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Appendix A. Computations using the method of images

The image calculations presented here summarize the treatment of Lamb (1932; Art. 98) in our dimensionless notation. Denoting by $c = 2h$ the center-to-center distance, one defines the following locations (f) and corresponding strength ratios (R) of successive dipoles. Thereby, each dipole exactly neutralizes around its own sphere the effect of the previous dipole, which emmanated from inside the other sphere. Starting with $f_0 = 0$, successive steps yield

$$f_{j-1} = c - \frac{1}{c - f_{j-2}}, \qquad R_{j-1} = \frac{1}{(c - f_{j-2})^3}$$
$$f_j = \frac{1}{f_{j-1}}, \qquad R_j = \frac{1}{f_{j-1}^3}$$

$(j = 2, 4, 6, \ldots)$. The actual dipole strengths are given by the successive relations

$$\mu_0 = 1, \qquad \mu_j = -\mu_{j-1} * R_j \qquad (j = 1, 2, 3, \ldots).$$

In terms of these quantities the virtual masses are obtained as follows:

$$\mathcal{M}_a^{\|} = 2\pi\,(1/3 + \mu_2 + \mu_4 + \mu_6 + \ldots),$$
$$\mathcal{M}_b^{\|} = 2\pi\,(\mu_1 + \mu_3 + \mu_5 + \ldots).$$

Appendix B. Integrands for the nonlinear drift coefficients

Below appear explicit expressions for the functions $\Lambda_i^{\|}(z,\rho)$, $\Lambda_i^{\perp}(z,\rho)$, $\Gamma_i^{\|}(z,\rho)$, $\Gamma_i^{\perp}(z,\rho)$ to be integrated according to (??) to obtain the respective drift force coefficients $A_i^{\|}$, $A_i^{\perp}$, $B_i^{\|}$, $B_i^{\perp}$. For brevity, we omit the arguments z, ρ from quantities appearing on the right-hand sides.

$$\Lambda_i^{\|}(z,\rho) = \left[\, T'_{zz} n_z + T'_{z\rho} n_\rho \right] \frac{\partial v_{zI}^{[2-i],\|}}{\partial z} + \left[T'_{z\rho}\, n_z + T'_{\rho\rho} n_\rho \right] \frac{\partial v_{\rho I}^{[2-i],\|}}{\partial z}$$
$$\Gamma_1^{\|}(z,\rho) = \sum_{C=R,I} \sum_{\alpha=1}^{2} \left\{ D'_{zz} \left[v_{zC}^{[\alpha],\|} \right]^2 + 2D'_{z\rho} v_{zC}^{[\alpha],\|} v_{\rho C}^{[\alpha],\|} + D_{\rho\rho} \left[v_{\rho C}^{[\alpha],\|} \right]^2 \right\},$$
$$\Gamma_2^{\|}(z,\rho) = \sum_{C=R,I} \left\{ D'_{zz} v_{zC}^{[1],\|} v_{zC}^{[2],\|} + D'_{z\rho} \left[v_{zC}^{[1],\|} v_{\rho C}^{[2],\|} + v_{\rho C}^{[1],\|} v_{zC}^{[2],\|} \right] + D'_{\rho\rho} v_{\rho C}^{[1],\|} v_{\rho C}^{[2],\|} \right\}, \tag{B 1}$$

$$\Lambda_i^{\perp}(z,\rho) = \left[\, T'_{zz} n_z + T'_{z\rho} n_\rho \right] \frac{\partial v_{zI}^{[2-i],\perp *}}{\partial \rho} + \left[T'_{z\rho}\, n_z + T'_{\rho\rho} n_\rho \right] \frac{\partial v_{\rho I}^{[2-i],\perp *}}{\partial \rho}$$
$$\Gamma_1^{\perp}(z,\rho) = \sum_{C=R,I} \sum_{\alpha=1}^{2} \left\{ D'_{zz} \left[v_{zC}^{[\alpha],\perp *} \right]^2 + 2D'_{z\rho} v_{zC}^{[\alpha],\perp *} v_{\rho C}^{[\alpha],\perp *} + D'_{\rho\rho} \left[v_{\rho C}^{[\alpha],\perp *} \right]^2 + D'_{\phi\phi} \left[v_{\phi C}^{[\alpha],\perp *} \right]^2 \right\},$$
$$\Gamma_2^{\perp}(z,\rho) = \sum_{C=R,I} \left\{ D'_{zz} v_{zC}^{[1],\perp *} v_{zC}^{[2],\perp *} + D'_{z\rho} \left[v_{zC}^{[1],\perp *} v_{\rho C}^{[2],\perp *} + v_{\rho C}^{[1],\perp *} v_{zC}^{[2],\perp *} \right] \right.$$
$$\left. + D'_{\rho\rho} v_{\rho C}^{[1],\perp *} v_{\rho C}^{[2],\perp *} + D'_{\phi\phi} v_{\phi C}^{[1],\perp *} v_{\phi C}^{[2],\perp *} \right\}. \tag{B 2}$$

In these equations the subscript "C" is used as an index to denote real or imaginary parts. Note that in finding $\Lambda_i^{\perp}$ the boundary conditions (??) were inserted into the derivative formula

$$\begin{aligned} \boldsymbol{b}^{\perp} \bullet \nabla \boldsymbol{v}^{[\alpha],\perp} \;=\; \frac{\partial \boldsymbol{v}^{[\alpha],\perp}}{\partial x} &= \left[\frac{\partial v_z^{[\alpha],\perp *}}{\partial \rho} \cos^2 \phi + \frac{v_z^{[\alpha],\perp}}{\rho} \sin^2 \phi \right] \boldsymbol{e}_z \\ &+ \left[\frac{\partial v_\rho^{[\alpha],\perp *}}{\partial \rho} \cos^2 \phi + \frac{v_\rho^{[\alpha],\perp} + v_\phi^{[\alpha],\perp}}{\rho} \sin^2 \phi \right] \boldsymbol{e}_\rho \\ &+ \left[\frac{\partial v_\phi^{[\alpha],\perp *}}{\partial \rho} - \frac{v_\rho^{[\alpha],\perp *} + v_\phi^{[\alpha],\perp *}}{\rho} \right] \sin \phi \cos \phi \, \boldsymbol{e}_\phi, \end{aligned}$$

thereby eliminating the second term in each bracketed expression.

Appendix C. Ring singularities for the unsteady Stokes equations

We list here the functions that must be integrated in the azimuthal angle ϕ' in order to obtain the singular basis functions $U_z^{(i),\|}(z,\rho;z_*,\rho_*)$, $U_\rho^{(i),\|}(z,\rho;z_*,\rho_*)$, $U_z^{(i),\perp *}(z,\rho;z_*,\rho_*)$, $U_\rho^{(i),\perp *}(z,\rho;z_*,\rho_*)$, $U_\phi^{(i),\perp *}(z,\rho;z_*,\rho_*)$; see §§4.2, 4.4. At each azimuthal position ϕ' we define

$$r \;=\; r(\phi'; z, \rho, z_*, \rho_*) \;=\; \sqrt{\rho^2 + r_n^2 - 2\rho\rho_n \cos\phi_j + (z - z_n)^2} \tag{C 1}$$

