Appendix C. Conservation laws

Equation of change for the singlet distribution function $f_1=f(\mathbf{x}, \mathbf{v}_1, \omega_1, t)$ in the absence of the external forces can be written in the form

$$\frac{\partial f_1}{\partial t}+\mathbf{v}_1\cdot\frac{\partial f_1}{\partial \mathbf{x}}=\frac{\partial_c f}{\partial t}, \tag{C.1}$$

where $\partial_c f_1/\partial t$ denotes the rate of change of f_1 due to particle collisions. In particular, this rate is equal to the difference between the gross rates at which collisions increase and diminish the number of particles within the space element $d\mathbf{x}d\mathbf{v}_1d\omega_1$ in the vicinity of the prescribed dynamic state. We can relate the former of these rates with the inverse collisions (A.8), while the latter rate is connected with the direct collisions (A.7).

Define the pair distribution function $f^{(2)}=f^{(2)}(\mathbf{x}_1, \mathbf{v}_1, \omega_1; \mathbf{x}_2, \mathbf{v}_2, \omega_2, t)$ in such a way, that $f^{(2)}d\mathbf{v}_1d\omega_1d\mathbf{x}_1d\mathbf{v}_2d\omega_2d\mathbf{x}_2$ is the number of pairs of particles, located at a time t in the volume elements $d\mathbf{x}_1$ and $d\mathbf{x}_2$, centered at $\mathbf{x}_1$ and $\mathbf{x}_2$ with velocities lying in the volume elements ($d\mathbf{v}_1$, $d\omega_1$) and ($d\mathbf{v}_2$, $d\omega_2$), centered at ($\mathbf{v}_1$, ω_1) and ($\mathbf{v}_2$, ω_2) in the velocity space. Noting, that in the moment of collision the center of the second particle is located in the point $\mathbf{x}_2=\mathbf{x}_1-\mathbf{k}\sigma$ (Fig. A1), and that the "collisional cylinder" volume $d\mathbf{x}_2=\sigma^2(\mathbf{k}\cdot\mathbf{v}_{21})d\mathbf{k}dt$ (Chapman & Cowling, 1970), with $d\mathbf{k}$ being the solid angle about the unit vector $\mathbf{k}$, one can show that the gross particles' diminution rate from the state ($\mathbf{x}$, $\mathbf{v}_1$, ω_1) is

$$\dot{N}_-=\int d^3v_2d^3\omega_2d^2kS(\mathbf{k}\cdot\mathbf{v}_{21})f^{(2)}(\mathbf{x}_1,\mathbf{v}_1,\omega_1;\mathbf{x}_1-\sigma\mathbf{k},\mathbf{v}_2,\omega_2;t). \tag{C.2a}$$

Here $S(\mathbf{k}\cdot\mathbf{v}_{21})=\sigma^2\theta(\mathbf{k}\cdot\mathbf{v}_{21})(\mathbf{k}\cdot\mathbf{v}_{21})$ and θ is the Heaviside function.

In an analogous fashion one can obtain the probability of inverse collisions (A.8)

$$f^{(2)}(\mathbf{x}_1'',\mathbf{v}_1'',\omega_1'';\mathbf{x}_1''-\sigma\mathbf{k}'',\mathbf{v}_2'',\omega_2'';t)d\mathbf{v}_1''d\omega_1''d\mathbf{x}_1d\mathbf{v}_2''d\omega_2''d\mathbf{x}_2'',$$

where $d\mathbf{x}_2''=\sigma^2(\mathbf{k}''\cdot\mathbf{v}_{21}'')d\mathbf{k}''dt$. Similarly, the gross rate of particle gain to the state ($\mathbf{x}_1$, $\mathbf{v}_1$, ω_1) may be obtained from the above expression by integrating it over the effectively infinite domains of variables $\mathbf{v}_1$, ω_1, $\mathbf{k}$:

