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Appendix to the paper "On the Creeping Motion of Two Arbitrary Sized Touching Spheres in a Linear Shear Field", by A. Nir and A. Acrivos.
Vol 59, 1973, p. 209

FM 2766

I. The Solution of the Equations of Motion for Arbitrary-Sized Spheres in a Linear Flow Field.

Here we describe the procedure used in the above paper for the solution of the functions $A_m(\nu)$ to $H_m(\nu)$.

Let V_r^N, $V_{\hat{\varphi}}^N$ and V_z^N, the cylindrical components of the velocities $\underset{\sim}{V}^N$ at the surface of the N^{th} sphere, be expanded as

$$\begin{pmatrix} V_r^N \\ V_z^N \end{pmatrix} = \sum_{m=o}^{\infty} \left\{ \begin{pmatrix} R_m^N \\ Z_m^N \end{pmatrix} \sin m\hat{\varphi} + \begin{pmatrix} R_{-m}^N \\ Z_{-m}^N \end{pmatrix} \cos m\hat{\varphi} \right\}$$

$$V_{\hat{\varphi}}^N = \sum_{m=o}^{\infty} \left\{ \frac{1}{m} \Phi_m^N \frac{\partial}{\partial\hat{\varphi}} \sin m\hat{\varphi} + \frac{1}{m} \Phi_{-m}^N \frac{\partial}{\partial\hat{\varphi}} \cos m\hat{\varphi} \right\} \qquad \text{(A1)}$$

where R_m^N, Φ_m^N and Z_m^N are only functions of η. When the components of (A1) are equated to the corresponding components of (2.5), evaluated on the surface $\xi = \xi_N$, the following relations for the various auxiliary functions replace the no-slip requirement (2.4)

$$\left.\begin{aligned} \pi_m &= \frac{2}{z} Z_m^N - \frac{2}{z} w_m \\ v_m &= R_m^N + \Phi_m^N - \frac{r}{z} Z_m^N + \frac{r}{z} w_m \\ u_m &= R_m^N - \Phi_m^N - \frac{r}{z} Z_m^N + \frac{r}{z} w_m \end{aligned}\right\} \text{ at } \xi = \xi_N . \qquad \text{(A2)}$$

These, together with (2.7) and its equivalents, constitute implicit relations between the various function coefficients A_m to H_m.

For convenience in further evaluations we define

$$P_A(\nu) = \frac{\sinh \lambda\nu\alpha \cosh \nu\alpha}{\sinh(\lambda+1)\nu\alpha}$$

$$P_B(\nu) = \frac{\sinh \nu\alpha \cosh \lambda\nu\alpha}{\sinh(\lambda+1)\nu\alpha}$$

$$Q_A(\nu) = \frac{\cosh \nu\alpha \cosh \lambda\nu\alpha}{\sinh(\lambda+1)\nu\alpha} \tag{A3}$$

$$Q_B(\nu) = \frac{\sinh \nu\alpha \sinh \lambda\nu\alpha}{\sinh(\lambda+1)\nu\alpha}$$

where λ is the size-ratio of the two spheres.

Rearrangement of the expressions in (A2) by use of the differential equation and recurrence relations satisfied by $J_m(\nu\eta)$, followed by an integration by parts and application of the Hankel transform from the η space to the ν domain, yield finally the desired functional dependence of C_m - H_m on A_m and B_m,

$$C_m = \mathcal{C}_m + \frac{2(1-m^2)}{\nu^2\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A_m + \left(\frac{\lambda+1}{\lambda}Q_B - \frac{\nu\alpha}{1-m^2}\right)B_m\right] - \frac{2}{\nu\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A'_m + \left(\frac{\lambda+1}{\lambda}Q_B - 2\nu\alpha\right)B'_m\right] + \frac{2}{\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A''_m + \frac{\lambda+1}{\lambda}Q_B B''_m\right]$$

$$E_m = \mathcal{E}_m + \frac{m-1}{\nu\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A_m + \left(\frac{\lambda+1}{\lambda}Q_B + \frac{\nu\alpha}{m-1}\right)B_m\right] + \frac{1}{\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A'_m + \frac{\lambda+1}{\lambda}Q_B B'_m\right] \tag{A4}$$

$$G_m = \mathcal{G}_m + \frac{m+1}{\nu\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A_m + \left(\frac{\lambda+1}{\lambda}Q_B - \frac{\nu\alpha}{m-1}\right)B_m\right] - \frac{1}{\alpha}\left[\left(1-\frac{\lambda+1}{\lambda}P_B\right)A'_m + \frac{\lambda+1}{\lambda}Q_B B'_m\right],$$

together with similar expressions for D_m, F_m and H_m in which the set $(\mathcal{C}_m, \mathcal{E}_m, \mathcal{G}_m, P_B, Q_B, A_m, B_m)$ is replaced by, respectively, $(\mathcal{D}_m, \mathcal{F}_m, \mathcal{H}_m, P_A, Q_A, B_m, A_m)$, and the prime denotes differentiation with respect to ν. (2.10) now follows readily.

