The Cameron-Martin-Wiener Method in Turbulence and in Burgers' Model: General Formulae, and Application to Late Decay

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The following are the complete texts of the equations that were omitted in the text of the above-named article as it was published in the Journal of Fluid Mechanics, 41 part 3, 593 (1970).

$$
\begin{aligned}
u_{(2)}(x,t) = {} & \frac{R_o^2}{24}\frac{\partial}{\partial x}\sqrt{\frac{R_o}{4\pi t}}\int e^{-(R_o/4t)z^2}\left(\int_{x_o}^{x+z} u(s;0)ds\right)^3 dz \\
& - \frac{R_o^2}{8}\frac{\partial}{\partial x}\left\{\sqrt{\frac{R_o}{4\pi t}}\int e^{-(R_o/4t)z^2}\left(\int_{x_o}^{x+z} u(s;0)ds\right)^2 dz\right\} \\
& \quad \left\{\sqrt{\frac{R_o}{4\pi t}}\int e^{-(R_o/4t)z^2}\left(\int_{x_o}^{x+z} u(s;0)ds\right) dz\right\} \\
& + \frac{R_o^2}{12}\frac{\partial}{\partial x}\left[\sqrt{\frac{R_o}{4\pi t}}\int e^{-(R_o/4t)z^2}\left(\int_{x_o}^{x+z} u(s;0)ds\right) dz\right]^3
\end{aligned}
\qquad (6.3c)
$$

$$
\begin{aligned}
\tilde{K}^{(1)}(k_1;t) = {} & -ik_1\tilde{L}^{(1)}(k_1;0)e^{-(t/R_o)k_1^2} \\
& + i\frac{R_o^2}{8\pi}k_1\tilde{L}^{(1)}(k_1;0)\int dk'\left(L^{(1)}(k';0)\right)^2 \\
& \quad \left(e^{-(t/R_o)(k_1-k')^2-(t/R_o)k'^2} - e^{-(t/R_o)k_1^2-(2t/R_o)k'^2}\right) \\
& + \mathcal{O}(R_o^3)
\end{aligned}
\qquad (6.9)
$$

$$\tilde{K}^{(2)}(k_1, k_2; t) = i \frac{R_o}{4} (k_1+k_2)\tilde{L}^{(1)}(k_1;0)\tilde{L}^{(1)}(k_2;0) \cdot$$

$$\left(e^{-(t/R_o)(k_1+k_2)^2} - e^{-(t/R_o)(k_1^2+k_2^2)}\right) + \mathcal{O}(R_o^3) \tag{6.10}$$

$$\tilde{K}^{(3)}(k_1, k_2, k_3; t) = -i \frac{R_o^2}{24} (k_1+k_2+k_3)\tilde{L}^{(1)}(k_1;0)\tilde{L}^{(1)}(k_2;0)\tilde{L}^{(1)}(k_3;0)$$

$$\left(e^{-(t/R_o)(k_1+k_2+k_3)^2} - e^{-(t/R_o)(k_1^2+(k_2+k_3)^2)} - e^{-(t/R_o)(k_2^2+(k_1+k_3)^2)}\right.$$

$$\left. - e^{-(t/R_o)(k_3^2+(k_1+k_2)^2)} + 2e^{-(t/R_o)(k_1^2+k_2^2+k_3^2)}\right) + \mathcal{O}(R_o^4) \tag{6.11}$$

$$\tilde{K}^{(m)}(k_1, \ldots, k_m; t) = \mathcal{O}(R_o^{m-1}) \quad . \tag{6.12}$$

(b) Non-Gaussian initial velocity field.

$$\tilde{K}^{(1)}(k_1; t) = -ik_1\tilde{L}^{(1)}(k_1;0)e^{-(t/R_o)k_1^2}$$

$$+ i \frac{R_o}{2\pi} k_1 e^{-(t/R_o)k_1^2} \int dk' \, \tilde{L}^{(1)}(k';0)\tilde{L}^{(2)}(k_1, -k';0)$$

$$\left(1-e^{-(2t/R_o)(k'^2-k_1k')}\right) + \mathcal{O}(R_o^2) \tag{6.13}$$

$$\tilde{K}^{(2)}(k_1, k_2; t) = -i (k_1+k_2)\tilde{L}^{(2)}(k_1, k_2;0) \, e^{-(t/R_o)(k_1+k_2)^2}$$

