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Supplement to

Heating a salinity gradient from a vertical side wall: nonlinear theory

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In this supplement the details of some of the analysis of §3 of Kerr(1990) are given. This analysis uses energy stability theory (cf. Joseph 1976a,b) to examine the stability of the background state that is found when a vertical salinity gradient is heated from a sidewall to finite amplitude disturbances. We use this method to find a lower bound for the value of the non-dimensional heating rate, Q, below which disturbances in some sense die away. The first part of this analysis follows a similar course to the analysis of the steady thermal boundary layer that occurs when a vertical temperature gradient is heated at a single vertical side wall (Dudis & Davis, 1971, and Joseph, 1976b, pp. 29-33.)

We take the full equations for perturbations to the background flow and non-dimensionalise them with respect to the following quantities:

$$T \quad \text{with respect to} \quad \Delta T \,, \tag{S.1a}$$

$$S \quad \text{with respect to} \quad \frac{\alpha \Delta T}{\beta} \,, \tag{S.1b}$$

$$\underset{\sim}{x} \quad \text{with respect to} \quad h = \frac{\alpha \Delta T}{(-\beta \overline{S}_z)} \,, \tag{S.1c}$$

$$t \quad \text{with respect to} \quad h^2/\kappa_T \,, \tag{S.1d}$$

$$\mathbf{u} \quad \text{with respect to} \quad \kappa_T/h \,, \tag{S.1e}$$

$$p \quad \text{with respect to} \quad \rho_0 \kappa_T^2/h^2 \,. \tag{S.1f}$$

These non-dimensionalisations differ from those used in Kerr (1989) and §2 of the paper. The resulting non-dimensional equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \overline{\mathbf{U}}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\overline{\mathbf{U}} = -\nabla p + \mathcal{R}(T-S)\hat{z} + \sigma\nabla^2\mathbf{u} \,, \tag{S.2a}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T + \overline{\mathbf{U}}\cdot\nabla T + \mathbf{u}\cdot\nabla\overline{T} = \nabla^2 T \,, \tag{S.2b}$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S + \overline{\mathbf{U}}\cdot\nabla S + \mathbf{u}\cdot\nabla\overline{S} = \tau\nabla^2 S \,, \tag{S.2c}$$

$$\nabla\cdot\mathbf{u} = 0 \,, \tag{S.2d}$$

where the background state is given by

$$\overline{U} = (0,0,\overline{W}(x,t)) \,, \tag{S.3a}$$

$$\overline{T} = f(x,t) \,, \tag{S.3b}$$

$$\overline{S} = f(x,t) - z \,. \tag{S.3c}$$

The Prandtl number, σ, and the salt/heat diffusivity ratio, τ, are defined as before. The new non-dimensional number to appear here is

$$\mathscr{R} = \frac{g\alpha\Delta T h^3}{\kappa_T{}^2} \,. \tag{S.4}$$

The boundary conditions are

$$\mathbf{u} = 0, \quad T = 0, \quad \frac{\partial S}{\partial x} = 0 \quad \text{at } x = 0 \,, \tag{S.5a}$$

and

$$\mathbf{u} \to 0, \quad T \to 0, \quad S \to 0 \quad \text{as } x \to \infty \,. \tag{S.5b}$$

We restrict ourselves to perturbations to the background state whose maximum amplitudes are bounded in the whole fluid region and which are absolutely integrable on $0 \le x < \infty$. We define the average of a quantity over y and z by

$$\overline{A}(x,t) = \lim_{K,L\to\infty} \frac{1}{4KL} \int_{-L}^{+L} \int_{-K}^{+K} A(x,y,z,t)\; dy\; dz \tag{S.6a}$$

and the brackets $\langle\ \rangle$ by

$$\langle A \rangle = \int_0^\infty \overline{A}(x,t)\; dx \,. \tag{S.6b}$$

It should be noted that if any scalar function ζ or vector function $\boldsymbol{\eta}$ vanishes at $x = 0$ and obeys the conditions for boundedness and absolute integrability then we have from the divergence theorem for volume integrals that

$$\langle \nabla\zeta \rangle = \mathbf{0} \tag{S.7a}$$

and

$$\langle \nabla\cdot\boldsymbol{\eta} \rangle = 0 . \tag{S.7b}$$

Taking products of (S.3a-c) with $\mathbf{u}$, T and S respectively and finding their averages we obtain

$$\frac{d}{dt}\langle \tfrac{1}{2}|\mathbf{u}|^2 \rangle + \langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\overline{\mathbf{U}}\rangle = \mathcal{R}\langle (T-S)w\rangle + \sigma\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\rangle , \tag{S.8a}$$

$$\frac{d}{dt}\langle \tfrac{1}{2}T^2 \rangle + \langle T\mathbf{u}\cdot\nabla\overline{T}\rangle = \langle T\nabla^2 T\rangle , \tag{S.8b}$$

$$\frac{d}{dt}\langle \tfrac{1}{2}S^2 \rangle + \langle S\mathbf{u}\cdot\nabla\overline{S}\rangle = \tau\langle S\nabla^2 S\rangle . \tag{S.8c}$$

