

# Projection-operator methods for classical transport in magnetized plasmas.

## Part 2. Nonlinear response and the Burnett equations — Supplement

John A. Krommes<sup>†</sup>

Princeton Plasma Physics Laboratory, P. O. Box 451, MS 28, Princeton, New Jersey  
08543-0451 USA

(Received xx; revised xx; accepted xx)

Some technical details of the reduction of the general Burnett equations of J. J. Brey *et al.* (Physica **109A**, 425–444 (1981)) to a one-component neutral fluid are given in order to support the results quoted by J. J. Brey in J. Chem. Phys. **79**, 4585–4598 (1983). The material is intended to supplement the paper of J. A. Krommes, ‘Projection-operator methods for classical transport in magnetized plasmas. Part 2. Nonlinear response and the Burnett equations’, J. Plasma Phys. **84**(6), <https://doi.org/10.1017/S0022377818000892>.

### CONTENTS

<b>1. Introduction</b>	1
1.1. Thermodynamic relations	2
1.2. The microscopic fluxes	2
1.3. The general formula for the dissipative fluxes	3
<b>2. Navier–Stokes terms (<math>\text{NS}_\beta^\alpha</math>)</b>	4
2.1. Navier–Stokes momentum flux ( $\text{NS}_P^P$ )	4
2.2. Navier–Stokes energy flux ( $\text{NS}_E^E$ )	5
<b>3. Burnett terms</b>	5
3.1. Linear Burnett terms ( $\text{B}_\beta^\alpha$ )	5
3.1.1. Linear Burnett — momentum ( $\text{B}_E^P$ )	5
3.1.2. Linear Burnett — energy ( $\text{B}_P^E$ )	6
3.2. Nonlinear Burnett terms ( $\text{B}_{\beta\gamma}^\alpha$ )	6
3.2.1. Nonlinear Burnett — momentum	7
3.2.2. Nonlinear Burnett — energy	12
3.3. Nonlinear Burnett terms — time derivatives	17
3.3.1. Nonlinear Burnett — momentum ( $\partial_t$ )	17
3.3.2. Nonlinear Burnett — energy ( $\partial_t$ )	19
<b>4. Summary of the dissipative fluxes</b>	23

### 1. Introduction

The following calculations are to be read in conjunction with the paper of Brey *et al.* (1981), which describes a general projection-operator formalism for obtaining the Burnett equations for an unmagnetized one-component fluid, and Appendix A of Brey (1983),

---

<sup>†</sup> Email address for correspondence: [krommes@princeton.edu](mailto:krommes@princeton.edu)

where the formulas are written out more explicitly. The purpose of this Supplement is to provide a reference for the details of that reduction, which were not given by Brey. Those details provide necessary background for the discussion by [Krommes \(2018\)](#) (Part 2 for short). The exposition will be somewhat more informal than the published Part 2.

### 1.1. Thermodynamic relations

Let us review various thermodynamic relations. With  $E$  being the total (mean) amount of energy in the system and  $S$  being the total entropy, the fundamental expression is

$$dE = T dS - p dV + \mu dN. \quad (1.1)$$

Here  $T$  is the temperature,  $p$  is the pressure,  $\mu$  is the chemical potential, and  $N$  is the total number of particles. One also has the Euler expression

$$E = TS - pV + \mu N, \quad (1.2)$$

which leads to the Gibbs–Duhem relation

$$0 = S dT - V dp + N d\mu. \quad (1.3)$$

It is convenient to recast these expressions in terms of the densities  $n \doteq N/V$ ,  $e \doteq E/V$ , and  $s \doteq S/V$ . Thus, (1.1) becomes

$$d(Ve) = T d(Vs) - p dV + \mu d(Vn), \quad (1.4)$$

or

$$de = T ds + \mu dn + (-e + Ts - p + n\mu)dV. \quad (1.5)$$

But dividing (1.2) by  $V$  gives

$$e = Ts - p + \mu n, \quad (1.6)$$

so (1.5) simplifies to

$$de = T ds + \mu dn, \quad (1.7)$$

and the Gibbs–Duhem relation is

$$0 = s dT - dp + n d\mu. \quad (1.8)$$

From (1.7), one has

$$T = \left( \frac{\partial e}{\partial s} \right)_n, \quad \mu = \left( \frac{\partial e}{\partial n} \right)_s. \quad (1.9)$$

This implies the Maxwell relation

$$\left( \frac{\partial T}{\partial n} \right)_s = \left( \frac{\partial \mu}{\partial s} \right)_n. \quad (1.10)$$

The Gibbs–Duhem relation (1.8) also leads to

$$\left( \frac{\partial \mu}{\partial T} \right)_p = -\frac{s}{n}, \quad \left( \frac{\partial \mu}{\partial p} \right)_T = \frac{1}{n}. \quad (1.11)$$

### 1.2. The microscopic fluxes

For a one-component system, the time derivatives of the microscopic densities can be written as the divergence of microscopic fluxes or currents according to

$$\partial_t \tilde{\mathbf{A}}(\mathbf{r}, t) = -\nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, t). \quad (1.12)$$

Those currents were discussed in §2:2.3. As a summary of the principle notation and results, one has

$$\partial_t \tilde{N} = -\nabla \cdot (m^{-1} \tilde{\mathbf{P}}), \quad (1.13a)$$

$$\partial_t \tilde{\mathbf{P}} = -\nabla \cdot \tilde{\boldsymbol{\tau}}, \quad (1.13b)$$

$$\partial_t \tilde{E} = -\nabla \cdot \tilde{\mathbf{J}}^E, \quad (1.13c)$$

where

$$\tilde{\boldsymbol{\tau}}(\mathbf{k}) \doteq \sum_{i=1}^{\mathcal{N}} [m \mathbf{v}_i \mathbf{v}_i + \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k})] e^{-i\mathbf{k} \cdot \mathbf{x}_i}, \quad (1.14a)$$

$$\tilde{\mathbf{J}}^E(\mathbf{k}) \doteq \sum_{i=1}^{\mathcal{N}} [E_i \mathbf{v}_i + \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k}) \cdot \mathbf{v}_i] e^{-i\mathbf{k} \cdot \mathbf{x}_i}, \quad (1.14b)$$

with  $\Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k})$  being defined by (2:B 6).

### 1.3. The general formula for the dissipative fluxes

Brey *et al.* (1981) show that the general expression for the fluxes through second order is

$$\begin{aligned} \langle \mathbf{J}^\alpha \rangle = & \langle \mathbf{J}^\alpha \rangle_{\text{Euler}} - \underbrace{\mathbf{k}_1^\beta [\mathbf{J}^\alpha] \cdot \nabla B_\beta(\mathbf{r}, t)}_{\text{NS}_\beta^\alpha} - \underbrace{\mathbf{g}_2^\beta [\mathbf{J}^\alpha] : \nabla \nabla B_\beta(\mathbf{r}, t)}_{\text{B}_\beta^\alpha} \\ & - \underbrace{\mathbf{h}_2^{\beta\gamma} [\mathbf{J}^\alpha] : \nabla B_\beta(\mathbf{r}, t) \nabla B_\gamma(\mathbf{r}, t)}_{\text{B}_{\beta\gamma}^\alpha} + \underbrace{\left[ \frac{\partial}{\partial t} \left( \mathbf{k}_2^\beta [\mathbf{J}^\alpha] \cdot \nabla B_\beta(\mathbf{r}, t) \right) \right]}_{\partial \text{B}_\beta^\alpha}^{(1)}. \end{aligned} \quad (1.15)$$

Here the various terms are tersely identified for future reference; NS and B stand for Navier–Stokes and Burnett, respectively. The indices  $\alpha$  and  $\beta$  refer to<sup>1</sup>  $N$ ,  $\mathbf{P}$ , or  $E$ . For example, in the momentum equation the linear Burnett term generates contributions  $\text{B}_\mathbf{P}^\mathbf{P}$  and  $\text{B}_E^\mathbf{P}$ . The various coefficients are defined as

$$\mathbf{k}_1^\beta [\tilde{\mathbf{J}}^\alpha](\mu, t) \doteq \int_0^\infty ds \int d\mathbf{r}' \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\mathbf{J}}^\beta(\mathbf{r}') \rangle_0 = \int_0^\infty ds \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\mathcal{J}}^\beta \rangle_0, \quad (1.16a)$$

$$\mathbf{k}_2^\beta [\tilde{\mathbf{J}}^\alpha](\mu, t) \doteq \int_0^\infty ds s \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\mathcal{J}}^\beta \rangle_0, \quad (1.16b)$$

$$\mathbf{g}_2^\beta [\tilde{\mathbf{J}}^\alpha](\mu, t) \doteq \int_0^\infty ds \int d\mathbf{r}' \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\mathbf{J}}^\beta(\mathbf{r}') \rangle_0 (\mathbf{r}' - \mathbf{r}), \quad (1.16c)$$

$$\mathbf{h}_2^{\beta\gamma} [\tilde{\mathbf{J}}^\alpha](\mu, t) \doteq \int_0^\infty ds \int d\mathbf{r}' \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \mathbf{Q} [\hat{\mathcal{J}}^\beta A'^\gamma(\mathbf{r}')] \rangle_0. \quad (1.16d)$$

