Appendix (1A)

$O(1)$:
$$\frac{\partial^4 \Omega_0}{\partial \eta^4} + 2v\,(l+m\sigma)\,\frac{\partial \Omega_0}{\partial \eta}\,\frac{\partial^2 \Omega_0}{\partial \eta^2} = 0$$

$$\Omega_0(\sigma, \pm 1) = \pm 1 \quad ; \quad \frac{\partial \Omega_0}{\partial \eta}(\sigma, \pm 1) = 0$$

$O(\varepsilon^{\frac{1}{2}})$:
$$\frac{\partial^4 \Omega_1}{\partial \eta^4} + 2v\,(l+m\sigma)\,\frac{\partial}{\partial \eta}\left(\frac{\partial \Omega_0}{\partial \eta}\,\frac{\partial \Omega_1}{\partial \eta}\right) = v\left(\frac{\partial \Omega_0}{\partial \eta}\,\frac{\partial^3 \Omega_0}{\partial \sigma \partial \eta^2} - \frac{\partial \Omega_0}{\partial \sigma}\,\frac{\partial^3 \Omega_0}{\partial \eta^3}\right)$$

$$\Omega_1(\sigma, \pm 1) = 0 \quad ; \quad \frac{\partial \Omega_1}{\partial \eta}(\sigma, \pm 1) = 0$$

$O(\varepsilon)$:
$$\frac{\partial^4 \Omega_2}{\partial \eta^4} + 2v\,(l+m\sigma)\,\frac{\partial}{\partial \eta}\left(\frac{\partial \Omega_0}{\partial \eta}\,\frac{\partial \Omega_2}{\partial \eta}\right) = v\left\{\frac{\partial \Omega_0}{\partial \eta}\,\frac{\partial^3 \Omega_1}{\partial \sigma \partial \eta^2} + \frac{\partial \Omega_1}{\partial \eta}\,\frac{\partial^3 \Omega_0}{\partial \sigma \partial \eta^2} - \frac{\partial \Omega_0}{\partial \sigma}\,\frac{\partial^3 \Omega_1}{\partial \eta^3} - \frac{\partial \Omega_1}{\partial \sigma}\,\frac{\partial^3 \Omega_0}{\partial \eta^3} - 2(l+m\sigma)\,\frac{\partial \Omega_1}{\partial \eta}\,\frac{\partial^2 \Omega_1}{\partial \eta^2}\right\} - 4(l+m\sigma)^2\,\frac{\partial^2 \Omega_0}{\partial \eta^2}$$

$$\Omega_2(\sigma, \pm 1) = 0 \quad ; \quad \frac{\partial \Omega_2}{\partial \eta}(\sigma, \pm 1) = 0$$

Appendix (1B)

The boundary conditions for each function is given by (23a) in the paper

$$G_1 : \frac{d^4 G_1}{d\eta^4} + 2vl \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dG_1}{d\eta}\right) = -2vm \frac{dG_0}{d\eta}\frac{d^2 G_0}{d\eta^2}$$

$$G_2 : \frac{d^4 G_2}{d\eta^4} + 2vl \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dG_2}{d\eta}\right) = -2vl \frac{dG_1}{d\eta}\frac{d^2 G_1}{d\eta^2} - 2vm \frac{dG_0}{d\eta}\frac{d^2 G_1}{d\eta^2}$$

$$- 2vm \frac{dG_1}{d\eta}\frac{d^2 G_0}{d\eta^2}$$

$$F_0 : \frac{d^4 F_0}{d\eta^4} + 2vl \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dF_0}{d\eta}\right) = v\left(\frac{dG_0}{d\eta}\frac{d^2 G_1}{d\eta^2} - G_1 \frac{d^3 G_0}{d\eta^3}\right)$$

$$F_1 : \frac{d^4 F_1}{d\eta^4} + 2vl \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dF_1}{d\eta}\right) = v\left\{ 2\frac{dG_0}{d\eta}\frac{d^2 G_2}{d\eta^2} + \frac{dG_1}{d\eta}\frac{d^2 G_1}{d\eta^2} - G_1 \frac{d^3 G_1}{d\eta^3}\right.$$

$$- 2G_2 \frac{d^3 G_0}{d\eta^3} - 2l \frac{dF_0}{d\eta}\frac{d^2 G_1}{d\eta^2} - 2l \frac{dG_1}{d\eta}\frac{d^2 F_0}{d\eta^2}$$

$$\left. - 2m \frac{dG_0}{d\eta}\frac{d^2 F_0}{d\eta^2} - 2m \frac{dF_0}{d\eta}\frac{d^2 G_0}{d\eta^2} \right\}$$

