Supplementary Information: Premelting increases the rate of regelation by an order of magnitude

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LINEARIZED ICE FLOW OVER WAVY TOPOGRAPHY

In this Supplementary Information, we elaborate upon the scaling argument given in the main manuscript and examine the extent to which spatial variations in effective stress affect the predicted relationship between sliding velocity v and basal shear stress τ. We anticipate that when the resistance to water flow is low enough that the liquid pressure can adjust easily to the melting and refreezing required by regelation, the variations in ice pressure along the basal interface should produce comparable variations in effective stress so that perturbations to the melting temperature are a factor of $\rho_l/(\rho_l-\rho_i)$ greater than those produced by pressure melting alone. Consistent with the scaling analysis, we demonstrate that the increased rate of heat flow driven by these enhanced temperature perturbations implies an order of magnitude increase to the predicted regelation velocity past small bumps. Moreover, we find that the transition wavelength $2\pi/\ell_0$ that marks the obstacle size for which regelation and plastic deformation are of equal importance is a factor of $\sqrt{\rho_l/(\rho_l-\rho_i)} \approx 3.5$ times longer than that predicted by conventional treatments (e.g. Weertman, 1957; Nye, 1969, 1970; Kamb, 1970; Gudmundsson, 1997).

Our goal in revisiting the detailed analysis provided by Kamb (1970) for regelation and deformation over a rough glacier bed is to examine the effects of making the minor extensions required to describe how the water pressure P_l varies independently from the ice pressure P_i. As in Weertman (1957) and Nye (1969), a central assumption to Kamb's Eq. (K18) (hereafter, equations from Kamb (1970) are cited as: K#), is that the pressures in each phase at the ice–liquid interface are identical (i.e. $P_i(x,y) = P_l(x,y) = P(x,y)$) and related to the offset from the average melting temperature T_m by

$$T(x,y) - T_m = -C_0\left[P(x,y) - P_m\right] \ , \tag{S1}$$

with P_m representing the average interfacial pressure over the bed, which is defined along in-plane coordinates x and y. As noted in Eq. (7) of the main manuscript, a more complete description of ice–liquid equilibrium that allows for differences between the ice and liquid pressures due to the wetting and curvature effects that together promote premelting (Dash and others, 2006) can be expressed as

$$T(x,y) - T_m = -C_0\left[P_l(x,y) - P_m + \frac{\rho_l}{\rho_l-\rho_i}N(x,y)\right] \ , \tag{S2}$$

where $N(x,y) = P_i(x,y) - P_l(x,y)$ is the spatially varying effective stress. To be precise, we note that in Eq. (S2) and elsewhere below, T_m is the bulk melting temperature when $P_i = P_l = P_m$ so that $N = 0$; in the more general case where $N \neq 0$ the average melting temperature of the interface deviates from T_m due to premelting. We note that small perturbations arising from the strain energy that accompanies nonhydrostatic stress distributions (e.g. Kamb, 1961) are neglected in Eq. (S2).

With ice flow in the x direction and the basal interface described by (cf. K3)

$$z_0(x,y) = \frac{A}{4\pi^2}\iint_{-\infty}^{\infty} a(h,k)e^{i(hx+ky)}\,\mathrm{d}h\mathrm{d}k \ , \tag{S3}$$

for area element A, small amplitude Fourier spectrum a, and horizontal wavenumbers h and k, the temperature distribution needed to satisfy heat balance constraints on regelation can be expressed as (cf. K15–K17)

$$T(x,y) - \bar{T} \approx -\frac{A}{4\pi^2}\frac{\rho_i\mathscr{L}}{2K}\iint_{-\infty}^{\infty}\frac{iha(h,k)v_R}{\ell}e^{i(hx+ky)}\,\mathrm{d}h\mathrm{d}k \ , \tag{S4}$$

where $\ell = \sqrt{h^2+k^2}$, $2K = K_i + K_s$ is determined by the thermal conductivities of ice (subscript i) and the underlying substrate (subscript s), and $\bar{T}$ is the average interfacial temperature. A detailed discussion of the approximations leading to this expression for the interfacial temperature is provided by Kamb (1970, see §5).

