Supplementary material for

Double-diffusive instabilities in a vertical slot

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The following was omitted from the paper due to its length and the feeling that its inclusion would not add extra insight into the behaviour of the instabilities in the small-α asymptotic limit.

1 Explicit solutions in the limit $\alpha \to 0$

The leading order approximation for the small α limit in §3.3 of Kerr & Tang yields the pair of equations for ψ_0 and S_0 (3.23a and 3.23c):

$$\psi_0'''' - Ra_S S_0' = 0,$$

$$\tau S_0'' + \psi_0' = i\overline{W}(x) S_{-1}$$

where S_{-1} is a constant, with boundary conditions

$$\psi_0 = \psi_0' = S_0' = 0 \quad \text{on} \quad x = \pm 1/2.$$

This has a solution for ψ_0 of

$$\psi = -iS_{-1} Ra_T \left(A + B\cosh Mx \cos Mx + C \sinh Mx \sin Mx \right.$$

$$\left. + Dx \sinh Mx \cos Mx + Ex \cosh Mx \sin Mx\right),$$

where A, B, C, D and E are given by

$$\begin{aligned} A &= \frac{-1}{4M^4}, \qquad (1) \\ B &= \frac{4(\sinh M + \sin M)^2 - \sinh\frac{M}{2}\sin\frac{M}{2}C_1}{16M^4(\sinh M + \sin M)^2\cosh\frac{M}{2}\cos\frac{M}{2}}, \\ C &= \frac{C_1}{16M^4(\sinh M + \sin M)^2}, \\ C_1 &= 8\cos\frac{M}{2}\cosh\frac{M}{2} - 4\cos\frac{3M}{2}\cosh\frac{M}{2} \\ &\quad +4\sin\frac{M}{2}\sinh\frac{3M}{2} - 4\sin\frac{3M}{2}\sinh\frac{M}{2} \end{aligned}$$

$$
\begin{aligned}
&-4\cos\frac{M}{2}\cosh\frac{3M}{2} - M\cos\frac{M}{2}\sinh\frac{3M}{2} \\
&-M\cos\frac{M}{2}\sinh\frac{M}{2} + M\sin\frac{3M}{2}\cosh\frac{M}{2} \\
&+M\sin\frac{M}{2}\cosh\frac{M}{2},
\end{aligned}
$$

$$
D = \frac{-\sin\frac{M}{2}\cosh\frac{M}{2}}{4M^3(\sinh M + \sin M)},
$$

$$
E = \frac{\sinh\frac{M}{2}\cos\frac{M}{2}}{4M^3(\sinh M + \sin M)},
$$

with a similar expression for S_0.

2 Exact solution for the small-α limit.

The integral (3.29) in Kerr & Tang can be rearranged to give the expression

$$
Ra_T^2 = \frac{-\tau^2}{\dfrac{F(M)}{4M^4} + \dfrac{G(M)}{4M^7(\sinh M + \sin M)}}. \tag{2}
$$