We also need the average of the sum of the product of T and (S.3c) and of the product of S and (S.3b):

$$\frac{d}{dt}\langle TS\rangle + \langle S\mathbf{u}\cdot\nabla\overline{T}\rangle + \langle T\mathbf{u}\cdot\nabla\overline{S}\rangle = \langle S\nabla^2 T\rangle + \tau\langle T\nabla^2 S\rangle . \tag{S.8d}$$

Substituting for $\overline{\mathbf{U}}$, $\overline{T}$ and $\overline{S}$ gives

$$\frac{d}{dt}\langle \tfrac{1}{2}|\mathbf{u}|^2 \rangle + \langle wu\frac{\partial}{\partial x}\overline{W}(x,t)\rangle = \mathcal{R}\langle Tw-Sw\rangle - \sigma\langle |\nabla\mathbf{u}|^2\rangle , \tag{S.9a}$$

$$\frac{d}{dt}\langle \tfrac{1}{2}T^2 \rangle + \langle Tu\frac{\partial}{\partial x}f(x,t)\rangle = -\langle |\nabla T|^2\rangle , \tag{S.9b}$$

$$\frac{d}{dt}\langle \tfrac{1}{2}S^2 \rangle + \langle Su\frac{\partial}{\partial x}f(x,t) \rangle - \langle Sw \rangle = -\tau\langle |\nabla S|^2 \rangle , \qquad (S.9c)$$

$$\frac{d}{dt}\langle TS \rangle + \langle (T+S)u\frac{\partial}{\partial x}f(x,t) \rangle - \langle Tw \rangle = -(1+\tau)\langle \nabla T\cdot\nabla S \rangle . \qquad (S.9d)$$

Taking the sum of these, with (S.9b-d) multiplied by weightings λ_b, λ_c and λ_d respectively, we obtain

$$\frac{d\mathcal{E}}{dt} = \mathcal{P} - \mathcal{D} \qquad (S.10a)$$

where

$$\mathcal{E} = \langle \tfrac{1}{2}|\mathbf{u}|^2 + \lambda_b\tfrac{1}{2}T^2 + \lambda_c\tfrac{1}{2}S^2 + \lambda_d TS \rangle , \qquad (S.10b)$$

$$\mathcal{P} = -\langle wu\frac{\partial}{\partial x}\overline{W}(x,t) \rangle + \mathcal{R}\langle Tw - Sw \rangle - \lambda_b\langle Tu\frac{\partial}{\partial x}f(x,t) \rangle$$

$$- \lambda_c\langle Su\frac{\partial}{\partial x}f(x,t) \rangle + \lambda_c\langle Sw \rangle - \lambda_d\langle (T+S)u\frac{\partial}{\partial x}f(x,t) \rangle + \lambda_d\langle Tw \rangle , \qquad (S.10c)$$

$$\mathcal{D} = \langle \sigma|\nabla\mathbf{u}|^2 + \lambda_b|\nabla T|^2 + \lambda_c\tau|\nabla S|^2 + \lambda_d(1+\tau)\nabla T\cdot\nabla S \rangle . \qquad (S.10d)$$

The values of λ_b, λ_c and λ_d are chosen so that both $\mathcal{E}$ and $\mathcal{D}$ are always positive definite.

