FM 8641

Non-axisymmetric motion of rigid closely-fitting particles in fluid-filled tubes

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Appendix: (Not intended for inclusion in published paper)
Evaluation of the resistance matrix for almost uniform gaps

Recall that the resistance matrix R_{ij} is given by

$$R_{ij} = 12 \int_S\int_{S^*} G(z,\theta;\ z^*,\theta^*)\ W_i(z,\theta)\ W_j(z^*,\theta^*)\ dS\ dS^* - \delta_{ij}\ W'_i \tag{A1}$$

where δ_{ij} denotes the Kronecker delta and W_i are given in Table 1 (with ξ_m replaced by $\delta\xi_{m1}$). We expand ξ_{m1} and h_1 as Fourier series:

$$\xi_{m1} = \frac{1}{2} \sum_{p=-\infty}^{\infty} X_p(z)\ e^{-ip\theta} \quad \text{and} \quad h_1 = \frac{1}{2} \sum_{p=-\infty}^{\infty} H_p(z)\ e^{-ip\theta} \tag{A2}$$

where X_p and H_p are complex for $p \neq 0$, with $X_{-p} = \overline{X}_p$ and $H_{-p} = \overline{H}_p$.

We now expand the resistance matrix in powers of δ, using the expansion of the Green's function in section 5. At leading order, we find:

$$R^{(0)}_{11} = R^{(0)}_{22} = -\kappa_1,\ \ R^{(0)}_{44} = R^{(0)}_{55} = -\kappa_2,\ \ R^{(0)}_{33} = R^{(0)}_{66} = -4\pi\ell_0 \tag{A3}$$

where
$$\kappa_1 = -12\pi^2 \iint g_1(z,z^*)\ dz\ dz^* = 24\pi\ [\ell_0 - \tanh\ell_0]$$

and
$$\kappa_2 = -12\pi^2 \iint g_1(z,z^*)\ z\ z^*\ dz\ dz^* = 24\pi\ell_0\ [\ell_0{}^2/3 - \ell_0\coth\ell_0 + 1] \tag{A4}$$

where the integrals here and throughout the Appendix are over $[-\ell_0,\ell_0]$. All other elements of $\mathbf{R}^{(0)}$ vanish. At leading order, $\mathbf{R}^{(0)}$ is diagonal.

At the next order (δ^1), we find contributions from the W'_i terms:

$$R^{(1)}_{33} = R^{(1)}_{66} = \pi \int H_0\ dz, \quad R^{(1)}_{36} = 0; \tag{A5}$$

contributions involving G_0:

$$R^{(1)}_{13} + iR^{(1)}_{23} = \int J_1\ dX_1/dz\ dz, \quad R^{(1)}_{16} + iR^{(1)}_{26} = -\ i\int J_1\ X_1\ dz,$$
$$R^{(1)}_{34} + iR^{(1)}_{35} = i\int J_2\ dX_1/dz\ dz, \quad R^{(1)}_{46} + iR^{(1)}_{56} = \int J_2\ X_1\ dz \tag{A6}$$

where
$$J_1(z) = -12\pi^2 \int g_1(z,z^*)\ dz^* = 12\pi\ [1 - \cosh z/\cosh\ell_0],$$
$$J_2(z) = -12\pi^2 \int g_1(z,z^*)\ z^*\ dz^* = 12\pi\ [z - \ell_0\ \sinh z/\sinh\ell_0]; \tag{A7}$$

and contributions involving G_1:

$$R^{(1)}_{11} + iR^{(1)}_{12} = \int (M_1\ H_0 + M_{-1}\ H_2)\ dz, \quad R^{(1)}_{22} - iR^{(1)}_{12} = \int (M_1\ H_0 - M_{-1}\ H_2)\ dz,$$
$$R^{(1)}_{15} + iR^{(1)}_{25} = \int (M_2\ H_0 + M_{-2}\ H_2)\ dz, \quad R^{(1)}_{24} - iR^{(1)}_{14} = \int (-M_2\ H_0 + M_{-2}\ H_2)\ dz,$$
$$R^{(1)}_{44} + iR^{(1)}_{45} = \int (M_3\ H_0 - M_{-3}\ H_2)\ dz, \quad R^{(1)}_{55} - iR^{(1)}_{45} = \int (M_3\ H_0 + M_{-3}\ H_2)\ dz, \tag{A8}$$