In (A4) the known integrals resulting from the transformation are

$$\begin{pmatrix}\mathcal{C}_m\\ \mathcal{D}_m\end{pmatrix} = 2\nu\begin{pmatrix}Q_B \operatorname{csch}\nu\alpha\\ Q_A \operatorname{sech}\nu\alpha\end{pmatrix}\int_0^\infty \frac{\eta\left(\alpha^2+\eta^2\right)^{\frac{1}{2}}}{\alpha} Z_m^I J_m(\nu\eta)d\eta$$

$$\mp 2\nu\begin{pmatrix}Q_B \operatorname{csch}\lambda\nu\alpha\\ Q_A \operatorname{sech}\lambda\nu\alpha\end{pmatrix}\int_0^\infty \frac{\eta\left(\lambda^2+\eta^2\right)^{\frac{1}{2}}}{\lambda\alpha} Z_m^{II} J_m(\nu\eta)d\eta$$

$$\begin{pmatrix}\mathcal{E}_m\\ \mathcal{F}_m\end{pmatrix} = \nu\begin{pmatrix}Q_B \operatorname{csch}\nu\alpha\\ Q_A \operatorname{sech}\nu\alpha\end{pmatrix}\int_0^\infty \frac{\eta}{\left(\alpha^2+\eta^2\right)^{\frac{1}{2}}}\left(R_m^I + \Phi_m^I - \frac{\eta}{\alpha} Z_m^I\right) J_{m-1}(\nu\eta)d\eta$$

$$\pm \nu\begin{pmatrix}Q_B \operatorname{csch}\lambda\nu\alpha\\ Q_A \operatorname{sech}\lambda\nu\alpha\end{pmatrix}\int_0^\infty \frac{\eta}{\left(\lambda^2\alpha^2+\eta^2\right)^{\frac{1}{2}}}\left(R_m^{II} + \Phi_m^{II} + \frac{\eta}{\lambda\alpha} Z_m^{II}\right) J_{m-1}(\nu\alpha)d\eta \qquad \text{(A5)}$$

$$\begin{pmatrix}\mathcal{G}_m\\ \mathcal{H}_m\end{pmatrix} = \nu\begin{pmatrix}Q_B \operatorname{csch}\nu\alpha\\ Q_A \operatorname{sech}\nu\alpha\end{pmatrix}\int_0^\infty \frac{\eta}{\left(\alpha^2+\eta^2\right)^{\frac{1}{2}}}\left(R_m^I - \Phi_m^I - \frac{\eta}{\alpha} Z_m^I\right) J_{m+1}(\nu\eta)d\eta$$

$$\pm \nu\begin{pmatrix}Q_B \operatorname{csch}\lambda\nu\alpha\\ Q_A \operatorname{sech}\lambda\nu\alpha\end{pmatrix}\int_0^\infty \frac{\eta}{\left(\lambda^2\alpha^2+\eta^2\right)^{\frac{1}{2}}}\left(R_m^I - \Phi_m^{II} + \frac{\eta}{\lambda\alpha} Z_m^{II}\right) J_{m+1}(\nu\alpha)d\eta \ .$$

Their explicit expressions, obtained with the use of the following definite integrals

$$\nu \int_0^\infty \frac{\eta J_0(\nu\eta)}{(\alpha^2+\eta^2)^{\frac{1}{2}}} d\eta = \frac{\nu^2}{1+\nu\alpha} \int_0^\infty \frac{\eta^2 J_1(\nu\eta)}{(\alpha^2+\eta^2)^{\frac{1}{2}}} d\eta = \frac{\nu^3}{3+3\nu\alpha+\nu^2\alpha^2} \int_0^\infty \frac{\eta^3 J_2(\nu\eta)}{(\alpha^2+\eta^2)^{\frac{1}{2}}} d\eta = e^{-\nu\alpha} \tag{A6}$$

and related expressions arrived at by differentiation of (A6) with respect to α or ν, are given in Table A1.