If we have the condition, over the set of all admissible functions $\mathbf{u}$, T and S, that

$$\sup\left\{ \frac{\mathcal{P}}{\mathcal{D}} \right\} \le A < 1 , \qquad (S.11)$$

where A is some constant, then

$$\frac{d\mathcal{E}}{dt} \le -(1-A)\mathcal{D} . \qquad (S.12)$$

As Dudis & Davis (1971) demonstrated, since the fluid region is unbounded there is no relationship of the form

$$\sup \left\{ \frac{\langle \psi^2 \rangle}{\langle |\nabla\psi|^2 \rangle} \right\} \leq B < \infty , \tag{S.13}$$

or its vector equivalent. Since this supremum is infinite, even if (S.12) holds we cannot show that $\mathcal{E} \to 0$ as $t \to \infty$. However, if (S.12) is integrated with respect to time then we obtain the result that

$$\mathcal{E}(t) - \mathcal{E}(0) \leq -(1-A) \int_0^t \mathfrak{D}(t') \, dt' . \tag{S.14}$$

If $\mathcal{E}(0)$ is initially finite then it follows that, if $A < 1$, $\mathcal{E}(t)$ remains bounded for all time and also, since $\mathcal{E}$ is never negative, that

$$\lim_{t\to\infty} \int_0^t \mathfrak{D}(t') \, dt' < \infty . \tag{S.15}$$

and so $\mathfrak{D} \to 0$ as $t \to \infty$.

From this we can conclude that the disturbance vorticity will decay to 0 for large time. Dudis & Davis then went on to show that, although this does not imply that $\mathcal{E} \to 0$, the energy of the disturbance contained between the wall and some arbitrary fixed distance from the wall will decay to 0, and so the energy of a perturbation is dispersed over an ever increasing volume.

Unlike the cases where energy stability analysis is applied to bodies of fluid that are bounded in some direction (e.g. flow between two plates), for a semi-infinite region

$$\sup_{\theta}\left\{\frac{\langle\cdot\,\theta^2\rangle}{\langle|\nabla\theta|^2\rangle}\right\} = \infty , \tag{S.16}$$

where θ is restricted to bounded functions that are absolutely integrable on $0 \le x < \infty$. For this reason we must choose λ_c and λ_d so that both the $\langle Tw\rangle$ and the $\langle Sw\rangle$ terms vanish from $\mathcal{F}$. This yields

$$\lambda_c = \mathfrak{R} \quad \text{and} \quad \lambda_d = -\mathfrak{R} . \tag{S.17}$$

At this point it is convenient to rescale λ_b, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{D}$ by factors of $\mathfrak{R}$ and $\mathbf{u}$ by a factor of $\mathfrak{R}^{\frac{1}{2}}$ so that the new (starred) quantities are given by

$$\mathfrak{R}\lambda_b^* = \lambda_b , \quad \mathfrak{R}\mathcal{E}^* = \mathcal{E} ,$$
$$\mathfrak{R}\mathcal{F}^* = \mathcal{F} , \quad \mathfrak{R}\mathcal{D}^* = \mathcal{D} ,$$
$$\mathfrak{R}^{\frac{1}{2}}\mathbf{u}^* = \mathbf{u} . \tag{S.18}$$

Dropping the stars gives

$$\mathcal{E} = \langle \tfrac{1}{2}|\mathbf{u}|^2 + \lambda_b\tfrac{1}{2}T^2 + \tfrac{1}{2}S^2 - TS \rangle , \tag{S.19a}$$

$$\mathcal{F} = -\langle wu\frac{\partial}{\partial x}\overline{W}(x,t)\rangle - (\lambda_b-1)\mathfrak{R}^{\frac{1}{2}}\langle Tu\frac{\partial}{\partial x}f(x,t)\rangle , \tag{S.19b}$$

$$\mathcal{D} = \langle \sigma|\nabla\mathbf{u}|^2 + \left[\lambda_b - \frac{(1+\tau)^2}{4\tau}\right]|\nabla T|^2 + \frac{1}{\tau}|\tau\nabla S - \frac{(1+\tau)}{2}\nabla T|^2 \rangle . \tag{S.19c}$$

From this we can see that the requirement that both $\mathcal{E}$ and $\mathcal{D}$ are positive definite gives the two restrictions on λ_b that

The value of λ_b is only restricted by (S.20b) so we are free to choose a value that minimises this bound for the supremum. The minimum value occurs when $\lambda_b = \frac{1+\tau^2}{2\tau}$ to give us the final bound, for this value of λ_b, that

$$(\lambda_b-1)\mathcal{R}^{\frac{1}{2}} \sup \left\{ \frac{-\langle Tu\frac{\partial}{\partial x}f(x,t)\rangle}{\mathcal{D}} \right\} \leq \frac{\mathcal{R}^{\frac{1}{2}}(1-\tau)}{2\delta^*\sqrt{\pi}\sqrt{\sigma\tau}}\, 1.4903 \ . \qquad (S.28)$$