$$\dot{N}_+=\int d^3v_2d^3\omega_2d^2(-k)f^{(2)}(\mathbf{x}_1'',\mathbf{v}_1'',\omega_1'';\mathbf{x}_1''+\sigma\mathbf{k},\mathbf{v}_2'',\omega_2'';t)S(-\mathbf{k}\cdot\mathbf{v}_{21}'')\frac{\partial(\mathbf{v}_1'',\mathbf{v}_2'',\omega_1'',\omega_2'')}{\partial(\mathbf{v}_1,\mathbf{v}_2,\omega_1,\omega_2)}$$

$$=\int d^3v_2d^3\omega_2d^2kf^{(2)}(\mathbf{x}_1'',\mathbf{v}_1'',\omega_1'';\mathbf{x}_1''+\sigma\mathbf{k},\mathbf{v}_2'',\omega_2'';t)S(\mathbf{k}\cdot\mathbf{v}_{21})\Lambda, \tag{C.2b}$$

$$\Lambda=-\partial(\mathbf{v}_1'',\mathbf{v}_2'',\omega_1'',\omega_2'')/\partial(\mathbf{v}_1,\mathbf{v}_2,\omega_1,\omega_2)\tilde{e}^{-1}. \tag{C.2c}$$

Formula (C.2b) was obtained using the relations $\mathbf{k}''=-\mathbf{k}$ (see Fig. A1) and $S(-\mathbf{k}\cdot\mathbf{v}_{21}'')=$ $S(\mathbf{k}\cdot\mathbf{v}_{21})/\tilde{e}$ [see (A.9)].

From (C.2a,b) it follows that

$$\frac{\partial_c f}{\partial t}=\dot{N}_+-\dot{N}_-=\int d^3v_2 d^3\omega_2 d^2kS(\mathbf{k}\cdot\mathbf{v}_{21})[\Lambda f^{(2)}(\mathbf{x},\tau_1'';\mathbf{x}+\sigma\mathbf{k},\tau_2'';t)$$
$$-f^{(2)}(\mathbf{x},\tau_1;\mathbf{x}-\sigma\mathbf{k},\tau_2;t)]. \quad \text{(C.3)}$$

To derive the conservation laws, multiply (C.1) with r.h.s. given by (C.3) by an arbitrary function $\psi_1=\psi(\mathbf{v}_1,\omega_1)$ and integrate over $\mathbf{v}_1$- and ω_1- domains, to obtain

$$\frac{\partial}{\partial t}<\psi>+\frac{\partial}{\partial \mathbf{x}}\cdot<\mathbf{v}_1\psi>=J_\psi(f,f), \quad \text{(C.4)}$$

where

$$<\psi>=\int d^6\tau_1 f_1\psi_1, \quad \text{(C.5)}$$

$$J_\psi(f,f)=\int d^6\tau_1\frac{\partial_c f}{\partial t}\psi_1=\int d^6\tau_1 d^6\tau_2 d^2kS(\mathbf{k}\cdot\mathbf{v}_{21})\psi_1[\Lambda f^{(2)}(\mathbf{x},\tau_1'';\mathbf{x}+\sigma\mathbf{k},\tau_2'';t)$$
$$-f^{(2)}(\mathbf{x},\tau_1;\mathbf{x}-\sigma\mathbf{k},\tau_2;t)], \quad \text{(C.6)}$$

and $d^6\tau_i=d^3\mathbf{v}_i d^3\omega_i$ is the element of the phase space $\tau_i=(\mathbf{v}_i,\omega_i)$. In the first term appearing in the square brackets of (C.6) interchange double primed and unprimed variables and replace $\mathbf{k}$ with $-\mathbf{k}''$. The resulting expression can be rewritten with the help of the relationship

$$d^2k''d^6\tau_1''d^6\tau_2''S(\mathbf{k}''\cdot\mathbf{v}_{21}'')=d^2kd^6\tau_1 d^6\tau_2 S(\mathbf{k}\cdot\mathbf{v}_{21})\Lambda$$

and recombined with the second term of (C.6), to obtain

$$J_\psi(f,f)=\int d^6\tau_1 d^6\tau_2 d^2kS(\mathbf{k}\cdot\mathbf{v}_{21})(\psi_1'-\psi_1)f^{(2)}(\mathbf{x},\tau_1;\mathbf{x}-\sigma\mathbf{k},\tau_2;t). \quad \text{(C.7a)}$$

