mathcal{T}_l^m|^2 \right) \right] r^2 \mathrm{d}r \tag{S.7}$$

$$= \sum_{l=0}^{l_{\max}} F_l^2, \tag{S.8}$$

where the prime denotes a halving of the $m = 0$ term in the sum. Here $b \sim O(E^{1/2})$ represents the boundary layer thickness; note that b can instead be set to zero to include boundary layers in the calculation. In this process we have defined

$$F_l^2 = 2 {\sum_{m=0}^{l}}' \int_{r_\mathrm{i}+b}^{r_\mathrm{o}-b} \left[|\mathcal{R}_l^m|^2 \;+\; l(l+1) \left(|\mathcal{S}_l^m|^2 + |\mathcal{T}_l^m|^2 \right) \right] r^2 \mathrm{d}r, \tag{S.9}$$

which gives the power spectra, F_l^2, of an individual *force* as a function of spherical harmonic degree, l.

2. Derivation of C_l

In an identical process to above, the curl of any force defined in (2.4) can be calculated and written in the same form such that

$$\mathbb{C}(\boldsymbol{F}) = \nabla \times \boldsymbol{F} = \hat{\mathcal{R}}\hat{\boldsymbol{r}} + r\nabla\hat{\mathcal{S}} + \boldsymbol{r} \times \nabla\hat{\mathcal{T}}. \tag{S.10}$$

The scalar potentials are hatted here simply to indicate they are not the same as those appearing in the expansion of force itself (cf. (S.1)). Following the same method used for forces we then have

$$C^2 \equiv \int [\mathbb{C}(\boldsymbol{F})]^2 \mathrm{d}V = \sum_{l=0}^{l_{\max}} C_l^2, \tag{S.11}$$

where

$$C_l^2 = 2\sum_{m=0}^{l}{}' \int_{r_\mathrm{i}+b}^{r_\mathrm{o}-b} \left[|\hat{\mathcal{R}}_l^m|^2 \; + \; l(l+1)\left(|\hat{\mathcal{S}}_l^m|^2 + |\hat{\mathcal{T}}_l^m|^2\right)\right] r^2 \mathrm{d}r, \tag{S.12}$$

giving the power spectra of an individual *curl of a force* as a function of spherical harmonic degree, l.

3. Derivation of P_l

For a given force, $\boldsymbol{F}$, if we write $\boldsymbol{F} = \nabla \times \boldsymbol{A} + \nabla\varphi$ for some potentials $\boldsymbol{A}$ and φ, then $\nabla \cdot \boldsymbol{F} = \nabla^2\varphi$. Then the projected force is defined as

$$\mathbb{P}(\boldsymbol{F}) = \boldsymbol{F} - \nabla\varphi. \tag{S.13}$$

The goal is to calculate φ so $\mathbb{P}(\boldsymbol{F})$ can be formed. Defining ψ to be the divergence of $\boldsymbol{F}$, we find

$$\boldsymbol{F} = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \mathcal{R}_l^m \boldsymbol{Y}_l^m + \mathcal{S}_l^m \boldsymbol{\Psi}_l^m + \mathcal{T}_l^m \boldsymbol{\Phi}_l^m \tag{S.14}$$

$$\Rightarrow \psi \equiv \nabla \cdot \boldsymbol{F} = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \left(\frac{\mathrm{d}\mathcal{R}_l^m}{\mathrm{d}r} + \frac{2}{r}\mathcal{R}_l^m - \frac{l(l+1)}{r}\mathcal{S}_l^m\right) Y_l^m = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \psi_l^m Y_l^m, \tag{S.15}$$

which defines ψ_l^m in terms of $\mathcal{R}_l^m$ and $\mathcal{S}_l^m$. Also expanding the scalar potential in terms of spherical harmonics we have

$$\varphi = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \varphi_l^m(r) Y_l^m(\theta, \phi) \tag{S.16}$$

$$\Rightarrow \nabla\varphi = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \frac{\mathrm{d}\varphi_l^m}{\mathrm{d}r}\boldsymbol{Y}_l^m + \frac{\varphi_l^m}{r}\boldsymbol{\Psi}_l^m \tag{S.17}$$

$$\Rightarrow \nabla^2\varphi = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2}\right)\varphi_l^m Y_l^m. \tag{S.18}$$

Hence, since $\psi = \nabla \cdot \boldsymbol{F} = \nabla^2\varphi$, for each l and m we have

$$\psi_l^m \equiv \frac{\mathrm{d}\mathcal{R}_l^m}{\mathrm{d}r} + \frac{2}{r}\mathcal{R}_l^m - \frac{l(l+1)}{r}\mathcal{S}_l^m = \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2}\right)\varphi_l^m, \tag{S.19}$$

which must be solved for each l and m subject to boundary conditions on φ_l^m. The boundary conditions applied are $\partial_n \varphi = \boldsymbol{F} \cdot \boldsymbol{n}$ where $\boldsymbol{n}$ is the unit vector normal to the boundary. As discussed in the main text, the potential, φ is then determined up to an arbitrary harmonic potential set by the boundary conditions.

Once φ_l^m is found the projection, $\mathbb{P}(\boldsymbol{F})$, is calculated according to (S.13). Following the same method as used above, the projection of any force defined in (2.4) can be decomposed as

$$\mathbb{P}(\boldsymbol{F}) = \tilde{\mathcal{R}}\hat{\boldsymbol{r}} + r\nabla\tilde{\mathcal{S}} + \boldsymbol{r} \times \nabla\tilde{\mathcal{T}}, \tag{S.20}$$

where the scalar potentials wear tildes here to indicate they are not the same as those appearing in the expansion of the original force itself (cf. (S.1)). In an identical procedure to previous sections, it follows that

$$P^2 \equiv \int [\mathbb{P}(\boldsymbol{F})]^2 dV = \sum_{l=0}^{l_{\max}} P_l^2, \tag{S.21}$$

where

$$P_l^2 = 2\sum_{m=0}^{l}{}' \int_{r_{\mathrm{i}}+b}^{r_{\mathrm{o}}-b} \left[|\tilde{\mathcal{R}}_l^m|^2 \ + \ l(l+1) \left(|\tilde{\mathcal{S}}_l^m|^2 + |\tilde{\mathcal{T}}_l^m|^2 \right) \right] r^2 \mathrm{d}r, \tag{S.22}$$

giving the power spectra of an individual *force projection* as a function of spherical harmonic degree, l.