**Supplementary material for:**

**The effects of basal topography and ice-sheet surface slope in a subglacial glaciofluvial deposition model**

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Material in this section is provided as a complement to the main text. Here we present a summary of the mathematical model, as well as some additional numerical simulations that reinforce the findings presented above.

# Mathematical Formulation

The following presents a condensed summary of the mathematical formulation used for the computational simulations presented in this work. For a complete description of the mathematical model see Hewitt and Creyts, (2019).

For a tunnel draining a catchment width $l\_{c}$ and length $l\_{a}$, fed by basal meltwater $m\_{b}$ and surface meltwater $m$, with a prescribed sediment supply $e$, the flux of water ($Q$) and sediment ($Q\_{s}$) evolves as:

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| --- | --- | --- |
|  | $$\frac{∂Q}{∂x}=l\_{c}\left(m\_{b}+m\right)$$ | (S1), |
|  | $$\frac{∂Q\_{s}}{∂x}=l\_{c}e-D$$ | (S2), |

where $x$ is distance along the channel and $D$ is the rate of sediment deposition within the channel. Cross sectional area of the deposited sediment, $A$, therefore evolves as:

|  |  |  |
| --- | --- | --- |
|  | $$\frac{∂A}{∂t}=\frac{D}{1-n\_{s}}$$ | (S3), |

where $n\_{s}$ is the porosity of the deposited sediment.

Cross-sectional area of the channel evolves due to ice wall melting and creep closure, with sediment deposition acting as an additional mechanism to constrict the channel. Evolution of the channel cross-sectional area, $S$, is modelled via:

|  |  |  |
| --- | --- | --- |
|  | $$\frac{∂S}{∂t}=\frac{Q\left(Ψ-βρ\_{w}gb\_{x}\right)}{ρ\_{i}\left(1+β\right)L}-\frac{2A\_{Glen}}{n^{n}}SN^{n}-\frac{D}{1-n\_{s}}$$ | (S4). |

In the above equation, the first term on the right hand side represents wall melting: $Ψ$ is the hydraulic potential gradient, $ρ\_{i}$ and $ρ\_{w}$ are the density of ice and water, $L$ is the latent heat, $β$ is a constant that accounts for the pressure dependence of the melting point, and $b\_{x}$ is the bed slope. The second term represents creep closure: $N$ is the effective pressure, and $A\_{Glen}$ and $n$ are constants in Glen’s flow law. The third term represents channel constriction due to the above-defined sediment deposition.

Sediment flux ($Q\_{s}$) is determined from the solution of the above four equations, along with an additional constraint; namely that sediment flux cannot exceed a carrying capacity $Q\_{eq}$. This leads to two potential scenarios: either the sediment supply is below the carrying capacity, in which case $D=0$; or the sediment supply is *at* the carrying capacity (i.e. $Q\_{s}=Q\_{eq}$). In the latter case $D>0$, and can be determined via equation S2. Negative values of D correspond to erosion or remobilisation of previously deposited sediment. Such scenarios are allowed by the model where pre-existing sediment has been deposited (i.e. $A>0$), however the underlying bedrock is not considered erosional.

The sediment carrying capacity ($Q\_{eq}$) is a function of water speed and channel width, and is modelled here in the form of the Meyer-Peter and Müller (1948) transport equation. This relation suggests that $Q\_{eq}\~\left(τ^{\*}-τ\_{c}^{\*}\right)^{{3}/{2}}$, where $τ^{\*}$ is the Shields stress, and $τ\_{c}^{\*}$ is a critical Shields stress required for sediment mobilisation. Accounting for the width of the channel being proportional to $S^{{1}/{2}}$, the sediment carrying capacity is computed as:

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| --- | --- | --- |
|   | $$Q\_{eq}\left(Q,S\right)=8\left(\frac{8Δρ\_{s}gd^{3}S}{πρ\_{w}}\right)^{{1}/{2}}max\left(\frac{fρ\_{w}Q^{2}}{Δρ\_{s}gdS^{2}}-τ^{\*},0\right)^{{3}/{2}}$$ | (S5). |

Here $ρ\_{s}$ is the sediment density, $Δρ=\left(ρ\_{s}-p\_{w}\right)$ is the buoyant density of sediment in water, $d$ is a representative grain size, and $f$ is a friction factor. The sediment transport equation represents a balance between the shear stresses produced by the water, and the amount and size of the available sediment. For a given type and quantity of sediment, the water velocity (i.e. ${Q}/{S}$) is the key factor in determining the carrying capacity.

The final equation required relates the water flux to the tunnel’s cross-sectional area, by considering a parameterisation of turbulent drag:

|  |  |  |
| --- | --- | --- |
|  | $$Q=K\_{c}S^{{5}/{4}}Ψ^{{1}/{2}}$$ | (S6). |

Here $K\_{c}$ is a constant, and $Ψ$ is the hydraulic potential gradient.