Denoting the set of arguments $\{z, \rho, z_*, \rho_*\}$ by the shorthand notation $\boldsymbol{x}_*$, and setting

$$\begin{aligned} A(x) &= 1 - (1 + x + x^2/3)e^{-x}, \\ B(x) &= 1 - (1 + x + x^2)e^{-x}, \end{aligned} \tag{C 2}$$

we have

$$\begin{aligned} f_z^{(1),\|}(\phi';\boldsymbol{x}_*) &= \frac{3A(\gamma r)}{\gamma^2 r^5}(\rho\cos\phi' - \rho_*)(z - z_*), \\ f_z^{(2),\|}(\phi';\boldsymbol{x}_*) &= \frac{3A(\gamma r)}{\gamma^2 r^5}(z - z_*)^2 - \frac{B(\gamma r)}{\gamma^2 r^3}, \\ f_z^{(2),\|}(\phi';\boldsymbol{x}_*) &= \frac{(z - z_*)}{r^3}, \end{aligned} \tag{C 3}$$

$$\begin{aligned} f_\rho^{(1),\|}(\phi';\boldsymbol{x}_*) &= \frac{3A(\gamma r)}{\gamma^2 r^5}(\rho\cos\phi' - \rho_*)(\rho - \rho_*\cos\phi') - \frac{B(\gamma r)}{\gamma^2 r^3}\cos\phi', \\ f_\rho^{(2),\|}(\phi';\boldsymbol{x}_*) &= \frac{3A(\gamma r)}{\gamma^2 r^5}(z - z_*)(\rho - \rho_*\cos\phi'), \\ f_\rho^{(3),\|}(\phi';\boldsymbol{x}_*) &= \frac{(\rho - \rho_*\cos\phi')}{r^3}, \end{aligned} \tag{C 4}$$

$$\begin{aligned} f_z^{(1),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{3A(\gamma r)}{\gamma^2 r^5}(z - z_*)(\rho\cos\phi' - \rho_*)\cos\phi', \\ f_z^{(2),\perp *}(\phi';\boldsymbol{x}_*) &= \left[\frac{3A(\gamma r)}{\gamma^2 r^5}(z - z_*)^2 - \frac{B(\gamma r)}{\gamma^2 r^3}\right]\cos\phi', \\ f_z^{(3),\perp *}(\phi';\boldsymbol{x}_*) &= -\frac{3A(\gamma r)}{\gamma^2 r^5}(z - z_*)\rho\sin^2\phi', \\ f_z^{(4),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\cos\phi'}{r^3}(z - z_*) \end{aligned} \tag{C 5}$$

$$f_\rho^{(1),\perp *}(\phi';\boldsymbol{x}_*) = \frac{3A(\gamma r)}{\gamma^2 r^5}(\rho - \rho_*\cos\phi')(\rho\cos\phi' - \rho_*)\cos\phi' - \frac{B(\gamma r)}{\gamma^2 r^3}\cos^2\phi',$$

$$f_\rho^{(2),\perp *}(\phi';\boldsymbol{x}_*) = \frac{3A(\gamma r)}{\gamma^2 r^5}(\rho - \rho_* \cos\phi')(z - z_*)\cos\phi',$$
$$f_\rho^{(3),\perp *}(\phi';\boldsymbol{x}_*) = \left[-\frac{3A(\gamma r)}{\gamma^2 r^5}(\rho - \rho_* \cos\phi')\rho \;+\; \frac{B(\gamma r)}{\gamma^2 r^3}\right]\sin^2\phi',$$
$$f_\rho^{(4),\perp *}(\phi';\boldsymbol{x}_*) = \frac{\cos\phi'}{r^3}(\rho - \rho_* \cos\phi'), \tag{C 6}$$
$$f_\phi^{(1),\perp *}(\phi';\boldsymbol{x}_*) = \left[\frac{3A(\gamma r)}{\gamma^2 r^5}\rho_*(\rho\cos\phi' - \rho_*) \;+\; \frac{B(\gamma r)}{\gamma^2 r^3}\right]\sin^2\phi',$$
$$f_\phi^{(2),\perp *}(\phi';\boldsymbol{x}_*) = \frac{3A(\gamma r)}{\gamma^2 r^5}\rho_*(z - z_*)\sin^2\phi',$$
$$f_\phi^{(3),\perp *}(\phi';\boldsymbol{x}_*) = \frac{3A(\gamma r)}{\gamma^2 r^5}\rho\rho_*\cos\phi'\sin^2\phi' \;-\; \frac{B(\gamma r)}{\gamma^2 r^3}\cos^2\phi',$$
$$f_\phi^{(4),\perp *}(\phi';\boldsymbol{x}_*) = \frac{\rho_*}{r^3}\sin^2\phi'. \tag{C 7}$$

For singularities lying on the z axis (i.e. the limit as $\rho_* \to 0$), the azimuthal integration becomes superfluous, and the basis functions are given directly by the following simpler formulas, where $\boldsymbol{x}_*^0 = \{z, \rho, z_*, 0\}$ and $r = \sqrt{\rho^2 + (z - z_*)^2}$:

$$U_z^{(1),\parallel}(\boldsymbol{x}_*^0) = 0,$$
$$U_z^{(2),\parallel}(\boldsymbol{x}_*^0) = \frac{3A(\mathrm{r})}{4\pi\gamma^2 \mathrm{r}^5}(z - z_*)^2 \;-\; \frac{B(\gamma r)}{4\pi\gamma^2 r^3},$$
$$U_z^{(3),\parallel}(\boldsymbol{x}_*^0) = \frac{z - z_*}{4\pi r^3}, \tag{C 8}$$
$$U_\rho^{(1),\parallel}(\boldsymbol{x}_*^0) = 0,$$
$$U_\rho^{(2),\parallel}(\boldsymbol{x}_*^0) = \frac{3A(\gamma r)}{4\pi\gamma^2 r^5}(z - z_*)\rho,$$
$$U_\rho^{(3),\parallel}(\boldsymbol{x}_*^0) = \frac{\rho}{4\pi r^3}, \tag{C 9}$$
$$U_z^{(1),\perp,*}(\boldsymbol{x}_*^0) = \frac{3A(\gamma r)}{8\pi\gamma^2 r^5}(z - z_*)\rho,$$
$$U_z^{(2),\perp *}(\boldsymbol{x}_*^0) = 0,$$
$$U_z^{(3),\perp *}(\boldsymbol{x}_*^0) = -U_z^{(1),\perp,*}(\boldsymbol{x}_*^0),$$
$$U_z^{(4),\perp *}(\boldsymbol{x}_*^0) = 0, \tag{C 10}$$
$$U_\rho^{(1),\perp *}(\boldsymbol{x}_*^0) = \frac{3A(\gamma r)\rho^2}{8\pi\gamma^2 r^5} \;-\; \frac{B(\gamma r)}{8\pi\gamma^2 r^3},$$
$$U_\rho^{(2),\perp *}(\boldsymbol{x}_*^0) = 0,$$
$$U_\rho^{(3),\perp *}(\boldsymbol{x}_*^0) = -U_\rho^{(1),\perp *}(\boldsymbol{x}_*^0),$$
$$U_\rho^{(4),\perp *}(\boldsymbol{x}_*^0) = 0, \tag{C 11}$$
$$U_\phi^{(1),\perp *}(\boldsymbol{x}_*^0) = \frac{B(\gamma r)}{8\pi\gamma^2 r^3},$$
$$U_\phi^{(2),\perp *}(\boldsymbol{x}_*^0) = 0,$$
$$U_\phi^{(3),\perp *}(\boldsymbol{x}_*^0) = -U_\phi^{(1),\perp *}(\boldsymbol{x}_*^0),$$
$$U_\phi^{(4),\perp *}(\boldsymbol{x}_*^0) = 0. \tag{C 12}$$