(Brey writes these formulas with  $e^{i\mathcal{L}s}$  on the left, which is a permissible manipulation with the Liouville operator.) As discussed in Part 2, the hats denote subtracted quantities. With previous notation, one has specifically  $\tilde{\mathbf{J}}^\alpha = (m^{-1} \tilde{\mathbf{P}}, \tilde{\boldsymbol{\tau}}, \tilde{\mathbf{J}}^E)^\text{T}$ . I shall now work out all of the contributions one by one. Note that all of the terms with  $\alpha = N$  vanish because  $\hat{\mathbf{J}}^N = \mathbf{0}$ . Also, one can make use of the isotropy of the reference state to conclude that various integrals vanish *to lowest order in the gradients* — e.g.,  $\langle \hat{\boldsymbol{\tau}} e^{-i\mathcal{L}s} \hat{\mathbf{J}}^E \rangle = O(\nabla)$ .

<sup>1</sup>In Part 2, lower case is used for these indices.

Finally, one has  $B_\beta = (\beta(\mu - \frac{1}{2}mu^2), \beta\mathbf{u}, -\beta)^T$ . (Here  $\beta$  is being used as both an index and as the inverse temperature.)

In the subsequent reductions, one makes use of the various integrals of correlation functions defined in Eq. (A7) of Brey (1983) and also tabulated in §2:4.2.

## 2. Navier–Stokes terms (NS $_\beta^\alpha$ )

### 2.1. Navier–Stokes momentum flux (NS $_P^P$ )

$$\langle \boldsymbol{\tau} \rangle_{\text{diss}}^{\text{NS}} = - \int_0^\infty ds \langle \widehat{\boldsymbol{\tau}}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}^P \rangle \cdot \nabla \mathbf{B}_P - \int_0^\infty ds \underbrace{\langle \widehat{\boldsymbol{\tau}}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}^E \rangle}_0 \cdot \nabla \mathbf{B}_E \quad (2.1a)$$

$$= - \int_0^\infty ds \langle \widehat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}_{kl} \rangle \nabla_l (\beta u_k) \quad (2.1b)$$

$$= - [K^{\text{I}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K^{\text{II}}\delta_{ij}\delta_{kl}] \frac{1}{T} \nabla_l u_k \quad (2.1c)$$

$$= - \frac{1}{T} \{ K^{\text{I}}[(\nabla \mathbf{u})^T + (\nabla \mathbf{u})] + K^{\text{II}}(\nabla \cdot \mathbf{u}) \mathbf{I} \} \quad (2.1d)$$

$$= - \frac{1}{T} \left[ K^{\text{I}} \left( (\nabla \mathbf{u})^T + (\nabla \mathbf{u}) - \frac{2}{d}(\nabla \cdot \mathbf{u}) \mathbf{I} \right) + \left( K^{\text{II}} + \frac{2}{d}K^{\text{I}} \right) (\nabla \cdot \mathbf{u}) \mathbf{I} \right] \quad (2.1e)$$

$$= -\eta \left( (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T - \frac{2}{d}(\nabla \cdot \mathbf{u}) \mathbf{I} \right) - \zeta (\nabla \cdot \mathbf{u}) \mathbf{I}, \quad (2.1f)$$

where the kinematic and bulk viscosities are

$$\boxed{\eta \doteq \frac{1}{T} K^{\text{I}}}, \quad \boxed{\zeta \doteq \frac{1}{T} \left( K^{\text{II}} + \frac{2}{d} K^{\text{I}} \right)}. \quad (2.2)$$

Notice that the kinematic,  $\eta$  contribution has been constructed to be traceless.

In the limit of weak coupling, it is well known that the bulk viscosity  $\zeta$  vanishes. To show this, one proceeds as follows. One has

$$\zeta \delta_{ij} = \frac{1}{dT} \int_0^\infty ds \langle \widehat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \text{Tr} \widehat{\mathcal{J}} \rangle. \quad (2.3)$$

Now

$$\widehat{\boldsymbol{\tau}}(\mathbf{r}') = \widetilde{\boldsymbol{\tau}}(\mathbf{r}') - \mathbf{I}[p(\mathbf{r}') + N'(\mathbf{r}')p_n + E'(\mathbf{r}')p_e]. \quad (2.4)$$

For the weakly coupled gas, one has  $p \approx nT$  and  $e \approx (d/2)nT$ , so  $(p_n)|_e = 0$  and  $(p_e)|_n = 2/d$ . Since those thermodynamic derivatives are constants, one has

$$\widehat{\mathcal{J}} = \int d\mathbf{r}' [\widetilde{\boldsymbol{\tau}}(\mathbf{r}') - p(\mathbf{r}') \mathbf{I}], \quad (2.5)$$

since  $\int d\mathbf{r}' N'(\mathbf{r}') = 0$  and similarly  $\int d\mathbf{r}' E'(\mathbf{r}') = 0$ . Also, one may ignore the internal-energy contributions to  $\widetilde{\boldsymbol{\tau}}$  and  $p$ . Thus,

$$\widehat{\mathcal{J}} \rightarrow \sum_{i=1}^{\mathcal{N}} m(\mathbf{w}_i \mathbf{w}_i - d^{-1} \langle w_i^2 \rangle \mathbf{I}) \quad (2.6)$$

and

$$\text{Tr} \widehat{\mathcal{J}} \rightarrow \sum_{i=1}^{\mathcal{N}} m(w_i^2 - \langle w_i^2 \rangle). \quad (2.7)$$

In the absence of particle correlations, all particles will be equivalent. The term  $mw^2 - (d/2)T$  is the orthogonalized single-particle kinetic energy — a null eigenfunction of the weakly coupled collision operator that lies in the hydrodynamic subspace. The remainder of the expression for  $\zeta$  will involve a weakly coupled Q; thus,  $\zeta$  vanishes in the limit of weak coupling.

## 2.2. Navier–Stokes energy flux ( $\text{NS}_E^E$ )

$$\langle \mathbf{J}^E \rangle_{\text{diss}}^{\text{NS}} = - \int_0^\infty ds \underbrace{\langle \widehat{\mathbf{J}}^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}^P \rangle_0}_0 \cdot \nabla B_P - \int_0^\infty ds \langle \widehat{\mathbf{J}}^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}^E \rangle_0 \cdot \nabla B_E \quad (2.8a)$$

$$= -(K^{\text{III}} \mathbf{I}) \cdot \nabla (-T^{-1}) \quad (2.8b)$$

$$= -\lambda \nabla T, \quad (2.8c)$$

where

$$\boxed{\lambda \doteq \frac{1}{T^2} K^{\text{III}}.} \quad (2.9)$$

## 3. Burnett terms

### 3.1. Linear Burnett terms ( $\text{B}_\beta^\alpha$ )

#### 3.1.1. Linear Burnett — momentum ( $\text{B}_E^P$ )

$$\begin{aligned} \langle \boldsymbol{\tau} \rangle_{\text{diss}}^{\text{B}_1} &= - \int_0^\infty ds \int d\mathbf{r}' \underbrace{\langle \widehat{\boldsymbol{\tau}}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathbf{J}}^P(\mathbf{r}') \rangle_0}_{0} (\mathbf{r}' - \mathbf{r}) : \nabla \nabla B_P \\ &\quad - \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{\boldsymbol{\tau}}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathbf{J}}^E(\mathbf{r}') \rangle (\mathbf{r}' - \mathbf{r}) : \nabla \nabla B_E \end{aligned} \quad (3.1a)$$

$$= -[K_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K_2\delta_{ij}\delta_{kl}] \nabla_k \nabla_l (-T^{-1}). \quad (3.1b)$$