In solving the above system of equations, $S$ and $A$ evolve according to equations (S3) and (S4), with Q and N determined from equations (S1) and (S.6), and $Q\_{s}$ and $D$ determined form equations (S2) and (S5). Other than the parameters described above, the model requires as input: meltwater and sediment source terms, and the topography of the bed and ice surface.

For the simulations presented here, we consider a constant basal meltwater supply of 5 mm a-1. The surface meltwater is considered to be zero above an ablation altitude of $s\_{a}=1000 m$, with surface meltwater supply increasing linearly below this altitude; i.e.

|  |  |  |
| --- | --- | --- |
|  | $$m=max\left(0, λ\left(s\_{a}-s\right)\right)$$ | (S7), |

where $s$ is the altitude, and λ = 3 ×10-3 a-1 represents the rate of increase for surface meltwater with distance below the ablation altitude.

|  |  |  |
| --- | --- | --- |
| **Parameter** | **Value** | **Description** |
| $$ρ\_{w}$$ | 1000 kg m-3  | Water density |
| $$ρ\_{i}$$ | 916 kg m-3  | Ice density |
| $$ρ\_{s}$$ | 2600 kg m-3  | Sediment density |
| $$L$$ | 3.3 $×10^{5}$ J kg-1 | Latent heat |
| $$n$$ | 3 | Glen’s law exponent |
| $$A\_{Glen}$$ | 2.4 $×10^{-24}$ s-1 Pa-3  | Glen’s law coefficient |
| $$K\_{c}$$ | $0.11 m^{{3}/{2}} kg^{{-1}/{2}}$ mm$Type equation here.$⁄S^3  | Turbulent drag coefficient |
| $$d$$ | 1 mm | Representative grain size |
| $$f$$ | 0.02 | Friction factor |
| $$τ\_{c}^{\*}$$ | 0.047 | Critical Shields stress |
| $$n\_{s}$$ | 0.3 | Sediment porosity |
| $$β$$ | 0.46 | Pressure melting coefficient |
| $$m\_{b}$$ |  5 mm a-1 | Basal melt rate |
| $$s\_{a}$$ | 1000 m | Ablation altitude |
| $$λ$$ | 3 ×10-3 a-1 | Surface melt scale factor |

Table S1: List of parameter values. Unless otherwise stated, the above parameter values are used for all numerical simulations presented in this work.

# Supplementary figures



Figure S1: Sediment deposition within the flushing zone. Large basal undulations are required to allow sediment deposition within the flushing zone between ~5km and ~20km from the margin. Here we show a Gaussian ridge of height 200m and variance 3km (left column; (a, b)), and variance 2km (right column; (c, d)). A very small region of deposition is encountered for the narrower ridge. Experiments suggest that basal gradient must be high to allow even small deposition within the flushing zone.



Figure S2: Basal undulations of magnitude 2 m. Even very small undulations can enhance sediment deposition, when ice thickness is low.

Figure S3: Multiple ridges within the submarginal deposition zone. Note that we consider only the final 10 km before the ice margin. Three ridges of height 75 m and variance 1km are placed at 4 km, 2 km and 0 km from the ice margin (left column; (a, b)), and at 5 km, 2 km, and 1 km from the ice margin (right column; (c, d)). Deposition is enhanced on the lee side of each ridge, separated by flushing zones on the upstream side of each ridge. Sediment deposition at the margin is enhanced by ridges that peak slightly inland of the ice margin.



Figure S4: Extended domain length with bumps of 10m amplitude. Beyond ~120 km from the ice margin deposition is negligible. The high pressure from the thick ice stifles channel growth in this region.



Figure S5: Variation of the critical Shields stress, $τ\_{c}^{\*}$, required for sediment mobilisation. A single-ridge configuration is used, of height 250 m located 50 km from the margin (a). The remaining plots show: on the left the sediment flux (solid line) and carrying capacity (dashed line), and on the right the rate of deposition. Figure b) takes the value of $τ\_{c}^{\*}=0.047$ used throughout this work. Figure c) reduces this by a factor of 10 to $τ\_{c}^{\*}=0.0047$. Figure c) increases by a factor of 10 to $τ\_{c}^{\*}=0.47$, and figure e) increases by a factor of 100 to $τ\_{c}^{\*}=4.7$. Note that we must increase the critical Shields stress for sediment mobilisation by more than one order of magnitude in order to observe notably increased deposition.

# Additional references

Meyer-Peter E and Müller R (1948) Formulas for bed-load transport. *IAHSR 2nd meeting, Stockholm, appendix 2*. IAHR. http://resolver.tudelft.nl/uuid:4fda9b61-be28-4703-ab06-43cdc2a21bd7