# Supplementary Information

T. Y. Wang, D.S. Dean, R. Zakine, S. Marbach

## Supplementary Discussion

**1 Boundary conditions in moving channels** **2**

**2 Flow field calculations** **3**

2.1 Notations . . . 3

2.2 Further simplifications: Lubrication approximation . . . 3

2.3 What is the velocity of the wall particles? . . . 4

2.4 Integration process to obtain the flow field . . . 4

**3 Long time transport coefficients for space and time varying diffusivity** **6**

# 1 Boundary conditions in moving channels

We first establish the boundary conditions at the upper and lower moving walls. To do this we consider the condition of conservation of particle number in the total system volume, denoted here by $V$. The total particle number is given by

$$N = \int_V \mathrm{d}^2\mathbf{r}\ \rho(\mathbf{r},t) = \int \mathrm{d}x \int_0^{h(x,t)} \mathrm{d}z\ \rho(x,z,t) \tag{S1}$$

where $\rho(x,z,t)$ is any density field (tracer density or density of particles) that satisfies a generic conservation equation

$$\frac{\partial \rho(x,z,t)}{\partial t} + \nabla \cdot \mathbf{j}(x,z,t) = 0 \tag{S2}$$

The condition $dN/dt = 0$ then gives

$$\begin{aligned}
0 &= \int \mathrm{d}x \frac{\partial h(x,t)}{\partial t}\rho(x,h(x,t)) + \int \mathrm{d}x \int_0^{h(x,t)} \mathrm{d}z \frac{\partial \rho(x,z,t)}{\partial t} \\
&= \int \mathrm{d}x \frac{\partial h(x,t)}{\partial t}\rho(x,h(x,t)) - \int \mathrm{d}x \int_0^{h(x,t)} \mathrm{d}z \nabla \cdot \mathbf{j}(x,z,t) \\
&= \int \mathrm{d}x \frac{\partial h(x,t)}{\partial t}\rho(x,h(x,t)) - \int_{\partial V} \mathbf{j}(\mathbf{r},t)\cdot \mathbf{n}\mathrm{d}S. 
\end{aligned} \tag{S3}$$

Here $\partial V$ denotes the surface of the volume $V$, $\mathbf{n}$ is the outward surface normal and $dS$ is the element of surface. We assume that the system is in such a state that the flux of particles through the surfaces at the horizontal edges, in $x = -L_{\max}/2$ and that in $x = +L_{\max}/2$, vanish or are equal to one another, which is the case with periodic boundary conditions as is performed in numerics. Here $L_{\max}$ is the horizontal size of the periodic simulation box.

The upper surface normal is given by

$$\mathbf{n}|_{z=h(x,t)} = \frac{(-\partial h(x,t)/\partial x, 1)}{\sqrt{(\partial h(x,t)/\partial x)^2 + 1}}, \tag{S4}$$

and $\mathrm{d}S = \sqrt{(\partial h(x,t)/\partial x)^2 + 1}\,\mathrm{d}x$. The conservation of particle number can thus be written as

$$\int \mathrm{d}x \left[ \frac{\partial h(x,t)}{\partial t}\rho(x,h(x,t)) - \left(\mathbf{j}(x,z,t)\cdot \mathbf{n}|_{z=h(x,t)} - \mathbf{j}(x,z,t)\cdot \mathbf{n}|_{z=0}\right) \sqrt{(\partial h(x,t)/\partial x)^2 + 1} \right] = 0. \tag{S5}$$

The boundary condition above must hold for any configuration of the density field and thus should hold at each point $\mathbf{x}$ on both the upper and lower surfaces. It should also hold for any functional form of the interface $h(x,t)$ which allows to split the terms into 2 boundary conditions

$$\mathbf{j}(x,z,t)\cdot \mathbf{n}|_{z=h(x,t)} \sqrt{(\partial h(x,t)/\partial x)^2 + 1} - \frac{\partial h(x,t)}{\partial t}\rho(x,h(x,t)) = 0 \tag{S6}$$

$$\mathbf{j}(x,z,t)\cdot \mathbf{n}|_{z=0} = 0. \tag{S7}$$

Using the expression of the upper surface normal (S4), then gives the upper boundary condition as

$$j_z(x,h(x,t),t) - \frac{\partial h(x,t)}{\partial x} j_x(x,h(x,t),t) = \frac{\partial h(x,t)}{\partial t}\rho(x,h(x,t)) = 0 \tag{S8}$$

that is reported in the main text.

## 2 Flow field calculations

Here we derive the expressions (1.17) and (1.18) from the main text used for the flow field within a channel with a fluctuating interface $h(x, t)$, when the embedded fluid is incompressible.