Now

$$\nabla \nabla (-T^{-1}) = \nabla (T^{-2} \nabla T) = T^{-2} \nabla \nabla T - 2T^{-3} (\nabla T) (\nabla T). \quad (3.2)$$

Thus,

$$\langle \boldsymbol{\tau} \rangle_{\text{diss}}^{\text{B}_1} = -[T^{-2} (K_1 \nabla \nabla T + K_2 \nabla^2 T \mathbf{I})] + 2T^{-3} [2K_1 (\nabla T) (\nabla T) + K_2 |\nabla T|^2 \mathbf{I}] \quad (3.3a)$$

$$= \eta_3 \nabla \nabla T + \eta_4 \nabla^2 T \mathbf{I} + \eta_5^{\text{B}_1} (\nabla T) (\nabla T) + \eta_6^{\text{B}_1} |\nabla T|^2 \mathbf{I}, \quad (3.3b)$$

where

$$\boxed{\eta_3 \doteq -2T^{-2} K_1,} \quad \boxed{\eta_4 \doteq -T^{-2} K_2,} \quad \boxed{\eta_5^{\text{B}_1} \doteq 4T^{-3} K_1,} \quad \boxed{\eta_6^{\text{B}_1} \doteq 2T^{-3} K_2.} \quad (3.4)$$

3.1.2. Linear Burnett — energy ( $B_P^E$ )

$$\begin{aligned} \langle \mathbf{J}^E \rangle_{\text{diss}}^{\text{B}_1} &= - \int_0^\infty ds \int d\mathbf{r}' \langle \hat{\mathbf{J}}^E(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\mathbf{J}}^P(\mathbf{r}') \rangle (\mathbf{r}' - \mathbf{r}) : \nabla \nabla B_P \\ &\quad - \int_0^\infty ds \int d\mathbf{r}' \underbrace{\langle \hat{\mathbf{J}}^E e^{-i\mathcal{L}s} \hat{\mathbf{J}}^E \rangle_0}_{0} (\mathbf{r}' - \mathbf{r}) : \nabla \nabla B_E \end{aligned} \quad (3.5a)$$

$$= - \int_0^\infty ds \int d\mathbf{r}' \langle \hat{J}_i^E(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\tau}_{jk}(\mathbf{r}') \rangle_0 (\mathbf{r}' - \mathbf{r})_l \nabla_k \nabla_l (T^{-1} u_j). \quad (3.5b)$$

Now

$$\nabla_k \nabla_l (T^{-1} u_j) = \nabla_k [T^{-1} \nabla_l u_j - T^{-2} (\nabla_l T) u_j] \quad (3.6a)$$

$$= T^{-1} \nabla_k \nabla_l u_j - T^{-2} [(\nabla_k T) (\nabla_l u_j) + (\nabla_l T) (\nabla_k u_j)]. \quad (3.6b)$$

Thus,

$$\begin{aligned} \langle \mathbf{J}^E \rangle_{\text{diss}}^{\text{B}_1} &= -[K_1(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki}) + K_2\delta_{jk}\delta_{il}] \\ &\quad \times [T^{-1} \nabla_l \nabla_k u_j - T^{-2} [(\nabla_l T) (\nabla_k u_j) + (\nabla_k T) (\nabla_l u_j)]] \end{aligned} \quad (3.7a)$$

$$\begin{aligned} &= -\{T^{-1} K_1 [\nabla^2 \mathbf{u} + \nabla (\nabla \cdot \mathbf{u})] + T^{-1} K_2 \nabla (\nabla \cdot \mathbf{u})\} \\ &\quad + T^{-2} K_1 \{2 \nabla T \cdot (\nabla \mathbf{u}) + [(\nabla \mathbf{u}) \cdot \nabla T] + \nabla T (\nabla \cdot \mathbf{u})\} \\ &\quad + T^{-2} K_2 [\nabla T (\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u}) \cdot \nabla T] \end{aligned} \quad (3.7b)$$

$$\begin{aligned} &= -T^{-1} K_1 \nabla^2 \mathbf{u} - T^{-1} (K_1 + K_2) \nabla (\nabla \cdot \mathbf{u}) \\ &\quad + T^{-2} \{K_1 [2(\nabla \mathbf{u})^T + (\nabla \mathbf{u})] + K_2 (\nabla \mathbf{u})\} \cdot \nabla T \\ &\quad + T^{-2} (K_1 + K_2) \nabla T (\nabla \cdot \mathbf{u}). \end{aligned} \quad (3.7c)$$

Now introduce

$$\mathbf{S} \doteq \frac{1}{2} [(\nabla \mathbf{u})^T + (\nabla \mathbf{u})], \quad \mathbf{\Omega} \doteq \frac{1}{2} [(\nabla \mathbf{u})^T - (\nabla \mathbf{u})] \quad (3.8)$$

such that

$$(\nabla \mathbf{u})^T = \mathbf{S} + \mathbf{\Omega}, \quad (\nabla \mathbf{u}) = \mathbf{S} - \mathbf{\Omega}. \quad (3.9)$$

The second line of the last equation then becomes

$$T^{-2} [(3K_1 + K_2) \mathbf{S} + (K_1 - K_2) \mathbf{\Omega}] \quad (3.10)$$

and one finds

$$\langle \mathbf{J}^E \rangle_{\text{diss}}^{\text{B}_1} = \lambda_2 \nabla (\nabla \cdot \mathbf{u}) + \lambda_3 \nabla^2 \mathbf{u} + \lambda_4^{\text{B}_1} (\nabla \cdot \mathbf{u}) \nabla T + \lambda_5^{\text{B}_1} \mathbf{S} \cdot \nabla T + \lambda_6^{\text{B}_1} \mathbf{\Omega} \cdot \nabla T, \quad (3.11)$$

where

$$\boxed{\lambda_2 \doteq -T^{-1} (K_1 + K_2)}, \quad \boxed{\lambda_3 \doteq -T^{-1} K_1}, \quad (3.12)$$

and

$$\boxed{\lambda_4^{\text{B}_1} \doteq T^{-2} (K_1 + K_2)}, \quad \boxed{\lambda_5^{\text{B}_1} \doteq T^{-2} (3K_1 + K_2)}, \quad \boxed{\lambda_6^{\text{B}_1} \doteq T^{-2} (K_1 - K_2)}. \quad (3.13)$$

3.2. Nonlinear Burnett terms ( $B_{\beta\gamma}^\alpha$ )

The terms involving  $(\nabla B)(\nabla B)$  stem from  $\mathbf{h}_2^{\beta\gamma}[\mathbf{J}^\alpha](\mu, t)$ , which contains  $Q = 1 - P$ . I shall work out the terms coming from the 1 and coming from the P separately.

I shall signify the various terms by  $B_{\beta\gamma}^\alpha \equiv (1 - P)(\alpha, \beta, \gamma)$ . (Of course, the  $P$  does not multiply the entire term; this is just a notation.) Note that the  $\beta$  component involves a total flux  $\mathcal{J}^\beta$ ; thus, there is no  $\beta = N$  contribution.

### 3.2.1. Nonlinear Burnett — momentum

Since  $\mathbf{P}$  is odd in velocity, the  $(\beta, \gamma)$  components must be either both even or both odd. Therefore,

$$\langle \boldsymbol{\tau} \rangle_{\text{diss}}^{B(2)} = (1 - P)[(\mathbf{P}, E, N) + (\mathbf{P}, E, E) + (\mathbf{P}, \mathbf{P}, \mathbf{P})]. \quad (3.14)$$

$(\mathbf{P}, E, N)$ :

$$(\mathbf{P}, E, N) = - \int_0^\infty ds \int d\mathbf{r}' \langle \hat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}_k^E N'(\mathbf{r}') \rangle (\mathbf{r}' - \mathbf{r})_l (\nabla_k B_E) (\nabla_l B_N) \quad (3.15a)$$

$$= -[K_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K_4\delta_{ij}\delta_{kl}](T^{-2}\nabla_k T)(\nabla_l B_N) \quad (3.15b)$$

$$= -T^{-2}[K_3(\nabla T \nabla B_N + \nabla B_N \nabla T) + K_4(\nabla T \cdot \nabla B_N)\mathbf{I}]. \quad (3.15c)$$

Now

$$\nabla B_N = \nabla \left[ \beta \left( \mu - \frac{1}{2} m u^2 \right) \right] \rightarrow \nabla(\beta \mu) \quad (3.16a)$$

$$= -T^{-2} \mu \nabla T + T^{-1} \nabla \mu. \quad (3.16b)$$

One can think of  $\mu$  as depending on  $T$  and  $P$ . Thus,

$$\nabla B_N = -\frac{1}{T} \left[ \frac{\mu}{T} - \left( \frac{\partial \mu}{\partial T} \right)_p \right] \nabla T + \frac{1}{T} \left[ \left( \frac{\partial \mu}{\partial p} \right)_T \right] \nabla p. \quad (3.17)$$

One only needs to evaluate the thermodynamic derivatives in the reference ensemble, so one can use the local thermodynamic relations discussed in §1.1. From (1.11), (3.17) becomes

$$\nabla B_N = -T^{-2}(\mu + T s n^{-1}) \nabla T + T^{-1} n^{-1} \nabla p \quad (3.18a)$$

$$= (nT)^{-1} (-hT^{-1} \nabla T + \nabla p), \quad (3.18b)$$

where again  $h \doteq e + p$  and the Euler equation (1.6) was used.