Appendix D. Ring singularities for the quasistatic Stokes equations

The formulas for the ring singularities in *quasistatic* Stokes flow are entirely analogous to those given in Appendix C for the unsteady Stokes equations. Because Nitsche and Brenner (1990, Appendix A) furnish expressions for the axisymmetric case, we confine our considerations to the case of motion perpendicular to the line of centers. With the singular basis functions defined as follows,

$$\begin{aligned} U_z^{(i),\perp *}(\boldsymbol{x}_*) &= \frac{1}{8\pi^2}\int_0^\pi f_z^{(i),\perp *}(\phi';\boldsymbol{x}_*)\,d\phi', \\ U_\rho^{(i),\perp *}(\boldsymbol{x}_*) &= \frac{1}{8\pi^2}\int_0^\pi f_\rho^{(i),\perp *}(\phi';\boldsymbol{x}_*)\,d\phi', \\ U_\phi^{(i),\perp *}(\boldsymbol{x}_*) &= \frac{1}{8\pi^2}\int_0^\pi f_\phi^{(i),\perp *}(\phi';\boldsymbol{x}_*)\,d\phi', \end{aligned} \tag{D 1}$$

we have

$$\begin{aligned} f_z^{(1),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\cos\phi'}{r^3}(z-z_*)(\rho\cos\phi'-\rho_*), \\ f_z^{(2),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\cos\phi'}{r^3}(z-z_*)^2 + \frac{\cos\phi'}{r}, \\ f_z^{(3),\perp *}(\phi';\boldsymbol{x}_*) &= -\frac{\sin^2\phi'}{r^3}(z-z_*)\rho, \\ f_z^{(4),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{2\cos\phi'}{r^3}(z-z_*), \end{aligned} \tag{D 2}$$

$$\begin{aligned} f_\rho^{(1),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\cos\phi'}{r^3}(\rho-\rho_*\cos\phi')(\rho\cos\phi'-\rho_*) + \frac{\cos^2\phi'}{r}, \\ f_\rho^{(2),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\cos\phi'}{r^3}(\rho-\rho_*\cos\phi')(z-z_*), \\ f_\rho^{(3),\perp *}(\phi';\boldsymbol{x}_*) &= -\frac{\sin^2\phi'}{r^3}(\rho-\rho_*\cos\phi')\rho - \frac{\sin^2\phi'}{r}, \\ f_\rho^{(4),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{2\cos\phi'}{r^3}(\rho-\rho_*\cos\phi'), \end{aligned} \tag{D 3}$$

$$\begin{aligned} f_\phi^{(1),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\sin^2\phi'}{r^3}\rho_*(\rho\cos\phi'-\rho_*) - \frac{\sin^2\phi'}{r}, \\ f_\phi^{(2),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\sin^2\phi'}{r^3}\rho_*(z-z_*), \\ f_\phi^{(3),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{\sin^2\phi'}{r^3}\rho\rho_*\cos\phi' + \frac{\cos^2\phi'}{r}, \\ f_\phi^{(4),\perp *}(\phi';\boldsymbol{x}_*) &= \frac{2\sin^2\phi'}{r^3}\rho_*. \end{aligned} \tag{D 4}$$

For singularities lying on the z axis the basis functions are given by

$$\begin{aligned} U_z^{(1),\perp,*}(\boldsymbol{x}_*^0) &= \frac{\rho(z-z_*)}{16\pi r^3}, \\ U_z^{(2),\perp *}(\boldsymbol{x}_*^0) &= 0, \\ U_z^{(3),\perp *}(\boldsymbol{x}_*^0) &= -U_z^{(1),\perp,*}(\boldsymbol{x}_*^0), \\ U_z^{(4),\perp *}(\boldsymbol{x}_*^0) &= 0, \end{aligned} \tag{D 5}$$

$$
\begin{aligned}
U_\rho^{(1),\perp *}(\boldsymbol{x}_*^0) &= \frac{1}{16\pi r}\left[\frac{\rho^2}{r^2} + 1\right],\\
U_\rho^{(2),\perp *}(\boldsymbol{x}_*^0) &= 0,\\
U_\rho^{(3),\perp *}(\boldsymbol{x}_*^0) &= -U_\rho^{(1),\perp *}(\boldsymbol{x}_*^0),\\
U_\rho^{(4),\perp *}(\boldsymbol{x}_*^0) &= 0,
\end{aligned} \tag{D 6}
$$

$$
\begin{aligned}
U_\phi^{(1),\perp *}(\boldsymbol{x}_*^0) &= -\frac{1}{16\pi r},\\
U_\phi^{(2),\perp *}(\boldsymbol{x}_*^0) &= 0,\\
U_\phi^{(3),\perp *}(\boldsymbol{x}_*^0) &= -U_\phi^{(1),\perp *}(\boldsymbol{x}_*^0),\\
U_\phi^{(4),\perp *}(\boldsymbol{x}_*^0) &= 0.
\end{aligned} \tag{D 7}
$$

Appendix E. Formulas for the first reflection at high frequencies

In §4.7 the first reflection for the unsteady Stokes equations is given through terms of order $O(\epsilon^5)$ by (4.22) and (4.23). This scaling applies whenever the Stokes layers of the two spheres are well separated. We collect here expressions for the various components of the corresponding velocity fields $\boldsymbol{V}^{(i),\parallel}(z,\rho)$ and $\boldsymbol{V}^{(i),\perp *}(z,\rho)$:

$$
\begin{aligned}
V_z^{(0),\parallel}(z,\rho) &= -\frac{1}{2}\left[\frac{G(\gamma)}{r^3} - 3H(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right] + \frac{3}{2}\left[\frac{G(\gamma)}{r^3} - G(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^2}{r^2},\\
V_\rho^{(0),\parallel}(z,\rho) &= \frac{3}{2}\left[\frac{G(\gamma)}{r^3} - G(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z\rho}{r^2},
\end{aligned} \tag{E 1}
$$

$$
\begin{aligned}
V_z^{(1),\parallel}(z,\rho) &= -9\left[\frac{I(\gamma)}{r^4} - \frac{5}{3}F_2(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z}{r} + 15\left[\frac{I(\gamma)}{r^4} - I(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^3}{r^3},\\
V_\rho^{(1),\parallel}(z,\rho) &= -3\left[\frac{I(\gamma)}{r^4} - 5\frac{G(\gamma r)}{\gamma r}\frac{e^{\gamma-\gamma r}}{r}\right]\frac{\rho}{r} + 15\left[\frac{I(\gamma)}{r^4} - I(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^2\rho}{r^3},
\end{aligned} \tag{E 2}
$$

$$
\begin{aligned}
V_z^{(2),\parallel}(z,\rho) &= -\frac{3}{5}\left[\frac{L(\gamma)}{r^5} - \frac{7}{3}F_3(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right] + 6\left[\frac{L(\gamma)}{r^5} - \frac{7}{5}F_4(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^2}{r^2}\\
&\qquad - 7\left[\frac{L(\gamma)}{r^5} - L(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^4}{r^4},\\
V_\rho^{(2),\parallel}(z,\rho) &= 3\left[\frac{L(\gamma)}{r^5} - \frac{7}{15}F_5(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z\rho}{r^2} - 7\left[\frac{L(\gamma)}{r^5} - L(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^3\rho}{r^4},
\end{aligned} \tag{E 3}
$$

$$
\begin{aligned}
V_z^{(0),\perp *}(z,\rho) &= \frac{3}{2}\left[\frac{G(\gamma)}{r^3} - G(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z\rho}{r^2},\\
V_\rho^{(0),\perp *}(z,\rho) &= -\frac{1}{2}\left[\frac{G(\gamma)}{r^3} - 3H(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right] + \frac{3}{2}\left[\frac{G(\gamma)}{r^3} - G(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{\rho^3}{r^3},\\
V_\phi^{(0),\perp *}(z,\rho) &= \frac{1}{2}\left[\frac{G(\gamma)}{r^3} - 3H(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right],
\end{aligned} \tag{E 4}
$$