### 2.1 Notations

The flow field $\boldsymbol{U} = (u, v)$ satisfies the Stokes equations and local conservation of mass (incompressibility)

$$\begin{cases} \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = -\frac{\partial P}{\partial x} \\ \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = -\frac{\partial P}{\partial y} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases} \tag{S9}$$

with boundary conditions valid at any boundaries:

$$\begin{cases} (\boldsymbol{U} - \boldsymbol{U}_{\mathrm{wall}}) \cdot \boldsymbol{t} = -2b[\boldsymbol{t} \cdot \boldsymbol{\sigma}(\boldsymbol{U}) \cdot \boldsymbol{n}], \\ (\boldsymbol{U} - \boldsymbol{U}_{\mathrm{wall}}) \cdot \boldsymbol{n} = 0, \end{cases} \tag{S10}$$

which correspond to partial slip on the tangential direction with slip length $b$, and no penetration on the normal direction, respectively. The stress tensor $\boldsymbol{\sigma}(\boldsymbol{U})$ of the fluid is defined as

$$\boldsymbol{\sigma}(\boldsymbol{U}) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{S11}$$

The boundary conditions have to be verified at each boundary (upper and lower walls). The velocity of the top wall/the boundary is $\boldsymbol{U}_{\mathrm{wall}}(x, y = h(x, t)) = (u_{\mathrm{wall,x}}, u_{\mathrm{wall,y}})$. And we will specify the values of the components later on, because there is an important distinction to be made. Note that here $\boldsymbol{U}_{\mathrm{wall}}$ corresponds to the velocity of a wall atom at position $(x, y = h(x, t))$.

The bottom wall is taken immobile and flat. We thus have $\boldsymbol{n} = (0, -1)$ (outwards normal) and $\boldsymbol{t} = (1, 0)$. Since the wall is impermeable, $v(x, y = 0) = 0$, and hence $\frac{\partial v}{\partial x}(x, y = 0) = 0$. As a consequence we find immediately the classical result at the bottom:

$$\begin{cases} u(x, y = 0) = b_0 \frac{\partial u}{\partial y}(x, y = 0) \\ v(x, y = 0) = 0. \end{cases} \tag{S12}$$

### 2.2 Further simplifications: Lubrication approximation

Now we use the lubrication approximation to find the flow field in its approximate form. In general we will consider that variations along the $x$ axis are "slow" whilst those along $y$ are fast. Formally this can be expressed by taking non dimensional scales $x \to \tilde{x}L$ and $y \to \tilde{y}H$ and $H/L = \epsilon \ll 1$. As a consequence (in non dimensional scales) $\frac{\partial =}{\partial x}O(1)$ while $\frac{\partial =}{\partial y}O(1/\epsilon)$. Since local conservation of mass (incompressibility) in the bulk needs to be satisfied at first order (otherwise both flow fields are 0) we have $\frac{\partial u}{\partial x} = O(\frac{\partial v}{\partial y})$ and hence $u/v = O(1/\epsilon)$. This is physically intuitive, the flow field in the vertical direction is just a small correction compared to the dominant flow field.

Let's now "list" all terms that we will need to compare. First, $\frac{\partial^2 u}{\partial y^2} / \frac{\partial^2 u}{\partial x^2} = O(1/\epsilon^2)$. Next, we need the higher order terms to be related, so we have $\frac{\partial^2 u}{\partial y^2} = O(-\frac{\partial P}{\partial x})$. Then, since $v/u = O(\epsilon)$, we find $\frac{\partial^2 v}{\partial y^2} / \frac{\partial^2 u}{\partial y^2} = O(\epsilon)$. And finally, using $\frac{\partial P}{\partial y} / \frac{\partial P}{\partial x} = O(1/\epsilon)$, we find $\frac{\partial P}{\partial y} / \frac{\partial^2 u}{\partial y^2} = O(1/\epsilon)$, such that the NS equations simplify at highest order to

$$\begin{cases} \frac{\partial^2 u}{\partial y^2} = -\frac{\partial P}{\partial x} \\ 0 = -\frac{\partial P}{\partial y} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases} \tag{S13}$$

These equations can be simply integrated, in the following order: $P = P(x,t)$ as there is no dependence on $y$ of pressure at lowest order. Then $u$ is integrated as $u = -(y^2/2)\frac{\partial P}{\partial x} + B(x)y + C(x)$ and finally $v = D + (y^3/6)\frac{\partial^2 P}{\partial x^2} - (y^2/2)\frac{\partial B}{\partial x} - \frac{\partial C}{\partial x}y$ where $B, C, D$ are to be determined, as well as $P$, from the boundary conditions.

For the top wall, $\boldsymbol{n} \simeq (-\frac{\partial h}{\partial x}, 1)$ (outwards normal) and $\boldsymbol{t} \simeq (1, \frac{\partial h}{\partial x})$ at lowest order in $\frac{\partial h}{\partial x} \sim h_0/L = O(\epsilon)$. Then we investigate the components of the stress tensor, when it is projected,

$$\begin{aligned} \boldsymbol{t}\cdot\boldsymbol{\sigma}(\boldsymbol{U})\cdot\boldsymbol{n} &= \boldsymbol{t}\cdot(-\frac{\partial u}{\partial x}\frac{\partial h}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), -\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial h}{\partial x} + \frac{\partial v}{\partial y}) \\ &= -\frac{\partial u}{\partial x}\frac{\partial h}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)(\frac{\partial h}{\partial x})^2 + \frac{\partial v}{\partial y}(\frac{\partial h}{\partial x}) \end{aligned} \tag{S14}$$

It is clear that in all of these terms the highest order term is in fact only $\frac{\partial u}{\partial y}/2$ hence the boundary conditions at the top boundary sum up to

$$\begin{cases} u(y=h) - u_{\text{wall,x}} = -b_h\frac{\partial u}{\partial y}(y=h) \\ v(y=h) - u(y=h)\frac{\partial h}{\partial x} = u_{\text{wall,y}} - u_{\text{wall,x}}\frac{\partial h}{\partial x}. \end{cases} \tag{S15}$$