In the limit of small amplitude bumps for the idealized case where ice rheology is treated as Newtonian with viscosity η, the stress distribution at the bed interface due to 'plastic' deformation at rate v_p can be written as (cf. K28–K29)

$$\tau_{zz}(x,y,0) = -\frac{A}{4\pi^2}2\eta\iint_{-\infty}^{\infty} ih\ell a(h,k)v_p e^{i(hx+ky)}\,\mathrm{d}h\mathrm{d}k \ . \tag{S5}$$

Following the arguments provided in §7 of Kamb (1970, cf. K30) we set $\tau_{zz}(x,y,0) = -(P_i(x,y) - \bar{P}_i)$, where $\bar{P}_i$ is the average interfacial ice pressure so that

$$P_i(x,y) - \bar{P}_i = \frac{A}{4\pi^2} 2\eta \iint_{-\infty}^{\infty} ih\ell a(h,k) v_p e^{i(hx+ky)}\, \mathrm{d}h\mathrm{d}k \ . \tag{S6}$$

In the limit that $N(x,y) = 0$ it is natural to identify $\bar{P}_i$ in Eq. (S6) with the reference pressure P_m and $\bar{T}$ in Eq. (S4) with the corresponding reference bulk melting temperature T_m, enabling comparison between these two expressions through direct substitution of Eq. (S1). In extending the treatment to allow for nonzero $N(x,y)$, we recognize that the average interfacial ice pressure must still support the glacier weight so that $\bar{P}_i = \rho_i g H$, where g is the acceleration of gravity and H is ice thickness (Kamb, 1970, see §19). Averaging the generalized equilibrium relation from Eq. (S2) we obtain

$$\bar{T} = T_m - C_0 \left(\bar{P}_i - P_m + \frac{\rho_i}{\rho_l - \rho_i} \bar{N} \right) , \tag{S7}$$

which predicts significant deviations between the average interfacial temperature and the bulk coexistence line when $\bar{N} \neq 0$. Upon substituting Eq. (S7) and using Eq. (S2) once again, the heat balance constraint on $T(x,y) - \bar{T}$ in Eq. (S4) can be written as

$$\begin{aligned} &P_i(x,y) - \bar{P}_i - \frac{\rho_i}{\rho_l}\left[P_l(x,y) + \bar{N} - \bar{P}_i\right] &\text{(S8)} \\ &= \frac{A}{4\pi^2} \frac{(\rho_l - \rho_i)\rho_i \mathscr{L}}{2\rho_l C_0 K} \iint_{-\infty}^{\infty} \frac{iha(h,k)v_R}{\ell} e^{i(hx+ky)}\, \mathrm{d}h\mathrm{d}k \ . \end{aligned}$$

To simplify the notation we define the modified transition wavenumber

$$\ell_0^2 = \frac{\rho_l - \rho_i}{\rho_l} \frac{\rho_i \mathscr{L}}{4\eta C_0 K} , \tag{S9}$$

where the ratio of the density difference to ρ_l makes ℓ_0 a factor of about 3.5 times smaller than that defined in (K36) so that, with $\bar{P}_l = \bar{P}_i - \bar{N}$, Eq. (S6) and (S8) imply

$$\begin{aligned} &P_l(x,y) - \bar{P}_l = \frac{A}{4\pi^2} \frac{2\eta\rho_l}{\rho_i} &\text{(S10)} \\ &\times \iint_{-\infty}^{\infty} ih \frac{a(h,k)v_p \ell^2 - a(h,k) v_R \ell_0^2}{\ell} e^{i(hx+ky)}\, \mathrm{d}h\mathrm{d}k \ . \end{aligned}$$

We need one more condition to constrain liquid pressure variations on the left side of Eq. (S10). Requiring the liquid flux $q(x,y)$ to balance melting and refreezing during regelation gives

$$\rho_l q(x,y) = -\rho_i v_R \frac{\partial z_0}{\partial x} \ . \tag{S11}$$

Where $q(x,y)$ in Eq. (S11) is positive the basal interface acts as a liquid sink, and where negative, a liquid source. In analogy with the arguments in Kamb (1970, §5) concerning regelation heat flow, we use Darcy's law to set $q \propto \nabla P_l$ and stipulate that $\nabla^2 P_l = 0$ in the lower half-space (neglecting englacial liquid transport for simplicity). Imposing the condition that $P_l - \bar{P}_l \to 0$ as $z \to -\infty$ leads to a solution in terms of the Fourier components D as

$$P_l(x,y) - \bar{P}_l = \frac{A}{4\pi^2} \iint_{-\infty}^{\infty} D(h,k) \exp(\ell z) e^{i(hx+ky)}\, \mathrm{d}h\mathrm{d}k \ . \tag{S12}$$

As for the scaling argument above, we assume a finite substrate permeability Π and assign the liquid viscosity as μ, so that Darcy's law gives

$$q(x,y) = -\frac{\Pi}{\mu} \frac{\partial P_l}{\partial z} \ . \tag{S13}$$

Combined with the liquid flux condition from Eq. (S11) and the interfacial geometry from Eq. (S3) this results in an expression for the liquid pressure along the interface as

$$P_l(x,y) - \bar{P}_l = \frac{A}{4\pi^2} \frac{\rho_i \mu}{\rho_l \Pi} \iint_{-\infty}^{\infty} \frac{ih}{\ell} v_R a(h,k) e^{i(hx+ky)}\, \mathrm{d}h\mathrm{d}k \ . \tag{S14}$$

