Indirect noise from weakly reacting inhomogeneities: Supplementary material

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(Received xx; revised xx; accepted xx)

1. Calculation of γ' and c_p'

We assume the flow at nozzle inlet comprises of two constituents, $\dot{m}^0_{air}$ kg/s air and $\dot{m}^0_{fuel}$ kg/s fuel. We assume $\dot{m}^0_{air} >> \dot{m}^0_{fuel}$, hence, total mass, $\dot{m}_T \approx \dot{m}^0_{air}$.
For combustion of methane,

$$\mathrm{CH_4} + 2O_2 \longrightarrow \mathrm{CO_2} + 2\mathrm{H_2O} \tag{1.1}$$

at a particular location in the nozzle, we assume a moles of fuel are burnt. We have four components, air, fuel, CO_2 and H_2O. Hence, a moles of CO_2 and $2a$ moles of H_2O are formed, while, a moles of CH_4 and $2a$ moles of O_2 are consumed. Hence, we have $\dot{m}^0_{fuel} - a * (W)_{fuel}$ kg/s fuel, $a * (W)_{\mathrm{CO_2}}$ kg/s CO_2 and $2a * (W)_{\mathrm{H_2O}}$ kg/s H_2O, where W is the molecular weight. We assume the mass of air remains constant. Therefore,

$$Y_{fuel} = \frac{\dot{m}^0_{fuel} - a * (W)_{fuel}}{\dot{m}_T} \tag{1.2}$$

$$Y_{air} = \frac{\dot{m}^0_{air} - 2a * (W)_{O_2}}{\dot{m}_T} \tag{1.3}$$

$$Y_{\mathrm{CO_2}} = \frac{a * (W)_{\mathrm{CO_2}}}{\dot{m}_T} \tag{1.4}$$

$$Y_{\mathrm{H_2O}} = \frac{2a * (W)_{\mathrm{H_2O}}}{\dot{m}_T} \tag{1.5}$$

The specific heat at constant pressure is defined as, $c_p = \sum_{i=1}^{N} c_{p,i} Y_i$, hence,

$$c_p = c_{p_{air}} Y_{air} + c_{p_{fuel}} Y_{fuel} + c_{p_{\mathrm{CO}_2}} Y_{\mathrm{CO}_2} + c_{p_{\mathrm{H}_2\mathrm{O}}} Y_{\mathrm{H}_2\mathrm{O}} \tag{1.6}$$

$$= c_{p_{air}} \frac{\dot{m}^0_{air} - 2a * (W)_{O_2}}{\dot{m}_T} + c_{p_{fuel}} \frac{\dot{m}^0_{fuel} - a * (W)_{fuel}}{\dot{m}_T} + c_{p_{\mathrm{CO}_2}} \frac{a * (W)_{\mathrm{CO}_2}}{\dot{m}_T} \dots$$

$$\dots + c_{p_{\mathrm{H}_2\mathrm{O}}} \frac{2a * (W)_{\mathrm{H}_2\mathrm{O}}}{\dot{m}_T} \tag{1.7}$$

Taking the derivative of c_p with respect to the mass fraction of species, by using

$$\frac{dc_p}{dY_i} = \frac{dc_p}{da} \frac{da}{dY_i} \tag{1.8}$$

yields

$$\frac{dc_p}{da} = c_{p_{fuel}} \frac{-(W)_{fuel}}{\dot{m}_T} + c_{p_{air}} \frac{-2(W)_{O_2}}{\dot{m}_T} + c_{p_{\mathrm{CO}_2}} \frac{(W)_{\mathrm{CO}_2}}{\dot{m}_T} + c_{p_{\mathrm{H}_2\mathrm{O}}} \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{\dot{m}_T} \tag{1.9}$$

and,

$$\frac{dY_{fuel}}{da} = \frac{-(W)_{fuel}}{\dot{m}_T} \tag{1.10}$$

$$\frac{dY_{air}}{da} = \frac{-2(W)_{O_2}}{\dot{m}_T} \tag{1.11}$$

$$\frac{dY_{\mathrm{CO}_2}}{da} = \frac{(W)_{\mathrm{CO}_2}}{\dot{m}_T} \tag{1.12}$$