$$
\begin{aligned}
V_z^{(1),\perp *}(z,\rho) &= 2\left[\frac{I(\gamma)}{r^4} - \frac{5}{2}F_1(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{\rho}{r} - 10\left[\frac{I(\gamma)}{r^4} - I(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^2\rho}{r^3},\\
V_\rho^{(1),\perp *}(z,\rho) &= 2\left[\frac{I(\gamma r)}{r^4} - \frac{5}{2}F_1(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z}{r} - 10\left[\frac{I(\gamma)}{r^4} - I(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z\rho^2}{r^3}
\end{aligned}
$$

$$V_\phi^{(1),\perp *}(z,\rho) = -2\left[\frac{I(\gamma)}{r^4} - \frac{5}{2}F_1(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z}{r}, \tag{E 5}$$

$$V_z^{(2),\perp *}(z,\rho) = -45\left[\frac{L(\gamma)}{r^5} - \frac{77}{45}F_6(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z\rho}{r^2} + 105\left[\frac{L(\gamma)}{r^5} - L(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^3\rho}{r^4},$$

$$V_\rho^{(2),\perp *}(z,\rho) = -12\left[\frac{L(\gamma)}{r^5} - 7\frac{I(\gamma r)}{\gamma r}\frac{e^{\gamma-\gamma r}}{r}\right] + 105\left[\frac{L(\gamma)}{r^5} - \frac{11}{15}F_7(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^2}{r^2}$$
$$- 105\left[\frac{L(\gamma)}{r^5} - L(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^4}{r^4},$$

$$V_\phi^{(2),\perp *}(z,\rho) = -3\left[\frac{L(\gamma)}{r^5} - \frac{7}{3}F_3(\gamma)\frac{e^{\gamma-\gamma r}}{r}\right] + 15\left[\frac{L(\gamma)}{r^5} - \frac{7}{3}F_3(\gamma r)\frac{e^{\gamma-\gamma r}}{r}\right]\frac{z^2}{r^2}. \tag{E 6}$$

In these formulas,

$$r = r(z,\rho) \stackrel{\text{def}}{=} \sqrt{z^2+\rho^2}. \tag{E 7}$$

The functions $G(x)$ through $L(x)$ are given by (3.36) and (3.40) in §3.7 above, while the $F_i(x)$ are defined as follows:

$$\begin{aligned}
F_1(x) &= 1 + 3x^{-1} + 6x^{-2} + 6x^{-3},\\
F_2(x) &= 1 + 4x^{-1} + 9x^{-2} + 9x^{-3},\\
F_3(x) &= 1 + 6x^{-1} + 21x^{-2} + 45x^{-3} + 45x^{-4},\\
F_4(x) &= 1 + 8x^{-1} + 33x^{-2} + 75x^{-3} + 75x^{-4},\\
F_5(x) &= 1 + 18x^{-1} + 93x^{-2} + 225x^{-3} + 225x^{-4},\\
F_6(x) &= 1 + \frac{78}{11}x^{-1} + \frac{303}{11}x^{-2} + \frac{675}{11}x^{-3} + \frac{675}{11}x^{-4},\\
F_7(x) &= 1 + \frac{138}{11}x^{-1} + \frac{663}{11}x^{-2} + \frac{1575}{11}x^{-3} + \frac{1575}{11}x^{-4}.
\end{aligned} \tag{E 8}$$

Tables

TABLE 1: Virtual masses and drift force coefficients as functions of the sphere-sphere gap L for both axial and transverse oscillations in potential flow.

L	$\mathcal{M}_a^{\parallel}$	$\mathcal{M}_b^{\parallel}$	$\hat{B}_1^{\parallel}$	$\hat{B}_2^{\parallel}$
1/128	2.3937	-0.9128	2.5772	-3.2271
$\sqrt{2}/128$	2.3858	-0.9028	2.3164	-2.9643
1/64	2.3758	-0.8899	2.0594	-2.7045
$\sqrt{2}/64$	2.3634	-0.8733	1.8072	-2.4484
1/32	2.3480	-0.8521	1.5613	-2.1969
$\sqrt{2}/32$	2.3295	-0.8254	1.3235	-1.9512
1/16	2.3075	-0.7920	1.0959	-1.7125
$\sqrt{2}/16$	2.2821	-0.7508	0.8813	-1.4823
1/8	2.2537	-0.7008	0.6828	-1.2621
$\sqrt{2}/8$	2.2233	-0.6412	0.5043	-1.0533
1/4	2.1925	-0.5718	0.3499	-0.8571
$\sqrt{2}/4$	2.1635	-0.4932	0.2236	-0.6748
1/2	2.1385	-0.4074	0.1285	-0.5080
$\sqrt{2}/2$	2.1194	-0.3188	0.6468×10^{-1}	-0.3598
1	2.1067	-0.2334	0.2780×10^{-1}	-0.2351
$\sqrt{2}$	2.0996	-0.1580	0.1000×10^{-1}	-0.1392
2	2.0963	-0.9821×10^{-1}	0.2981×10^{-2}	-0.7372×10^{-1}
$2\sqrt{2}$	2.0950	-0.5582×10^{-1}	0.7344×10^{-3}	-0.3469×10^{-1}
4	2.0945	-0.2909×10^{-1}	0.1507×10^{-3}	-0.1454×10^{-1}
$4\sqrt{2}$	2.0944	-0.1400×10^{-1}	0.2617×10^{-4}	-0.5484×10^{-2}
8	2.0944	-0.6283×10^{-2}	0.3924×10^{-5}	-0.1885×10^{-2}

TABLE 1 (continued).

L	$\mathcal{M}_a^\perp$	$\mathcal{M}_b^\perp$	$\hat{B}_1^\perp$	$\hat{B}_2^\perp$
1/16	2.1552	0.3730	0.2841	0.6498
$\sqrt{2}/16$	2.1484	0.3570	0.2387	0.5931
1/8	2.1406	0.3365	0.1916	0.5304
$\sqrt{2}/8$	2.1320	0.3108	0.1454	0.4631
1/4	2.1230	0.2796	0.1027	0.3921
$\sqrt{2}/4$	2.1144	0.2430	0.6616×10^{-1}	0.3190
1/2	2.1071	0.2020	0.3790×10^{-1}	0.2460
$\sqrt{2}/2$	2.1014	0.1587	0.1882×10^{-1}	0.1770
1	2.0978	0.1165	0.7915×10^{-2}	0.1168
$\sqrt{2}$	2.0958	0.7896×10^{-1}	0.2776×10^{-2}	0.6944×10^{-1}
2	2.0949	0.4909×10^{-1}	0.8059×10^{-3}	0.3683×10^{-1}
$2\sqrt{2}$	2.0945	0.2791×10^{-1}	0.1940×10^{-3}	0.1734×10^{-1}
4	2.0944	0.1454×10^{-1}	0.3907×10^{-4}	0.7272×10^{-2}
$4\sqrt{2}$	2.0944	0.6998×10^{-2}	0.6692×10^{-5}	0.2742×10^{-2}
8	2.0944	0.3142×10^{-2}	0.9942×10^{-6}	0.9425×10^{-3}

TABLE 2: Computational parameters for the LSBSM numerics.

Parameter set, "PS"	1	2
Intervals for azimuthal integration	12	24
Singularity points (N)	18	34
Boundary points	78	150
Matrix: axial modes, viscous flow	156×50	300×98
transverse modes, viscous flow	234×60	450×124
transverse modes, potential flow	79×14	151×30

TABLE 3: Computational parameters, and accuracy of the LSBSM numerics for axial modes in viscous flow at various frequencies and separations: optimal radius R_* of auxiliary boundary; Simpson's-rule approximation $\bar{\mathcal{E}}$ of the square-integral boundary criterion $\mathcal{E}$, Eq. (4.7); maximum boundary residual $|\Delta V|_{\max}$; net source $|\Sigma Q|_{\max}$; and largest coefficients $F_{\max} = \max\{|G_n^{(z)}|, |G_n^{(\rho)}|\}$, $Q_{\max} = \max\{|G_n^{(q)}|\}$.