Thus,

$$(\mathbf{P}, E, N) = -(\bar{n}T^3)^{-1} \{ K_3[\nabla T(-T^{-1}h \nabla T + \nabla p) + (-T^{-1}h \nabla T + \nabla p) \nabla T] \\ + K_4 \nabla T \cdot (-T^{-1}h \nabla T + \nabla p) \mathbf{I} \} \quad (3.19a)$$

$$= (\bar{n}T^4)^{-1} [2K_3 h \nabla T \nabla T + K_4 h |\nabla T|^2 \mathbf{I}] \\ - (\bar{n}T^3)^{-1} [K_3(\nabla T \nabla p + \nabla p \nabla T) + K_4 \nabla T \cdot \nabla p \mathbf{I}] \quad (3.19b)$$

$$= \eta_5^{B_{EN}^{(2)}} \nabla T \nabla T + \eta_6^{B_{EN}^{(2)}} |\nabla T|^2 + \eta_7^{B_{EN}^{(2)}} (\nabla T \nabla p + \nabla p \nabla T) + \eta_8^{B_{EN}^{(2)}} \nabla T \cdot \nabla p \mathbf{I}, \quad (3.19c)$$

where

$$\boxed{\eta_5^{B_{EN}^{(2)}} \doteq 2(\bar{n}T^4)^{-1} h K_3,} \quad \boxed{\eta_6^{B_{EN}^{(2)}} \doteq (\bar{n}T^4)^{-1} h K_4,} \quad (3.20)$$

and

$$\boxed{\eta_7^{B_{EN}^{(2)}} \doteq -(\bar{n}T^3)^{-1} K_3,} \quad \boxed{\eta_8^{B_{EN}^{(2)}} \doteq -(\bar{n}T^3)^{-1} K_4.} \quad (3.21)$$

---

$-\mathcal{P}(\mathbf{P}, E, N)$ :

$$-\mathcal{P}(\mathbf{P}, E, N) = \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \mathcal{P}[\widehat{\mathcal{J}}_k^E N'(\mathbf{r}')] \rangle (\mathbf{r}' - \mathbf{r})_l (\nabla_k B_E) (\nabla_l B_N). \quad (3.22)$$

Now

$$\begin{aligned} \mathcal{P}[\widehat{\mathcal{J}}_k^E N'(\mathbf{r}')] &= \underbrace{\langle \widehat{\mathcal{J}}_k^E N'(\mathbf{r}') \rangle}_0 + (N' E') * \mathcal{M}_2^{-1} * \underbrace{\langle (N' E')^T \widehat{\mathcal{J}}_k^E N'(\mathbf{r}') \rangle}_0 \\ &\quad + \mathbf{P}' * \mathcal{M}_{\mathbf{P}\mathbf{P}}^{-1} * \langle \mathbf{P}' \widehat{\mathcal{J}}_k^E N'(\mathbf{r}') \rangle. \end{aligned} \quad (3.23)$$

The averages vanish because they are vectors, so they must be proportional to  $\mathbf{u}$ , which vanishes in the local frame. Because  $\mathcal{M}_{\mathbf{P}\mathbf{P}}(\bar{\mathbf{x}}, \bar{\mathbf{x}}') = m\bar{n}T\delta(\bar{\mathbf{x}} - \bar{\mathbf{x}}')$ , the last term is

$$\begin{aligned} &\int d\bar{\mathbf{x}} d\bar{\mathbf{x}}' \mathbf{P}'(\bar{\mathbf{x}}) \cdot \mathcal{M}_{\mathbf{P}\mathbf{P}}^{-1}(\bar{\mathbf{x}}, \bar{\mathbf{x}}') \cdot \langle \mathbf{P}'(\bar{\mathbf{x}}') \widehat{\mathcal{J}}^E N'(\mathbf{r}') \rangle_0 \\ &= (m\bar{n}T)^{-1} \int d\bar{\mathbf{x}} \mathbf{P}'(\bar{\mathbf{x}}) \cdot \langle \mathbf{P}'(\bar{\mathbf{x}}) \widehat{\mathcal{J}}^E N'(\mathbf{r}') \rangle_0. \end{aligned} \quad (3.24)$$

Now

$$\begin{aligned} \langle \mathbf{P}'(\bar{\mathbf{x}}) \widehat{\mathcal{J}}^E N'(\mathbf{r}') \rangle &= \left\langle \left( \sum_{i=1}^{\mathcal{N}} m\mathbf{w}_i \delta(\bar{\mathbf{x}} - \mathbf{x}_i) \right) \right. \\ &\quad \times \left. \left[ \sum_{j=1}^{\mathcal{N}} E_j \mathbf{w}_j + \Delta \tilde{\boldsymbol{\tau}}_j \cdot \mathbf{w}_j - \left( \frac{h}{n} \right)_i \mathbf{w}_j \right] N'(\mathbf{r}') \right\rangle_0. \end{aligned} \quad (3.25)$$

Averaging over velocity restricts  $j$  to  $i$ . Thus,

$$\langle \mathbf{P}'(\bar{\mathbf{x}}) \widehat{\mathcal{J}}^E N'(\mathbf{r}') \rangle = \sum_{i=1}^{\mathcal{N}} \left\langle \delta(\bar{\mathbf{x}} - \mathbf{x}_i) m\mathbf{w}_i \left[ E_i \mathbf{I} + \Delta \tilde{\boldsymbol{\tau}}_i - \left( \frac{h}{n} \right)_i \mathbf{I} \right] \cdot \mathbf{w}_i N'(\mathbf{r}') \right\rangle_0 \quad (3.26a)$$

$$= T \mathbf{I} \sum_{i=1}^{\mathcal{N}} \left\langle \delta(\bar{\mathbf{x}} - \mathbf{x}_i) \left[ \frac{5}{2} T + U_i + \frac{1}{3} \text{Tr} \Delta \tilde{\boldsymbol{\tau}}_i - \left( \frac{h}{n} \right)_i \right] N'(\mathbf{r}') \right\rangle_0 \quad (3.26b)$$

$$= T \mathbf{I} \left\langle \left( \frac{h}{n} \right)' (\bar{\mathbf{x}}) N'(\mathbf{r}') \right\rangle_0. \quad (3.26c)$$

One may replace  $\langle \dots \rangle_0 \approx \langle \dots \rangle_{\mathbf{B}}$ . Then

$$\langle \mathbf{P}'(\bar{\mathbf{x}}) \widehat{\mathcal{J}}^E N'(\mathbf{r}') \rangle = \bar{n} T \mathbf{I} \frac{\delta h(\bar{\mathbf{x}})}{\delta[(\beta\mu)(\mathbf{r}')]}. \quad (3.27)$$

The functional derivative essentially gives back a  $\delta(\bar{\mathbf{x}} - \mathbf{r}')$ , so performing the integral in the projection brings one to

$$\begin{aligned} -\mathcal{P}(\mathbf{P}, E, E) &= [K_{20}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K_{21}\delta_{ij}\delta_{kl}] \\ &\quad \times \left[ \frac{\partial}{\partial(\beta\mu)} \left( \frac{h}{mn} \right) \right]_{\beta} (T^{-2} \nabla_k T) [(nT)^{-1} (-hT^{-1} \nabla_l T + \nabla_l p)]. \end{aligned} \quad (3.28)$$

From the Gibbs–Duhem relation

$$s \, dT - dp + n \, d\mu = 0 \quad (3.29)$$

at constant  $T$ , one finds

$$\left(\frac{\partial\chi}{\partial(\beta\mu)}\right)_\beta = nT \left(\frac{\partial\chi}{\partial p}\right)_T. \quad (3.30)$$

Thus, the pressure term gives a contribution

$$-P(\mathbf{P}, E, N)_{\nabla p} = \eta_7^{\text{B}_{PEN}^{(2)}} (\nabla T \nabla p + \nabla p \nabla T) + \eta_8^{\text{B}_{PEN}^{(2)}} (\nabla T \cdot \nabla p), \quad (3.31)$$

where

$$\boxed{\eta_7^{\text{B}_{PEN}^{(2)}} \doteq T^{-2} \frac{\partial}{\partial p} \left( \frac{h}{mn} \right) K_{20}}, \quad \boxed{\eta_8^{\text{B}_{PEN}^{(2)}} \doteq T^{-2} \frac{\partial}{\partial p} \left( \frac{h}{mn} \right) K_{21}}. \quad (3.32)$$

The remaining term in  $-h\nabla T$  will be canceled by a contribution from  $-P(\mathbf{P}, E, E)$ .

