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Renormalized expansions in the theory of turbulence with the use of Lagrangian position function.

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Note that the author would like to have all requests for copies of this appendix referred to him, so that he can send them the most up-to-date information.

Appendix.

Here are given algebraic details to derive (2.35) ∿ (2.43) and (2.48) ∿ (2.52).

[A-I]

To derive (2.36), it is convenient to write (2.16) in another form as;

$$\frac{\partial}{\partial s} u_i(\underset{\sim}{x},t|s) = L_i(\underset{\sim}{x},t|s)$$

$$= \int d^3\underset{\sim}{x}''[-\frac{1}{\rho}\frac{\partial}{\partial x_i''} p(\underset{\sim}{x}'',s) + \nu\nabla^2_{\underset{\sim}{x}''}u_i(\underset{\sim}{x}'',s)]\psi(\underset{\sim}{x}'',s;\underset{\sim}{x},t), \quad \text{(A 1a)}$$

where

$$-\frac{1}{\rho}\frac{\partial}{\partial x_i} p(\underset{\sim}{x},s) = \frac{\lambda}{2} S_{imn}(\nabla_{\underset{\sim}{x}})[u_m(\underset{\sim}{x},s)u_n(\underset{\sim}{x},s)], \quad \text{(A 1b)}$$

$$S_{imn}(\nabla_{\underset{\sim}{x}}) \equiv [\Pi_{im}(\nabla_{\underset{\sim}{x}})\frac{\partial}{\partial x_n} + \Pi_{in}(\nabla_{\underset{\sim}{x}})\frac{\partial}{\partial x_m}]. \quad \text{(A 1c)}$$

Equation (A 1) can be verified by noting the physical meaning of the MTD $(\partial/\partial s)u_i(\underset{\sim}{x},t|s)$; which is equivalent to the usual Lagrangian time derivative. Equation (A 1) can, of course, be verified also directly from (2.16). For, by applying Gauss' theorem, we can transform the following volume integral;

$$\int d^3\underset{\sim}{x}''\{\frac{1}{2}[(\delta_{in}\frac{\partial}{\partial x_m''} + \delta_{im}\frac{\partial}{\partial x_n''})(u_m(\underset{\sim}{x}'',s)u_n(\underset{\sim}{x}'',s))]\psi(\underset{\sim}{x}'',s;\underset{\sim}{x},t)$$

$$+ u_i(\underset{\sim}{x}'',s)u_m(\underset{\sim}{x}'',s)\frac{\partial}{\partial x_m''}\psi(\underset{\sim}{x}'',s;\underset{\sim}{x},t)\}$$

$$= \int d^3\underset{\sim}{x}''\frac{\partial}{\partial x_m''}\{u_i(\underset{\sim}{x}'',s)u_m(\underset{\sim}{x}'',s)\psi(\underset{\sim}{x}'',s;\underset{\sim}{x},t)\} \quad , \quad \text{(A 2)}$$

into a surface integral over the boundary surface S (say). It is clear that this surface integral is zero as far as the position vector $y(\underset{\sim}{x},t|s)$ at time s (see (1.2)) of the fluid element whose space-time trajectory passes through $(\underset{\sim}{x},t)$ does not lie on S. By putting the integral of (A.2) equal to zero and using (2.4), we can easily obtain (A 1) from (2.16).

Hence we can write B_{ij} in (2.19) as

$$B_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = B^{\nu}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') + B^{\pi}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t'), \tag{A 3a}$$

where

$$B^{\nu}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \nu\left\langle\int d^3\underset{\sim}{x}''[\nabla^2_{\underset{\sim}{x}''}u_i(\underset{\sim}{x}'',t)]\psi(\underset{\sim}{x}'',t;\underset{\sim}{x},t')\cdot u_j(\underset{\sim}{x}',t')\right\rangle, \tag{A 3b}$$

and

$$B^{\pi}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t')$$
$$= \frac{\lambda}{2}\left\langle\int d^3\underset{\sim}{x}''\{S_{imn}(\nabla_{\underset{\sim}{x}''})[u_m(\underset{\sim}{x}'',t)u_n(\underset{\sim}{x}'',t)]\}\psi(\underset{\sim}{x}'',t;\underset{\sim}{x},t')\cdot u_j(\underset{\sim}{x}',t')\right\rangle. \tag{A 3c}$$