## 2.3 What is the velocity of the wall particles?

We consider in the simulations that the boundary's position obeys is $h(x,t) = H + h_0\cos(kx - \omega t) = H + h_0\cos\left(\frac{2\pi}{L}(x - v_{\text{wall}}t)\right)$. While this explicitly gives the *position* of the boundary it *does not* give the expression for the velocity of the wall atoms, which is what is needed to specify the boundary conditions via $\boldsymbol{U}_{\text{wall}}$. There are two possibilities. A first possibility would be that the wall atoms perform peristalsis or pumping (1, 2), *e.g.* atoms do not move in average but jiggle vertically. There is no average mass displacement in that case. This situation is quite common for phonons on carbon nanotubes (3) or peristaltic waves of intestine walls (4) or in vasculature networks (5). In that case $\boldsymbol{U}_{\text{wall}} = (0, \partial_t h)$. The other possibility is that there is average mass displacement of wall atoms, but the atoms do not move relative to one another. In that case, $\boldsymbol{U}_{\text{wall}} = (v_{\text{wall}}, 0)$. We will focus on peristalsis here.

## 2.4 Integration process to obtain the flow field

The boundary conditions thus obey

$$\text{partial slip:} \begin{cases} u(y=0) = b_0\frac{\partial u}{\partial y}(y=0) \\ u(y=h) = -b_h\frac{\partial u}{\partial y}(y=h) \end{cases} \tag{S16}$$

$$\text{non penetration:} \begin{cases} v(y=h) - u(y=h)\frac{\partial h}{\partial x} = \partial_t h \\ v(y=0) = 0 \end{cases} \tag{S17}$$

which correspond respectively to partial slip on both walls, and non penetration boundary conditions. The quantities $b_0$ (respectively $b_h$) indicate the slip lengths on the bottom (respectively top) walls.

Integrating for $u$ yields

$$u(x,y,t) = 6U_0(x,t)\frac{\frac{b_0}{h}\left(\frac{2b_h}{h}+1\right) + \left(\frac{2b_h}{h}+1\right)\frac{y}{h} - \left(\frac{b_0}{h}+\frac{b_h}{h}+1\right)\frac{y^2}{h^2}}{12\frac{b_0}{h}\frac{b_h}{h} + 4\left(\frac{b_0}{h}+\frac{b_h}{h}\right) + 1} = 6U_0(x,t)f(h(x,t),y) \tag{S18}$$

where $U_0(x,t) = \frac{1}{h}\int_0^h u dy$ is the average flow field in the $x$ direction. The expression for $u$ is slightly cumbersome but it has natural limits with limiting regimes of the slip lengths $b_0$ and $b_h$. In particular, in our case where $b_0 = b_h = \infty$ (perfect slip at both walls) we obtain (1.17) . Because of mass conservation we also have

$$\frac{\partial(hU_0)}{\partial x} = -\frac{\partial h}{\partial t} \tag{S19}$$

such that $\frac{\partial U}{\partial x}_0 = -\frac{1}{h}(\frac{\partial h}{\partial t} + U_0\frac{\partial h}{\partial x})$. We can therefore express

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\left(6\frac{\partial U_0}{\partial x}f(h(x,t),y) + 6U_0(x,t)\frac{\partial h}{\partial x}\frac{\partial f(h(x,t),y)}{\partial h}\right) \tag{S20}$$

and integration and use of boundary conditions finally yield a general expression for $v(x, y, t)$ that is quite lengthy and that we do not report here. Again, for $b_0 = b_h \to \infty$, we obtain (1.18) .

Now what about the value of $U_0(x, t)$? With conservation of mass we obtain

$$U_0(x,t) = U_0(x=0,t)\frac{h(x=0,t)}{h(x,t)} + \frac{h_0}{h}v_{\text{wall}}\left[\cos\left(\frac{2\pi}{L}(x - v_{\text{wall}}t)\right) - \cos\left(\frac{2\pi}{L}v_{\text{wall}}t\right)\right] \tag{S21}$$

but we are missing the boundary term $U_0(x = 0, t)$. To be coherent with our assumption that the wall atoms do not move in average, namely of peristalsis, then it is natural to expect that there is no pressure driven flow and hence that the average pressure drop has to vanish $\int_0^L \frac{\partial P}{\partial x}dx = 0$ (the average force on the fluid is 0) (2). Expanding $\int_0^L \frac{\partial P}{\partial x}dx$ in powers of $\epsilon$ we simply obtain at lowest order that $\int_0^L U_0(x,t)dx = 0$. After an easy integration we obtain that $U_0(x = 0, t) = v_{\text{wall}}$ and hence $U_0(x,t) \simeq v_{\text{wall}}\frac{h_0}{H}\cos\left(\frac{2\pi}{L}(x - v_{\text{wall}}t)\right)$.

## 3 Long time transport coefficients for space and time varying diffusivity

In this section we derive the long time transport coefficients for tracers in a bath of interacting particle when we do not neglect the contribution of the mean density to the local diffusion coefficient.