The function $F(M)$ is given by

$$
\begin{aligned}
F(M) = &\\
&A + B\left(\frac{\cosh\frac{M}{2}\sin\frac{M}{2} + \sinh\frac{M}{2}\cos\frac{M}{2}}{M}\right) \\
&+C\left(\frac{\cosh\frac{M}{2}\sin\frac{M}{2} - \sinh\frac{M}{2}\cos\frac{M}{2}}{M}\right) \\
&+D\left(\frac{\cosh\frac{M}{2}\cos\frac{M}{2} + \sinh\frac{M}{2}\sin\frac{M}{2}}{2M} - \frac{\cosh\frac{M}{2}\sin\frac{M}{2}}{M^2}\right) \\
&+E\left(\frac{-\cosh\frac{M}{2}\cos\frac{M}{2} + \sinh\frac{M}{2}\sin\frac{M}{2}}{2M} + \frac{\sinh\frac{M}{2}\cos\frac{M}{2}}{M^2}\right) \\
&-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M + \sin M}A\left(\frac{\cosh\frac{M}{2}\sin\frac{M}{2} - \sinh\frac{M}{2}\cos\frac{M}{2}}{M}\right)
\end{aligned}
$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}B\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}C\left(-\frac{\cosh M\sin M}{8M}-\frac{\sinh M\cos M}{8M}+\frac{\sinh M}{4M}-\frac{1}{4}+\frac{\sin M}{4M}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}D\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}-\frac{\sin^2\frac{M}{2}}{4M}+\frac{1}{8M}-\frac{\sin M}{8M^2}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}E\left(-\frac{\cosh M\cos M-2\cosh M+\sinh M\sin M}{16M}+\frac{\cosh M\sin M-2\sinh M}{16M^2}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}A\left(\frac{\cosh\frac{M}{2}\sin\frac{M}{2}+\sinh\frac{M}{2}\cos\frac{M}{2}}{M}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}B\left(\frac{\cosh M\sin M+\sinh M\cos M+2\sinh M}{8M}+\frac{\sin M}{4M}+\frac{1}{4}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}C\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}D\left(\frac{\cosh M\cos M+2\cosh M+\sinh M\sin M}{16M}-\frac{\cosh M\sin M+2\sinh M}{16M^2}\right)$$

$$-\frac{2\sin\frac{M}{2}\cosh\frac{M}{2}}{\sinh M+\sin M}E\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}+\frac{\sin^2\frac{M}{2}}{4M}-\frac{1}{8M}+\frac{\sin M}{8M^2}\right)$$

$$-\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}A\left(\frac{\cosh\frac{M}{2}\sin\frac{M}{2}+\sinh\frac{M}{2}\cos\frac{M}{2}}{M}\right)$$

$$-\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}B\left(\frac{\cosh M\sin M+\sinh M\cos M+2\sinh M}{8M}+\frac{\sin M}{4M}+\frac{1}{4}\right)$$

$$-\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}C\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$-\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}D\left(\frac{\cosh M\cos M+2\cosh M+\sinh M\sin M}{16M}\right.$$
$$\left.-\frac{\cosh M\sin M+2\sinh M}{16M^2}\right)$$

$$-\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}E\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}\right.$$
$$\left.+\frac{\sin^2\frac{M}{2}}{4M}-\frac{1}{8M}+\frac{\sin M}{8M^2}\right)$$

$$+\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}A\left(\frac{\cosh\frac{M}{2}\sin\frac{M}{2}-\sinh\frac{M}{2}\cos\frac{M}{2}}{M}\right)$$

$$+\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}B\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$+\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}C\left(-\frac{\cosh M\sin M}{8M}-\frac{\sinh M\cos M}{8M}+\frac{\sinh M}{4M}-\frac{1}{4}+\frac{\sin M}{4M}\right)$$

$$+\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}D\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}\right.$$
$$\left.-\frac{\sin^2\frac{M}{2}}{4M}+\frac{1}{8M}-\frac{\sin M}{8M^2}\right)$$

$$+\frac{2\sinh\frac{M}{2}\cos\frac{M}{2}}{\sinh M+\sin M}E\left(-\frac{\cosh M\cos M-2\cosh M+\sinh M\sin M}{16M}\right.$$
$$\left.+\frac{\cosh M\sin M-2\sinh M}{16M^2}\right),$$

The function $G(M)$ is given by

$$G(M)=$$
$$-\sinh\frac{M}{2}\cos\frac{M}{2}2BM^3\left(-\frac{\sinh M\cos M}{8M}-\frac{\cosh M\sin M}{8M}+\frac{\sinh M}{4M}+\frac{1}{4}-\frac{\sin M}{4M}\right)$$

$$-\sinh\frac{M}{2}\cos\frac{M}{2}2BM^3\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$+\sinh\frac{M}{2}\cos\frac{M}{2}2CM^3\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$-\sinh\frac{M}{2}\cos\frac{M}{2}2CM^3\left(-\frac{\sinh M\cos M}{8M}-\frac{\cosh M\sin M}{8M}+\frac{\sinh M}{4M}+\frac{1}{4}-\frac{\sin M}{4M}\right)$$