Similary, by applying Gauss' theorem, we can simplify a few terms in (2.17a); if we write $C_{ij}(\equiv\langle\hat{C}_{ij}\rangle)$ as

$$C_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \langle(\nu\text{-terms})\rangle + \langle(\Psi\text{-terms})\rangle + \langle\hat{C}^{\pi}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\rangle, \tag{A 4a}$$

where $\hat{C}^{\pi}_{ij}$ represents the 2nd, 3rd and last terms of (2.17a) (i. e. those which do not contain ν nor Ψ), (ν-terms) represents the 1st and 4th terms, and (Ψ-terms) represents the 5th and 6th terms of (2.17a), then we can write $\langle\hat{C}^{\pi}_{ij}\rangle$ as

$$<\hat{C}^{\pi}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t')> = \lambda < \int d^3\underset{\sim}{x}''\{S_{imn}(\nabla_{\underset{\sim}{x}''})[u_m(\underset{\sim}{x}'',t)\hat{G}^E_{nj}(\underset{\sim}{x}'',t;\underset{\sim}{x}',t')]\}$$

$$\times\ \psi(\underset{\sim}{x}'',t;\underset{\sim}{x},t')> . \qquad \text{(A 4b)}$$

[A-II]

From (2.1) and (2.2), we obtain the expansions of u_i and ψ in powers of λ as;

$$u_i(\underset{\sim}{x},t) = u_i^0(\underset{\sim}{x},t) - \frac{\lambda}{2}\int_{t_0}^{t} ds \int d^3\underset{\sim}{x}'\{\hat{G}^{E0}_{ij}(\underset{\sim}{x},t,\underset{\sim}{x}',s)$$

$$\times\ P_{jmn}(\nabla_{\underset{\sim}{x}'})[u_m^0(\underset{\sim}{x}',s)u_n^0(\underset{\sim}{x}',s)]\} + \ldots, \qquad \text{(A 5)}$$

$$\psi(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \psi^0(\underset{\sim}{x},t;\underset{\sim}{x}',t') - \lambda\int_{t'}^{t} ds\{u_j^0(\underset{\sim}{x},s)\frac{\partial}{\partial x_j}\psi^0(\underset{\sim}{x},s,\underset{\sim}{x}',t')\}$$

$$+ \ldots\ldots \quad . \qquad \text{(A 6)}$$

The expansions of $\hat{G}^E$, Ψ and $\hat{G}$ are also obtained similary from (2.1),(2.2),(2.12) $\sim$ (2.15) in terms of $\underset{\sim}{u}^0$ $(\equiv u_i^0(\underset{\sim}{x},t))$, $\hat{G}^{E0}$ and ψ^0. Here, from (2.24), ψ^0 is known to be equal to a δ-function, and from (2.25) $\hat{G}^{E0}=G^0$. Hence we can obtain expansions of $\underset{\sim}{u}$, ψ, $\hat{G}^E$, Ψ and $\hat{G}$ in terms of $\underset{\sim}{u}^0$ and G^0. By substituting these expansions into A, $\overline{B}$ and $\overline{C}$ (see (2.18),(2.32) $\sim$ (2.34),(A 3) and (A 4)), we obtain expansions of A, $\overline{B}$ and $\overline{C}$ in terms of $\underset{\sim}{u}^0$ and G^0.