When the above expressions, (A5), together with (A4) are substituted into (2.9), the explicit form of (2.11) is found

$$\begin{aligned} &\left[\nu P'_A + \left(\frac{\lambda}{\lambda+1} - P_A\right)\right] B''_m + \left[-\nu Q'_A + Q_A\right] A''_m + \left[\nu P''_A + P'_A - \frac{1}{\nu}\left(\frac{\lambda}{\lambda+1} - P_A\right)\right] B'_m \\ &+ \left[-\nu Q''_A - Q'_A - \frac{Q_A}{\nu} + \frac{2\lambda\alpha}{\lambda+1}\right] A'_m + \left[-P''_A - \frac{1}{\nu}\left(1+m^2\right) P'_A + \frac{1}{\nu^2}\left(1-m^2\right)\left(\frac{\lambda}{1+\lambda} - P_A\right)\right] B_m \\ &+ \left[Q''_A + \frac{1}{\nu}\left(1+m^2\right) Q'_A + \frac{1}{\nu^2}\left(1-m^2\right) Q_A - \frac{2\lambda\alpha}{\nu(\lambda+1)}\right] A_m = \frac{\lambda\alpha}{2(\lambda+1)} \mathcal{A}_m \end{aligned} \tag{A7}$$

together with a similar equation in which the set $(A_m, B_m, P_A, Q_A, \mathcal{A}_m)$ is replaced by, respectively, $(B_m, A_m, P_B, Q_B, \mathcal{B}_m)$. The various inhomogeneous terms are given in Table A1.

To solve (A7) we examine the asymptotic behavior of its solutions. As $\nu \to 0$ the form of the equations is

$$\frac{4}{\lambda\alpha^2}\left[A''_m - \frac{1}{\nu} A'_m + \frac{(1-m^2)}{\nu^2} A_m\right] - \frac{2(\lambda-1)\alpha}{3}\left[\nu^3 B''_m + 5\nu^2 B'_m - (5+m^2)\nu B_m\right] = K_m(\nu) + O(\nu^n)$$

and (A8)

$$\frac{2(\lambda-1)}{\lambda\alpha}\left[\frac{1}{\nu} A''_m - \frac{1}{\nu^2} A'_m + \frac{(1-m^2)}{\nu^3} A_m\right] + \frac{4\lambda\alpha^2}{3}\left[\nu^2 B''_m + 5\nu B'_m - (5+m^2) B_m\right] = L_m(\nu) + O(\nu^n)$$

while, as $\nu \to \infty$

$$\frac{\lambda+1}{\lambda\alpha}\left[A''_m+\frac{4\lambda\alpha}{\lambda+1}A'_m-\frac{4\lambda\alpha}{\lambda+1}A_m\right]+\frac{\lambda-1}{\lambda\alpha}\left[B''_m-\frac{1}{\nu}B'_m+\frac{1-m^2}{\nu^2}B_m\right]=S_m(\nu)e^{-2\nu\alpha}+O(e^{-2\lambda\nu\alpha})+O(e^{-4\nu\alpha})$$

and (A9)

$$\frac{\lambda-1}{\lambda\alpha}\left[A''_m-\frac{1}{\nu}A'_m+\frac{(1-m^2)}{\nu^2}A_m\right]+\frac{\lambda+1}{\lambda\alpha}\left[B''_m+\frac{4\lambda\alpha}{\lambda+1}B'_m-\frac{4\lambda\alpha}{\lambda+1}B_m\right]=T_m(\nu)e^{-2\nu\alpha}+O(e^{-2\lambda\alpha})+O(e^{-4\nu\alpha})$$

where the coefficients of the polynomials K, L, S and T depend on e_{ij} and $(\Omega_i-\omega_i)$, and n is determined by their truncation. The leading terms in the first order approximation to the homogeneous solutions (A_{mh}, B_{mh}) and the particular solutions (A_{mp}, B_{mp})

$$A_{mh}=\nu^{1\pm m},\quad A_{op}=O(\nu),\quad A_{\pm 1p}=O(\nu),\quad A_{\pm 2}=O(\nu^5)$$