L	Ω	PS	$R_*^{\parallel}$	$\bar{\mathcal{E}}^{\parallel}$	$\lvert\Delta V\rvert_{\max}$	$\lvert\Sigma Q\rvert$	$F_{\max}$	$Q_{\max}$
1.0	0	1	0.40	7.0×10^{-10}	3.2×10^{-5}	9.7×10^{-9}	18.8	9.3
1.0	1/2048	1	0.4	7.2×10^{-10}	3.2×10^{-5}	9.8×10^{-9}	18.8	9.4
1.0	1/1024	1	0.4	7.3×10^{-10}	3.3×10^{-5}	9.8×10^{-9}	18.8	9.4
1.0	1/512	1	0.4	7.5×10^{-10}	3.3×10^{-5}	9.9×10^{-9}	18.8	9.4
1.0	1/256	1	0.4	7.8×10^{-10}	3.3×10^{-5}	1.0×10^{-8}	18.9	9.4
1.0	1/128	1	0.4	8.1×10^{-10}	3.4×10^{-5}	1.0×10^{-8}	18.9	9.4
1.0	1/64	1	0.4	8.7×10^{-10}	3.5×10^{-5}	1.0×10^{-8}	18.9	9.4
1.0	1/32	1	0.4	9.6×10^{-10}	3.7×10^{-5}	1.1×10^{-8}	18.9	9.4
1.0	1/16	1	0.4	1.1×10^{-9}	3.9×10^{-5}	1.1×10^{-8}	19.0	9.4
1.0	1/8	1	0.4	1.4×10^{-9}	4.4×10^{-5}	1.2×10^{-8}	19.1	9.5
1.0	1/4	1	0.4	1.9×10^{-9}	5.2×10^{-5}	1.4×10^{-8}	19.3	9.5
1.0	1/2	1	0.4	3.1×10^{-9}	6.6×10^{-5}	1.8×10^{-8}	19.6	9.7
1.0	1	1	0.4	6.0×10^{-9}	9.2×10^{-5}	2.5×10^{-8}	20.4	10.1
1.0	2	1	0.4	1.3×10^{-8}	1.4×10^{-4}	3.7×10^{-8}	22.3	10.9
1.0	4	1	0.4	2.8×10^{-8}	2.0×10^{-4}	5.4×10^{-8}	26.0	12.6
1.0	8	1	0.4	5.0×10^{-8}	2.6×10^{-4}	7.0×10^{-8}	30.8	14.8
1.0	16	1	0.4	5.2×10^{-8}	2.6×10^{-4}	6.0×10^{-8}	48.4	15.3
1.0	32	1	0.43	1.8×10^{-8}	1.6×10^{-4}	2.5×10^{-8}	149	23.2
1.0	64	1	0.37	2.2×10^{-8}	2.0×10^{-4}	5.6×10^{-8}	692	60.4
1.0	128	1	0.33	2.3×10^{-8}	1.9×10^{-4}	3.7×10^{-8}	5430	235
1.0	256	1	0.29	1.0×10^{-7}	2.9×10^{-4}	3.7×10^{-8}	100000	2380

TABLE 3 (continued).

L	Ω	PS	$R_*^{\parallel}$	$\bar{\mathcal{E}}^{\parallel}$	$\lvert\Delta V\rvert_{\max}$	$\lvert\Sigma Q\rvert$	$F_{\max}$	$Q_{\max}$
1/4	0	2	0.62	7.1×10^{-11}	2.5×10^{-5}	2.6×10^{-9}	62.0	15.2
$\sqrt{2}/4$	0	2	0.57	7.6×10^{-13}	2.5×10^{-6}	1.4×10^{-9}	46.7	13.3
1/2	0	2	0.5	6.0×10^{-13}	2.6×10^{-6}	5.6×10^{-7}	889	32.2
$\sqrt{2}/2$	0	2	0.45	1.0×10^{-15}	1.1×10^{-7}	7.6×10^{-7}	1330	36.4
$\sqrt{2}$	0	1	0.33	6.2×10^{-11}	9.3×10^{-6}	1.8×10^{-9}	19.7	12.2
2	0	1	0.3	8.0×10^{-12}	3.6×10^{-6}	9.1×10^{-11}	14.1	7.9
1/4	1/16	2	0.62	7.6×10^{-11}	2.6×10^{-5}	1.7×10^{-9}	62.1	15.2
$\sqrt{2}/4$	1/16	2	0.57	9.0×10^{-13}	2.8×10^{-6}	1.6×10^{-9}	46.8	13.3
1/2	1/16	2	0.5	6.6×10^{-13}	2.8×10^{-6}	5.0×10^{-7}	962	34.1
$\sqrt{2}/2$	1/16	2	0.45	1.6×10^{-15}	1.4×10^{-7}	8.7×10^{-7}	1630	43.7
$\sqrt{2}$	1/16	1	0.33	9.9×10^{-11}	1.1×10^{-5}	2.0×10^{-9}	20.5	12.5
2	1/16	1	0.3	1.0×10^{-11}	4.1×10^{-6}	7.8×10^{-11}	15.1	8.4
1/4	4	2	0.62	2.2×10^{-10}	3.8×10^{-5}	3.7×10^{-9}	62.8	15.4
$\sqrt{2}/4$	4	2	0.57	4.3×10^{-11}	1.4×10^{-5}	5.5×10^{-9}	50.6	13.4
1/2	4	2	0.5	2.9×10^{-12}	5.7×10^{-6}	3.6×10^{-7}	2000	61.8
$\sqrt{2}/2$	4	2	0.44	9.6×10^{-15}	3.4×10^{-7}	2.0×10^{-6}	9370	236
$\sqrt{2}$	4	1	0.31	9.6×10^{-10}	3.1×10^{-5}	5.3×10^{-9}	60.8	31.1
2	4	1	0.25	8.8×10^{-11}	1.2×10^{-5}	5.6×10^{-8}	1160	69.0

TABLE 4: Computational parameters, and accuracy of the LSBSM numerics for transverse modes in viscous flow at various frequencies and separations: optimal radius R_* of auxiliary boundary; Simpson's-rule approximation $\bar{\mathcal{E}}$ of the square-integral boundary criterion $\mathcal{E}$, Eq. (4.11); maximum bondary residual $|\Delta V|_{\max}$; and largest coefficients $F_{\max} = \max\{|G_n^{(z)}|, |G_n^{(\rho)}|\}, |G_n^{(\phi)}|\}$; $Q_{\max} = \max\{|G_n^{(q)}|\}$.