Now let us first consider about B_{ij} in (A.3). Because the distribution over the ensemble of the initial velocity field $u_i(\underset{\sim}{x},t_0)$ is assumed to be Gaussian with zero mean, B^{π}_{ij} (see (A 3c)) yields zero in $O(\lambda)$. In $O(\lambda^2)$, it yields

$$(u_m \to u_m^1) + (u_n \to u_n^1) + (\psi \to \psi^1) + (u_j \to u_j^1), \qquad \text{(A 7)}$$

where e.g. $(\psi \to \psi^1)$ means that ψ in (A.3c) is to be replaced by the first order term ψ^1 of (A.6) and the other u-terms in it are to be replaced by the zeroth order term $\underset{\sim}{u}^0$ of (A.5), i.e.

$$(\psi \to \psi^1) \equiv \frac{\lambda}{2} < \int d^3\underset{\sim}{x}''\{[S_{imn}(\nabla_{\underset{\sim}{x}''})[u_m^0(\underset{\sim}{x}'',t)u_n^0(\underset{\sim}{x}'',t)]]$$

$$\times [-\lambda \int_{t'}^{t} ds\ u_a^0(\underset{\sim}{x}'',s)\frac{\partial}{\partial x_a''}\psi^0(\underset{\sim}{x}'',s;\underset{\sim}{x},t')]\}\cdot u_j^0(\underset{\sim}{x}',t')>$$

$$\equiv B_{ij}^{\pi I}(\underset{\sim}{x},t;\underset{\sim}{x}',t'). \tag{A.8}$$

By using (2.24) and integrating (A.8) by parts, we obtain

$$B_{ij}^{\pi I}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \frac{\lambda^2}{2}\int_{t'}^{t} ds < \frac{\partial}{\partial x_a}\{[S_{imn}(\nabla_{\underset{\sim}{x}})[u_m^0(\underset{\sim}{x},t)u_n^0(\underset{\sim}{x},t)]]$$

$$\times u_a^0(\underset{\sim}{x},s)\}\cdot u_j^0(\underset{\sim}{x}',t')> . \tag{A.9}$$

The remaining three terms of (A.7) give after substitution of (2.24)

$$\frac{\lambda}{2}<[S_{imn}(\nabla_{\underset{\sim}{x}})(u_m^1(\underset{\sim}{x},t)u_n^0(\underset{\sim}{x},t)+u_m^0(\underset{\sim}{x},t)u_n^1(\underset{\sim}{x},t))]\cdot u_j^0(\underset{\sim}{x}',t')$$

$$+ [S_{imn}(\nabla_{\underset{\sim}{x}})(u_m^0(\underset{\sim}{x},t)u_n^0(\underset{\sim}{x},t))]\cdot u_j^1(\underset{\sim}{x}',t')> \equiv B_{ij}^{\pi II}(\underset{\sim}{x},t;\underset{\sim}{x}',t'), \tag{A.10}$$

where u^1 denotes the first order term of (A.5).

The fourth order moment $<\underset{\sim}{u}^0\underset{\sim}{u}^0\underset{\sim}{u}^0\underset{\sim}{u}^0>$ in (A.9) and (A.10) can be expressed in terms of $<\underset{\sim}{u}^0\underset{\sim}{u}^0>=U^{E0}$, i.e. by virtue of (2.26) in terms of U^0. By transforming (A.9) and (A.10) into the wave-vector space defined similary to (2.27), we can calculate the contributions of these terms to $\bar{B}_{ij}(\underset{\sim}{k},t,t') \equiv P_{ib}(\underset{\sim}{k})B_{bj}(\underset{\sim}{k},t,t')$. By noting (2.31), we obtain

$$P_{ib}(\underset{\sim}{k})B^{\pi I}_{bj}(\underset{\sim}{k},t,t')=\frac{-\lambda^2}{2}\int_{t'}^{t}ds\sum_{\underset{\sim}{p},\underset{\sim}{r}}^{\Delta}P_{ib}(\underset{\sim}{k})k_a S_{bmn}(\underset{\sim}{p})$$

$$\times\ [Q^0_{ma}(-\underset{\sim}{r},t,s)Q^0_{nj}(\underset{\sim}{k},t,t') + Q^0_{mj}(\underset{\sim}{k},t,t')Q^0_{na}(-\underset{\sim}{r},t,s)]$$

$$= -\lambda^2[\int_{t'}^{t}ds\sum_{\underset{\sim}{p},\underset{\sim}{r}}^{\Delta}P_{ib}(\underset{\sim}{k})k_a S_{bmn}(\underset{\sim}{p})Q^0_{ma}(-\underset{\sim}{r},t,s)]Q^0_{nj}(\underset{\sim}{k},t,t')$$