In a similar fashion we find, using the upwelling velocity appropriate to the large time asymptotics for the error function temperature profile, that

$$\sup \left\{ \frac{\langle wu\frac{\partial}{\partial x}\overline{W}(x,t)\rangle}{\mathcal{D}} \right\} \leq \frac{(1-\tau)\delta^*}{4\sigma\sqrt{\pi}}\, 2.2283 \ . \qquad (S.29)$$

The fluid will be stable to finite amplitude disturbances if $\sup\left\{\frac{\mathcal{I}}{\mathcal{D}}\right\} < 1$. Combining the results (S.28) and (S.29) gives the condition that the background state is stable to arbitrary disturbances if

$$\sup\left\{\frac{\mathcal{I}}{\mathcal{D}}\right\} \leq \frac{(1-\tau)\delta^*}{4\sigma\sqrt{\pi}}\, 2.2283 + \frac{\mathcal{R}^{\frac{1}{2}}(1-\tau)}{2\delta^*\sqrt{\pi}\sqrt{\sigma\tau}}\, 1.4903 < 1 \ . \qquad (S.30)$$

Expressing this in terms of Q and the original definition of δ used in I and § 2 we obtain the condition, for small δ, that the fluid is stable to all disturbances of arbitrary amplitude if

$$Q < \frac{4\pi\delta^4}{(1-\tau)^2}\frac{1}{(1.4903)^2}\left[1 - \frac{\delta\ 2.2283}{4\sigma\sqrt{\pi}}\right]^2 \ . \qquad (S.31)$$

This result is (3.14) of the paper.

The second part of the energy stability analysis uses the extra constaint on the allowed disturbances by only considering disturbances that are periodic in the vertical, with a period of Δz. Thus

$$(\mathbf{u},\ T,\ S)(x,y,z+\Delta z,t) \ = \ (\mathbf{u},\ T,\ S)(x,y,z,t)\ . \tag{S.32}$$

We split $\mathbf{u}$, T and S into parts that are independent of x and y and parts that have zero mean value when averaged over the vertical. We define

$$\mathbf{u} \ = \ \mathbf{u}_p(x,y,z,t) + \mathbf{u}_i(x,t)\ , \tag{S.33}$$

where

$$\mathbf{u}_i(x,t) \ = \ \overline{\mathbf{u}(x,y,z,t)} \quad \text{and} \quad \mathbf{u}_p \ = \ \mathbf{u} - \mathbf{u}_i\ , \tag{S.34}$$

with similar definitions for T_i, T_p, S_i and S_p. It is straight forward to show that $u_i = 0$ for all x and t.

We split the terms in $\mathcal{F}$ into their mean and oscillatory parts, obtaining

$$\mathcal{F} \ = \ -\langle (w_p+w_i)u_p\frac{\partial}{\partial x}\overline{W}(x,t)\rangle \ - \ (\lambda_b-1)\mathcal{R}^{\frac{1}{2}}\langle (T_p+T_i)u_p\frac{\partial}{\partial x}f(x,t)\rangle\ . \tag{S.35}$$

But

$$\overline{A_pA_i} \ = \ \overline{A_p}A_i \ = \ 0\ , \tag{S.36}$$

and so

$$\mathcal{F} \ = \ -\langle w_pu_p\frac{\partial}{\partial x}\overline{W}(x,t)\rangle \ - \ (\lambda_b-1)\mathcal{R}^{\frac{1}{2}}\langle T_pu_p\frac{\partial}{\partial x}f(x,t)\rangle\ . \tag{S.37}$$

Hence $\mathcal{F}$ is independent of the mean part of the disturbances. The choice of λ_c and λ_d , (S.17), is again made to remove the $\langle Tw\rangle$ and $\langle Sw\rangle$ from $\mathcal{F}$, as these terms lead to an unbounded supremum as before.

For any quantity A_p with period Δz in the vertical and no mean part the following inequalities hold:

$$\left[\frac{2\pi}{\Delta z}\right]^2 \overline{(A_p)^2} \le \overline{\left[\frac{\partial A}{\partial z}p\right]^2} \le \overline{|\nabla A_p|^2} . \tag{S.38}$$