$$H_0 : \frac{d^4 H_0}{d\eta^4} + 2vl \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dH_0}{d\eta}\right) = v\left\{\frac{dG_0}{d\eta}\frac{d^2 F_1}{d\eta^2} + \frac{dF_0}{d\eta}\frac{d^2 G_1}{d\eta^2}\right.$$

$$\left. - G_1 \frac{d^3 F_0}{d\eta^3} - F_1 \frac{d^3 G_0}{d\eta^3} - 2l \frac{dF_0}{d\eta}\frac{d^2 F_0}{d\eta^2}\right\}$$

$$- 4l^2 \frac{d^2 G_0}{d\eta^2}$$

Appendix (2A)

$$L_1 \equiv -\frac{4ik}{R}\left(D_1^2 - k^2\right) + 2k\beta_I + \frac{dG_0}{d\eta}\left(D_1^2 - 3k^2\right) - \frac{d^3G_0}{d\eta^3}$$

$$L_2 \equiv \frac{-2i}{R}\left(D_1^2 - 3k^2\right) + \beta_I - 3k\frac{dG_0}{d\eta}$$

$$L_3 \equiv \frac{4ikl}{R}\left(D_1^2 - k^2\right) - \left(2l\frac{dG_0}{d\eta} + \beta_I\, im\,\sigma_1^2\right)\left(D_1^2 - k^2\right) - 2l\frac{d^2G_0}{d\eta^2}D_1$$

$$L_4 \equiv ik\left(\frac{dF_0}{d\eta} + \sigma_1\frac{dG_1}{d\eta}\right)\left(D_1^2 - k^2\right) - ik\left(\frac{d^3F_0}{d\eta^3} + \sigma_1\frac{d^3G_1}{d\eta^3}\right)$$

$H(\sigma_1)$ in (36) is given by

$$H(\sigma_1) = \left(C_2 + C_3 + C_4 + C_5\right) / C_1$$

$$C_1(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_1 f_0 \, d\eta$$

$$C_2(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_1 \frac{\partial f_0}{\partial \eta} \, d\eta$$

$$C_3(\sigma_1) = \frac{dk}{d\sigma_1}\int_{-1}^{1} \tilde{f}_0 L_2 f_0 \, d\eta$$

$$C_4(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_3 f_0 \, d\eta$$

$$C_5(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_4 f_0 \, d\eta$$

Appendix (2B)

$$M_1 \equiv \frac{4l}{R}\left(D_1^2 - 3k^2\right) - \left(4ikl\frac{dG_0}{d\eta} - 2\beta_I km\sigma_1^2\right)$$

$$M_2 \equiv \left(\frac{dF_0}{d\eta} + \sigma_1\frac{dG_1}{d\eta}\right)\left(D_1^2 - 3k^2\right) - \left(\frac{d^3F_0}{d\eta^3} + \sigma_1\frac{d^3G_1}{d\eta^3}\right)$$

$$M_3 \equiv \frac{-12kl}{R} - i\left(2l\frac{dG_0}{d\eta} + i\beta_I m\sigma_1^2\right)$$

$$M_4 \equiv -3k\left(\frac{dF_0}{d\eta} + \sigma_1\frac{dG_1}{d\eta}\right)$$

$$M_5 \equiv \frac{12k}{R} + 3i\frac{dG_0}{d\eta}$$

$$M_6 \equiv -\frac{4l^2}{R}\left(D_1^2 - k^2\right) + \left(\frac{4m\sigma_1 ik}{R} - 2m\sigma_1\frac{dG_0}{d\eta} - i\frac{\beta_I m^2\sigma_1^4}{2}\right)\left(D_1^2 - k^2\right) - 2m\sigma_1\frac{d^2G_0}{d\eta^2}D_1$$

$$M_7 \equiv -\left\{2l\left(\frac{dF_0}{d\eta} + \sigma_1\frac{dG_1}{d\eta}\right) + G_1 D_1\right\}\left(D_1^2 - k^2\right) + \left\{\frac{d^2G_1}{d\eta^2} - 2l\left(\frac{d^2F_0}{d\eta^2} + \sigma_1\frac{d^2G_0}{d\eta^2}\right)\right\}D_1$$

$$M_8 \equiv ik\left(\frac{dH_0}{d\eta} + \sigma_1\frac{dF_1}{d\eta} + \sigma_1^2\frac{dG_2}{d\eta}\right)\left(D_1^2 - k^2\right) - ik\left(\frac{d^3H_0}{d\eta^3} + \sigma_1\frac{d^3F_1}{d\eta^3} + \sigma_1^2\frac{d^3G_2}{d\eta^3}\right)$$