Our starting point, as in the main paper, is to obtain the stationary density of particles $\overline{\rho}(x)$ in the frame of reference where the wall is immobile, but now without neglecting the dependence of the diffusion coefficient $D(\overline{\rho}(x))$ on density. The steady state flux of particles is obtained by assuming in (3.7) of the main text that $\overline{\rho}(x,y) \simeq \overline{\rho}(x)$, such that

$$J = h(x)\left(-D(x)\frac{\partial\overline{\rho}(x)}{\partial x} - \upsilon_0 D(x)\overline{\rho}(x)\frac{\partial\overline{\rho}(x)}{\partial x} + \overline{\rho}(x)v_{\text{wall}}\right) \tag{S22}$$

where we reversed the direction of $v_{\text{wall}}$ compared to the main text to obtain $J > 0$ and where we introduced $\upsilon_0 \equiv E_0/k_BT\rho_0$ a characteristic interaction volume. For simplicity we also introduce $f(x)$ such that $h(x) = H(1+\epsilon f(x))$ (in the frame of reference where the channel is fixed), and where $\int_0^L f(x)\mathrm{d}x = 0$ by definition. To pursue the derivation we will need to obtain the expression of $J$ at order $\epsilon^2$ as well as that of $\overline{\rho}(x) = \rho_0 + \epsilon\rho_1(x) + \epsilon^2\rho_2(x)$. Notice, compared to the main text, that now we assume $D(x)$ depends on space as

$$D(x) = D(\rho_0) + D'(\rho_0)(\epsilon\rho_1(x) + \epsilon^2\rho_2(x))) + \frac{1}{2}D''(\rho_0)(\epsilon\rho_1(x) + \epsilon^2\rho_2(x)))^2 + ... \tag{S23}$$

In all of these cases note that the solutions $\rho_i(x)$ should still be periodic. We will also make use of the conservation of mass

$$\int_0^L \rho(x)h(x)\mathrm{d}x = \int_0^L (\rho_0 + \epsilon\rho_1(x) + \epsilon^2\rho_2(x))H(1+\epsilon f(x))\mathrm{d}x = \rho_0 LH \tag{S24}$$

that we can expand at all orders in $\epsilon$ to yield (using the fact that $\int f(x)\mathrm{d}x = 0$),

$$\int_0^L \rho_1(x)\mathrm{d}x = 0\ ,\quad \int_0^L \left(\rho_2(x) + \rho_1(x)f(x)\right)\mathrm{d}x = 0. \tag{S25}$$

In the following we will only need integrals $\int_0^L \rho_2(x)\mathrm{d}x$, and not the expression of $\rho_2(x)$, which means we do not have to explicitly calculate $\rho_2(x)$ and rather we can simply express its integral with respect to $\rho_1(x)$.

Now we also expand $J = J_0 + \epsilon J_1 + \epsilon^2 J_2$. Carefully expanding at all orders in $\epsilon$, we obtain

$$\begin{cases} J_0 &= H\rho_0 v_{\text{wall}}, \\ J_1 &= Hf(x)\rho_0 v_{\text{wall}} - HD_0\rho_1'(x) - H\upsilon_0 D_0\rho_0\rho_1'(x) + H\rho_1(x)v_{\text{wall}}, \\ J_2 &= Hf(x)\rho_1(x)v_{\text{wall}} - Hf(x)D_0\rho_1'(x) - Hf(x)\upsilon_0 D_0\rho_0\rho_1'(x) \\ & \quad -HD'(\rho_0)\rho_1\rho_1'(x) - H\upsilon_0 D'(\rho_0)\rho_1\rho_0\rho_1'(x) - H\upsilon_0 D_0\rho_1(x)\rho_1'(x) \\ & \quad -HD_0\rho_2'(x) - H\upsilon_0 D_0\rho_0\rho_2'(x) + H\rho_2(x)v_{\text{wall}}. \end{cases} \tag{S26}$$

We can simplify these expressions by taking their averages over a period, and using periodicity and conservation of mass, we find

$$\begin{cases} J_0 &= H\rho_0 v_{\text{wall}}, \\ J_1 &= 0, \\ J_2 &= -\dfrac{H}{L}D_0^\star \displaystyle\int_0^L \mathrm{d}x f(x)\rho_1'(x) \end{cases} \tag{S27}$$

where we recall that $D_0^\star = D_0(1+\upsilon_0\rho_0)$.

We can obtain the analytical expression for $\rho_1(x)$ by solving the equation for $J_1$ before doing the average,

$$J_1 = 0 = Hf(x)\rho_0 v_{\text{wall}} - HD_0^\star \rho_1'(x) + H\rho_1(x)v_{\text{wall}}, \tag{S28}$$

which does not change compared to the main text and is simply

$$\rho_1(x) = \rho_0 \frac{\text{Pe}^\star}{1+(\text{Pe}^\star)^2}\Big(\sin(k_0 x) - \text{Pe}^\star\cos(k_0 x)\Big). \tag{S29}$$

Notice that compared to the solution of the main text some signs are reversed that correspond to the reversal of $v_{\text{wall}}$.

