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Appendices to: "The motion of axisymmetric dipolar particles in homogeneous shear flows

by

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Appendix A: Particle orbits under the action of a strong ($\lambda \gg 1$) external field.

Here we look for an asymptotic solution of (3.1b) for $\theta(\tau)$. Substituting

$$\tan\frac{\theta}{2} = g e^{-\lambda\tau} \tag{A.1}$$

into (3.1b), we obtain

$$\frac{dg}{d\tau} = \frac{1}{2} g B \sin 2\phi - \frac{B g^3 e^{-2\lambda\tau}}{1+g^2 e^{-2\lambda\tau}} \sin 2\phi \quad . \tag{A.2}$$

Define

$$g = g_0 + g_1 , \tag{A.3}$$

wherein g_0 satisfies the linear part of (A.2), namely

$$\frac{dg_0}{d\tau} = \frac{1}{2} g_0 B \sin 2\phi , \tag{A.4}$$

which is readily integrated to yield

$$g_0 = \frac{C}{(1+B\cos 2\phi)^{1/2}} \tag{A.5}$$

when use is made of (3.1a). Eq. (A.2) is now formally integrated by variation of parameters leading to

$$g = g_0\left(1 - B\int_0^{\tau} \frac{g^3(\tau_1) e^{-\lambda\tau_1}}{1+g^2(\tau_1) e^{-2\lambda\tau_1}} \frac{\sin 2\phi}{g_0(\tau_1)} d\tau_1\right) \quad . \tag{A.6}$$

From (A.2) it follows that $\left|\frac{dg}{d\tau}\right| \le \alpha g$, where α is some positive constant. Hence,

$$0 \le g \le g(0)e^{\alpha\tau} . \tag{A.7}$$

Making use of the latter relation and noting that $g_0 \neq 0$ for all τ ($|B| < 1$), one can readily establish for all τ the bound

$$\left| \int_0^{\tau} \frac{Bg^3(\tau_1)e^{-2\lambda\tau_1}}{1+g^2(\tau_1)e^{-2\lambda\tau_1}} \frac{\sin 2\phi}{g_0(\tau_1)} d\tau_1 \right| \le \frac{\overline{C}}{\lambda} \tag{A.8}$$

in which $\overline{C}$ is some constant (independent of both λ and τ), thereby obtaining the approximation (3.2). A similar, exponentially rapid convergence to the stable equilibrium orientation is also expected in the presence of a strong ($\lambda \gg 1$) external field acting in an arbitrary direction ($\overline{\theta} \neq 0$).

Appendix B: Nearly periodic motions

B.1 *Slightly deformed spheres* ($B \cong o(1)$)

Following Hinch & Leal (1972), the scalar components of $\underset{\sim}{e}$ in the Cartesian axes (x_1, x_2, x_3) are

$$\lambda e_1 = (\frac{1}{2}\cos\alpha + A\cos\beta)\sin\alpha , \tag{B.1a}$$

$$\lambda e_2 = -\frac{1}{2}\sin^2\alpha + A\cos\alpha\cos\beta , \tag{B.1b}$$

and

$$\lambda e_3 = A\sin\beta . \tag{B.1c}$$

The corresponding components of the equation of motion (2.1) are, respectively

$$\dot{e}_1 = -\lambda e_1 e_3 + \frac{1}{2}Be_1[2e_2e_3\cos 2\overline{\phi} + \left(e_2^2 - e_3^2\right)\sin 2\overline{\phi}] , \tag{B.2a}$$

$$\dot{e}_2 = -\frac{1}{2}e_3 - \lambda e_2 e_3 - \frac{1}{2}B[e_3(1-2e_2^2)\cos 2\overline{\phi} + e_2(e_1^2 + 2e_3^2\_\sin 2\overline{\phi}] \;\; , \tag{B.2b}$$

$$\dot{e}_3 = \frac{1}{2}e_2 + \lambda\left(e_2^2 + e_3^2\right) + \frac{1}{2}B[-e_2(1-2e_3^2)\cos\overline{\phi} + e_3(e_1^2 + 2e_2^2)\sin 2\overline{\phi}]. \tag{B.2c}$$

Taking the time derivatives of (B.1) and eliminating the components of $\underset{\sim}{\dot{e}}$ between the resulting equations and (B.2), we obtain equations (4.12).

The O(B) balance in (4.12a) results in

$$\frac{\partial\alpha_1}{\partial\tau} = \frac{1}{2}\frac{\sin\alpha_0}{e_1^{(0)}}\{[e_1^{(0)}\cos\alpha_0 - (e_1^{(0)2} + 2e_3^{(0)2})(e_1^{(0)}\cos\alpha_0 - e_2^{(0)}\sin\alpha_0)]\sin 2\overline{\phi}$$

$$+e_3^{(0)}[2e_1^{(0)}e_2^{(0)}\cos\alpha_0 + (1-2e_2^{(0)2})\sin\alpha_0]\cos 2\overline{\phi}\} - \frac{d\alpha_0}{d\tau_1}, \quad \text{(B.3)}$$

wherein $e_i^{(0)}(i = 1,2,3)$ are the expressions resulting from the substitution of α_0 and β_0 into (B.1). In order that α_1 be a bounded function of τ we require that

$$\int_0^T \frac{\partial\alpha_1}{\partial\tau} d\tau = 0 \quad \text{(B.4)}$$

in which T denotes the period of β_0 (uniform for all orbits). Substitute in the latter condition the right-hand side of (B.3) while making use of the identity

$$e_1\cos\alpha - e_2\sin\alpha = \frac{\sin\alpha}{2\lambda},$$

obtained from (B.1), together with the relation

$$\frac{\partial\beta_0}{\partial\tau} = \frac{\lambda e_1^{(0)}}{\sin\alpha_0},$$

resulting from (4.15b) in conjunction with (B.1a), to obtain

$$\frac{\partial\alpha_0}{\partial\tau_1} = \frac{1}{2}\sin 2\overline{\phi}\sin\alpha_0\left[-\frac{\sin^2\alpha_0}{2\lambda^2 T}\int_0^{2\pi}\frac{e_1^{(0)2} + 2e_3^{(0)2}}{e_1^{(0)2}}d\beta_0 + \cos\alpha_0\right]. \quad \text{(B.5)}$$