$(\mathbf{P}, E, E)$ :

$$(\mathbf{P}, E, E) = - \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}_k^E E'(\mathbf{r}') \rangle (\mathbf{r}' - \mathbf{r})_l (\nabla_k B_E) (\nabla_l B_E) \quad (3.33a)$$

$$= -[K_5(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K_6\delta_{ij}\delta_{kl}] T^{-4} \nabla_k T \nabla_l T \quad (3.33b)$$

$$= -T^{-4} [2K_5 \nabla T \nabla T + K_6 |\nabla T|^2 \mathbf{I}] \quad (3.33c)$$

$$= \eta_5^{\text{B}_{EE}^{(2)}} \nabla T \nabla T + \eta_6^{\text{B}_{EE}^{(2)}} |\nabla T|^2 \mathbf{I}, \quad (3.33d)$$

where

$$\boxed{\eta_5^{\text{B}_{EE}^{(2)}} \doteq -2T^{-4} K_5}, \quad \boxed{\eta_6^{\text{B}_{EE}^{(2)}} \doteq -T^{-4} K_6}. \quad (3.34)$$

$-P(\mathbf{P}, E, E)$ :

The evaluation of this term proceeds along the same lines as for  $-P(\mathbf{P}, E, N)$ , and leads to a contribution proportional to

$$\frac{\delta}{\delta[-\beta(\mathbf{r}')] } \left( \frac{h(\overline{\mathbf{x}})}{mn} \right) T^{-4} \nabla T \nabla T. \quad (3.35)$$

Now

$$\left( \frac{\partial}{\partial(-\beta)} \right)_{\beta\mu} = \left( \frac{\partial}{\partial(-\beta)} \right)_p + \left( \frac{\partial p}{\partial(-\beta)} \right)_{\beta\mu} \left( \frac{\partial}{\partial p} \right)_T \quad (3.36a)$$

$$= T^2 \left( \frac{\partial h}{\partial T} \right)_p + T^2 \left( \frac{\partial p}{\partial T} \right)_{\beta\mu} \left( \frac{\partial h}{\partial p} \right)_T. \quad (3.36b)$$

To evaluate  $\partial p/\partial T$ , rewrite the Gibbs-Duhem relation

$$s dT - dp + n d\mu = 0 \quad (3.37)$$

as

$$s dT - dp + n\beta^{-1} d(\beta\mu) - n\mu \underbrace{\beta d\beta}_{dT/T} = 0 \quad (3.38)$$

or at constant  $\beta\mu$ ,

$$\left( \frac{\partial p}{\partial T} \right)_{\beta\mu} = \beta(Ts + n\mu) = \beta h. \quad (3.39)$$

The  $\partial/\partial p$  term cancels with the  $\nabla T$  term in  $-P(\mathbf{P}, E, N)$ . Thus, one gets a contribution

$$-P(\mathbf{P}, E, E)_{\nabla T} = \eta_5^{B_{PEE}^{(2)}} \nabla T \nabla T + \eta_6^{B_{PEE}^{(2)}} |\nabla T|^2, \quad (3.40)$$

where

$$\boxed{\eta_5^{B_{PEE}^{(2)}} \doteq T^{-2} \left[ \frac{\partial}{\partial T} \left( \frac{h}{mn} \right) \right]_p K_{20}}, \quad \boxed{\eta_6^{B_{PEE}^{(2)}} \doteq T^{-2} \left[ \frac{\partial}{\partial T} \left( \frac{h}{mn} \right) \right]_p K_{21}}. \quad (3.41)$$

$(\mathbf{P}, \mathbf{P}, \mathbf{P})$ :

$$(\mathbf{P}, \mathbf{P}, \mathbf{P}) = - \int_0^\infty ds \int d\mathbf{r}' \langle \hat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{T}}_{kl} G'_m(\mathbf{r}') \rangle_0 (\mathbf{r}' - \mathbf{r})_n \nabla_l (T^{-1} u_k) \nabla_n (T^{-1} u_m) \quad (3.42a)$$

$$\begin{aligned} &= -\{K_7 \delta_{ij} \delta_{kl} \delta_{mn} + K_8 \delta_{ij} (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}) \\ &\quad + K_9 \delta_{kl} (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) + K_{10} \delta_{mn} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ &\quad + K_{11} [\delta_{km} (\delta_{il} \delta_{jn} + \delta_{jl} \delta_{in}) + \delta_{lm} (\delta_{ik} \delta_{jn} + \delta_{jk} \delta_{in})] \\ &\quad + K_{12} [\delta_{kn} (\delta_{il} \delta_{jm} + \delta_{jl} \delta_{im}) + \delta_{ln} (\delta_{ik} \delta_{jm} + \delta_{jk} \delta_{im})]\} \\ &\quad \times T^{-2} (\nabla_l u_k) (\nabla_n u_m) \end{aligned} \quad (3.42b)$$

$$\begin{aligned} &= -T^{-2} K_7 (\nabla \cdot \mathbf{u})^2 \mathbf{I} - T^{-2} K_8 (\nabla_l u_k) (\nabla_l u_k + \nabla_k u_l) \mathbf{I} \\ &\quad - T^{-2} \{K_9 (\nabla \cdot \mathbf{u}) [(\nabla \mathbf{u})^T + (\nabla \mathbf{u})] + K_{10} (\nabla \cdot \mathbf{u}) [(\nabla \mathbf{u})^T + (\nabla \mathbf{u})]\} \\ &\quad - T^{-2} K_{11} [(\nabla_i u_k) (\nabla_j u_k) + (\nabla_j u_k) (\nabla_i u_k) \\ &\quad \quad + (\nabla_l u_i) (\nabla_j u_l) + (\nabla_l u_j) (\nabla_i u_l)] \\ &\quad - T^{-2} K_{12} [(\nabla_i u_k) (\nabla_k u_j) + (\nabla_j u_k) (\nabla_k u_i) \\ &\quad \quad + (\nabla_l u_i) (\nabla_l u_j) + (\nabla_l u_j) (\nabla_l u_i)]. \end{aligned} \quad (3.42c)$$

$$\begin{aligned} &= -T^{-2} K_7 (\nabla \cdot \mathbf{u})^2 \mathbf{I} - T^{-2} K_8 (\mathbf{S} + \mathbf{\Omega}) : (2\mathbf{S}) \mathbf{I} - T^{-2} (K_9 + K_{10}) (\nabla \cdot \mathbf{u}) (2\mathbf{S}) \\ &\quad - T^{-2} \{K_{11} [2(\nabla \mathbf{u}) \cdot (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T \cdot (\nabla \mathbf{u})^T + (\nabla \mathbf{u}) \cdot (\nabla \mathbf{u})] \\ &\quad + K_{12} [(\nabla \mathbf{u}) \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \cdot (\nabla \mathbf{u})^T + 2(\nabla \mathbf{u})^T \cdot (\nabla \mathbf{u})]\}. \end{aligned} \quad (3.42d)$$