$$\frac{dY_{\mathrm{H}_2\mathrm{O}}}{da} = \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{\dot{m}_T} \tag{1.13}$$

$$\frac{dc_p}{dY_{fuel}} = c_{p_{fuel}} + c_{p_{air}} \frac{(W)_{O_2}}{(W)_{fuel}} - c_{p_{\mathrm{CO}_2}} \frac{(W)_{\mathrm{CO}_2}}{(W)_{fuel}} - c_{p_{\mathrm{H}_2\mathrm{O}}} \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{(W)_{fuel}} \tag{1.14}$$

$$\frac{dc_p}{dY_{air}} = c_{p_{fuel}} \frac{(W)_{fuel}}{2 * (W)_{O_2}} + c_{p_{air}} - c_{p_{\mathrm{CO}_2}} \frac{(W)_{\mathrm{CO}_2}}{2 * (W)_{O_2}} - c_{p_{\mathrm{H}_2\mathrm{O}}} \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{2 * (W)_{O_2}} \tag{1.15}$$

$$\frac{dc_p}{dY_{\mathrm{CO}_2}} = c_{p_{fuel}} \frac{-(W)_{fuel}}{(W)_{\mathrm{CO}_2}} + c_{p_{air}} \frac{-2 * (W)_{O_2}}{(W)_{\mathrm{CO}_2}} + c_{p_{\mathrm{CO}_2}} + c_{p_{\mathrm{H}_2\mathrm{O}}} \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{(W)_{\mathrm{CO}_2}} \tag{1.16}$$

$$\frac{dc_p}{dY_{\mathrm{H}_2\mathrm{O}}} = c_{p_{fuel}} \frac{-(W)_{fuel}}{2 * (W)_{\mathrm{H}_2\mathrm{O}}} + c_{p_{air}} \frac{-(W)_{O_2}}{(W)_{\mathrm{H}_2\mathrm{O}}} + c_{p_{\mathrm{CO}_2}} \frac{(W)_{\mathrm{CO}_2}}{2 * (W)_{\mathrm{H}_2\mathrm{O}}} + c_{p_{\mathrm{H}_2\mathrm{O}}} \tag{1.17}$$

The derivative of the ratio of specific heats, γ, is

$$\frac{d \log \gamma}{dY_i} = \frac{d \log c_p}{dY_i} - \frac{d \log c_v}{dY_i} \tag{1.18}$$

Thus,

$$\frac{d\gamma}{dY_{fuel}} = \gamma \frac{d \log \gamma}{dY_{fuel}} = \frac{\gamma}{c_p} \left(c_{p_{fuel}} + c_{p_{air}} \frac{2 * (W)_{O_2}}{(W)_{fuel}} - c_{p_{\mathrm{CO}_2}} \frac{(W)_{\mathrm{CO}_2}}{(W)_{fuel}} - c_{p_{\mathrm{H}_2\mathrm{O}}} \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{(W)_{fuel}} \right) \dots$$

$$\dots - \frac{\gamma}{c_v} \left(c_{v_{fuel}} + c_{v_{air}} \frac{2 * (W)_{O_2}}{(W)_{fuel}} - c_{v_{\mathrm{CO}_2}} \frac{(W)_{\mathrm{CO}_2}}{(W)_{fuel}} - c_{v_{\mathrm{H}_2\mathrm{O}}} \frac{2 * (W)_{\mathrm{H}_2\mathrm{O}}}{(W)_{fuel}} \right) \tag{1.19}$$

We assume the flow to be weakly reacting $a << 1$ and $\dot{m}^0_{air} >> \dot{m}^0_{fuel}$. Hence, $c_p \approx c_{p_{air}}$.

The mass fractions of fuel and products are estimated using the species balance equation. The mass fraction of air can be calculated via

$$Y_{air} = 1 - \left(Y_{fuel} + Y_{\mathrm{CO_2}} + Y_{\mathrm{H_2O}}\right) \tag{1.20}$$

The compositional noise factors are calculated using the relations in the next sections.

