Supplementary material for "Grounding-line flux conditions for marine ice-sheet systems under effective-pressure-dependent and hybrid friction laws"

1. Flux conditions for hybrid friction laws: the (T) and (RC2) friction laws

In this section, we derive flux conditions for the (T) and (RC2) friction laws, in a similar way to what has been presented in section 4 of the article for the (RC1) friction law. Because the (T) and (RC2) friction laws are qualitatively similar – the (RC2) friction law can be thought of as the regularisation of the (T) friction law – both friction laws are considered together.

1.1. *Derivation*

Introducing the parameter $\tilde{\upsilon}$, which is a scaled version of A_s as

$$\tilde{\upsilon} \equiv \begin{cases} (\rho g)^{1/p}(2\rho g)^{-n} C^{(p+1)/p} A^{-1} h_{\mathrm{gl}}^{[1+p(n-1)]/p}\, A_s & (\mathrm{N_A\ model}), \\ (\rho g)^{1/p}(2\rho g)^{-n} [C(1-c)]^{(p+1)/p} A^{-1} h_{\mathrm{gl}}^{[1+p(n-1)]/p}\, A_s & (\mathrm{N_B\ model}), \end{cases} \tag{S1.1}$$

the following problem is obtained, in terms of the variables $(\tilde{U}, \tilde{W})$:

$$\begin{cases} \dfrac{\mathrm{d}\tilde{U}}{\mathrm{d}\tilde{X}} = -|\tilde{W}|^{n-1}\tilde{W}, \quad \text{for } \tilde{X} > 0, & \text{(S1.2}a\text{)} \\ \dfrac{\mathrm{d}\tilde{W}}{\mathrm{d}\tilde{X}} = -\dfrac{|\tilde{W}|^{n+1}}{\tilde{U}} - \dfrac{1}{4}\tilde{\Phi}(\tilde{U};\tilde{Q}_{\mathrm{gl}})\,\mathrm{sgn}(\tilde{U}) + \dfrac{\tilde{Q}_{\mathrm{gl}}|\tilde{W}|^{n-1}\tilde{W}}{4\,\tilde{U}^2}, \quad \text{for } \tilde{X} > 0, & \text{(S1.2}b\text{)} \\ (\tilde{U}, \tilde{W}) = (\tilde{Q}_{\mathrm{gl}}, \delta/8), \quad \text{at } \tilde{X} = 0, & \text{(S1.2}c\text{)} \\ (\tilde{U}, \tilde{W}) \to (0, 0), \quad \text{as } \tilde{X} \to +\infty, & \text{(S1.2}d\text{)} \end{cases}$$

with $\tilde{\Phi}$ defined, for the (T) friction law, as

$$\tilde{\Phi}(\tilde{U};\tilde{Q}_{\mathrm{gl}}) = \min\left(1 - 1_{\mathrm{A}}\frac{\tilde{U}}{\tilde{Q}_{\mathrm{gl}}}, \frac{\tilde{U}}{\tilde{Q}_{\mathrm{gl}}}\left(\frac{\tilde{U}}{\tilde{\upsilon}}\right)^p\right), \tag{S1.3}$$

and, for the (RC2) friction law, as

$$\tilde{\Phi}(\tilde{U};\tilde{Q}_{\mathrm{gl}}) = \left(\frac{|\tilde{U}|}{|\tilde{U}| + \tilde{\upsilon}\left(\tilde{Q}_{\mathrm{gl}}/\tilde{U} - 1_{\mathrm{A}}\right)^{\frac{1}{p}}}\right)^p \left(1 - 1_{\mathrm{A}}\frac{\tilde{U}}{\tilde{Q}_{\mathrm{gl}}}\right). \tag{S1.4}$$