The last two lines are

$$\begin{aligned} &-T^{-2} \{K_{11} [2(\mathbf{S} - \mathbf{\Omega}) \cdot (\mathbf{S} + \mathbf{\Omega}) + (\mathbf{S} + \mathbf{\Omega}) \cdot (\mathbf{S} + \mathbf{\Omega}) + (\mathbf{S} - \mathbf{\Omega}) \cdot (\mathbf{S} - \mathbf{\Omega})] \\ &\quad + K_{12} [(\mathbf{S} - \mathbf{\Omega}) \cdot (\mathbf{S} - \mathbf{\Omega}) + (\mathbf{S} + \mathbf{\Omega}) \cdot (\mathbf{S} + \mathbf{\Omega}) + 2(\mathbf{S} + \mathbf{\Omega}) \cdot (\mathbf{S} - \mathbf{\Omega})] \end{aligned} \quad (3.43a)$$

$$= -T^{-2} [4(K_{11} + K_{12}) \mathbf{S} \cdot \mathbf{S} + 2(K_{11} - K_{12}) (\mathbf{S} \cdot \mathbf{\Omega} - \mathbf{\Omega} \cdot \mathbf{S})]. \quad (3.43b)$$

Thus,

$$\begin{aligned} (\mathbf{P}, \mathbf{P}, \mathbf{P}) &= \eta_9^{B_{PP}^{(2)}} (\nabla \cdot \mathbf{u}) \mathbf{I} + \eta_{10}^{B_{PP}^{(2)}} \mathbf{S} : \mathbf{S} + \eta_{11}^{B_{PP}^{(2)}} (\nabla \cdot \mathbf{u}) \mathbf{S} \\ &\quad + \eta_{12}^{B_{PP}^{(2)}} \mathbf{S} \cdot \mathbf{S} + \eta_{13}^{B_{PP}^{(2)}} (\mathbf{S} \cdot \mathbf{\Omega} - \mathbf{\Omega} \cdot \mathbf{S}), \end{aligned} \quad (3.44a)$$

where

$$\boxed{\eta_9^{B_{PP}^{(2)}} \doteq -T^{-2} K_7}, \quad \boxed{\eta_{10}^{B_{PP}^{(2)}} \doteq -2T^{-2} K_8}, \quad \boxed{\eta_{11}^{B_{PP}^{(2)}} \doteq -2T^{-2} (K_9 + K_{10})}, \quad (3.45)$$

and

$$\boxed{\eta_{12}^{B_{PP}^{(2)}} \doteq -4T^{-2} (K_{11} + K_{12})}, \quad \boxed{\eta_{13}^{B_{PP}^{(2)}} \doteq -2T^{-2} (K_{11} - K_{12})}. \quad (3.46)$$

---

$-\mathbf{P}(\mathbf{P}, \mathbf{P}, \mathbf{P})$ :

$$-\mathbf{P}(\mathbf{P}, \mathbf{P}, \mathbf{P}) = \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{\tau}_{ij}(\mathbf{r}) e^{-i\mathcal{L}s} \mathbf{P} \widehat{\mathcal{T}}_{mn} G'_s(\mathbf{r}') \rangle (\mathbf{r}' - \mathbf{r})_l T^{-2} (\nabla_n u_m) (\nabla_l u_s). \quad (3.47)$$

Symmetry implies that only  $\mathbf{P}'$  enters the projection. Thus, one needs to calculate  $\langle \mathbf{P}'(\mathbf{x}) \widehat{\mathcal{T}} \mathbf{P}'(\mathbf{r}') \rangle_{kmns}$ . Recall that

$$\widehat{\boldsymbol{\tau}} = \boldsymbol{\tau} - \mathbf{I}(p + N'p_n + E'p_e). \quad (3.48)$$

Therefore,

$$\widehat{\mathcal{T}} = \mathcal{T} - \mathbf{I} \left[ \int d\mathbf{y} p(\mathbf{y}) + \underbrace{\left( \widetilde{\mathcal{E}} - \int d\mathbf{y} e(\mathbf{y}) \right) \left( \frac{\partial p}{\partial e} \right)_n}_{p_e} \right]. \quad (3.49)$$

[Note  $\int d\mathbf{y} N'(\mathbf{y}) = 0$ .] One thus has

$$\begin{aligned} \langle \mathbf{P}'(\mathbf{x}) \widehat{\mathcal{T}} \mathbf{P}'(\mathbf{r}') \rangle &= m \sum_{ijk} \left\langle \delta(\mathbf{x} - \mathbf{x}_i) \delta(\mathbf{r}' - \mathbf{x}_k) \right. \\ &\quad \times \left[ m\mathbf{w}_i(m\mathbf{w}_j\mathbf{w}_j + \Delta\boldsymbol{\tau}_j) - \mathbf{I}N^{-1} \left( \int p + \sum_l (E_l - \langle E_l \rangle) \right) \right] \mathbf{w}_k \Big\rangle. \end{aligned} \quad (3.50a)$$

Consider the integration over velocity directions. If  $k \neq i$ , this vanishes. There are then the possibilities  $j \neq i$  (with  $l = j$  and  $\lambda_{\neq j}$ ) and  $j = i$ . For  $j \neq i$ , one has

$$\left\langle m\mathbf{w}_j\mathbf{w}_j + \Delta\boldsymbol{\tau}_j - \mathbf{I}N - 1 \int p \right\rangle = 0. \quad (3.51)$$

The energy terms contribute

$$-\mathbf{I}_{mn} \langle m\mathbf{w}_i\mathbf{w}_i (E_i - \langle E_i \rangle) \rangle = -a\delta_{ks}\delta_{mn}, \quad (3.52)$$

where

$$a = \frac{1}{3} \left\langle mw^2 \left( \frac{1}{2}m(w^2 - \langle w^2 \rangle) \right) \right\rangle = \frac{1}{6}T^2(15 - 9) = T^2. \quad (3.53)$$

Remaining is

$$m^2 \left\langle \mathbf{w} \left( \mathbf{w}\mathbf{w} - \frac{1}{3}\langle w^2 \rangle \mathbf{I} \right) \mathbf{w} \right\rangle_{kmns} = a\delta_{ks}\delta_{mn} + b(\delta_{km}\delta_{ns} + \delta_{kn}\delta_{ms}). \quad (3.54)$$

Taking traces with  $(ks)$  and  $(mn)$  gives

$$3(3a + 2b) = m^2 \langle w^2(w^2 - \langle w^2 \rangle) \rangle = 6. \quad (3.55)$$

Taking traces with  $(km)$  and  $(ns)$  gives

$$3a + 12b = \langle w^2 w^2 - \frac{1}{3}\langle w^2 \rangle w^2 \rangle = 15 - 3 = 12. \quad (3.56)$$

The solution of the system

$$3a + 2b = 2, \quad (3.57a)$$

$$3a + 12b = 12 \quad (3.57b)$$

is  $a = 0$  and  $b = 1$ . Therefore,

$$\langle \mathbf{P}'(\bar{\mathbf{x}}) \widehat{\mathcal{T}} \mathbf{P}'(\mathbf{r}') \rangle_{kmns} = mnT^2 \delta(\bar{\mathbf{x}} - \mathbf{r}') (\delta_{km} \delta_{ns} + \delta_{kn} \delta_{ms} - p_e \delta_{ks} \delta_{mn}). \quad (3.58)$$

Upon dividing by  $m\bar{n}T$  (from  $\mathcal{M}_{PP}$ ) and performing the integral over  $\bar{\mathbf{x}}$ , one has

$$\begin{aligned} & -P(\mathbf{P}, \mathbf{P}, \mathbf{P}) \\ &= T^{-1} [K_{20}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + K_{21} \delta_{ij} \delta_{kl}] \\ & \quad \times [\delta_{km} \delta_{ns} + \delta_{kn} \delta_{ms} - p_e \delta_{ks} \delta_{mn}] (\nabla_n u_m) (\nabla_l u_s) \end{aligned} \quad (3.59a)$$

$$\begin{aligned} &= T^{-1} [K_{20}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + K_{21} \delta_{ij} \delta_{kl}] \\ & \quad \times [(\nabla_n u_k) (\nabla_l u_n) + (\nabla_k u_m) (\nabla_l u_m) - p_e (\nabla \cdot \mathbf{u}) (\nabla_l u_k)] \end{aligned} \quad (3.59b)$$

$$\begin{aligned} &= T^{-1} K_{20} \{ (\nabla_n u_i) (\nabla_j u_n) + (\nabla_n u_j) (\nabla_i u_n) + (\nabla_i u_m) (\nabla_j u_m) + (\nabla_j u_m) (\nabla_i u_m) \\ & \quad - p_e (\nabla \cdot \mathbf{u}) [(\nabla_j u_i) + (\nabla_i u_j)] \} \\ & \quad + T^{-1} K_{21} \delta_{ij} [(\nabla_n u_l) (\nabla_l u_n) + (\nabla_k u_m) (\nabla_k u_m) - p_e (\nabla \cdot \mathbf{u})^2] \end{aligned} \quad (3.59c)$$