(Owing to the occurrence of $e_3^{(0)}$ (cf. (B.1c)), the coefficient of $\cos 2\overline{\phi}$ on the right-hand side of (B.3) is an odd function of β_0 with respect to $\beta_0 = \pi$ and therefore disappears after the integration (B.4).) Substitution of $e_1^{(0)}$ and $e_3^{(0)}$ and integration yield (4.16).

B.2 Weak external field ($\lambda \cong o(1)$)

In the O(λ), we obtain from (2.7) the system of equations

$$\frac{\partial \phi_1}{\partial \tau} = -\phi_1 E \sin 2\phi_0 - \frac{\sin \overline{\theta}}{\sin \theta_0} \sin(\phi_0 - \overline{\phi}) - \frac{\partial \phi_0}{\partial \tau_1} \tag{B.6a}$$

and

$$\frac{\partial \theta_1}{\partial \tau} = \phi_1 (\frac{1}{2} B \cos 2\phi_0 \sin 2\theta_0) + \theta_1 (\frac{1}{2} B \sin 2\phi_0 \cos 2\theta_0) +$$

$$+ \sin \overline{\theta} \cos \theta_0 \cos(\phi_0 - \overline{\phi}) - \cos \overline{\theta} \sin \theta_0 - \frac{\partial \theta}{\partial \tau_1} \ . \tag{B.6b}$$

The general solution of the assoc ated homogeneous problem is

$$\begin{pmatrix} \phi_1^h \\ \theta_1^h \end{pmatrix} = \alpha(\tau_1) \begin{bmatrix} \frac{1}{2}(1 + B\cos 2\phi_0) \\ \frac{1}{4} B \sin 2\theta_0 \sin 2\phi_0 \end{bmatrix} + \beta(\tau_1) \begin{bmatrix} 0 \\ \frac{2\cos^2 \theta_0}{(1 + B\cos 2\phi_0)^{1/2}} \end{bmatrix} \ . \tag{B.7}$$

(As usual in this type of problems, one of the solutions is proportional to $\left(\frac{\partial \phi_0}{\partial \tau}, \frac{\partial \theta_0}{\partial \tau}\right)$; cf. Kevorkian & Cole 1981.) Cbviously, this general solution is τ-periodic and thus bounded in τ. We seek a particu ar solution of the original inhomogeneous problem by variation of parameters, i.e.

$$\begin{pmatrix} \phi_1^p \\ \theta_1^p \end{pmatrix} = \gamma(\tau) \begin{bmatrix} \frac{1}{2}(1 - B\cos 2\phi_0) \\ \frac{1}{4} B \sin 2\theta_0 \sin 2\phi_0 \end{bmatrix} + \delta(\tau) \begin{bmatrix} 0 \\ \frac{2\cos^2 \theta_0}{(1 + B\cos 2\phi_0)^{1/2}} \end{bmatrix} \ , \tag{B.8}$$

to obtain

$$\frac{d\gamma}{d\tau} = \frac{d\tilde{\tau}}{d\tau_1} - 2 \frac{\sin \overline{\theta}}{\sin \theta_0} \frac{\sin(\phi_0 - \overline{\phi})}{1 + B\cos 2\phi_0} \ .$$

In order that (B.8) be periodic ir the fast time variable we require that

$$\frac{d\tilde{\tau}}{d\tau_1} = \frac{2}{T} \int_{\tilde{\tau}}^{\tilde{\tau}+T} \frac{\sin \overline{\theta}}{\sin \theta_0} \frac{\sin(\phi_0 - \overline{\phi})}{(1 + B\cos 2\phi_0)} d\tau =$$

$$= \frac{4}{T} \int_0^{2\pi} \frac{\sin \overline{\theta}}{\sin \theta_0} \frac{\sin(\phi_0 - \overline{\phi})}{(1 + B\cos 2\phi_0)^2} d\phi_0 = 0 \ , \tag{B.9}$$

when use is made of (5.4a) and the fact that the latter intergrand is odd with respect to $\phi_0 = \pi$. The other coefficient in (B. 8) similarly satisfies

$$\frac{d\delta}{d\tau}\frac{2\cos^2\theta_0}{(1+B\cos 2\phi_0)^{1/2}} = -\frac{d\gamma}{d\tau}(\frac{1}{4}B\sin 2\theta_0 \sin 2\phi_0) - \frac{1}{2C}\frac{dC}{d\tau_1}\sin 2\theta_0 +$$

$$+\sin\overline{\theta}\cos\theta_0\cos(\phi_0 - \overline{\phi}) - \cos\overline{\theta}\sin\theta_0 \quad .$$

Substituting the above expression for $\frac{d\gamma}{d\tau}$ while making use of the symmetry properties of the various functions we obtain from the periodicity requirement imposed on $\delta(\tau)$ the relation (5.6b).

References

Kevorkian, J. & Cole, J.D. 1981 *Perturbation Methods in Applied Mathematics.* Springer