$$-\sinh\frac{M}{2}\cos\frac{M}{2}6DM^2\left(-\frac{\sinh M\cos M}{8M}-\frac{\cosh M\sin M}{8M}+\frac{\sinh M}{4M}+\frac{1}{4}-\frac{\sin M}{4M}\right)$$

$$-\sinh\frac{M}{2}\cos\frac{M}{2}2DM^3\left(-\frac{\cosh M\cos M-2\cosh M+\sinh M\sin M}{16M}\right.$$
$$\left.+\frac{\cosh M\sin M-2\sinh M}{16M^2}\right)$$

$$-\sinh\frac{M}{2}\cos\frac{M}{2}2DM^3\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}\right.$$
$$\left.+\frac{\sin^2\frac{M}{2}}{4M}-\frac{1}{8M}+\frac{\sin M}{8M^2}\right)$$

$$+\sinh\frac{M}{2}\cos\frac{M}{2}6EM^2\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$+\sinh\frac{M}{2}\cos\frac{M}{2}2EM^3\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}\right.$$
$$\left.+\frac{\sin^2\frac{M}{2}}{4M}-\frac{1}{8M}+\frac{\sin M}{8M^2}\right)$$

$$-\sinh\frac{M}{2}\cos\frac{M}{2}2EM^3\left(-\frac{\cosh M\cos M-2\cosh M+\sinh M\sin M}{16M}\right.$$
$$\left.+\frac{\cosh M\sin M-2\sinh M}{16M^2}\right)$$

$$+\sin\frac{M}{2}\cosh\frac{M}{2}2BM^3\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)$$

$$+\sin\frac{M}{2}\cosh\frac{M}{2}2BM^3\left(\frac{\sinh M\cos M+\cosh M\sin M+2\sinh M}{8M}-\frac{1}{4}-\frac{\sin M}{4M}\right)$$

$$
\begin{aligned}
&-\sin\frac{M}{2}\cosh\frac{M}{2}2CM^3\left(\frac{\sinh M\cos M+\cosh M\sin M+2\sinh M}{8M}-\frac{1}{4}-\frac{\sin M}{4M}\right)\\
&+\sin\frac{M}{2}\cosh\frac{M}{2}2CM^3\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)\\
&+\sin\frac{M}{2}\cosh\frac{M}{2}6DM^2\left(\frac{\cosh M\sin M-\sinh M\cos M}{8M}\right)\\
&+\sin\frac{M}{2}\cosh\frac{M}{2}2DM^3\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}\right.\\
&\qquad\left.-\frac{\sin^2\frac{M}{2}}{4M}+\frac{1}{8M}-\frac{\sin M}{8M^2}\right)\\
&+\sin\frac{M}{2}\cosh\frac{M}{2}2DM^3\left(\frac{\cosh M\cos M+2\cosh M+\sinh M\sin M}{16M}\right.\\
&\qquad\left.-\frac{\cosh M\sin M+2\sinh M}{16M^2}\right)\\
&-\sin\frac{M}{2}\cosh\frac{M}{2}6EM^2\left(\frac{\sinh M\cos M+\cosh M\sin M+2\sinh M}{8M}-\frac{1}{4}-\frac{\sin M}{4M}\right)\\
&-\sin\frac{M}{2}\cosh\frac{M}{2}2EM^3\left(\frac{\cosh M\cos M+2\cosh M+\sinh M\sin M}{16M}\right.\\
&\qquad\left.-\frac{\cosh M\sin M+2\sinh M}{16M^2}\right)\\
&+\sin\frac{M}{2}\cosh\frac{M}{2}2EM^3\left(\frac{\sinh M\sin M-\cosh M\cos M}{16M}+\frac{\sinh M\cos M}{16M^2}\right.\\
&\qquad\left.-\frac{\sin^2\frac{M}{2}}{4M}+\frac{1}{8M}-\frac{\sin M}{8M^2}\right).
\end{aligned}
$$