Finally, we can insert the expression for $\rho_1(x)$ to obtain the expression of the flux at second order

$$J = H\rho_0 v_{\text{wall}} - \frac{H}{L}D_0^\star \int_0^L \mathrm{d}x f(x)\rho_1'(x) = H\rho_0 v_{\text{wall}}\left(1 - \frac{\epsilon^2}{2}\frac{1}{1+(\text{Pe}^\star)^2}\right). \tag{S30}$$

The subsequent step is to analyze equation (3.14) of the main manuscript, on the marginal density $p_{\text{tr}}(x,t)$ to find the tracer, in the frame of reference where the wall is fixed

$$\frac{\partial p_{\text{tr}}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D_0(\overline{\rho})\left[\frac{\partial p_{\text{tr}}}{\partial x} - p_{\text{tr}}\frac{\partial \ln h(x,t)}{\partial x} + \frac{E_0}{k_B T}\frac{p_{\text{tr}}(x,t)}{\rho_0}\frac{\partial \overline{\rho}}{\partial x}\right] - v_{\text{wall}} p_{\text{tr}}\right). \tag{S31}$$

Notice that compared to (3.14) we have reversed the sign of $v_{\text{wall}}$.

Now we will first recast the above equation with easier notations

$$\frac{\partial p_{\text{tr}}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x)\frac{\partial p_{\text{tr}}(x,t)}{\partial x} - D(x)p_{\text{tr}}(x,t)\frac{\partial \ln h(x)}{\partial x} + D(x)v_0 p_{\text{tr}}(x,t)\frac{\partial \rho(x)}{\partial x} - v p_{\text{tr}}(x,t)\right) \tag{S32}$$

where $D(x)$ contains all the spatial dependence of the diffusion coefficient induced by the locally varying density and we abbreviate $v \equiv v_{\text{wall}}$.

Now recall that

$$J \equiv h(x)\left(D(x)\frac{\partial \rho(x)}{\partial x} + v_0 D(x)\rho(x)\frac{\partial \rho(x)}{\partial x} - \rho(x)v\right) \tag{S33}$$

is the integrated longitudinal flux of the steady state local density. We can then recast (S32) again into

$$\frac{\partial p_{\text{tr}}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x)\frac{\partial p_{\text{tr}}(x,t)}{\partial x} - D(x)p_{\text{tr}}(x,t)\frac{\partial \ln h(x)\rho(x)}{\partial x} - \frac{J}{\rho(x)h(x)}p_{\text{tr}}(x,t)\right). \tag{S34}$$

To obtain the long time diffusion and drift of this equation, we finally need to put it under the form used in Ref. (6)

$$\frac{\partial p_{\text{tr}}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\kappa(x)p_{\text{tr}}(x,t)\right) - u(x)p_{\text{tr}}(x,t)\right). \tag{S35}$$

We see immediately that $\kappa(x) \equiv D(x)$ and

$$u(x) \equiv \frac{\partial D(x)}{\partial x} + \frac{J}{\rho(x)h(x)} + D(x)\frac{\partial \ln(\rho(x)h(x))}{\partial x} = \frac{1}{\rho(x)h(x)}\left(J + \frac{\partial D(x)\rho(x)h(x)}{\partial x}\right). \tag{S36}$$

We now can use exact results for one dimensional systems in Ref. (6), which give

$$V_{\text{eff}} = \frac{L}{\int_0^L dx I_+(x)} \tag{S37}$$

and

$$D_{\text{eff}} = \frac{L^2\int_0^L dx D(x) I_+(x)^2 I_-(x)}{[\int_0^L dx I_+(x)]^3}, \tag{S38}$$

where

$$I_+(x) = \frac{1}{\kappa(x)} e^{\Gamma(x)} \int_x^{+\infty} e^{-\Gamma(x')} \mathrm{d}x' \tag{S39}$$

and symmetrically

$$I_-(x) = \frac{1}{\kappa(x)} e^{-\Gamma(x)} \int_{-\infty}^{x} e^{\Gamma(x')} \mathrm{d}x' \tag{S40}$$

with

$$\Gamma(x) = \int_0^x \frac{u(x')}{\kappa(x')} \mathrm{d}x'. \tag{S41}$$

Notice that the above integrals indeed converge because the average direction of the drift is towards the right ($u$ is positive). In the case where the drift is reversed, one would simply have to swap integral bounds in the definitions of $I_+(x)$ and $I_-(x)$.

Our aim is noe the drift is w to simplify the above expressions for $V_{\text{eff}}$ and $D_{\text{eff}}$. We start by simplifying the expressions for $I_+(x)$ and $I_-(x)$. Notice that

$$\begin{aligned} \Gamma(x) &= \int_0^x \frac{1}{D(x')\rho(x')h(x')} \left( J + \frac{\partial D(x')\rho(x')h(x')}{\partial x'} \right) \mathrm{d}x' \\ &= \ln\left( \frac{D(x)\rho(x)h(x)}{D(0)\rho(0)h(0)} \right) + \int_0^x \frac{J}{D(x')\rho(x')h(x')} \mathrm{d}x' \end{aligned} \tag{S42}$$

and therefore we can simplify

$$\begin{aligned} I_+(x) =& \frac{1}{D(x)} \frac{D(x)\rho(x)h(x)}{D(0)\rho(0)h(0)} \exp\left( J \int_0^x \frac{\mathrm{d}x'}{h(x')\rho(x')D(x')} \right) \times \\ & \int_x^\infty \mathrm{d}x' \frac{D(0)\rho(0)h(0)}{h(x')\rho(x')D(x')} \exp\left( -J \int_0^{x'} \frac{dx''}{h(x'')\rho(x'')D(x'')} \right) \\ =& \rho(x)h(x) \exp\left( J \int_0^x \frac{\mathrm{d}x'}{h(x')\rho(x')D(x')} \right) \times \\ & \int_x^\infty \mathrm{d}x' \frac{1}{h(x')\rho(x')D(x')} \exp\left( -J \int_0^{x'} \frac{dx''}{h(x'')\rho(x'')D(x'')} \right) \\ =& -\frac{1}{J} \rho(x)h(x) \exp\left( J \int_0^x \frac{\mathrm{d}x'}{h(x')\rho(x')D(x')} \right) \times \\ & \left[ \exp\left( -J \int_0^\infty \frac{\mathrm{d}x''}{h(x'')\rho(x'')D(x'')} \right) - \exp\left( -J \int_0^x \frac{\mathrm{d}x''}{h(x'')\rho(x'')D(x'')} \right) \right]. \end{aligned} \tag{S43}$$