$$\equiv \lambda^2 I^0_{ij}(\underset{\sim}{k},t,t'), \tag{A.11}$$

where

$$S_{bmn}(\underset{\sim}{p}) = 2p_b p_m p_n/p^2 ,$$

and we have used $S_{bmn}(\underset{\sim}{p}) = S_{bnm}(\underset{\sim}{p})$. While, because of the presence of the factor $\partial/\partial x_i$ in $S_{imn}(\nabla_{\underset{\sim}{x}})$ (see (A.1c) and (2.5)), it can be shown that $B^{\pi II}_{ij}(\underset{\sim}{k},t,t')$ contain the factor k_i and consequently

$$P_{ib}(\underset{\sim}{k})B^{\pi II}_{bj}(\underset{\sim}{k},t,t') = 0 , \tag{A.12}$$

for $P_{ib}(\underset{\sim}{k})k_b = 0$. Thus $B^{\pi II}_{bj}(\underset{\sim}{k})$ does not contribute to $\overline{B}_{ij}(\underset{\sim}{k}) \equiv P_{ib}(\underset{\sim}{k})B_{bj}(\underset{\sim}{k})$. Moreover we readily see that

$$P_{ib}(\underset{\sim}{k})B^{\nu}_{bj}(\underset{\sim}{k},t,t') = -\nu[k^2 Q^0_{ij}(\underset{\sim}{k},t,t') + O(\lambda)]. \tag{A.13}$$

From (A.3), (A.7) $\sim$ (A.13), we have

$$\overline{B}_{ij}(\underset{\sim}{k},t,t') = -\nu X_{ij}(\underset{\sim}{k},t,t') + \lambda^2 I_{ij}(\underset{\sim}{k},t,t'), \tag{A.14a}$$

with

$$X_{ij}(\underset{\sim}{k},t,t') = k^2 Q^0_{ij}(\underset{\sim}{k},t,t') + O(\lambda), \tag{A.14b}$$

$$I_{ij}(\underset{\sim}{k},t,t') = I^0_{ij}(\underset{\sim}{k},t,t') + O(\lambda), \tag{A.14c}$$

where I^0_{ij} is given by (A.11).

Next let us consider about C_{ij} in (A.4). By substituting the primitive expansions of u, ψ and $\hat{G}^E$ into (A,4b), we obtain expansions of $C^{\pi}_{ij}(\equiv\langle\hat{C}^{\pi}_{ij}\rangle)$. In $O(\lambda)$, it gives zero. In $O(\lambda^2)$, it yields

$$(u_m \to u^1_m) + (\hat{G}^E \to \hat{G}^{E1}) + (\psi \to \psi^1) \quad . \tag{A.15}$$

The meanings of these terms should be understood analogously to (A.7). The third term $(\psi\to\psi^1)$ yields after substitution of (2.24), (cf. (A.8) and (A.9)),

$$(\psi\to\psi^1) = \lambda^2 \int_{t'}^{t} ds \left\langle \frac{\partial}{\partial x_a}\{(S_{imn}(\nabla_{\underset{\sim}{x}})[u^0_m(\underset{\sim}{x},t)G^0_{nj}(\underset{\sim}{x},t;\underset{\sim}{x}',t')])u^0_a(\underset{\sim}{x},s)\}\right\rangle$$

$$\equiv C^{\pi I}_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') \quad . \tag{A.16}$$

By using $\langle\underset{\sim}{u}^0\underset{\sim}{u}^0\rangle = U^0$ and (2.31), and transforming (A.16) into the wavevector space defined similary to (2.28), we obtain

$$P_{ib}(\underset{\sim}{k})C^{\pi I}_{bc}(\underset{\sim}{k},t,t')P_{cj}(\underset{\sim}{k}) = -\lambda^2[[\int_{t'}^{t} ds \sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{r}} P_{ib}(\underset{\sim}{k})k_a S_{bmn}(\underset{\sim}{p})Q^0_{ma}(-\underset{\sim}{r},t,s)]$$

$$\times F^0_{nj}(\underset{\sim}{k},t,t') \equiv \lambda^2 J^0_{ij}(\underset{\sim}{k},t,t') \quad . \tag{A.17}$$