L	Ω	PS	$R_*^\perp$	$\bar{\mathcal{E}}^\perp$	$\|\Delta V\|_{\max}$	$F_{\max}$	$Q_{\max}$
1.0	0	1	0.49	3.7×10^{-9}	4.4×10^{-5}	9.6	3.3
1.0	1/2048	1	0.49	3.8×10^{-9}	4.5×10^{-5}	9.7	3.4
1.0	1/1024	1	0.49	3.8×10^{-9}	4.5×10^{-5}	9.8	3.4
1.0	1/512	1	0.49	3.9×10^{-9}	4.6×10^{-5}	9.8	3.4
1.0	1/256	1	0.49	4.0×10^{-9}	4.7×10^{-5}	9.9	3.4
1.0	1/128	1	0.49	4.1×10^{-9}	4.8×10^{-5}	10.0	3.4
1.0	1/64	1	0.49	4.4×10^{-9}	5.0×10^{-5}	10.1	3.5
1.0	1/32	1	0.49	4.8×10^{-9}	5.3×10^{-5}	10.4	3.6
1.0	1/16	1	0.50	5.5×10^{-9}	5.7×10^{-5}	10.7	3.6
1.0	1/8	1	0.50	7.0×10^{-9}	6.8×10^{-5}	11.3	3.8
1.0	1/4	1	0.51	1.0×10^{-8}	8.6×10^{-5}	12.3	4.1
1.0	1/2	1	0.52	1.7×10^{-8}	1.2×10^{-4}	14.0	4.5
1.0	1	1	0.52	3.3×10^{-8}	1.8×10^{-4}	16.7	5.4
1.0	2	1	0.52	6.9×10^{-8}	2.5×10^{-4}	21.2	6.7
1.0	4	1	0.51	1.6×10^{-7}	3.3×10^{-4}	29.0	9.2
1.0	8	1	0.51	4.5×10^{-7}	4.9×10^{-4}	42.8	12.8
1.0	16	1	0.51	1.4×10^{-6}	9.3×10^{-4}	70.4	18.9
1.0	32	1	0.47	2.4×10^{-6}	1.2×10^{-3}	185	35.5
1.0	64	1	0.33	9.3×10^{-7}	7.8×10^{-4}	1270	133
1.0	128	1	0.33	7.0×10^{-7}	6.4×10^{-4}	6190	400

TABLE 4 (continued).

L	Ω	PS	$R_*^\perp$	$\bar{\mathcal{E}}^\perp$	$\lvert\Delta V\rvert_{\max}$	$F_{\max}$	$Q_{\max}$
1/4	0	2	0.63	4.3×10^{-11}	1.1×10^{-5}	6.5	1.4
$\sqrt{2}/4$	0	2	0.57	3.6×10^{-12}	2.2×10^{-6}	9.2	1.9
1/2	0	2	0.55	2.6×10^{-13}	7.9×10^{-7}	8.4	1.6
$\sqrt{2}/2$	0	2	0.56	7.6×10^{-15}	1.4×10^{-7}	5.4	1.4
$\sqrt{2}$	0	1	0.46	4.8×10^{-10}	2.2×10^{-5}	9.3	3.6
2	0	1	0.4	7.4×10^{-11}	8.1×10^{-6}	8.9	4.2
1/4	1/16	2	0.64	5.9×10^{-11}	9.5×10^{-6}	6.3	1.3
$\sqrt{2}/4$	1/16	2	0.57	6.3×10^{-12}	2.7×10^{-6}	10.6	2.0
1/2	1/16	2	0.55	3.8×10^{-13}	8.8×10^{-7}	9.1	1.8
$\sqrt{2}/2$	1/16	2	0.55	8.2×10^{-15}	1.6×10^{-7}	6.0	1.6
$\sqrt{2}$	1/16	1	0.47	6.4×10^{-10}	2.4×10^{-5}	10.4	4.0
2	1/16	1	0.4	9.1×10^{-11}	9.1×10^{-6}	10.2	4.9
1/4	4	2	0.66	3.0×10^{-9}	8.1×10^{-5}	13.5	2.7
$\sqrt{2}/4$	4	2	0.60	3.9×10^{-10}	2.9×10^{-5}	22.1	3.3
1/2	4	2	0.55	1.3×10^{-11}	4.6×10^{-6}	40.7	4.8
$\sqrt{2}/2$	4	2	0.53	3.0×10^{-13}	5.9×10^{-7}	27.1	4.7
$\sqrt{2}$	4	1	0.47	1.5×10^{-8}	1.0×10^{-4}	29.2	10.5
2	4	1	0.4	1.9×10^{-9}	3.6×10^{-5}	29.1	13.2

TABLE 5: Computational parameters, and accuracy of the LSBSM numerics for transverse modes in potential flow: optimal radius R_* of auxiliary boundary; Simpson's-rule aproximation $\bar{\mathcal{E}}$ of the square-integral boundary criterion $\mathcal{E}$, Eq. (4.18); maximum boundary residual $|\Delta V_n|_{\max}$; and largest coefficient $Q_{\max} = \max\{|G_n^{(q)}|\}$.

L	PS	$R_*^\perp$	$\bar{\mathcal{E}}^\perp$	$\|\Delta V_n\|_{\max}$	$Q_{\max}$
$1/16$	2	0.78	3.6×10^{-5}	1.6×10^{-2}	2.4
$\sqrt{2}/16$	2	0.76	7.8×10^{-6}	7.8×10^{-3}	2.4
$1/8$	2	0.74	1.1×10^{-6}	3.1×10^{-3}	2.5
$\sqrt{2}/8$	2	0.72	1.0×10^{-7}	9.4×10^{-4}	2.6
$1/4$	2	0.67	8.3×10^{-9}	2.5×10^{-4}	2.8
$\sqrt{2}/4$	2	0.61	5.5×10^{-10}	5.4×10^{-5}	3.0
$1/2$	2	0.55	1.6×10^{-11}	7.8×10^{-6}	3.4
$\sqrt{2}/2$	2	0.49	1.6×10^{-13}	6.9×10^{-7}	3.8
1	1	0.47	1.0×10^{-7}	4.4×10^{-4}	7.9
$\sqrt{2}$	1	0.41	6.5×10^{-9}	9.9×10^{-5}	9.0
2	1	0.34	2.7×10^{-10}	1.8×10^{-5}	10.9
$2\sqrt{2}$	1	0.27	6.6×10^{-12}	2.4×10^{-6}	13.7
4	1	0.21	9.5×10^{-14}	2.5×10^{-7}	17.7
$4\sqrt{2}$	1	0.15	8.1×10^{-16}	2.3×10^{-8}	24.7
8	1	0.11	4.1×10^{-18}	1.5×10^{-8}	33.7

New tables 6, 7 ∈ app.

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TABLE 6: Comparison of quasistatic Stokes friction coefficients computed with (i) the present singularity method, "LSBSM," vs (ii) the boundary-multipole collocation technique, "BMC," of Kim and Mifflin (1985).

L	$\zeta_a^{\parallel}/(6\pi)$ LSBSM	$\zeta_a^{\parallel}/(6\pi)$ BMC	$\zeta_b^{\parallel}/(6\pi)$ LSBSM	$\zeta_b^{\parallel}/(6\pi)$ BMC
0.2	2.64105	2.6410482	-1.98449	-1.9844938
0.5	1.7226671	1.7226671	-1.0497516	-1.0497516
1.0	1.3684796	1.3684796	-0.6701750	-0.67017504
2.0	1.1694701	1.1694701	-0.4272118	-0.42721186

L	$\zeta_a^{\perp}/(6\pi)$ LSBSM	$\zeta_a^{\perp}/(6\pi)$ BMC	$\zeta_b^{\perp}/(6\pi)$ LSBSM	$\zeta_b^{\perp}/(6\pi)$ BMC
0.2	1.2880376	1.2880376	-0.5467510	-0.54675095
0.5	1.1648647	1.1648647	-0.4010789	-0.40107891
1.0	1.0921017	1.0921018	-0.2971758	-0.29717586
2.0	1.0433030	1.0433030	-0.2044768	-0.20447686

TABLE 7: Drag coefficients for Brinkman flow at $L = 1$ as computed with the present singularity method, "LSBSM," are compared with the results obtained by Kim and Russel (1985), "KR," using the method of reflections and the boundary-multipole collocation technique.