$$+ i \frac{R_o}{4} (k_1+k_2)\tilde{L}^{(1)}(k_1;0)\tilde{L}^{(1)}(k_2;0) \left(e^{-(t/R_o)(k_1+k_2)^2} - e^{-(t/R_o)(k_1^2+k_2^2)}\right)$$

$$+ i \frac{R_o}{2\pi}(k_1+k_2)\int dk'\tilde{L}^{(2)}(k_1, k';0)\tilde{L}^{(2)}(k_2, -k' = 0) \cdot$$

$$\left(e^{-(t/R_o)(k_1+k_2)^2} - e^{-(t/R_o)(k_1+k_2)^2-(2t/R_o)(k'+k_1)(k'-k_2)}\right)$$

$$+\mathcal{O}(R_o^2) \qquad (6.14)$$

$$\tilde{K}^{(3)}(k_1, k_2, k_3; t)$$

$$= i \frac{R_o}{6}(k_1+k_2+k_3)\left\{\tilde{L}^{(1)}(k_1;0)\tilde{L}^{(2)}(k_2, k_3;0)\left(e^{-(t/R_o)(k_1+k_2+k_3)^2}\right.\right.$$

$$\left. - e^{-(t/R_o)(k_1^2+(k_2+k_3)^2)}\right) + \tilde{L}^{(1)}(k_2;0)\tilde{L}^{(2)}(k_1, k_3;0)$$

$$\left(e^{-(t/R_o)(k_1+k_2+k_3)^2} - e^{-(t/R_o)(k_2^2+(k_1+k_3)^2)}\right)$$

$$+ \tilde{L}^{(1)}(k_3;0)\tilde{L}^{(2)}(k_1, k_2;0)\left(e^{-(t/R_o)(k_1+k_2+k_3)^2} - e^{-(t/R_o)(k_3^2+(k_1+k_2)^2)}\right)$$

$$+\mathcal{O}(R_o^2) \qquad (6.15)$$

$$\tilde{K}^{(4)}(k_1, k_2, k_3, k_4; t)$$

$$= i \frac{R_o}{12}(k_1+k_2+k_3+k_4) \cdot$$

$$\left\{\tilde{L}^{(2)}(k_1, k_3;0)\tilde{L}^{(2)}(k_3, k_4;0)\left(e^{-(t/R_o)(k_1+k_2+k_3+k_4)^2} - e^{-(t/R_o)((k_1+k_2)^2+(k_3+k_4)^2)}\right)\right.$$

$$+\tilde{L}^{(2)}(k_1, k_3;0)L^{(2)}(k_2, k_4;0)\left(e^{-(t/R_o)(k_1+k_2+k_3+k_4)^2} - e^{-(t/R_o)((k_1+k_3)^2+(k_2+k_4)^2)}\right)$$

$$+\tilde{L}^{(2)}(k_1, k_4; 0)\tilde{L}^{(2)}(k_2, k_3; 0)\left(e^{-(t/R_o)(k_1+k_2+k_3+k_4)^2} - e^{-(t/R_o)((k_1+k_3)^2+(k_2+k_3)^2)}\right)\Bigg\} + \mathcal{O}(R_o^2) \tag{6.16}$$

$$\tilde{K}^{(5)}(k_1, k_2, k_3, k_4, k_5; t) = \mathcal{O}(R_o^2) \tag{6.17}$$

$$\begin{aligned} E(k,t) = {} & k^2\left(\tilde{L}^{(1)}(k,0)\right)^2 e^{-(2t/R_o)k^2} \\ & + \frac{R_o^2}{4\pi}\, k^2\left(\tilde{L}^{(1)}(k,0)\right)^2 e^{-(t/R_o)k^2}\int dk'\left(\tilde{L}^{(1)}(k';0)\right)^2 \cdot \\ & \qquad \left(e^{-(t/R_o)(k^2+2k'^2)} - e^{-(t/R_o)((k-k')^2+k'^2)}\right) \\ & + \frac{R_o^2}{16\pi}\, k^2\int dk'\left(\tilde{L}^{(1)}(k';0)\tilde{L}^{(1)}(k-k')^2\right) \\ & \qquad \left(e^{-(2t/R_o)k^2} - 2e^{-(t/R_o)k^2-(t/R_o)(k'^2+(k-k')^2)} + e^{-(2t/R_o)(k'^2+(k-k')^2)}\right) \\ & + \mathcal{O}(R_o^4) \quad . \end{aligned} \tag{6.25}$$