Finding the supremum of $\mathcal{I}/\mathcal{D}$ for each of the two parts of $\mathcal{I}$ we find that

$$\sup \left\{ \frac{\langle T_p u_p \frac{\partial}{\partial x} f(x,t)\rangle}{\mathcal{D}} \right\} \le$$

$$\frac{1}{2} \left[\sigma(\lambda_b - \frac{(1+\tau)^2}{4\tau}) \right]^{-\frac{1}{2}} \sup \left\{ \frac{\langle (\sigma u_p{}^2 + (\lambda_b - \frac{(1+\tau)^2}{4\tau}) T_p{}^2) |\frac{\partial}{\partial x} f(x,t)| \rangle}{\langle \sigma |\nabla u_p|^2 + (\lambda_b - \frac{(1+\tau)^2}{4\tau}) |\nabla T_p|^2 \rangle} \right\}$$

$$\le \frac{1}{2} \left[\frac{\Delta z}{2\pi}\right]^2 \left[\sigma(\lambda_b - \frac{(1+\tau)^2}{4\tau}) \right]^{-\frac{1}{2}} \sup \left\{ \frac{\langle (\sigma u_p{}^2 + (\lambda_b - \frac{(1+\tau)^2}{4\tau}) T_p{}^2) |\frac{\partial}{\partial x} f(x,t)| \rangle}{\langle \sigma (u_p)^2 + (\lambda_b - \frac{(1+\tau)^2}{4\tau}) (T_p)^2 \rangle} \right\} . \tag{S.39}$$

But for any function g(x)

$$\sup_{\phi} \left\{ \frac{\langle \phi^2 |\frac{dg}{dx}| \rangle}{\langle \phi^2 \rangle} \right\} = \sup_{x} \left\{ |\frac{dg}{dx}| \right\} , \tag{S.40}$$

and so

$$(\lambda_b - 1)\mathcal{R}^{\frac{1}{2}} \sup \left\{ \frac{\langle T_p u_p \frac{\partial}{\partial x} f(x,t)\rangle}{\mathcal{D}} \right\} \le$$

$$(\lambda_b - 1)\mathcal{R}^{\frac{1}{2}} \frac{1}{2} \left[\sigma(\lambda_b - \frac{(1+\tau)^2}{4\tau}) \right]^{-\frac{1}{2}} |\frac{\partial}{\partial x} f(0,t)| . \tag{S.41}$$

Minimising the right-hand side as before gives, for $\lambda_b = \frac{1+\tau^2}{2\tau}$,

$$(\lambda_b - 1)\mathcal{R}^{\frac{1}{2}} \sup \left\{ \frac{\langle T_p u_p \frac{\partial}{\partial x} f(x,t)\rangle}{\mathcal{D}} \right\} \le \frac{(1-\tau)}{2\sqrt{\sigma\tau}} \mathcal{R}^{\frac{1}{2}} \left[\frac{\Delta z}{2\pi}\right]^2 \frac{\delta^*}{\sqrt{\pi}} . \tag{S.42}$$

In a similar fashion we find that

$$\sup \left\{ \frac{\langle w_p u_p \frac{\partial}{\partial x} \bar{W}(x,t) \rangle}{\mathscr{D}} \right\} \leq \frac{1}{2} \left[\frac{\Delta z}{2\pi}\right]^2 \frac{(1-\tau)}{2\sqrt{\pi}} \delta^{*3} . \tag{3.43}$$

Combining these results gives us the result that the background state is stable to arbitrary periodic disturbances of vertical period Δz if

$$\frac{1}{2} \left[\frac{\Delta z}{2\pi}\right]^2 \frac{(1-\tau)}{2\sqrt{\pi}} \delta^{*3} + \frac{(1-\tau)}{2\sqrt{\sigma\tau}} \mathscr{R}^{\frac{1}{2}} \left[\frac{\Delta z}{2\pi}\right]^2 \frac{\delta^*}{\sqrt{\pi}} < 1 . \tag{S.44}$$

Re-expressing this in terms of Q and δ gives the condition that the background state is stable, for small values of δ, if

$$Q < 4\pi \left[\frac{2\pi}{\Delta z}\right]^4 (1-\tau)^4 \left\{ 1 - \frac{\Delta z^2 \delta^3}{16\sqrt{\pi}(1-\tau)^2\pi^2} \right\} . \tag{S.45}$$

The experiments of Chen, Briggs & Wirtz (1971), Huppert & Josberger (1980) and Huppert & Turner (1980) always have that $\Delta z < 1$. If we use this value then we get the condition that the fluid is stable to periodic disturbances with non-dimensional period less that 1 if

$$Q < 64\pi^5 (1-\tau)^4 \{1-0(\delta^3)\} \approx 19{,}585(1-\tau)^4 . \tag{S.46}$$

This is result (3.21) of the paper.