$$\begin{aligned} &= T^{-1} K_{20} [(\nabla \mathbf{u})^T \cdot (\nabla \mathbf{u})^T + (\nabla \mathbf{u}) \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u}) \cdot (\nabla \mathbf{u})^T + (\nabla \mathbf{u}) \cdot (\nabla \mathbf{u})^T \\ & \quad - p_e (\nabla \cdot \mathbf{u}) ((\nabla \mathbf{u})^T + (\nabla \mathbf{u}))] \\ & \quad + T^{-1} K_{21} \mathbf{I} \{ \text{Tr}[(\nabla \mathbf{u})^T + (\nabla \mathbf{u})] \cdot (\nabla \mathbf{u}) - p_e (\nabla \cdot \mathbf{u})^2 \} \end{aligned} \quad (3.59d)$$

$$= T^{-1} K_{20} [4\mathbf{S} \cdot \mathbf{S} + 2(\mathbf{S} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \mathbf{S}) - 2p_e (\nabla \cdot \mathbf{u}) \mathbf{S}] + T^{-1} K_{21} \mathbf{I} [2\mathbf{S} : \mathbf{S} - p_e (\nabla \cdot \mathbf{u})^2] \quad (3.59e)$$

$$\begin{aligned} &= \eta_{12}^{B_{PPP}^{(2)}} \mathbf{S} \cdot \mathbf{S} + \eta_{13}^{B_{PPP}^{(2)}} (\mathbf{S} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \mathbf{S}) + \eta_{11}^{B_{PPP}^{(2)}} (\nabla \cdot \mathbf{u}) \mathbf{S} \\ & \quad + \eta_{10}^{B_{PPP}^{(2)}} \mathbf{S} : \mathbf{S} + \eta_9^{B_{PPP}^{(2)}} (\nabla \cdot \mathbf{u})^2 \mathbf{I}, \end{aligned} \quad (3.59f)$$

where

$$\boxed{\eta_9^{B_{PPP}^{(2)}} \doteq -T^{-1} p_e K_{21}}, \quad \boxed{\eta_{10}^{B_{PPP}^{(2)}} \doteq 2T^{-1} K_{21}}, \quad (3.60)$$

and

$$\boxed{\eta_{11}^{B_{PPP}^{(2)}} \doteq -2T^{-1} p_e K_{20}}, \quad \boxed{\eta_{12}^{B_{PPP}^{(2)}} \doteq 4T^{-1} K_{20}}, \quad \boxed{\eta_{13}^{B_{PPP}^{(2)}} \doteq 2T^{-1} K_{20}}. \quad (3.61)$$

### 3.2.2. Nonlinear Burnett — energy

Again there are no  $\beta = N$  terms, so

$$\langle \mathbf{J}^E \rangle_{\text{diss}}^{B^{(2)}} = (E, \mathbf{P}, N) + (E, \mathbf{P}, E) + (E, E, \mathbf{P}). \quad (3.62)$$

$(E, E, \mathbf{P})$ :

$$\begin{aligned} (E, E, \mathbf{P}) &= - \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{J}_i^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}' - \mathbf{r})_l [\nabla_j (-T^{-1})] (\nabla_l (T^{-1} u_k)) \rangle \\ & \quad (3.63a) \end{aligned}$$

$$= -(K_{13} \delta_{ij} \delta_{kl} + K_{14} \delta_{ik} \delta_{jl} + K_{15} \delta_{il} \delta_{jk}) T^{-3} (\nabla_j T) (\nabla_l u_k) \quad (3.63b)$$

$$= -T^{-3} [K_{13} (\nabla \cdot \mathbf{u}) \nabla T + K_{14} (\nabla \mathbf{u})^T \cdot \nabla T + K_{15} (\nabla \mathbf{u}) \cdot \nabla T] \quad (3.63c)$$

$$= \lambda_4^{B_{EP}^{(2)}} (\nabla \cdot \mathbf{u}) \nabla T + \lambda_5^{B_{EP}^{(2)}} \mathbf{S} \cdot \nabla T + \lambda_6^{B_{EP}^{(2)}} \boldsymbol{\Omega} \cdot \nabla T, \quad (3.63d)$$

where

$$\boxed{\lambda_4^{\text{B}_{EP}^{(2)}} \doteq -T^{-3}K_{13}}, \quad \boxed{\lambda_5^{\text{B}_{EP}^{(2)}} \doteq -T^{-3}(K_{14} + K_{15})}, \quad \boxed{\lambda_6^{\text{B}_{EP}^{(2)}} \doteq -T^{-3}(K_{14} - K_{15})}. \quad (3.64)$$

$-P(E, E, \mathbf{P})$ :

$$-P(E, E, \mathbf{P}) = T^{-3} \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{J}_i^E(\mathbf{r}) e^{-i\mathcal{L}s} P[\widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}')] \rangle (\mathbf{r}' - \mathbf{r})_l (\nabla_j T) (\nabla_l u_k). \quad (3.65a)$$

By symmetry, only the scalar quantities enter the projection. One thus needs to work out  $\langle N'(\overline{\mathbf{x}}) \widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}') \rangle$  and  $\langle E'(\overline{\mathbf{x}}) \widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}') \rangle$ . Then

$$\begin{aligned} -P(E, E, \mathbf{P}) = T^{-3} & \left( \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{J}_i^E e^{-i\mathcal{L}s} N'(\overline{\mathbf{x}}) \rangle (\mathbf{r}' - \mathbf{r})_l \right)^T \\ & \cdot \mathcal{M}_2^{-1}(\overline{\mathbf{x}}, \overline{\mathbf{x}}') \cdot \begin{pmatrix} \langle N'(\overline{\mathbf{x}}') \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}') \rangle \\ \langle E'(\overline{\mathbf{x}}') \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}') \rangle \end{pmatrix} (\nabla_j T) (\nabla_l u_k). \end{aligned} \quad (3.66a)$$

From Novikov's theorem (Krommes 2015, appendix B, and references therein) for arbitrary  $A$  ( $= N'$  or  $E'$ ),

$$\mathcal{M}_2^{-1}(\overline{\mathbf{x}}, \overline{\mathbf{x}}') \cdot \begin{pmatrix} \langle N'(\overline{\mathbf{x}}') \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}') \rangle \\ \langle E'(\overline{\mathbf{x}}') \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}') \rangle \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\delta[\widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}')] }{\delta N'(\overline{\mathbf{x}})} \right\rangle \\ \left\langle \frac{\delta[\widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}')] }{\delta E'(\overline{\mathbf{x}})} \right\rangle \end{pmatrix}. \quad (3.67)$$

Now

$$\widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}') = \sum_{i=1}^{\mathcal{N}} [E_i \mathbf{v}_i + \Delta \boldsymbol{\tau}_i \cdot \mathbf{v}_i - (h/n) \mathbf{v}_i] \sum_{j=1}^{\mathcal{N}} m \mathbf{v}_j \delta(\mathbf{r}' - \mathbf{x}_j). \quad (3.68)$$

Because  $N'$  and  $E'$  are scalars, one can average over the velocity angles. Individual particles are uncorrelated, so  $j = i$ . Isotropy makes the result proportional to the unit tensor. Thus,

$$\langle \widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}') \rangle_\Omega = \sum_{i=1}^{\mathcal{N}} \left[ \left( \frac{1}{2} m v_i^2 + U_i \right) \frac{1}{3} m v_i^2 \mathbf{I} + \Delta \boldsymbol{\tau} \left( \frac{1}{3} m v_i^2 \right) - \left( \frac{h}{n} \right) \left( \frac{1}{3} m v_i^2 \right) \mathbf{I} \right] \delta(\mathbf{r}' - \mathbf{x}_i). \quad (3.69)$$

Now average over velocity magnitudes:

$$\langle \widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}') \rangle_v = \sum_{i=1}^{\mathcal{N}} \left[ \left( \frac{5}{2} T^2 + T U_i \right) \mathbf{I} + \Delta \boldsymbol{\tau}_i - \left( \frac{h}{n} \right) T \mathbf{I} \right] \delta(\mathbf{r}' - \mathbf{x}_i). \quad (3.70)$$