Now combining this with the constraint on P_l arrived at in Eq. (S10) from the regelation heat balance and plastic deformation leaves us with

$$\frac{2\eta\rho_l}{\rho_i}\left[a(h,k)v_p\ell^2 - a(h,k)v_R\ell_0^2\right] - \frac{\rho_i\mu}{\rho_l\Pi} v_R a(h,k) = 0 \ . \tag{S15}$$

Kamb (1970, see §7) argued for the equivalence of considering the total ice velocity to arise from a combination of sliding and regelation components over a fixed bed topography defined by Fourier components $a(h,k)$, and the alternative view of equating the sliding and regelation velocities over differing fractional amplitudes of the total topography. For simplicity here we adopt the former representation and set $v_p + v_R = v$ so that

$$v_R = \frac{\ell^2}{\ell^2 + \ell_0^2 + \ell_\Pi^2} v , \tag{S16}$$

where v is the total sliding velocity and

$$\ell_\Pi^2 = \frac{\rho_i^2 \mu}{\rho_l^2 2\eta \Pi} \ . \tag{S17}$$

The characteristic wavenumber ℓ_Π gauges the length scale $\lambda_\Pi = 2\pi/\ell_\Pi$ over which the liquid transport needed to accommodate the regelation process imposes a significant impediment. For topographic features at large wavenumbers so that $\ell^2 \gg \ell_0^2 + \ell_\Pi^2$, slip is almost entirely accomplished by regelation and Eq. (S16) reduces to $v_R \approx v$. Moreover, a comparison between ℓ_0 from Eq. (S9) and ℓ_Π from Eq. (S17) reveals that $\ell_\Pi^2 \ll \ell_0^2$ when $\Pi \gg \rho_i \Pi_0/[(\rho_l - \rho_i)]$ with Π_0 defined as in Eq. (6) of the main manuscript. As anticipated from the scaling analysis, this suggests that for reasonable estimates of Π a comparison of the limiting wave numbers yields $\ell_\Pi^2 \ll \ell_0^2$, implying that liquid transport plays a negligible role in restricting v_R. Hence, as in conventional treatments of regelation due to pressure melting (Weertman, 1957; Nye, 1969; Kamb, 1970), the relative importance of plastic deformation to regelation in controlling sliding speed depends on a comparison between the topographic wavelength $2\pi/\ell$ and the transition wavelength $2\pi/\ell_0$. Importantly, the transition wavelength is governed by considerations of heat flow, and because liquid transport is relatively efficient, variations in ice pressure produce commensurate variations in effective stress. The resultant perturbations to the melting temperature (see Fig. 1 in the main manuscript) cause the transition wavelength to be approximately 3.5 times larger than would be expected if the phase behavior were governed by pressure melting alone.

Finally, to arrive at a relationship between sliding velocity and shear stress we follow the arguments given by Kamb (1970, see §8) to write

$$\tau = \left\langle (P_i(x, y) - \bar{P}_i)\frac{\partial z_0}{\partial x} \right\rangle , \tag{S18}$$

where the notation $\langle \cdot \rangle$ on the right denotes the in-plane average. Substituting in from the expressions in Eq. (S6) for $P_i(x, y) - \bar{P}_i$ and Eq. (S3) for z_0 while making judicious use of the Fourier Inversion Theorem leads to (cf. K42)

$$\tau = \frac{A}{4\pi^2} 2\eta v \iint_{-\infty}^{\infty} h^2 \ell a^2(h, k) \frac{\ell_0^2 + \ell_\Pi^2}{\ell^2 + \ell_0^2 + \ell_\Pi^2} \, \mathrm{d}h \mathrm{d}k \ . \tag{S19}$$

This is almost identical to the result from Kamb's analysis, notably with ℓ_0 defined slightly differently in Eq. (S9) from its counterpart in (K36). In fact, in the limit that $\ell_\Pi^2 \ll \ell_0^2$ the only difference is in the factor of $\sqrt{\rho_l/(\rho_l - \rho_i)} \approx 3.5$ enhancement of the transition wavelength $2\pi/\ell_0$ in comparison with that defined by Kamb. Note that the average effective stress $\bar{N}$ does not appear in the relationship between τ and v – even in the limit that $\bar{N} \to 0$. This is consistent with the inference that melting temperature perturbations across obstacles are primarily caused by changes in $N(x, y)$ rather than by changes to $P_i(x, y)$ alone.

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