1.1. *Calculation of* $\aleph_1$

The heat capacity factor is given by (for simplicity, we assume $T^0 = 0$),

$$\aleph_{1,i} = \frac{1}{\gamma - 1}\frac{d\log\gamma}{dY_i} \tag{1.21}$$

$$\aleph_1 = \sum_{i=1}^{N} \aleph_{1,i} Y_i \tag{1.22}$$

$$\aleph_1 = \frac{1}{\bar{\gamma} - 1}\left(\frac{d\log\gamma}{dY_{fuel}}Y_{fuel} + \frac{d\log\gamma}{dY_{air}}Y_{air} + \frac{d\log\gamma}{dY_{\mathrm{CO_2}}}Y_{\mathrm{CO_2}} + \frac{d\log\gamma}{dY_{\mathrm{H_2O}}}Y_{\mathrm{H_2O}}\right) \tag{1.23}$$

$$\aleph_1 = \frac{1}{\bar{\gamma} - 1}\left(\frac{d\log\gamma}{dY_{air}} + \left(\frac{d\log\gamma}{dY_{fuel}} - \frac{d\log\gamma}{dY_{air}}\right)Y_{fuel} \ldots\right.$$

$$\left.\ldots + \left(\frac{d\log\gamma}{dY_{\mathrm{CO_2}}} - \frac{d\log\gamma}{dY_{air}}\right)Y_{\mathrm{CO_2}} + \left(\frac{d\log\gamma}{dY_{\mathrm{H_2O}}} - \frac{d\log\gamma}{dY_{air}}\right)Y_{\mathrm{H_2O}}\right) \tag{1.24}$$

For a general hydrocarbon combustion reaction

$$C_mH_n + \left(m + \frac{n}{4}\right)O_2 \longrightarrow m\mathrm{CO_2} + \frac{n}{2}\mathrm{H_2O} \tag{1.25}$$

$$\frac{dc_p}{dY_{fuel}} = c_{p_{fuel}} - c_{p_{\mathrm{CO_2}}}\frac{m(W)_{\mathrm{CO_2}}}{(W)_{fuel}} - c_{p_{\mathrm{H_2O}}}\frac{\frac{n}{4} * (W)_{\mathrm{H_2O}}}{(W)_{fuel}} \tag{1.26}$$

$$\frac{dc_p}{dY_{\mathrm{CO_2}}} = c_{p_{fuel}}\frac{-(W)_{fuel}}{m(W)_{\mathrm{CO_2}}} + c_{p_{\mathrm{CO_2}}} + c_{p_{\mathrm{H_2O}}}\frac{\frac{n}{4} * (W)_{\mathrm{H_2O}}}{m(W)_{\mathrm{CO_2}}} \tag{1.27}$$

$$\frac{dc_p}{dY_{\mathrm{H_2O}}} = c_{p_{fuel}}\frac{-(W)_{fuel}}{\frac{n}{4} * (W)_{\mathrm{H_2O}}} + c_{p_{\mathrm{CO_2}}}\frac{m(W)_{\mathrm{CO_2}}}{\frac{n}{4} * (W)_{\mathrm{H_2O}}} + c_{p_{\mathrm{H_2O}}} \tag{1.28}$$

and

$$\frac{d\gamma}{dY_{fuel}} = \gamma\frac{d\log\gamma}{dY_{fuel}} = \frac{\gamma}{c_p}\left(c_{p_{fuel}} - c_{p_{\mathrm{CO_2}}}\frac{m(W)_{\mathrm{CO_2}}}{(W)_{fuel}} - c_{p_{\mathrm{H_2O}}}\frac{\frac{n}{4}(W)_{\mathrm{H_2O}}}{(W)_{fuel}}\right)\ldots$$

$$\ldots - \frac{\gamma}{c_v}\left(c_{v_{fuel}} - c_{v_{\mathrm{CO_2}}}\frac{m(W)_{\mathrm{CO_2}}}{(W)_{fuel}} - c_{v_{\mathrm{H_2O}}}\frac{\frac{n}{4}(W)_{\mathrm{H_2O}}}{(W)_{fuel}}\right) \tag{1.29}$$