$$E'(k,t) = k^2(\tilde{L}^{(1)}(k;0))^2 e^{-(2t/R_o)k^2}$$

$$+ \frac{R_o^2}{16\pi} k^2 e^{-(2t/R_o)k^2} \left\{ \int dk' \left(\tilde{L}^{(1)}(k';0)\tilde{L}^{(1)}(k-k';0) \right)^2 \right\} + \mathcal{O}(R_o^4) \tag{6.26}$$

$$\mathcal{E}(k,t) = \frac{R_o^2}{4\pi} k^2 \left(\tilde{L}^{(1)}(k';0) \right)^2 \; e^{-(t/R_o)\, k^2}$$

$$\int dk' \left(\tilde{L}^{(1)}(k';0) \right)^2 \left(e^{-(t/R_o)(k^2+2k'^2)} - e^{-(t/R_o)((k-k')^2+k'^2)} \right)$$

$$+ \frac{R_o}{16\pi} k^2 \int dk' \left(\tilde{L}^{(1)}(k';0)\tilde{L}^{(1)}(k-k';0) \right)^2$$

$$\left(e^{-(2t/R_o)(k'^2+(k-k')^2)} - 2e^{-(t/R_o)(k^2+k'^2)-(t/R_o)(k-k')^2} \right)$$

$$+ \mathcal{O}(R_o^4) \tag{6.27}$$

$$T(k,t) = - \frac{R_o}{4\pi} k^2 \int dk'k'(k'-k) \left(\tilde{L}^{(1)}(k';0)\tilde{L}^{(1)}(k'-k;0) \right)^2$$

$$e^{-(t/R_o)(k'^2+(k'-k)^2)} \left(e^{-(t/R_o)k^2} - e^{-(t/R_o)(k'^2+(k-k')^2)} \right)$$

$$- \frac{R_o}{2\pi} k^2 \left(\tilde{L}^{(1)}(k,0) \right)^2 e^{-(t/R_o)k^2} \int dk'k'(k'-k) \left(\tilde{L}^{(1)}(k';0) \right)^2 e^{-(t/R_o)(k'^2+(k-k')^2)}$$

$$+ \frac{R_o}{2\pi} k^2 \left(\tilde{L}^{(1)}(k;0) \right)^2 e^{-(2t/R_o)k^2} \int dk'k'^2 \left(\tilde{L}^{(1)}(k';0) \right)^2 e^{-(2t/R_o)k'^2}$$

$$+ \mathcal{O}(R_o^3) \quad . \tag{6.28}$$

$$E(k,t) = k^2\left(\tilde{L}^{(1)}(k,0)\right)^2 e^{-(2t/R_o)k^2} + \frac{1}{\pi} k^2 e^{-(2t/R_o)k^2} \int dk' \left(\tilde{L}^{(2)}(k',k-k';0)\right)^2$$

$$- \frac{R_o}{\pi} k^2 \tilde{L}^{(1)}(k,0) \int dk' \tilde{L}^{(1)}(k';0) \tilde{L}^{(2)}(k,-k';0) \cdot$$

$$\left(e^{-(2t/R_o)k^2} - e^{-(2t/R_o)(k'-(k/2))^2-(3t/2R_o)k^2}\right)$$

$$- \frac{R_o}{2\pi} k^2 \int dk' \tilde{L}^{(1)}(k';0) \tilde{L}^{(1)}(k-k';0) \tilde{L}^{(2)}(k',k-k';0) \cdot$$

$$\left(e^{-(2t/R_o)k^2} - e^{-(2t/R_o)(k'-(k/2))^2-(3/2)(t/R_o)k^2}\right)$$

$$- \frac{R_o}{\pi^2} k^2 \iint dk_1' dk_2' \tilde{L}^{(2)}(k_1', k-k_1'; 0) \tilde{L}^{(2)}(k_1', k_2'; 0) \tilde{L}^{(2)}(k-k_1', -k_2'; 0) \cdot$$