where
$$M_{\pm 1}(z) = 18\pi^2 \iint \Gamma_{1\mp 1}\ (z',z,z^*)\ dz'\ dz^* = [(dJ_1/dz)^2 \pm J_1{}^2]/8\pi,$$

$$M_{\pm 2}(z) = 18\pi^2 \iint \Gamma_{1\mp 1}\ (z', z, z^*)\ z'\ dz'\ dz^* = [dJ_1/dz\ dJ_2/dz \pm J_1\ J_2]/8\pi,$$

$$M_{\pm 3}(z) = 18\pi^2 \iint \Gamma_{1\mp 1}\ (z', z, z^*)\ z'\ z^*\ dz'\ dz^* = [(dJ_2/dz)^2 \pm J_2{}^2]/8\pi \qquad \text{(A9)}$$

Several conclusions may be drawn concerning the $O(\delta)$ expansion of the resistance matrix:

(i) $R^{(1)}$ is independent of third and higher Fourier components of the particle shape.

(ii) The transverse forces and torques (F_x, F_y, T_x, T_y) resulting from axial motion of the particle (V_z) depend only on the first Fourier coefficient of ξ_{m1}. If the particle is axisymmetric, these forces and torques depend on the position of the particle but not on its shape.

(iii) From (A8), the transverse forces and torques resulting from transverse particle motion (V_x, V_y, Ω_x, Ω_y) depend only on the zeroth and second Fourier coefficients of the gap width h_1. These forces and torques depend on the shape of the particle, but, from (4.2), are independent of its position.

From the resistance matrix, we may compute the motion of a non-axisymmetric particle driven by an axial pressure difference, for which

$$X_1 = [X_{p1} - (a_1{}' + \alpha_1{}'z) - i(b_1{}' + \beta_1{}'z)]/2 \qquad \text{(A10)}$$

where $X_{p1}(z)$ is the first complex Fourier coefficient of the particle shape ξ_{p1}, and $a' = \delta a_1{}'$, etc. At first order in δ, (6.8) and (A6) give

$$\frac{d}{dt}(a_1{}' + ib_1{}') = \frac{1}{2\kappa_1}\int_{-\ell_0}^{\ell_0} K_1(z)\ \frac{dX_{p1}}{dz}\, dz + \frac{\alpha_1{}' + i\beta_1{}'}{2},$$

$$\frac{d}{dt}(\alpha_1{}' + i\beta_1{}') = \frac{1}{2\kappa_2}\int_{-\ell_0}^{\ell_0} K_2(z)\ \frac{dX_{p1}}{dz}\, dz \qquad \text{(A11)}$$

In this approximation, the particle rotates with a constant angular velocity, and moves transversely with a velocity that depends linearly on its angle to the tube axis, giving a parabolic trajectory. If the particle is axisymmetric ($X_{p1} = 0$), it moves in a straight line along the bisector of the particle and tube axes, independent of particle shape. The motion is neutrally stable with regard to exponential solutions. In general, the particle would eventually collide with the wall, in the absence of higher order effects. These findings indicate the need to pursue the expansion to $O(\delta^2)$. In particular, we are interested in the effects of a particle's shape on its trajectory [cf. (ii) above].