	X_F		Y_F	
γ	LSBSM	KR	LSBSM	KR
0.1	0.72437893	0.72437893	0.8386335	0.83863349
1.0	0.85529790	0.855297887	1.019740	1.01973959
10.0	0.9541943	0.9541943384	1.025741	1.025741179

TABLE 8: Drift-force coefficients $B_i^{\parallel}(0,1)$ and $B_i^{\perp}(0,1)$ computed with three different integration meshes that are described in §4.8. Note that the second and third columns provide a check of consistency with Eq. (3.45).

Mesh 1 $\leftarrow$ Regular mesh

Mesh 2 $\leftarrow$ Far-field mesh, $\Omega = 1/256$

Mesh 3 $\leftarrow$ Boundary-layer mesh, $\Omega = 128$

Mesh	$B_1^{\parallel}$	$B_2^{\parallel}$	$B_1^{\perp}$	$B_2^{\perp}$
1	-1.2248	-1.2249	1.2993	0.1437
2	-1.2218	-1.2218	1.2998	0.1442
3	-1.2272	-1.2275	1.3001	0.1446

TABLE 9: Drift-force coefficients for potential flow $\hat{B}_i^{\|}$, $\hat{B}_i^{\perp}$ calculated for $L = 1$ in two different ways: (i) from the virtual mass coefficients; and (ii) by integrating over the fluid domain. The integration meshes are defined in Table 8.

Method	$\hat{B}_1^{\|}$	$\hat{B}_2^{\|}$	$\hat{B}_1^{\perp}$	$\hat{B}_2^{\perp}$
Virtual mass	0.027798	-0.235147	0.007915	0.116759
Mesh 1	0.027799	-0.235147	0.007914	0.116759
Mesh 2	0.027797	-0.235147	0.007915	0.116759
Mesh 3	0.027802	-0.235147	0.007914	0.116759

TABLE 10: Consistency of the Stokes friction coefficients $\zeta_a^{\{0\},\|}$, $\zeta_b^{\{0\},\|}$, $\zeta_a^{\{0\},\perp}$, $\zeta_b^{\{0\},\perp}$ with numerically evaluated surface integrals involving the Stokes velocity fields $\boldsymbol{u}^{[1]}(\boldsymbol{r})$, $\boldsymbol{u}^{[2]}(\boldsymbol{r})$. Here we calculate

$$\left[\frac{d}{ds}\Delta F_z^{\mathrm{rel}}\right]^{\|}, \qquad \left[\frac{d}{ds}\Delta F_z^{\mathrm{rel}}\right]^{\perp}$$

at $L = 1$ by three different methods, as given by Eqs. (6.21) and (6.22).

Equation	Calculation method	$\left[\frac{d}{ds}\Delta F_z^{\mathrm{rel}}\right]^{\|}$	$\left[\frac{d}{ds}\Delta F_z^{\mathrm{rel}}\right]^{\perp}$
Eq. (6.21)	1st surface integral	-61.0645	16.3215
Eq. (6.21)	2nd surface integral	-61.0645	16.3201
Eq. (6.22)	Friction coefficients	-61.0646	16.3206

TABLE 11: Frequency-dependent friction coefficients for $L = 1$.

Ω	$\Re\{\zeta_a^\parallel\}$	$\Im\{\zeta_a^\parallel\}$	$\Re\{\zeta_b^\parallel\}$	$\Im\{\zeta_b^\parallel\}$	$\Re\{\zeta_a^\perp\}$	$\Im\{\zeta_a^\perp\}$	$\Re\{\zeta_b^\perp\}$	$\Im\{\zeta_b^\perp\}$
1/2048	25.939	0.152	-12.489	0.137	20.772	0.195	-5.416	0.179
1/1024	25.999	0.219	-12.430	0.190	20.849	0.280	-5.339	0.250
1/512	26.084	0.318	-12.346	0.216	20.959	0.406	-5.231	0.346
1/256	26.205	0.467	-12.230	0.354	21.116	0.592	-5.079	0.475
1/128	26.379	0.692	-12.067	0.474	21.342	0.871	-4.867	0.645
1/64	26.632	1.039	-11.843	0.620	21.669	1.295	-4.575	0.863
1/32	27.005	1.579	-11.541	0.787	22.150	1.943	-4.176	1.135
1/16	27.556	2.425	-11.146	0.959	22.869	2.938	-3.644	1.462
1/8	28.425	3.750	-10.649	1.105	23.962	4.462	-2.949	1.841
1/4	29.757	5.822	-10.057	1.178	25.637	6.781	-2.060	2.259
1/2	31.824	9.043	-9.396	1.110	28.199	10.278	-0.946	2.687
1	35.011	14.047	-8.724	0.804	32.073	15.514	0.416	3.051
2	39.860	21.854	-8.132	0.125	37.786	23.364	1.972	3.214
4	47.122	34.180	-7.758	-1.119	45.904	35.367	3.462	3.065
8	57.816	54.033	-7.777	-3.629	57.094	54.579	4.420	2.866
16	73.302	86.891	-8.378	-6.670	72.607	86.841	4.696	3.525
32	95.431	143.02	-9.717	-12.420	94.601	142.56	4.932	6.226
64	126.82	241.97	-11.890	-22.451	125.86	241.03	5.874	11.184
128	171.24	421.31	-15.073	-40.819	170.10	419.56	7.487	20.365
256	234.07	753.86	-19.650	-75.444				
64	126.8	242.0	-11.81	-22.42	125.8	241.0	5.886	11.20
128	171.2	421.3	-14.98	-40.74	170.1	419.5	7.468	20.35
256	234.0	753.8	-19.54	-75.30	232.6	750.6	9.745	37.60
512	322.9	1382	-26.05	-141.6	321.1	1376	12.99	70.70
1024	448.6	2586	-35.29	-270.2	446.3	2575	17.60	134.9
2048	626.3	4921	-48.38	-522.0	623.2	4900	24.14	260.7
4096	877.7	9487	-66.92	-1018	873.5	9447	33.39	508.3
8192	1233	18470	-93.15	-1999	1227	18390	46.48	998.2
16384	1736	36230	-130.3	-3945	1728	36080	64.99	1970
32768	2447	71450	-182.7	-7815	2436	71160	91.18	3903

20.960

-3.219

6.076

-75.443

TABLE 12: Drift-force coefficients as functions of frequency for axial modes at $L = 1$. The upper results for $\Omega \leq 256$ are from the LSBSM numerical method, while the lower results for $\Omega \geq 64$ are from the reflection solution.

Ω	$A_1^{\parallel}$	$A_2^{\parallel}$	$B_1^{\parallel}$	$B_2^{\parallel}$	$B_3^{\parallel}$	$B_4^{\parallel}$
1/2048	0.330	5.742	-1.209	-1.209	-1.209	-1.208
1/1024	0.370	5.672	-1.204	-1.204	-1.204	-1.203
1/512	0.427	5.574	-1.198	-1.197	-1.197	-1.196
1/256	0.506	5.437	-1.190	-1.189	-1.187	-1.186
1/128	0.614	5.246	-1.180	-1.177	-1.176	-1.172
1/64	0.759	4.984	-1.170	-1.162	-1.161	-1.152
1/32	0.946	4.633	-1.160	-1.143	-1.144	-1.125
1/16	1.172	4.178	-1.153	-1.117	-1.124	-1.086
1/8	1.420	3.614	-1.154	-1.082	-1.104	-1.033
1/4	1.651	2.959	-1.165	-1.034	-1.082	-0.963
1/2	1.817	2.254	-1.184	-0.968	-1.055	-0.879
1	1.871	1.564	-1.201	-0.883	-1.016	-0.785
2	1.793	0.957	-1.197	-0.779	-0.957	-0.685
4	1.592	0.484	-1.146	-0.658	-0.873	-0.580
8	1.312	0.167	-1.032	-0.531	-0.760	-0.474
16	1.011	-0.006	-0.863	-0.416	-0.627	-0.380
32	0.742	-0.066	-0.676	-0.338	-0.495	-0.316
64	0.531	-0.068	-0.505	-0.296	-0.378	-0.285
128	0.378	-0.054	-0.364	-0.274	-0.282	-0.269
256	0.267	-0.040	-0.254	-0.260	-0.204	-0.258
64	0.529	-0.064	-0.502	-0.295	-0.376	-0.285
128	0.375	-0.051	-0.362	-0.272	-0.281	-0.268
256	0.265	-0.038	-0.253	-0.259	-0.203	-0.257
512	0.188	-0.028	-0.171	-0.250	-0.142	-0.250
1024	0.133	-0.020	-0.112	-0.245	-0.096	-0.245
2048	0.094	-0.014	-0.070	-0.241	-0.061	-0.241
4096	0.066	-0.010	-0.040	-0.239	-0.036	-0.239
8192	0.047	-0.007	-0.020	-0.237	-0.017	-0.237
16384	0.033	-0.005	-0.005	-0.236	-0.004	-0.236
32768	0.023	-0.004	0.005	-0.235	0.005	-0.235