After interchanging subscripts 1 and 2 and replacing vectors $\mathbf{v}_{21}$, $\mathbf{k}$ by $-\mathbf{v}_{21}$, $-\mathbf{k}$ one can rewrite (C.7a) in the form

$$J_\psi(f,f)=\int d^6\tau_1 d^6\tau_2 d^2kS(\mathbf{k}\cdot\mathbf{v}_{21})(\psi_2'-\psi_2)f^{(2)}(\mathbf{x},\tau_1;\mathbf{x}+\sigma\mathbf{k},\tau_2;t). \quad \text{(C.7b)}$$

Combining (C.7a,b) and using the identity

$$f^{(2)}(\mathbf{x},\tau_1;\mathbf{x}-\sigma\mathbf{k},\tau_2;t)-f^{(2)}(\mathbf{x}+\sigma\mathbf{k},\tau_1;\mathbf{x},\tau_2;t)=$$

$$\int_0^1\frac{\partial}{\partial\lambda}\left[f^{(2)}(\mathbf{x}+\sigma\mathbf{k}(1-\lambda),\tau_1;\mathbf{x}-\lambda\sigma\mathbf{k},\tau_2;t)\right]d\lambda=$$

$$-\sigma\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{x}}\int_0^1 f^{(2)}(\mathbf{x}+\sigma\mathbf{k}(1-\lambda),\tau_1;\mathbf{x}-\lambda\sigma\mathbf{k},\tau_2;t)d\lambda, \quad \text{(C.8)}$$

one can show that

$$J_\psi(f,f)=-\frac{\partial}{\partial\mathbf{x}}\cdot J^{(c)}(\psi)+I(\psi), \quad \text{(C.9)}$$

where

$$J^{(c)}(\psi) \equiv \frac{\sigma}{4}\int d^6\tau_1 d^6\tau_2 d^2k \int_0^1 d\lambda S(\mathbf{k}\cdot\mathbf{v}_{21})\mathbf{k}\Delta'\psi f^{(2)}[\mathbf{x}+\sigma\mathbf{k}(1-\lambda),\tau_1;\mathbf{x}-\lambda\sigma\mathbf{k},\tau_2;t], \qquad \text{(C.10a)}$$

$$I(\psi) = \frac{1}{2}\int d^6\tau_1 d^6\tau_2 d^2k S(\mathbf{k}\cdot\mathbf{v}_{21})\Delta\psi f^{(2)}(\mathbf{x}+\sigma\mathbf{k},\tau_1;\mathbf{x},\tau_2;t), \qquad \text{(C.10b)}$$

and where $\Delta'\psi = \Delta\psi_1 - \Delta\psi_2$, $\Delta\psi = \Delta\psi_1 + \Delta\psi_2$, $\Delta\psi_i \equiv \psi'_i - \psi_i$.

The first term in r.h.s. of (C.9) may be identified with the collisional transfer contribution to the flux of quantity ψ, whereas the second one is the source (sink) term, which clearly vanishes if ψ is one of the summational invariants. For the collisional model (A.4) the particle mass, m and momentum, $m\mathbf{v}$ are conserved. Replacing ψ in (C.4), (C.10a) by the above properties, we, therefore, obtain

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\mathbf{x}}\cdot(\rho\mathbf{u}) = 0, \qquad \frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}t_{ij}(\mathbf{x},f) = 0. \qquad \text{(C.11a,b)}$$

Using in (C.4), (C.9), (C.10) $\psi = E = mv^2/2 + I\omega^2/2$ we similarly get an equation

$$\frac{\partial}{\partial t}[ne_0 + \frac{\rho u^2}{2}] + \frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{q}(\mathbf{x},f) = I(E), \qquad \text{(C.12)}$$

clearly stating that the total mechanical energy is not a conserved property. In the above, ρ, $\mathbf{u}$ and e_0 are the mean density, linear velocity and total energy of particle random motion $\rho = mn = <m>$, $\rho u = <m\mathbf{v}>$, $ne_0 = <m|\mathbf{v}-\mathbf{u}|^2 + I\omega^2>/2 = ne_{0t} + ne_{0r}$.