We consider the case of an axisymmetric particle driven by an axial pressure difference. We assume that the particle has radius $r_0[1 - \epsilon + \epsilon\delta s(z)]$ where $s(z)$ is a prescribed function describing the particle shape, with zero mean. Then

$$\xi_{p1} = s(z) \ \text{ and } \ \xi_{w1} = [a_1 + \alpha_1(z - c)]\cos\theta + [b_1 + \beta_1(z - c)]\sin\theta \qquad \text{(A12)}$$

The only non-zero Fourier components of h_1 and ξ_{m1} are then

$$H_{\pm 1}(z) = [a_1 + \alpha_1(z - c)] \pm i\,[b_1 + \beta_1(z - c)] = 2X_{\pm 1}(z)$$

$$H_0(z) = -2s(z) = -2X_0(z) \qquad \text{(A13)}$$

The components of $\mathbf{R}^{(1)}$ take simpler forms:

$$R^{(1)}_{13} = \alpha_1\kappa_1/2,\ \ R^{(1)}_{23} = \beta_1\kappa_1/2,\ \ R^{(1)}_{46} = \alpha_1\kappa_2/2,\ \ R^{(1)}_{56} = \beta_1\kappa_2/2$$
$$R^{(1)}_{16} = (b_1 - \beta_1 c)\kappa_1/2,\ \ R^{(1)}_{26} = -\ (a_1 - \alpha_1 c)\kappa_1/2$$
$$R^{(1)}_{33} = R^{(1)}_{66} = R^{(1)}_{34} = R^{(1)}_{35} = R^{(1)}_{36} = 0 \tag{A14}$$
$$R^{(1)}_{11} = R^{(1)}_{22} = -2\nu_1, \quad R^{(1)}_{15} = -R^{(1)}_{24} = -2\nu_2,$$
$$R^{(1)}_{44} = R^{(1)}_{55} = -2\nu_3, \quad R^{(1)}_{12} = R^{(1)}_{14} = R^{(1)}_{25} = R^{(1)}_{45} = 0 \tag{A15}$$

where
$$\nu_i = \int M_i(z)\ s(z)\ dz,\ \ i=1,2,3 \tag{A16}$$

At second order in δ, only the components $R^{(2)}_{i3}$ are required in order to compute up to $O(\delta^2)$ the motion of particle driven by an axial pressure difference. Using (A1) and (5.10), we find that

$$R^{(2)}_{13} = -\lambda_1\kappa_1(a_1 - \alpha_1 c) - (\nu_1 + \lambda_3\kappa_1)\alpha_1,\ \ R^{(2)}_{23} = -\lambda_1\kappa_1(b_1 - \beta_1 c) - (\nu_1 + \lambda_3\kappa_1)\beta_1$$
$$R^{(2)}_{34} = \lambda_2\kappa_2(b_1 - \beta_1 c) + (\nu_2 + \lambda_4\kappa_2)\beta_1,\ \ R^{(2)}_{35} = -\lambda_2\kappa_2(a_1 - \alpha_1 c) - (\nu_2 + \lambda_4\kappa_2)\alpha_1$$
$$R^{(2)}_{36} = -\kappa_1(\alpha_1 b_1 - \beta_1 a_1)/4 \tag{A17}$$

where
$$\lambda_i = 1/\kappa_i \int K_i(z^*)\ ds/dz^*\ dz^*,\ \ i = 1,2,3,4$$

with $\kappa_3 = \kappa_1$, $\kappa_4 = \kappa_2$, and

$$K_1(z^*) = 18\pi^3 \iint \Gamma_{10}(z,z',z^*)\ dz\ dz' = 18\pi\ [1 - \cosh z^*/\cosh\ell_0]\ ,$$
$$K_2(z^*) = 18\pi^3 \iint \Gamma_{10}(z,z',z^*)\ z\ dz\ dz' = 18\pi\ [z^* - \ell_0 \sinh z^*/\sinh\ell_0]\ ,$$
$$K_3(z^*) = 18\pi^3 \iint \Gamma_{10}(z,z',z^*)\ z'\ dz\ dz' = z^*K_1(z^*) - \tanh\ell_0\ K_2(z^*)/\ell_0\ ,$$
$$K_4(z^*) = 18\pi^3 \iint \Gamma_{10}(z,z',z^*)\ z\ z'\ dz\ dz'$$
$$= z^*K_2(z^*) - \ell_0 K_1(z^*)/\tanh\ell_0 + 9\pi(\ell_0{}^2 - z^{*2})\ . \tag{A18}$$