The function $\mathcal{J}(\sigma_1)$ in (40) is given by

$$\mathcal{J}(\sigma_1) = \frac{i\,d^2A_0}{d\sigma_1^2}C_{25} - \frac{dA_0}{d\sigma_1}\left(C_{10} + C_{11} + C_{16} - 2i\,C_{24}\right) - A_0\left[\sum_{r=12}^{22} C_r - i23\right]$$
$$- \left(C_6 + C_7 + C_8 + C_9\right)\Big/C_1$$

where C_1 is the same as that given in Appendix (2A) and

$$C_6(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_1 \frac{\partial f_1}{\partial \sigma_1} d\eta$$

$$C_7(\sigma_1) = \frac{dk}{d\sigma_1} \int_{-1}^{1} \tilde{f}_0 L_2 f_1 \, d\eta$$

$$C_8(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_3 f_1 \, d\eta$$

$$C_9(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_4 f_1 \, d\eta$$

$$C_{10}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_1 f_1 \, d\eta$$

$$C_{11}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_2 f_0 \, d\eta$$

$$C_{12}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_1 \frac{\partial f_0}{\partial \sigma_1} d\eta$$

$$C_{13}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_2 \frac{\partial f_0}{\partial \sigma_1} d\eta$$

$$C_{14}(\sigma_1) = \frac{dk}{d\sigma_1} \int_{-1}^{1} \tilde{f}_0 M_3 f_0 \, d\eta$$

$$C_{15}(\sigma_1) = \frac{dk}{d\sigma_1} \int_{-1}^{1} \tilde{f}_0 M_4 f_0 \, d\eta$$

$$C_{16}(\sigma_1) = \frac{dk}{d\sigma_1} \int_{-1}^{1} \tilde{f}_0 M_5 f_0 \, d\eta$$

$$C_{17}(\sigma_1) = \frac{dk}{d\sigma_1} \int_{-1}^{1} \tilde{f}_0 M_5 \frac{\partial f_0}{\partial \sigma_1} d\eta$$

$$C_{18}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_6 f_0 \, d\eta$$

$$C_{19}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_7 f_0 \, d\eta$$

$$C_{20}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 M_8 f_0 \, d\eta$$

$$C_{21}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 \left[\frac{d^2 k}{d\sigma_1^2} \left(\frac{4k}{R} + i \frac{dG_0}{d\eta} \right) \right] f_0 \, d\eta$$

$$C_{22}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 \frac{3}{R} \left(\frac{dk}{d\sigma_1} \right)^2 f_0 \, d\eta$$

$$C_{23}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_2 \frac{\partial^2 f_0}{\partial \sigma_1^2} d\eta$$

$$C_{24}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_2 \frac{\partial f_0}{\partial \sigma_1} d\eta$$

$$C_{25}(\sigma_1) = \int_{-1}^{1} \tilde{f}_0 L_2 f_0 \, d\eta$$

Appendix (3A)

The boundary conditions for the following functions are the same as before.

$$G_0 : \quad \frac{d^4 G_0}{d\eta^4} + 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{dG_0}{d\eta}\frac{d^2 G_0}{d\eta^2} = 0$$

$$G_1 : \quad \frac{d^4 G_1}{d\eta^4} + 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dG_1}{d\eta}\right) = 4\upsilon m_1 \operatorname{sech}^2\sigma_0 \operatorname{Tanh}\sigma_0 \frac{dG_0}{d\eta}\frac{d^2 G_0}{d\eta^2}$$

$$\begin{aligned} G_2 : \quad \frac{d^4 G_2}{d\eta^4} + 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dG_2}{d\eta}\right) = & -2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{dG_1}{d\eta}\frac{d^2 G_1}{d\eta^2} \\ & + 4\upsilon m_1 \operatorname{sech}^2\sigma_0 \operatorname{Tanh}\sigma_0 \frac{dG_0}{d\eta}\frac{d^2 G_1}{d\eta^2} \\ & + 4\upsilon m_1 \operatorname{sech}^2\sigma_0 \operatorname{Tanh}\sigma_0 \frac{dG_1}{d\eta}\frac{d^2 G_0}{d\eta} \\ & - 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \left(3\operatorname{Tanh}^2\sigma_0 - 1\right)\frac{dG_0}{d\eta}\frac{d^2 G_0}{d\eta^2} \end{aligned}$$

$$F_0 : \quad \frac{d^4 F_0}{d\eta^4} + 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dF_0}{d\eta}\right) = \upsilon\left(\frac{dG_0}{d\eta}\frac{d^2 G_1}{d\eta^2} - G_1\frac{d^3 G_0}{d\eta^3}\right)$$