$$B_{mh}=\nu^{-2\pm\sqrt{9+m^2}},\quad B_{op}=-\frac{24\lambda}{(\lambda+1)^3},\quad B_{\pm 1p}=-\frac{2}{3}\frac{(\lambda-1)}{(\lambda+1)^2},\quad B_{\pm 2}=O(\nu^2) \qquad \text{(A10)}$$

for (A8), and

$$A_{mh}=B_{mh}=(\nu,\ e^{-2\nu\alpha}),\quad A_{mp}=O(S_m e^{-2\nu\alpha}),\quad B_{mp}=O(T_m e^{-2\nu\alpha}) \qquad \text{(A11)}$$

for (A9) provide then the appropriate boundary conditions for the numerical solution in view of (2. 10) and the requirement that both A_m and B_m decay exponentially for large ν.

The above system was solved numerically as a boundary value problem by representing the derivatives in the usual finite differences scheme which, in turn, yields

$$\underset{\approx}{\aleph}\,\underset{\sim}{Y}+\underset{\approx}{\beth}\,\underset{\sim}{Z}=\underset{\sim}{\text{ס}}$$

$$\underset{\approx}{\gimel}\,\underset{\sim}{Y}+\underset{\approx}{\daleth}\,\underset{\sim}{Z}=\underset{\sim}{\text{ש}} \qquad \text{(A12)}$$

where $\underset{\sim}{Y}$, $\underset{\sim}{Z}$, $\underset{\sim}{\text{מ}}$ and $\underset{\sim}{\text{ש}}$ are the vectors of the values of A_m, B_m, $\mathcal{A}_m$ and $\mathcal{B}_m$ at the grid points and $\underset{\approx}{\aleph}$, $\underset{\approx}{\beth}$, $\underset{\approx}{\gimel}$ and $\underset{\approx}{\daleth}$ represent the matrix forms of the linear operators of (A7). It was found possible to avoid a straightforward solution of (A12), which would have involved inverting a complicated matrix, and to by-pass numerical instabilities, which, in view of (A8), arise in the regions $\lambda \to 1$ and $\lambda \to \infty$ as $\nu \to 0$, by the use of the simple iterative procedure

$$\left(\underset{\approx}{\gimel} - \frac{\nu\lambda}{1-\lambda}\underset{\approx}{\aleph}\right)\underset{\sim}{Y}^{(n+1)} = \left(\underset{\sim}{\text{ש}} - \frac{\nu\lambda}{1-\lambda}\underset{\sim}{\text{מ}}\right) - \left(\underset{\approx}{\daleth} - \frac{\nu\lambda}{1-\lambda}\underset{\approx}{\beth}\right)\underset{\sim}{Z}^{(n)}$$

$$\underset{\approx}{\daleth}\,\underset{\sim}{Z}^{(n+1)} = \underset{\sim}{\text{ש}} - \underset{\approx}{\gimel}\,\underset{\sim}{Y}^{(n+1)}$$

where n is the iteration number.

Table A1: The explicit expressions for the integrals in (A5) and the inhomogeneous terms in (A7) (for $U_3 = -\zeta e_{33}$)*

$$\mathcal{A}_o = \left[10\nu(P'_B - P'_A) + 3\nu^2(P''_B - P''_A)\right] e_{33}$$

$$\mathcal{B}_o = (20\nu Q'_B + 6\nu^2 Q''_B)\, e_{33}$$

$$\mathcal{C}_o = -2(1-2Q_B)\, e_{33}$$

$$\mathcal{D}_o = -2(P_B - P_A)\, e_{33}$$

$$\mathcal{E}_o = -\nu(1-2Q_B)(\Omega_3 - \omega_3 + \tfrac{3}{2}e_{33})$$

$$\mathcal{F}_o = \nu(P_B - P_A)(\Omega_3 - \omega_3 + \tfrac{3}{2}e_{33})$$

$$\mathcal{G}_o = -\nu(1-2Q_B)(\Omega_3 - \omega_3 - \tfrac{3}{2}e_{33})$$

$$\mathcal{H}_o = \nu(P_B - P_A)(\Omega_3 - \omega_3 - \tfrac{3}{2}e_{33})$$

$$\mathcal{A}_{-1} = -\left[\frac{2}{\nu\alpha}(1-2Q_A) + 4(\nu+\frac{1}{\alpha})Q'_A - \frac{2}{\nu\alpha}\left(\frac{\lambda-1}{\lambda}\right)(P_A - Q_A - \nu P'_A + \nu Q'_A) - 2\zeta\nu(P''_B - P''_A)\right](\Omega_2 - \omega_2)$$
$$-\left[\frac{2}{\nu\alpha}(1-2Q_A) + 4(5\nu+\frac{1}{\alpha})Q'_A + 8\nu^2 Q''_A - \frac{2}{\nu\alpha}\left(\frac{\lambda-1}{\lambda}\right)(P_A - Q_A - \nu P'_A + \nu Q'_A) + 2\zeta\nu(P''_B - P''_A)\right] e_{13}$$