TABLE 13: Drift-force coefficients as functions of frequency for transverse modes at $L = 1$. The upper results for $\Omega \leq 128$ are from the LSBSM numerical method, while the lower results for $\Omega \geq 64$ are from the reflection solution.

Ω	$A_1^\perp$	$A_2^\perp$	$B_1^\perp$	$B_2^\perp$	$B_3^\perp$	$B_4^\perp$
1/2048	-2.420	-1.208	1.290	0.135	1.290	0.135
1/1024	-2.414	-1.198	1.287	0.131	1.287	0.131
1/512	-2.408	-1.186	1.282	0.127	1.282	0.127
1/256	-2.400	-1.169	1.275	0.121	1.274	0.121
1/128	-2.390	-1.147	1.266	0.114	1.264	0.113
1/64	-2.374	-1.117	1.253	0.106	1.250	0.104
1/32	-2.352	-1.076	1.235	0.100	1.228	0.094
1/16	-2.315	-1.020	1.211	0.098	1.196	0.085
1/8	-2.255	-0.947	1.180	0.105	1.150	0.078
1/4	-2.156	-0.850	1.141	0.126	1.084	0.074
1/2	-1.998	-0.726	1.093	0.162	0.995	0.071
1	-1.764	-0.571	1.032	0.206	0.879	0.068
2	-1.454	-0.391	0.946	0.242	0.740	0.070
4	-1.104	-0.211	0.824	0.254	0.598	0.094
8	-0.782	-0.076	0.672	0.240	0.481	0.152
16	-0.541	-0.008	0.522	0.217	0.382	0.202
32	-0.377	0.011	0.397	0.193	0.295	0.198
64	-0.266	0.013	0.297	0.172	0.227	0.166
128	-0.187	0.010	0.219	0.156	0.174	0.148
64	-0.266	0.011	0.296	0.172	0.227	0.166
128	-0.188	0.009	0.217	0.155	0.174	0.148
256	-0.133	0.007	0.158	0.143	0.131	0.139
512	-0.094	0.005	0.114	0.135	0.098	0.132
1024	-0.066	0.004	0.082	0.129	0.073	0.128
2048	-0.047	0.003	0.059	0.125	0.055	0.124
4096	-0.033	0.002	0.044	0.123	0.041	0.122
8192	-0.023	0.001	0.033	0.121	0.031	0.120
16384	-0.016	0.001	0.025	0.119	0.024	0.119
32768	-0.012	0.001	0.020	0.118	0.019	0.118

TABLE 14: Drift-force coefficients as functions of separation for axial modes at $\Omega = 1/16$ and $\Omega = 4$.

$\Omega = 1/16$:

L	$A_1^{\parallel}$	$A_2^{\parallel}$	$B_1^{\parallel}$	$B_2^{\parallel}$	$B_3^{\parallel}$	$B_4^{\parallel}$
$1/4$	1.730	3.569	-1.169	-1.142	-1.151	-1.122
$\sqrt{2}/4$	1.542	3.770	-1.169	-1.141	-1.149	-1.119
$1/2$	1.385	3.943	-1.168	-1.138	-1.146	-1.114
$\sqrt{2}/2$	1.261	4.082	-1.164	-1.131	-1.139	-1.105
1	1.172	4.178	-1.153	-1.117	-1.124	-1.086
$\sqrt{2}$	1.115	4.213	-1.128	-1.086	-1.093	-1.049
2	1.078	4.158	-1.075	-1.026	-1.031	-0.980

$\Omega = 4$:

L	$A_1^{\parallel}$	$A_2^{\parallel}$	$B_1^{\parallel}$	$B_2^{\parallel}$	$B_3^{\parallel}$	$B_4^{\parallel}$
$1/4$	2.646	0.251	-1.372	-0.895	-1.079	-0.833
$\sqrt{2}/4$	2.393	0.395	-1.351	-0.868	-1.059	-0.803
$1/2$	2.141	0.490	-1.315	-0.827	-1.025	-0.757
$\sqrt{2}/2$	1.880	0.524	-1.254	-0.762	-0.969	-0.688
1	1.592	0.484	-1.146	-0.658	-0.873	-0.580
$\sqrt{2}$	1.262	0.365	-0.970	-0.501	-0.716	-0.420
2	0.904	0.202	-0.731	-0.310	-0.510	-0.231

TABLE 15: Drift-force coefficients as functions of separation for transverse modes at $\Omega = 1/16$ and $\Omega = 4$.

$\Omega = 1/16$:

L	$A_1^\perp$	$A_2^\perp$	$B_1^\perp$	$B_2^\perp$	$B_3^\perp$	$B_4^\perp$
1/4	-3.379	0.065	1.596	-0.161	1.589	-0.166
$\sqrt{2}/4$	-3.166	-0.161	1.525	-0.098	1.517	-0.105
1/2	-2.915	-0.425	1.439	-0.031	1.429	-0.038
$\sqrt{2}/2$	-2.628	-0.718	1.335	0.037	1.323	0.028
1	-2.315	-1.020	1.211	0.098	1.196	0.085
$\sqrt{2}$	-1.983	-1.298	1.066	0.143	1.045	0.124
2	-1.649	-1.512	0.902	0.164	0.871	0.136

$\Omega = 4$:

L	$A_1^\perp$	$A_2^\perp$	$B_1^\perp$	$B_2^\perp$	$B_3^\perp$	$B_4^\perp$
1/4	-2.247	-0.044	1.452	0.192	1.274	0.030
$\sqrt{2}/4$	-2.042	-0.136	1.352	0.238	1.160	0.066
1/2	-1.785	-0.210	1.219	0.271	1.007	0.087
$\sqrt{2}/2$	-1.467	-0.240	1.042	0.280	0.810	0.089
1	-1.104	-0.211	0.824	0.254	0.598	0.094
$\sqrt{2}$	-0.754	-0.143	0.595	0.201	0.413	0.120
2	-0.482	-0.074	0.398	0.135	0.265	0.126

TABLE 16: Inertial drift forces $\langle \Delta \mathcal{F} \rangle^*_{(\mathcal{I})}$ and $\langle \Delta \mathcal{F} \rangle^*_{(\mathcal{M})}$ as functions of particle/fluid density ratio λ at $L = 1$.

λ	$\langle \Delta \mathcal{F} \rangle^*_{(\mathcal{I})}$	$\langle \Delta \mathcal{F} \rangle^*_{(\mathcal{M})}$
0	0.26	0.0022
1/2	0.24	0.0057
1	0.22	0.0049
2	0.20	0.0034
5	0.14	0.0017