It follows from (C.4), (C.9) that the momentum flux $t_{ij}(\mathbf{x},f) = t_{ij}^{(k)} + t_{ij}^{(c)}$ and the energy flux $q_i(\mathbf{x},f) = q_i^{(k)} + q_i^{(c)}$ are each composed of the kinetic part and the collisional - transfer part, labeled respectively with the superscripts k and c. In turn, each part of these fluxes consists of macroscopic (convective) and microscopic (diffusive) contributions. Denote the microscopic contributions to the momentum flux as $\mathbf{P}$ (pressure tensor) and the comparable contributions to the kinetic energy flux as $\mathbf{j}$ (heat diffusion flux):

$$t_{ij}(\mathbf{x},f) = \rho u_i u_j + P_{ij}(\mathbf{x},f), \qquad \text{(C.13a)}$$

$$q_i(\mathbf{x},f) = u_i n(e_0 + \frac{mu^2}{2}) + u_j P_{ij}(x,f) + j_i(\mathbf{x},f). \qquad \text{(C.13b)}$$

The kinetic parts of the pressure tensor and of the heat flux vector are independent of particle collisional properties

$$P_{ij}^{(k)} = m<C_i C_j>, \qquad j_i^{(k)} = \frac{m}{2}<C_i(C^2 + I\omega^2/m)>, \qquad \text{(C.14a,b)}$$

where **C**=**v**-**u** denotes the peculiar velocity of translation. The collisional-transfer parts of these values are determined by $J^{(c)}$ [cf. eq. (C.10a)], with the concomitant replacement of ψ by $m\mathbf{v}$:

$$\mathbf{j}^{(c)}(\mathbf{x},f) = J^{(c)}\left(\frac{mC^2}{2} + I\frac{\omega^2}{2}\right), \quad \mathrm{P}^{(c)}(\mathbf{x},f) = J^{(c)}(m\mathbf{v}), \tag{C.15a,b}$$

where

$$\Delta'(m\mathbf{v}) = 2m\{\eta_2[\mathbf{v}_{21} - \frac{\sigma}{2}\mathbf{k}\times(\omega_1+\omega_2)] + (\eta_1 - \eta_2)\mathbf{k}(\mathbf{v}_{21}\cdot\mathbf{k})\}, \tag{C.15c}$$

$$\Delta'\left(\frac{mC^2}{2} + \frac{I\omega^2}{2}\right) = m\Big\{(\eta_1 - \eta_2)\left[(\mathbf{k}\cdot\mathbf{C}_2)^2 - (\mathbf{k}\cdot\mathbf{C}_1)^2\right] + \eta_2\Big[C_2^2 - C_1^2 -$$

$$\mathbf{k}\cdot(\omega_2\times\mathbf{C}_2 + \omega_1\times\mathbf{C}_1)\Big]\Big\} + m\eta_2\frac{\sigma^2}{4}\left[\omega_2^2 - \omega_1^2 - (\mathbf{k}\cdot\omega_2)^2 + (\mathbf{k}\cdot\omega_1)^2\right], \tag{C.15d}$$

and $\mathbf{C}_i=\mathbf{v}_i-\mathbf{u}$, i=1,2.