Finally, notice that $\int_0^\infty \frac{\mathrm{d}x''}{h(x'')\rho(x'')D(x'')} \to +\infty$ since all of the functions $h(x)$, $\rho(x)$ and $D(x)$ are all positive and periodic. We therefore obtain, since $J > 0$ (since we reversed the sign of $v_{\text{wall}}$ in this entire section),

$$I_+(x) = +\frac{\rho(x)h(x)}{J}. \tag{S44}$$

The term $I_+(x)$ is thus proportional to the steady state density.

On the other hand we have

$$I_-(x) = \frac{1}{D(x)h(x)\rho(x)} \exp\left( -J \int_0^x \frac{\mathrm{d}x'}{h(x')\rho(x')D(x')} \right) \int_{-\infty}^x \mathrm{d}x' h(x')\rho(x') \exp\left( J \int_0^{x'} \frac{\mathrm{d}x''}{h(x'')\rho(x'')D(x'')} \right), \tag{S45}$$

which does not simplify as easily as $I_+(x)$.

We can now turn to simplifying $V_{\text{eff}}$. It is obtained via (S37) and (S44)

$$V_{\text{eff}} = \frac{LJ}{\int_0^L dx h(x)\rho(x)} = \frac{J}{\rho_0 H}, \tag{S46}$$

which is clearly a result that can be argued from physical grounds. Naturally, the flux of particles $J = \rho_0 H V_{\text{eff}}$, is the mean velocity multiplied by the average density. This result can be written using (S30) as

$$V_{\text{eff}} = v_{\text{wall}}\left(1 - \frac{\epsilon^2}{2}\frac{1}{1+(\text{Pe}^\star)^2}\right), \tag{S47}$$

which means that going back to the rest frame of the fluid we find

$$V_{\text{eff}} = v_{\text{wall}}\frac{\epsilon^2}{2}\frac{1}{1+(\text{Pe}^\star)^2}. \tag{S48}$$

We now make progress on the calculation of $D_{\text{eff}}$. From (S38) and our expanded expression for $I_+(x)$ we obtain

$$D_{\text{eff}} = \frac{JL^2\int_0^L \mathrm{d}x\, D(x)\rho^2(x)h^2(x)I_-(x)}{[\int_0^L \mathrm{d}x\,\rho(x)h(x)]^3} = \frac{J\int_0^L \mathrm{d}x\, D(x)\rho^2(x)h^2(x)I_-(x)}{L[\rho_0 H]^3}. \tag{S49}$$

We must make more progress on $I_-(x)$ to obtain a more explicit result. Let

$$R(x) \equiv \frac{J}{h(x)\rho(x)D(x)}. \tag{S50}$$

$R(x)$ is not periodic however we can split it up between a periodic and a non periodic part by defining

$$\overline{R} = \frac{1}{L}\int_0^L R(x)\mathrm{d}x \text{ and } r(x) = \int_0^x (R(x') - \overline{R})\mathrm{d}x' \tag{S51}$$

such that

$$\int_0^x \mathrm{d}x' R(x') = r(x) + \overline{R}x. \tag{S52}$$

The function $r(x)$ is now clearly periodic. Now we use

$$\begin{aligned}
I_-(x)\frac{J}{R(x)} &= \exp\left(-r(x) - \overline{R}x\right)\int_{-\infty}^x \mathrm{d}x' h(x')\rho(x')\exp\left(r(x') + \overline{R}x'\right)\\
&= \exp\left(-r(x) - \overline{R}x\right)\sum_{n=0}^{\infty}\int_{x-(n+1)L}^{x-nL} \mathrm{d}x' h(x')\rho(x')\exp\left(r(x') + \overline{R}x'\right)\\
&= \exp\left(-r(x) - \overline{R}x\right)\sum_{n=0}^{\infty}\int_{x}^{x+L} \mathrm{d}x' h(x')\rho(x')\exp\left(r(x') + \overline{R}(x' - (n+1)L)\right)\\
&= \exp\left(-r(x) - \overline{R}x\right)\int_{x}^{x+L} \mathrm{d}x' h(x')\rho(x')\exp\left(r(x') + \overline{R}x'\right)\sum_{n=0}^{\infty} e^{-\overline{R}(n+1)L}\\
&= \exp\left(-r(x) - \overline{R}x\right)\int_{x}^{x+L} \mathrm{d}x' h(x')\rho(x')\exp\left(r(x') + \overline{R}x'\right)\frac{1}{\exp(\overline{R}L) - 1}\\
&= \frac{1}{\exp(\overline{R}L) - 1}\exp\left(-r(x) - \overline{R}x\right)\int_0^L \mathrm{d}x' h(x+x')\rho(x+x')\exp\left(r(x+x') + \overline{R}(x+x')\right)\\
&= \frac{1}{\exp(\overline{R}L) - 1}\int_0^L \mathrm{d}x' h(x+x')\rho(x+x')\exp\left(r(x+x') - r(x) + \overline{R}x'\right).
\end{aligned} \tag{S53}$$