The first two terms of (A.15) does not contribute to $\overline{C}_{ij}(\underset{\sim}{k}) \equiv P_{ib}(\underset{\sim}{k})C_{bc}(\underset{\sim}{k})P_{cj}(\underset{\sim}{k})$ by the same reason as $B^{\pi II}_{ij}$ does not to $\overline{B}_{ij}(\underset{\sim}{k})$. It is not difficult to see that (Ψ-terms) gives zero in $O(\lambda^2)$ and to calculate (ν-terms) in $O(\lambda^0)$. Thus we obtain

$$\overline{C}_{ij}(\underset{\sim}{k},t,t') \equiv P_{ib}(\underset{\sim}{k})C_{bc}(\underset{\sim}{k},t,t')P_{cj}(\underset{\sim}{k}) = -\nu Y_{ij}(\underset{\sim}{k},t,t') + \lambda^2 J_{ij}(\underset{\sim}{k}), \tag{A.18a}$$

with

$$Y_{ij}(\underset{\sim}{k},t,t')=k^2F^0_{ij}(\underset{\sim}{k},t,t') + O(\lambda), \tag{A.18b}$$

$$J_{ij}(\underset{\sim}{k},t,t')= J^0_{ij}(\underset{\sim}{k},t,t') + O(\lambda), \tag{A.18c}$$

where J^0_{ij} is given by (A.17).

The expansion of A (see (2.18)) can be obtained in a similar manner to that of of $\overline{B}$ and $\overline{C}$.

As is clear from the above calculations, we can obtain the expansions of A, $\overline{B}$ and $\overline{C}$ in terms of Q^0 and F^0. In the same way as for A, $\overline{B}$ and $\overline{C}$, we can expand also Q and F in terms of Q^0 and F^0. (These Q, F, Q^0 and F^0 have respectively only two, not four, time arguments.) By reverting these expansions of Q and F, we obtain expansions of Q^0 and F^0 in functional powers of Q and F. By the substitutions of these expansions of Q^0 and F^0 into A, $\overline{B}$ and $\overline{C}$ obtained above, we can express A, $\overline{B}$ and $\overline{C}$ in functional powers of Q and F.

It is worthwhile to note that in the lowest order in λ

$$Q^0=Q \quad \text{and} \quad F^0=F.$$

In order to obtain the lowest order terms of X, I, Y and J in (A.14) and (A.18), we have only to replace Q^0 and F^0 in (A.14b,c), (A.11),(A.18b,c) and (A.17) by Q and F respectively. Thus we can obtain (2.36),(2.37) with (2.39),(2.40),(2.42),(2.43). Similary we can verify (2.35) with (2.38),(2.41). The structure of H_{ij} in (2.41) is the same as that which appears in DIA.

Finally, (2.48b),(2.48c) with (2.50),(2.52) can be obtained from (2.36) and (2.37) by using (2.47) and by noting

$$P_{ib}(\underset{\sim}{k})k_a S_{bmn}(\underset{\sim}{p})P_{ma}(r)P_{ni}(\underset{\sim}{k})$$

$$= 2k_a p_b p_m p_n P_{bn}(\underset{\sim}{k})P_{ma}(\underset{\sim}{r})/p^2 = 2k_a k_m P_{ma}(\underset{\sim}{r})p_b p_n P_{bn}(\underset{\sim}{k})/p^2$$

$$= 2k^2(1-y^2)(1-z^2), \qquad \text{(A.19)}$$

where we have used $p_m P_{ma}(\underset{\sim}{r})=k_m P_{ma}(\underset{\sim}{r})$, because $\underset{\sim}{k}=\underset{\sim}{p}+\underset{\sim}{r}$ and $r_m P_{ma}(\underset{\sim}{r})=0$.

As for the derivation of (2.48a) with (2.49),(2.51), the reader may consult Leslie's book (1973).