Appendix D. Existence and uniqueness of the hydrodynamic solution

In section 3.2 it was shown that the hydrodynamic solution (23) of the Boltzmann - Enskog equation, describing spatially homogeneous state of a system composed of perfectly smooth granules, is fully determined by function $F_1 = F(V_1^2, e)$. This function must be found as a solution of eq. (27), rewritten here in the form

$$-K_e(F,F)\left(\frac{3}{2}F_1 + V_1^2\frac{\partial F_1}{\partial V_1^2}\right) = J(F,F), \tag{D.1}$$

where K_e, J are the following operators:

$$K_e(F,F) = -\frac{\pi(1-e^2)}{16}\int d^3V_1 d^3V_2 F_1 F_2 V_{21}^3, \tag{D.2a}$$

$$J(f,g) \equiv \int d^3v_2 d^2k\,\theta(\mathbf{k}\cdot\mathbf{V}_{21})(\mathbf{k}\cdot\mathbf{V}_{21})[e^{-2}(f_1''g_2'' + f_2''g_1'') - f_1g_2 - f_2g_1]. \tag{D.2b}$$

F_1 is subjected to the normalization conditions

$$\int d^3V_1 F_1 = 1, \quad \int d^3V_1 F_1 V_1^2 = 2. \tag{D.3}$$

We will prove the existence and uniqueness of solution for eqs. (D.1) - (D.2) with conditions (D.3) for the case of slightly inelastic collisions, characterized by $\varepsilon = 1 - e \ll 1$.

Solution F_1 will be constructed by series expansion in terms of small parameter ε

$$F_1 = \sum_{i=0}^{\infty} F_1^{(i)}\varepsilon^i. \tag{D.4}$$

Introducing (D.4) into eq. (D.1) and expanding both sides of the latter equation in series with respect to ε, one obtains the following nonlinear homogeneous integral equation governing the zero-order function $F^{(0)}$

$$J(F^{(0)},F^{(0)})\Big|_{e=1}=0, \qquad \text{(D.5)}$$

and linear nonhomogeneous equations for the higher order functions in (D.4)

$$J(F^{(0)},F^{(i)})\Big|_{e=1}=W^{(i)} \qquad i=1,2,\ldots, \qquad \text{(D.6)}$$

where $W_1^{(i)}= W^{(i)}(V_1, F^{(n)})$, $n = 0, 1, \ldots, i-1$.

Expansion (D.4) and normalization conditions (D.3) may be used to obtain the following subsidiary conditions for $F_1^{(i)}$

$$\int d^3V_1F_1^{(0)}=\frac{1}{2}\int d^3V_1V_1^2F_1^{(0)}=1 \qquad \text{(D.7a)}$$

$$\int d^3V_1F_1^{(i)}=\int d^3V_1V_1^2F_1^{(i)}=0 \quad i=1,2,\ldots \;. \qquad \text{(D.7b)}$$

$J(F,F)\Big|_{e=1}$ is the collisional integral appearing in the kinetic equation, describing the dilute smooth elastic spheres gas (β=-1, e=1). Therefore, (see Chapman & Cowling, 1970), eq. (D.5) subject to conditions (D.7a) possesses a unique solution which is the Maxwell-Boltzmann function, written here in the form

$$F_1^{(0)}=(\pi\alpha_t)^{-3/2}\exp\left(-V_1^2/\alpha_t\right), \qquad \text{(D.8)}$$

where $\alpha_t = 4/3$. It follows, thus, that the expressions $J(F^{(0)},F^{(i)})\Big|_{e=1}$ are identical to the linearized collisional integrals obtaining in the case (β=-1, e=1).

It has been proven in the kinetic theory of dilute gases (see e.g., Chapman & Cowling, 1970) that each of the equations of the type (D.6) subject to conditions (D.7b) possesses a unique solution if and only if $W_1^{(i)}$, appearing in r. h. s. of (B.6) are orthogonal to all summational invariants (in our notation 1, $\mathbf{V}_1$, V_1^2).

Note, that eqs. (27) and (D.1) possess the property, that for any F, multiplication of both parts of this equation by 1, $\mathbf{V}_1$, V_1^2 and integration over the velocity space convert them to identities. Consequently, eq. (D.6), obtained by expanding (D.1) in power series of ε, are characterized by the same property. Bearing in mind that the operator $J(F^{(0)},F^{(i)})\Big|_{e=1}$ is orthogonal to 1, $\mathbf{V}_1$, V_1^2, one can prove that functions $W_1^{(i)}$ appearing in r.h.s. of eq. (D.6) also possess the same orthogonality property. The existence of the integrals