$$\mathcal{B}_{-1} = -\left[\frac{2}{\nu\alpha}(P_B - P_A) - 2(\nu+\frac{1}{\alpha})(P'_B - P'_A) - \frac{2}{\nu\alpha}\left(\frac{\lambda-1}{\lambda}\right)(P_B - Q_B - \nu P'_B + \nu Q'_B) - 4\zeta\nu Q''_B\right](\Omega_2 - \omega_2)$$
$$-\left[\frac{2}{\nu\alpha}(P_B - P_A) - 2(5\nu+\frac{1}{\alpha})(P'_B - P'_A) - 4\nu^2(P''_B - P''_A) - \frac{2}{\nu\alpha}\left(\frac{\lambda-1}{\lambda}\right)(P_B - Q_B - \nu P'_B + \nu Q'_B) + 4\zeta\nu Q''_B\right] e$$

*for m = + 1 replace $\Omega_2 - \omega_2$ by $-\Omega_1 + \omega_1$ and e_{13} by e_{23} in expressions with m = - 1; similarly, for m = + 2 replace $e_{11} - e_{22}$ by $2e_{12}$ in m = - 2.

Table A1: Continued

$$\mathcal{C}_{-1} = 2\left[\left(1 + \frac{1}{\nu\alpha}\right)(P_B - P_A) - \frac{1}{\nu\alpha}\left(\frac{\lambda-1}{\lambda}\right)(P_B - Q_B)\right](\Omega_2 - \omega_2 + e_{13})$$

$$\mathcal{D}_{-1} = 2\left[\left(1 + \frac{1}{\nu\alpha}\right)(1 - 2Q_A) - \frac{1}{\nu\alpha}\left(\frac{\lambda-1}{\lambda}\right)(P_A - Q_A)\right](\Omega_2 - \omega_2 + e_{13})$$

$$\mathcal{E}_{-1} = -\left[\left(\nu + \frac{1}{\alpha}\right)(P_B - P_A) - \frac{\lambda-1}{\lambda\alpha}(P_B - Q_B) + 2\zeta(1 - 2Q_B)\right](\Omega_2 - \omega_2)$$

$$+\left[\left(3\nu - \frac{1}{\alpha}\right)(P_B - P_A) + \frac{\lambda-1}{\lambda\alpha}(P_B - Q_B) + 2\zeta(1 - 2Q_B)\right]e_{13}$$

$$\mathcal{F}_{-1} = -\left[\left(\nu + \frac{1}{\alpha}\right)(1 - 2Q_A) - \frac{\lambda-1}{\lambda\alpha}(P_A - Q_A) - 2\zeta(P_B - P_A)\right](\Omega_2 - \omega_2)$$

$$+\left[\left(3\nu - \frac{1}{\alpha}\right)(1 - 2Q_A) + \frac{\lambda-1}{\lambda\alpha}(P_A - Q_A) - 2\zeta(P_B - P_A)\right]e_{13}$$

$$\mathcal{G}_{-1} = -\left[\left(\nu + \frac{1}{\alpha}\right)(P_B - P_A) - \frac{\lambda-1}{\lambda\alpha}(P_B - Q_B)\right](\Omega_2 - \omega_2 + e_{13})$$

$$\mathcal{H}_{-1} = -\left[\left(\nu + \frac{1}{\alpha}\right)(1 - 2Q_A) - \frac{\lambda-1}{\lambda\alpha}(P_A - Q_A)\right](\Omega_2 - \omega_2 + e_{13})$$

$$\mathcal{A}_{-2} = \nu^2(P''_B - P''_A)(e_{11} - e_{22})$$

$$\mathcal{B}_{-2} = 2\nu^2 Q''_B(e_{11} - e_{22})$$

$$\mathcal{E}_{-2} = \nu(1 - 2Q_B)(e_{11} - e_{22})$$

$$\mathcal{F}_{-2} = \nu(P_B - P_A)(e_{11} - e_{22})$$