We obtain similar expressions for the products. We can assume that $\gamma \approx \bar{\gamma}$, $c_p \approx \bar{c}_p$, and $c_v \approx \bar{c}_v$ Hence,

$$\frac{d\gamma}{dY_{fuel}} = \gamma\frac{d\log\gamma}{dY_{fuel}} = \frac{\bar{\gamma}}{\bar{c}_p}\left(c_{p_{fuel}} - c_{p_{\mathrm{CO_2}}}\frac{m(W)_{\mathrm{CO_2}}}{(W)_{fuel}} - c_{p_{\mathrm{H_2O}}}\frac{\frac{n}{4}(W)_{\mathrm{H_2O}}}{(W)_{fuel}}\right)\ldots$$

$$\ldots - \frac{\bar{\gamma}}{\bar{c}_v}\left(c_{v_{fuel}} - c_{v_{\mathrm{CO_2}}}\frac{m(W)_{\mathrm{CO_2}}}{(W)_{fuel}} - c_{v_{\mathrm{H_2O}}}\frac{\frac{n}{4}(W)_{\mathrm{H_2O}}}{(W)_{fuel}}\right) \tag{1.30}$$

and

$$\gamma' = \sum_{i=1}^{N} \frac{d\gamma}{dY_i} Y_i' \tag{1.31}$$

1.2. *Calculation of $\bar{\psi}$*

The specific Gibbs energy can be written as

$$g_i = h_i - T s_i \tag{1.32}$$

also, using first-order homogeneity, $\mu_i = W_i g_i$ (**?**)

$$\frac{\mu_i}{W_i} = h_i - T s_i \tag{1.33}$$

$$\frac{\mu_i}{W_i} = c_{p_i}(T - T^o) + h_{chem} - T s_i \tag{1.34}$$

Assuming $T^o = 0$

$$\frac{\mu_i}{W_i} = T\left(c_{p_i} - s_i\right) + \Delta h^o_{f,i} \tag{1.35}$$

$$\bar{\psi}_1 = \frac{1}{\bar{c}_p \bar{T}} \sum_{i=1}^{N} \left(\frac{\bar{\mu}_i}{W_i} - \Delta h^o_{f,i}\right) Y_i \tag{1.36}$$

$$\bar{\psi}_1 = \frac{1}{\bar{c}_p} \sum_{i=1}^{N} \left(c_{p_i} - s_i\right) Y_i \tag{1.37}$$

$$\bar{\psi}_1 = \sum_{i=1}^{N} \psi_{1,i} Y_i \tag{1.38}$$

$$\bar{\psi}_1 = \psi_{air} + \left(\psi_{fuel} - \psi_{air}\right) Y_{fuel} + \left(\psi_{\mathrm{CO_2}} - \psi_{air}\right) Y_{\mathrm{CO_2}} + \left(\psi_{\mathrm{H_2O}} - \psi_{air}\right) Y_{\mathrm{H_2O}} \tag{1.39}$$

1.3. *Calculation of $\bar{\phi}$*

$$\bar{\phi}_{1,i} = \frac{d \log \gamma}{dY_i} \log \bar{p}^{\frac{1}{\bar{\gamma}}} \tag{1.40}$$

$$\bar{\phi}_{1,i} = \log \bar{p}^{\frac{1}{\bar{\gamma}}} \left(\bar{\gamma} - 1\right) \bar{\aleph}_{1,i} \tag{1.41}$$

$$\bar{\phi}_1 = \log \bar{p}^{\frac{1}{\bar{\gamma}}} \left(\bar{\gamma} - 1\right) \sum_{i=1}^{N} \left(\aleph_{1,i} Y_i\right) \tag{1.42}$$

$$\bar{\phi}_1 = \log \bar{p}^{\frac{1}{\bar{\gamma}}} \left(\bar{\gamma} - 1\right) \aleph_1 \tag{1.43}$$