The system (S1.2) can be solved for several values of $\tilde{\upsilon}$ in order to approximate the mapping $\tilde{\upsilon} \mapsto \tilde{Q}_{\mathrm{gl}}(\tilde{\upsilon})$. We can also guess the way in which $\tilde{Q}_{\mathrm{gl}}$ depends on δ, as the (T) and (RC2) friction laws can be associated with a (W) friction law for low values of $\tilde{\upsilon}$, and a (C) friction law for large values of $\tilde{\upsilon}$. This suggests to introduce $\check{Q}_{\mathrm{gl}}$ and $\check{\upsilon}$ as

$$\check{Q}_{\mathrm{gl}} \equiv \begin{cases} \tilde{Q}_{\mathrm{gl}}\,(\delta/8)^{n-1} \\ \tilde{Q}_{\mathrm{gl}}\,(\delta/8)^{n} \end{cases} \quad \text{and} \quad \check{\upsilon} \equiv \begin{cases} \tilde{\upsilon}\,(\delta/8)^{(p+1-np)/p} & (\mathrm{N_A\ model}), \\ \tilde{\upsilon}\,(\delta/8)^{-n} & (\mathrm{N_B\ model}), \end{cases} \tag{S1.5}$$

so that the flux conditions will take the form

$$
\begin{cases}
q_{\mathrm{gl}} = \check{Q}_{\mathrm{gl}}(\check{v})\ (\delta/8)^{n-1}\,(2\rho g)^n C^{-1} A\, h_{\mathrm{gl}}^{n+2}, & \text{(S1.6}a\text{)}\\
\text{with}\quad \check{v} \equiv \left(\dfrac{\delta}{8}\right)^{\frac{p+1-np}{p}} \left[\dfrac{(\rho g)^{\frac{1}{p}}}{(2\rho g)^n} C^{\frac{p+1}{p}} A^{-1} h_{\mathrm{gl}}^{\frac{1}{p}-1+n}\right] A_s, & \text{(S1.6}b\text{)}
\end{cases}
$$

for the N_A model, and

$$
\begin{cases}
q_{\mathrm{gl}} = \check{Q}_{\mathrm{gl}}(\check{v})\ (\delta/8)^{n}\,(2\rho g)^n [C(1-c)]^{-1} A\, h_{\mathrm{g}}^{n+2}, & \text{(S1.7}a\text{)}\\
\text{with}\quad \check{v} \equiv \left(\dfrac{\delta}{8}\right)^{-n} \left[\dfrac{(\rho g)^{\frac{1}{p}}}{(2\rho g)^n} [C(1-c)]^{\frac{p+1}{p}} A^{-1} h_{\mathrm{gl}}^{\frac{1}{p}-1+n}\right] A_s, & \text{(S1.7}b\text{)}
\end{cases}
$$

for the N_B model. The mappings $\check{v} \mapsto \check{Q}_{\mathrm{gl}}(\check{v})$ for the (RC2) and (T) friction laws are approximated following table S1. The following additional approximation of the maximum function has been introduced:

$$
m_{\epsilon_1,\epsilon_2}(a, b, x) = a\,(\log[\exp(x - b/a) + \epsilon_1] + \epsilon_2) + b. \qquad \text{(S1.8)}
$$

The comparison between the approximations of the mappings $\check{v} \mapsto \check{Q}_{\mathrm{gl}}(\check{v})$ and the numerical results is shown in figure S1.

1.2. *Comments*

It can remarked that these mappings do not systematically consist in a smooth transition between a constant value $\check{Q}_{\mathrm{gl}} = \check{Q}_{\mathrm{gl}}|_{\check{v}=0}$ and a curve $\check{Q}_{\mathrm{gl}} = \check{v}^{p/(p+1)}$, as opposed to the (RC1) case.

For the (RC2) and (T) friction laws combined with the N_A model, the values of $\check{Q}_{\mathrm{gl}}$ are above the curve $\check{Q}_{\mathrm{gl}} = \check{v}^{p/(p+1)}$, with some offset that cannot be neglected. This is explained as follows: even if $\check{v}$ is very large, there is always a region where there will be Coulomb friction. This means that a pure Weertman behaviour can never be reached, which yields this difference.

For the (T) friction law with the N_B model, the transition from the constant value to the curve is abrupt. As explained in section 3.3 of the article, if one considers the N_B effective-pressure model, the membrane-stress divergence stays small compared to the friction and gravity stresses, and the stress distribution is almost the same everywhere in the domain, with gravity essentially balancing friction. Then, the boundary condition at the grounding line can be used to directly obtain the flux condition. This means that the minimum function will be "transferred" directly to the flux condition. This analysis is not valid for the N_A model, as in that case the stress distribution is not almost constant throughout the domain.