$$\left(e^{-(2t/R_o)k^2} - e^{-(2t/R_o)k^2-(2t/R_o)(k_1'+k_2')(k_1'+k_2'-k)}\right)$$

$$+ \mathcal{O}(R_o^2) \quad . \tag{6.37}$$

$$E'\left(\sqrt{\frac{R_o}{t}}\, q, t\right) \approx \left(\frac{R_o}{t}\right)\left\{\left(\tilde{L}^{(1)}(0,0)\right)^2 + \frac{1}{\pi}\int dk' \left(\tilde{L}^{(2)}(k',-k';0)\right)^2\right\} \cdot q^2 e^{-2q^2}$$

$$- \frac{R_o}{\pi}\left(\frac{R_o}{t}\right) q^2 e^{-2q^2} \left\{ \tilde{L}^{(1)}(0,0) \int dk' \tilde{L}^{(1)}(k',0) \tilde{L}^{(2)}(0,k';0)\right.$$

$$+ \frac{1}{2} \int dk' \tilde{L}^{(1)}(k';0) \tilde{L}^{(1)}(-k';0) \tilde{L}^{(2)}(k',-k';0)$$

$$\left. + \frac{1}{\pi} \iint dl_1' dk_2' \tilde{L}^{(2)}(k_1', -k_1'; 0) \tilde{L}^{(2)}(k_1', k_2'; 0) \tilde{L}^{(2)}(-k_1', -k_2'; 0) \right\}$$

$$\tag{6.38}$$

and

$$\mathcal{E}\left(\sqrt{\frac{R_o}{t}} \cdot q, t\right) \approx \frac{R_o}{2\sqrt{2\pi}} \left\{ 3\left(\tilde{L}^{(1)}(0,0)\right)^2 \left(\tilde{L}^{(2)}(0,0;0)\right) + \frac{2}{\pi} \int dk' \left(\tilde{L}^{(2)}(k',-k';0)\right)^3 \right\}$$

$$\cdot \left(\frac{R_o}{t}\right)^{3/2} q^2 e^{-(3/2)q^2} \quad , \tag{6.39}$$

$$T\left(\sqrt{\frac{R_o}{t}}\ q,t\right) \approx \frac{1}{4}\frac{1}{\sqrt{2\pi}}\left\{3\left(\tilde{L}^{(1)}(0,0)\right)^2\left(\tilde{L}^{(2)}(0,0;0)\right)+\frac{2}{\pi}\int dk'\right.$$

$$\left.\left(\tilde{L}^{(2)}(k',-k';0)\right)^3\right\}\cdot\left(\frac{R_o}{t}\right)^{5/2} q^2(1-q^2)e^{-(3/2)q^2} \quad . \qquad (6.40)$$

$$\left\langle\left(u^{(n)}(x,t)\right)^2\right\rangle \approx \frac{1}{2\sqrt{2\pi}}\left(\tilde{L}^{(1)}(0,0)\right)^2 \frac{(2n+1)!!}{4^{n+1}}\left(\frac{R_o}{t}\right)^{n+(3/2)} \quad . \qquad (6.46)$$

The third order moment is given by

$$\left\langle\left(u^{(n)}(x,t)\right)^3\right\rangle \approx (-)^{(n-1)/2}\,\frac{3}{2}\,\frac{R_o}{\pi}\left(\tilde{L}^{(1)}(0,0)\right)^4\cdot\left(\frac{R_o}{t}\right)^{(3/2)(n+1)+1}$$

$$\cdot\left\{\sum_{r=0}^{n+1}(-)^r\binom{n+1}{r}\frac{(2n-2r+1)!!\,(n+2r)!!}{4^{n+2}\cdot 3^{r+(n/2)+1}}\right.$$

$$\left.-\sum_{r=0,2,4,\ldots}^{n+1}\binom{n+1}{r}\frac{(2n-r+1)!!\,(n+r)!!}{2^{3(n+2)}}\right\} \quad , \qquad (6.47)$$

$$N_4^{(n)}(t) \approx \frac{3\, R_o^2}{32\pi^{3/2}} \left(\tilde{L}^{(1)}(0,0)\right)^6 \left(\frac{R_o}{t}\right)^{2n+(7/2)} \cdot$$