$$\begin{aligned} F_1 : \quad \frac{d^4 F_1}{d\eta^4} + 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dF_1}{d\eta}\right) = \upsilon \Bigg\{ & 2\frac{dG_0}{d\eta}\frac{d^2 G_2}{d\eta^2} + \frac{dG_1}{d\eta}\frac{d^2 G_1}{d\eta^2} - G_1\frac{d^3 G_1}{d\eta^3} \\ & - 2G_2\frac{d^3 G_0}{d\eta^3} - 2m_1 \operatorname{sech}^2\sigma_0 \frac{dF_0}{d\eta}\frac{d^2 G_1}{d\eta^2} \\ & - 2m_1 \operatorname{sech}^2\sigma_0 \frac{dG_1}{d\eta}\frac{d^2 F_0}{d\eta^2} + 4m_1 \operatorname{sech}^2\sigma_0 \operatorname{Tanh}\sigma_0 \left(\frac{dG_0}{d\eta}\frac{d^2 F_0}{d\eta^2} + \frac{dF_0}{d\eta}\frac{d^2 G_0}{d\eta^2}\right)\Bigg\} \end{aligned}$$

$$\begin{aligned} H_0 : \quad \frac{d^4 H_0}{d\eta^4} + 2\upsilon m_1 \operatorname{sech}^2\sigma_0 \frac{d}{d\eta}\left(\frac{dG_0}{d\eta}\frac{dH_0}{d\eta}\right) = \upsilon \Bigg\{ & \frac{dG_0}{d\eta}\frac{d^2 F_1}{d\eta^2} + \frac{dF_0}{d\eta}\frac{d^2 G_1}{d\eta^2} - G_1\frac{d^3 F_0}{d\eta^3} - F_1\frac{d^3 G_0}{d\eta^3} \\ & - 2m_1 \operatorname{sech}^2\sigma_0 \frac{dF_0}{d\eta}\frac{d^2 F_0}{d\eta^2}\Bigg\} - 4m_1^2 \operatorname{sech}^2\sigma_0 \frac{d^2 G_0}{d\eta^2} \end{aligned}$$

Appendix 3B

The equations for ϕ_0, ϕ_1, and ϕ_2 are identical in form to those given in (30), (31) and (32) respectively. The differences appear in the operators:

$O(1)$: The difference is in $\beta_I = \beta \exp\left(2m_1\sigma_1 \operatorname{sech}^2\sigma_0\right)$

$O(\varepsilon^{\frac{1}{2}})$: L_1, L_2 and L_4 are unchanged in form. The difference is in the new β_I. L_3 has become

$$L_3 \equiv m_1 \operatorname{sech}^2\sigma_0 \left[4ik\left(D_1^2 - k^2\right) - \left(2\frac{dG_0}{d\eta} - 2i\beta_I\sigma_1^2 \tanh\sigma_0\right)\left(D_1^2 - k^2\right) - 2\frac{d^2G_0}{d\eta^2}D_1 \right]$$

$O(\varepsilon)$: M_2, M_4, M_5 and M_8 remain unchanged, they do not contain l or m. M_1, M_3, M_6, M_7 become

$$M_1 \equiv \frac{4m_1\operatorname{sech}^2\sigma_0}{R}\left(D_1^2 - 3k^2\right) - 4m_1\operatorname{sech}^2\sigma_0\left(ik\frac{dG_0}{d\eta} + \beta_I k\sigma_1 \tanh\sigma_0\right)$$

$$M_3 \equiv -m_1\operatorname{sech}^2\sigma_0\left(\frac{12k}{R} + 2i\frac{dG_0}{d\eta} + 2\sigma_1^2\tanh\sigma_0\right)\beta_I$$

$$M_6 \equiv 4m_1\operatorname{sech}^2\sigma_0\left(\sigma_1\frac{dG_0}{d\eta}\tanh\sigma_0 - \frac{m_1}{R}\operatorname{sech}^2\sigma_0 - \frac{2\sigma_1}{R}\tanh\sigma_0\, ik\right)\left(D_1^2 - k^2\right)$$
$$- i\beta_I m_1\left(\frac{2}{3}\sigma_1^3\left(1 + 2\tanh^2\sigma_0\right) + 2m_1\sigma_1^4\tanh\sigma_0\operatorname{sech}^4\sigma_0\right)\left(D_1^2 - k^2\right)$$
$$+ 4m_1\sigma_1\operatorname{sech}\sigma_0\tanh\sigma_0\frac{d^2G_0}{d\eta^2}D_1$$

$$M_7 \equiv -\left(2m_1\operatorname{sech}^2\sigma_0\left(\frac{dF_0}{d\eta} + \sigma_1\frac{dG_1}{d\eta}\right) + G_1D_1\right)\left(D_1^2 - k^2\right)$$
$$+ \left(\frac{d^2G_1}{d\eta^2} - 2m_1\operatorname{sech}^2\sigma_0\left(\frac{d^2F_0}{d\eta^2} + \sigma_1\frac{d^2G_0}{d\eta^2}\right)\right)D_1$$