Friction law	Effective pressure	$\check{Q}(\check{v})$	Free parameters
(RC2)	N_A $(1_A = 1)$	$m_{\epsilon_1,\epsilon_2}\left(\check{Q}_{gl}^{(B)}, \check{Q}_{gl}^{(C)}, \check{v}^{p/(p+1)}\right)$	$(\epsilon_1, \epsilon_2) = (0.255, 0.223)$
	N_B $(1_A = 0)$	$m_{\epsilon}\left(\check{Q}_{gl}^{(B)}, \check{Q}_{gl}^{(C)}, \check{v}^{p/(p+1)}\right)$	$\epsilon = 3.046$
(T)	N_A $(1_A = 1)$	$m_{\epsilon_1,\epsilon_2}\left(\check{Q}_{gl}^{(B)}, \check{Q}_{gl}^{(C)}, \check{v}^{p/(p+1)}\right)$	$(\epsilon_1, \epsilon_2) = (0.255, 0.223)$
	N_B $(1_A = 0)$	$m_{\epsilon}\left(\check{Q}_{gl}^{(B)}, \check{Q}_{gl}^{(C)}, \check{v}^{p/(p+1)}\right)$	$\epsilon = +\infty$

Table S1: Functions $\check{v} \mapsto \check{Q}(\check{v})$ used in the flux condition of the (RC2) and (T) friction laws.

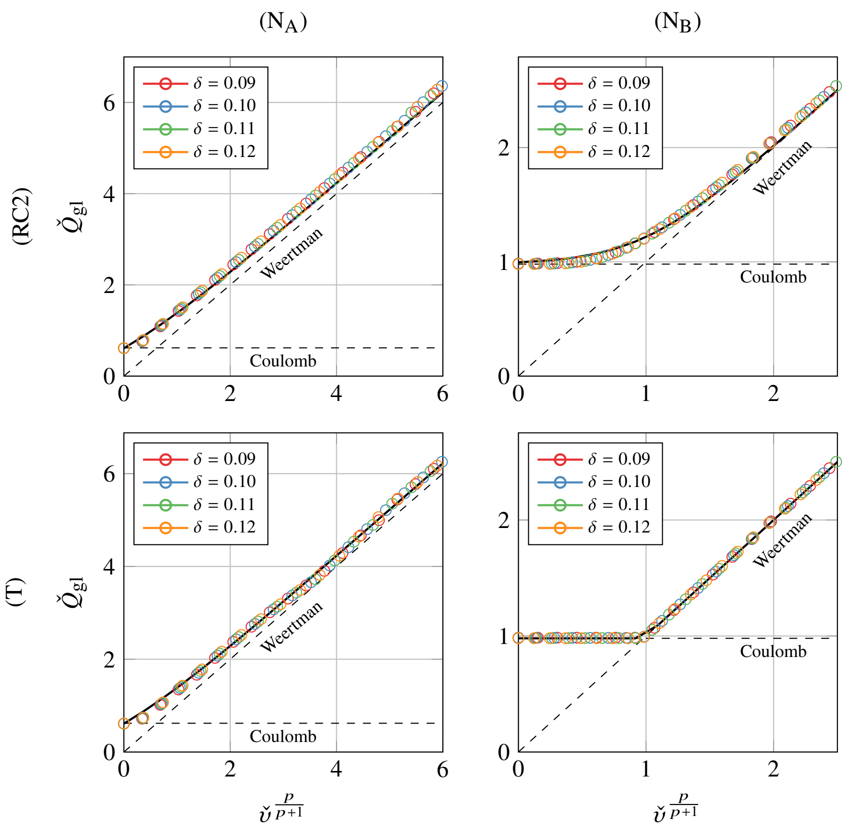

Figure S1: Relation between $\check{v}$ and $\check{Q}_{gl}$ for the (RC2) and (T) friction laws combined with the N_A and N_B effective-pressure models. The circles correspond to values obtained numerically (using the numerical method described in the appendix B of the article), and the continuous lines correspond to the approximations described in table S1.