We can now go back to $D_{\text{eff}}$,

$$D_{\text{eff}} = J\frac{\int_0^L dx\int_0^L dx' h(x)\rho(x)h(x+x')\rho(x+x')\exp\left(r(x+x') - r(x) - \overline{R}x'\right)}{L\rho_0^3 H^3(\exp(\overline{R}L) - 1)}. \tag{S54}$$

We must now expand $D_{\text{eff}}$ with respect to $\epsilon$. Before moving on, we observe that

$$\begin{aligned}
\overline{R} &= \frac{1}{L}\int_0^L \frac{V_{\text{eff}}}{D(x)}\frac{\rho_0 H}{\rho(x)h(x)}\mathrm{d}x \\
&= \ldots \\
&= \frac{v_{\text{wall}}}{D_0}\left[1 - \frac{\epsilon^2}{2}\frac{(\mathrm{Pe}^\star)^2}{1+(\mathrm{Pe}^\star)^2}\left(\epsilon_D + \epsilon_{D,2} - \epsilon_D^2\right)\right] \\
&\equiv \mathrm{Pe}\frac{2\pi}{L}\left(1+\epsilon^2 R_2\right)
\end{aligned} \tag{S55}$$

where we used any formal analysis software to obtain the systematic expansion of $1/(D(x)\rho(x)h(x))$ with respect to $\epsilon$ and then its integral and introduced

$$\epsilon_D = \frac{D'(\rho_0)\rho_0}{D(\rho_0)} \text{ and } \epsilon_{D,2} = \frac{D''(\rho_0)\rho_0^2}{2D(\rho_0)}. \tag{S56}$$

It is thus clear that $r(x)$ is of order $\epsilon$ at least, and we can write

$$\begin{aligned}
r(x) &= -\epsilon\frac{v_{\text{wall}}}{D_0}\int_0^x \mathrm{d}x'\left(f(x') + \frac{\rho_1(x')}{\rho_0} + \frac{D'(\rho_0)}{D_0}\rho_1(x') + O(\epsilon)\right) \\
&= -\epsilon\frac{\mathrm{Pe}}{1+(\mathrm{Pe}^\star)^2}\left(-2(1+\epsilon_D)\mathrm{Pe}^\star\sin(\pi x/L)^2 + (\epsilon_D(\mathrm{Pe}^\star)^2 - 1)\sin(2\pi x/L)\right) + O(\epsilon) \\
&\equiv \epsilon r_1(x) + O(\epsilon^2).
\end{aligned} \tag{S57}$$

In what follows, we show that it is sufficient to work at order $O(\epsilon)$ on $r(x)$ to obtain the relevant results. Using this $O(\epsilon)$ approximation on $r_1$ we obtain

$$D_{\text{eff}} = \frac{J}{L\rho_0^3 H^3(\exp(\overline{R}L)-1)}\int_0^L \mathrm{d}x\int_0^L \mathrm{d}x'\, h(x)\rho(x)h(x+x')\rho(x+x')\exp(\epsilon r_1(x+x') - \epsilon r_1(x) + \overline{R}x'). \tag{S58}$$

Let $I(x,x') = h(x)\rho(x)h(x+x')\rho(x+x')\exp(\epsilon r_1(x+x') - \epsilon r_1(x) + \overline{R}x')$ be the integrand, and we now expand it to order $\epsilon^2$,

$$\begin{aligned}
I(x,x') &= H^2(1+\epsilon f(x))(1+\epsilon f(x+x'))(\rho_0 + \epsilon\rho_1(x) + \epsilon^2\rho_2(x))(\rho_0 + \epsilon\rho_1(x+x') + \epsilon^2\rho_2(x+x')) \\
&\quad \times\exp(\overline{R}x')\left(1 + \epsilon(r_1(x+x') - r_1(x)) + \frac{\epsilon^2}{2}(r_1(x+x') - r_1(x))^2\right).
\end{aligned} \tag{S59}$$

Before expanding more, to make progress we will already remove all the terms that integrate to 0, keeping in mind that

$$0 = \int_0^L \mathrm{d}x f(x+a) = \int_0^L \mathrm{d}x\,\rho_1(x+a) = \int_0^L \mathrm{d}x\,(\rho_1(x+a)f(x+a) + \rho_2(x+a)) \tag{S60}$$