$$\int d^3V_1W_1^{(i)},\ \int d^3V_1\mathbf{V}_1W_1^{(i)},\ \int d^3V_1V_1^2W_1^{(i)}$$

may be proven by showing that (i) all function $F_1^{(i)}$ possess infinite number of derivatives and (ii) integrals of the type $\int d^3V_1 V_1^n F_1^{(i)}$ exist for $n = 0,1,\ldots$. Since $F_1^{(0)}$, given by formula (D.8) satisfies the above properties (i) and (ii), the same is true for $F_1^{(i)}$ (i=1, 2, ...), which may be easily proven by invoking the mathematical induction arguments.

Appendix E. Estimation of the Maxwell-Boltzmann approximation

The goal of this appendix is the estimation of the accuracy of the Maxwell-Boltzmann approximation (D.8) of eq. (27) with normalization conditions (29). Towards this end expand function F_1 in the Sonine polynomials series (30) and restrict ourselves by the following approximation

$$F_1 = F_1^{(0)}\left[a_0 S_{1/2}^{(0)}(\tilde{V}_1^2) + a_1 S_{1/2}^{(1)}(\tilde{V}_1^2) + a_2 S_{1/2}^{(2)}(\tilde{V}_1^2)\right], \tag{E.1}$$

where

$$S_{1/2}^{(0)}(\tilde{V}_1^2) = 1,\ S_{1/2}^{(1)}(\tilde{V}_1^2) = \frac{3}{2} - \tilde{V}_1^2,\ S_{1/2}^{(2)}(\tilde{V}_1^2) = \frac{15}{8} - \frac{5}{2}\tilde{V}_1^2 + \frac{1}{2}\tilde{V}_1^4. \tag{E.2}$$

Upon introducing approximation (E.1) into condition (29) and using normalization conditions (31) imposed on $S_m^{(n)}(x)$, one obtains the first two coefficients :

$$a_0 = 1,\ a_1 = 0. \tag{E.3}$$

The coefficient a_2 in the approximation (E.1) will be found by method moments (see Condiff *et. al.*, 1965) using assumption a_2<<1 together with following moment equation:

$$-K_e(F,F)\int d^3V_1\psi_1\left(\frac{3}{2}F_1 + V_1^2\frac{dF_1}{dV_1^2}\right) = \frac{1}{2}\int d^3V_1 d^3V_2 F_1 F_2 \int d^2k q(\mathbf{k}\cdot\mathbf{V}_{21})(\mathbf{k}\cdot\mathbf{V}_{21})\Delta\psi. \tag{E.4}$$

wherein $K_e(F,F)$ is given by eq. (24), $\Delta\psi = \psi_1 - \psi_2 - \psi_1' - \psi_2'$ and ψ_i is an arbitrary function of V_i (i=1,2). The latter equation follows from eqs. (27), (28) and the properties (C.7a,b) of the collisional integral $\tilde{J}(F,F)$.

Equations (27), (28) were constructed such a way that the choices $\psi_1 = 1$, V_1^2 turn equation (E.4) into identity. Due to the linearity of eq.(E.4) with respect to ψ, the latter is also true for an arbitrary linear combinations of 1 and V_1^2 and, consequently, (see eq. (E.2)) it is true for $S_{1/2}^{(0)}(\tilde{V}_1^2)$, $S_{1/2}^{(1)}(\tilde{V}_1^2)$. Following the main idea of the moment method (see Condiff *et. al.*, 1965) we will choose a_2 in such a way, so as to satisfy eq. (E.4) with F_1 given by the approximation (E.1) by $\psi_1 = S_{1/2}^{(2)}(\tilde{V}_1^2)$, or, bearing in mind the above properties of eq. (E.4), by the function $\psi_1 = \tilde{V}_1^4$. Using eqs. (A.4), (A.5) for the particular case of smooth particles (β=-1), considered now, one obtains:

$$\Delta\tilde{V}^4 = (1+e)^2\left(\mathbf{k}\cdot\tilde{\mathbf{V}}_{21}\right)^2\left[\left(\tilde{\mathbf{V}}_1\cdot\mathbf{k}\right)^2+\left(\tilde{\mathbf{V}}_2\cdot\mathbf{k}\right)^2\right]+2(1+e)\left(\mathbf{k}\cdot\tilde{\mathbf{V}}_{21}\right)\left[\tilde{V}_1^2\left(\tilde{\mathbf{V}}_1\cdot\mathbf{k}\right)^2-\tilde{V}_2^2\left(\tilde{\mathbf{V}}_2\cdot\mathbf{k}\right)^2\right]+$$

$$2\left(\frac{1+e}{2}\right)^2\left(\tilde{V}_1^2+\tilde{V}_2^2\right)\left(\mathbf{k}\cdot\tilde{\mathbf{V}}_{21}\right)^2-\frac{(1+e)^3(3-e)}{8}\left(\mathbf{k}\cdot\tilde{\mathbf{V}}_{21}\right)^4. \tag{E.5}$$

Upon integrating the latter value over the **k** space, one obtains:

$$\int d^2kS(\mathbf{k}\cdot\mathbf{V}_{21})\Delta\tilde{V}^4 = \sqrt{\alpha_t}\left[C_1\tilde{V}_{21}(\tilde{V}_{21}\cdot\tilde{\mathbf{V}})^2+C_2\tilde{V}_{21}^5+C_3\tilde{V}_{21}^3\tilde{V}^2\right], \tag{E.6}$$

where $\tilde{\mathbf{V}}_{21}=\tilde{\mathbf{V}}_2-\tilde{\mathbf{V}}_1$, $\tilde{\mathbf{V}}=\left(\tilde{\mathbf{V}}_2+\tilde{\mathbf{V}}_1\right)/2$, $\alpha_t=4/3$, and where

$$C_1=\pi(1+e)(e-3)/2,\quad C_2=-\pi\left(1-e^2\right)\left(e^2+2\right)/24,\quad C_3=\pi(1+e)(2e-1)/3. \tag{E.7a,b,c}$$

Equations (E.1), (E.3) together with assumptions of smallness of a_2 yield:

$$F_1F_2=F_1^{(0)}F_2^{(0)}\left\{1+a_2\left[S_{1/2}^{(2)}(\tilde{V}_1^2)+S_{1/2}^{(2)}(\tilde{V}_2^2)\right]\right\}. \tag{E.8}$$

After substituting eqs. (E.6), (E.8) into r.h.s. of eq. (E.4), one evaluates:

$$\int d^3V_1d^3V_2F_1F_2\int d^2kq(\mathbf{k}\cdot\mathbf{V}_{21})(\mathbf{k}\cdot\mathbf{V}_{21})\Delta\tilde{V}^4=\sqrt{\frac{\alpha_t}{2\pi}}(C_1A_1+C_2A_2+C_3A_3), \tag{E.9}$$

where

$$A_1=4+\frac{55}{4}a_2,\quad A_2=96+570a_2,\quad A_3=12+\frac{69}{4}a_2. \tag{E.10}$$

The l.h.s. of eq. (E.4) is the product of two integrals. For their evaluation it is necessary to introduce approximation (E.8) into eq. (24) (see eq. (34)):

$$K_e(F,F)=-\sqrt{\frac{2\pi}{3}}\frac{4}{3}\left(1-e^2\right)\left(1+\frac{3a_2}{16}\right). \tag{E.11}$$

The second integral is evaluated by invoking the following identity:

$$\int d^3V_1\tilde{V}_1^4\left(\frac{3}{2}F_1+V_1^2\frac{dF_1}{dV_1^2}\right)=-2\int d^3V_1\tilde{V}_1^4F_1=-\frac{15}{2}(1+a_2). \tag{E.12}$$

Equations (E.4), (E.7), (E.9) - (E.12) after some algebraic transformations yield a linear equation with respect to a_2, the solution of which is (cf. eq. (33))

$$a_2=\frac{16(1-e)(1-2e^2)}{64+(1-e)\left[(1+e^2)190+147\right]}. \tag{E.13}$$