$$\cdot \left\{ \sum_{\substack{r=0 \\ n:\ \text{even} \\ s+\ell:\ \text{odd}}}^{n+1} \sum_{s=0}^{2r} \sum_{\ell=0}^{n+1} \binom{n+1}{r}\binom{2r}{s}\binom{n+1}{\ell}(-)^{r+\ell+1} \right.$$

$$\cdot \frac{(2n-2r+1)!!\ (2r+n-s-\ell)!!\ (n+s+\ell)!!}{2^{3n+\ell+3+(1/2)(n+s+\ell+1)}\ 3^r}$$

$$+ \sum_{r=0}^{n+1} \sum_{s=0,2,4\ldots}^{n+1} \binom{n+1}{r}\binom{n+1}{s}(-)^{r+1} \frac{(2n-2r+1)!!\ (2n-s+1)!!\ (2r+s-1)!!}{2^{4n-s+3+(1/2)} \cdot 3^{r+(s/2)+(1/2)}}$$

$$+ \sum_{r=0,2,\ldots}^{n+1} \sum_{s=0,2,\ldots}^{n+1} \binom{n+1}{r}\binom{n+1}{s} \frac{(2n-r+1)!!\ (r+s-1)!!\ (2n-s+1)!!}{2^{2(n+1)+(3/2)}}$$

$$+ \sum_{\substack{r=0,2,\ldots \\ n+s+\ell+1:\ \text{even}}}^{n+1} \sum_{s=0}^{r} \sum_{\ell=0}^{n+1} (-)^{\ell} \binom{n+1}{r}\binom{r}{s}\binom{n+1}{\ell} \frac{(2n-r+1)!!\ (n+r-s-\ell)!!(n+s+\ell)!!}{2^{3n+3+(1/2)} \cdot 3^{(1/2)(n+s+\ell)}}$$

$$\left. + \sum_{\substack{r=0,2,\ldots \\ n+s+1:\ \text{even}}}^{n+1} \sum_{s=0}^{r} \binom{n+1}{r}\binom{r}{s} \frac{(2n-r+1)!!\ (n+r-s)!!\ (n+s)!!}{2^{4n+4+(1/2)}} \right\} . \qquad (6.48)$$

$$\left\langle \left(u^{(n)}(x,t) \right)^2 \right\rangle \approx \frac{1}{2\sqrt{2\pi}} \left[\left(\tilde{L}^{(1)}(0,0) \right)^2 + \frac{1}{\pi} \int dk' \left(\tilde{L}^{(2)}(-k',k';0) \right)^2 \right] \cdot \frac{(2n+1)!!}{4^{n+1}} \left(\frac{R_o}{t} \right)^{n+(3/2)} . \tag{6.51}$$

The third order moment is given by

$$\left\langle \left(u^{(n)}(x,t) \right)^3 \right\rangle \approx \frac{\sqrt{3}}{2\pi} (-)^{(n+1)/2} \left(\tilde{L}^{(1)}(0,0) \right)^2 \left(\tilde{L}^{(2)}(0,0;0) \right) \cdot \left(\frac{R_o}{t} \right)^{(3/2)(n+1)+1} \cdot$$

$$\cdot \sum_{r=0}^{n+1} (-)^r \binom{n+1}{r} \frac{(2n-2r+1)!! \, (n+2r)!!}{4^n \cdot 3^{r+(n+1)/2}}$$

$$= 0 \quad \text{if} \quad n \quad \text{is even} \quad . \tag{6.52}$$

$$N_4^{(n)}(t) \approx \frac{12}{\pi^{(3/2)}} \left(\tilde{L}^{(1)}(0,0) \right)^2 \left(\tilde{L}^{(2)}(0,0;0) \right) \cdot \sum_{r=0}^{n+1} \sum_{s=0}^{n+1} (-)^{r+s} \binom{n+1}{r} \binom{n+1}{s}$$

$$\frac{(2n-2r+1)!! \, (2n-2s+1)!! \, (2r+2s-1)!!}{2^{4(n+1)} \cdot 2^{r+s}} \left(\frac{R_o}{t} \right)^{2n+(7/2)} . \tag{6.53}$$