for any $a$ as well as the identity

$$\int_0^L \mathrm{d}x(g(x) - g(x+a)) = 0, \tag{S61}$$

for any periodic function $g$, and in particular for $r_1(x)$,

$$\begin{aligned}
I(x,x') &= H^2\exp(\overline{R}x')\Big(\rho_0^2 + \epsilon^2\Big[\rho_0^2 f(x)f(x+x') + \rho_1(x)\rho_1(x+x') + \rho_1(x)\rho_0 f(x+x') + \rho_1(x+x')\rho_0 f(x) \\
&\quad + (f(x)\rho_0 + f(x+x')\rho_0 + \rho_1(x) + \rho_1(x+x'))\,(r_1(x+x') - r_1(x)) \\
&\quad + \frac{1}{2}(r_1(x+x') - r_1(x))^2\Big]\Big) \\
&= H^2\exp(\overline{R}x')\Big(\rho_0^2 + \epsilon^2\Big[\rho_0^2 f(x)f(x+x') + \rho_1(x)\rho_1(x+x') + \rho_1(x)\rho_0 f(x+x') + \rho_1(x+x')\rho_0 f(x) \\
&\quad + (f(x)\rho_0 + \rho_1(x))\,r_1(x+x') - (f(x+x')\rho_0 + \rho_1(x+x'))r_1(x) \\
&\quad + \frac{1}{2}(r_1(x+x') - r_1(x))^2\Big]\Big) \\
&\equiv H^2\rho_0^2\exp(\overline{R}x')\left(1 + \epsilon^2\mathcal{I}(x,x')\right).
\end{aligned} \tag{S62}$$

This approach eliminates the appearance of the function $\rho_2(x)$. In addition, we find that all terms in the expansion of $r_1(x)$ appear at order $\epsilon^2$ in $I(x, x')$ and therefore we indeed only need the order 1 terms in $r(x)$.

$$D_{\text{eff}} = \frac{J}{\rho_0 H}\frac{1}{\overline{R}} + \frac{J}{\rho_0 H}\epsilon^2 \frac{\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x' \mathcal{I}(x, x')}{L\left(\exp(\overline{R}L) - 1\right)}. \tag{S63}$$

We recall that

$$\frac{J}{\rho_0 H} = V_{\text{eff}} = v_{\text{wall}}\left(1 - \frac{\epsilon^2}{2}\frac{1}{1 + (\text{Pe}^\star)^2}\right) \tag{S64}$$

and $\overline{R} = \frac{v_{\text{wall}}}{D_0}\left(1 + \epsilon^2 R_2\right)$ such that

$$D_{\text{eff}} = D_0\left(1 - \epsilon^2 R_2 - \frac{\epsilon^2}{2}\frac{1}{1 + (\text{Pe}^\star)^2}\right) + v_{\text{wall}}\epsilon^2 \frac{1}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x' \mathcal{I}(x, x'). \tag{S65}$$

Carrying on the integrals with the use of an automated software for example, it is quite straightforward to show

$$D_{\text{eff}} = D_0\left(1 + \frac{\epsilon^2}{2}\frac{3\text{Pe}^2 - 1 - (\text{Pe}^\star)^2\epsilon_D^2 + (1 + \text{Pe}^2)(\text{Pe}^\star)^2\left[\epsilon_D + \epsilon_{D,2}\right]}{(1 + (\text{Pe}^\star)^2)(1 + \text{Pe}^2)}\right) \tag{S66}$$

where we recall that

$$\epsilon_D = \frac{D'(\rho_0)\rho_0}{D(\rho_0)} \text{ and } \epsilon_{D,2} = \frac{D''(\rho_0)\rho_0^2}{2D(\rho_0)}. \tag{S67}$$

In this compact form notice that we recover the limit where $D_{\text{eff}}$ is given by (3.19) of the main text when the diffusion coefficient is assumed to be uniform in space, namely when $\epsilon_D = 0$ and $\epsilon_{D,2} = 0$.

In summary we found that the effective drift is not changed by a varying diffusion coefficient in this context. Likely the largest contribution to the effective diffusion coefficient is that proportional to $\epsilon_D$, such that (S66) can be approximated by

$$D_{\text{eff}} \simeq D_0\left(1 + \frac{\epsilon^2}{2}\frac{3\text{Pe}^2 - 1}{(1 + (\text{Pe}^\star)^2)(1 + \text{Pe}^2)} + \epsilon_D\frac{\epsilon^2}{2}\frac{(\text{Pe}^\star)^2}{1 + (\text{Pe}^\star)^2}\right). \tag{S68}$$

In our simulations, at most $\epsilon_D \simeq 0.1$ which yields negligible corrections numerically. Notice that, based on a general perturbation theory that derives the long-time diffusion coefficient with spatiotemporal varying local drift and diffusivity, we can recover the same results found here. The details of this perturbation theory will be published elsewhere.

## Supplementary References


[1] Marbach, S. & Alim, K. Active control of dispersion within a channel with flow and pulsating walls. *Physical Review Fluids* **4**, 114202 (2019).

[2] Chakrabarti, B. & Saintillan, D. Shear-induced dispersion in peristaltic flow. *Physics of Fluids* **32**, 113102 (2020). URL http://aip.scitation.org/doi/10.1063/5.0030569.

[3] Ma, M. *et al.* Water transport inside carbon nanotubes mediated by phonon-induced oscillating friction. *Nature nanotechnology* **10**, 692–695 (2015).

[4] Codutti, A., Cremer, J. & Alim, K. Changing flows balance nutrient absorption and bacterial growth along the gut. *bioRxiv* (2022).

[5] Alim, K., Amselem, G., Peaudecerf, F., Brenner, M. P. & Pringle, A. Random network peristalsis in physarum polycephalum organizes fluid flows across an individual. *Proceedings of the National Academy of Sciences* **110**, 13306–13311 (2013).

[6] Guérin, T. & Dean, D. S. Kubo formulas for dispersion in heterogeneous periodic nonequilibrium systems. *Physical Review E* **92**, 062103 (2015).