After the remaining average, the entire result will be proportional to  $\mathbf{I}$ , so  $\Delta \boldsymbol{\tau}_i \rightarrow \frac{1}{3} \text{Tr } \Delta \boldsymbol{\tau}$ . Note that the last,  $h$  term is proportional to  $\tilde{N}(\mathbf{r}')$ . One has

$$- \left( \frac{h}{n} \right) \tilde{N} = - \left( \frac{h}{n} \right) N' - h, \quad (3.71)$$

so

$$\langle \widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}') \rangle_v = T \left[ \underbrace{\frac{3}{2} N'(\mathbf{r}') T + U'(\mathbf{r}') + P'(\mathbf{r}')}_{E'} - \left( \frac{h}{n} \right) N'(\mathbf{r}') \right], \quad (3.72)$$

where the identification with  $E'$  is made at constant  $T$ . Thus,

$$\begin{aligned} \mathcal{M}_2^{-1}(\overline{\mathbf{x}}, \overline{\mathbf{x}}') \cdot \begin{pmatrix} \langle N'(\overline{\mathbf{x}}') \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}') \rangle \\ \langle E'(\overline{\mathbf{x}}') \widehat{\mathcal{J}}_j^E G'_k(\mathbf{r}') \rangle \end{pmatrix} &= \begin{pmatrix} \left\langle \frac{\delta[\widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}')] }{\delta N'(\overline{\mathbf{x}})} \right\rangle \\ \left\langle \frac{\delta[\widehat{\mathcal{J}}^E \mathbf{P}'(\mathbf{r}')] }{\delta E'(\overline{\mathbf{x}})} \right\rangle \end{pmatrix} \\ &= T \begin{pmatrix} \left\langle \frac{\delta}{\delta N'(\overline{\mathbf{x}})} \left[ E'(\mathbf{r}') + P'(\mathbf{r}') - \left( \frac{h}{n} \right) N'(\mathbf{r}') \right] \right\rangle \\ \left\langle \frac{\delta}{\delta E'(\overline{\mathbf{x}})} \left[ E'(\mathbf{r}') + P'(\mathbf{r}') - \left( \frac{h}{n} \right) N'(\mathbf{r}') \right] \right\rangle \end{pmatrix} \end{aligned} \quad (3.73a)$$

$$= T \begin{pmatrix} \left\langle \frac{\delta P'(\mathbf{r}')}{\delta N'(\overline{\mathbf{x}})} - \left( \frac{h}{n} \right) \delta(\overline{\mathbf{x}} - \mathbf{r}') \right\rangle \\ \delta(\overline{\mathbf{x}} - \mathbf{r}') + \left\langle \frac{\delta P'(\mathbf{r}')}{\delta E'(\overline{\mathbf{x}})} \right\rangle \end{pmatrix} \quad (3.73c)$$

and

$$-P(E, E, \mathbf{P}) = T^{-2} \left[ K_{22} \left( \frac{\partial p}{\partial n} - \frac{h}{n} \right) + K_{23} \left( 1 + \frac{\partial p}{\partial e} \right) \right] (\nabla_j T) \underbrace{(\nabla_i u_j)}_{\mathbf{S} - \boldsymbol{\Omega}}. \quad (3.74a)$$

$$= \lambda_5^{\text{B}_{PEP}^{(2)}} \mathbf{S} \cdot \boldsymbol{\nabla} T + \lambda_6^{\text{B}_{PEP}^{(2)}} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} T, \quad (3.74b)$$

where

$$\boxed{\lambda_5^{\text{B}_{PEP}^{(2)}} \doteq T^{-2} K_{22} \left( \frac{\partial p}{\partial n} - \frac{h}{n} \right) + K_{23} \left( 1 + \frac{\partial p}{\partial e} \right)}, \quad (3.75a)$$

$$\boxed{\lambda_6^{\text{B}_{PEP}^{(2)}} \doteq -T^{-2} \left[ K_{22} \left( \frac{\partial p}{\partial n} - \frac{h}{n} \right) + K_{23} \left( 1 + \frac{\partial p}{\partial e} \right) \right]}. \quad (3.75b)$$

$(E, \mathbf{P}, N)$ :

$$(E, \mathbf{P}, N) = - \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{J}_i^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{T}}_{jk} N' \rangle (\mathbf{r}' - \mathbf{r})_l [\nabla_k (T^{-1} u_j)] (\nabla_l B_N) \quad (3.76a)$$

$$= -T^{-1} [K_{16}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + K_{17}\delta_{il}\delta_{jk}] (\nabla_k u_j) (\nabla_l B_N) \quad (3.76b)$$

$$= -T^{-1} [2K_{16} \mathbf{S} \cdot \nabla B_N + K_{17} (\nabla \cdot \mathbf{u}) \nabla B_N] \quad (3.76c)$$

$$= -(\overline{n}T^2)^{-1} [2K_{16} \mathbf{S} \cdot (-hT^{-1} \nabla T + \nabla p) + K_{17} (\nabla \cdot \mathbf{u}) (-hT^{-1} \nabla T + \nabla p)] \quad (3.76d)$$

$$= \lambda_4^{\text{B}_{\mathbf{P}N}^{(2)}} (\nabla \cdot \mathbf{u}) \nabla T + \lambda_5^{\text{B}_{\mathbf{P}N}^{(2)}} \mathbf{S} \cdot \nabla T + \lambda_7^{\text{B}_{\mathbf{P}N}^{(2)}} \mathbf{S} \cdot \nabla p + \lambda_8^{\text{B}_{\mathbf{P}N}^{(2)}} (\nabla \cdot \mathbf{u}) \nabla p, \quad (3.76e)$$

where

$$\boxed{\lambda_4^{\text{B}_{\mathbf{P}N}^{(2)}} \doteq (\overline{n}T^3)^{-1} h K_{17}}, \quad \boxed{\lambda_5^{\text{B}_{\mathbf{P}N}^{(2)}} \doteq 2(\overline{n}T^3)^{-1} h K_{16}}, \quad (3.77)$$

and

$$\boxed{\lambda_7^{\text{B}_{\mathbf{P}N}^{(2)}} \doteq -2(\overline{n}T^2)^{-1} K_{16}}, \quad \boxed{\lambda_8^{\text{B}_{\mathbf{P}N}^{(2)}} \doteq -(\overline{n}T^2)^{-1} K_{17}}. \quad (3.78)$$

$-P(E, \mathbf{P}, N)$ :

$$-P(E, \mathbf{P}, N) = T^{-1} \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{J}_i^E(\mathbf{r}) e^{-i\mathcal{L}s} P \widehat{\mathcal{T}}_{jk} N' \rangle (\mathbf{r}' - \mathbf{r})_l (\nabla_k u_j) (\nabla_l B_N) \quad (3.79a)$$

$$\begin{aligned} &= (\overline{n}T^2)^{-1} \left( \int_0^\infty ds \int d\mathbf{r}' \langle \widehat{J}_i^E e^{-i\mathcal{L}s} N'(\overline{\mathbf{x}}) \rangle (\mathbf{r}' - \mathbf{r})_l \right)^\text{T} \\ &\quad \cdot \mathcal{M}_2^{-1}(\overline{\mathbf{x}}, \overline{\mathbf{x}}') \cdot \left( \begin{array}{c} \langle N'(\overline{\mathbf{x}}') \widehat{\mathcal{T}}_{jk} N'(\mathbf{r}') \rangle \\ \langle E'(\overline{\mathbf{x}}') \widehat{\mathcal{T}}_{jk} N'(\mathbf{r}') \rangle \end{array} \right) (\nabla_k u_j) (-hT^{-1} \nabla_l T + \nabla_l p). \end{aligned} \quad (3.79b)$$

By symmetry,

$$\langle N'(\overline{\mathbf{x}}) \widehat{\mathcal{T}}_{jk} N'(\mathbf{r}') \rangle = \frac{1}{3} \langle N'(\overline{\mathbf{x}}') \text{Tr} \widehat{\mathcal{T}} N'(\mathbf{r}') \rangle \delta_{jk} \quad (3.80a)$$

$$= - \left\langle N'(\overline{\mathbf{x}}') \left( \int d\mathbf{y} [N'(\mathbf{y}) p_n(\mathbf{y}) + E'(\mathbf{y}) p_e(\mathbf{y})] \right) N'(\mathbf{r}') \right\rangle \delta_{jk}. \quad (3.80b)$$

By Novikov's theorem for arbitrary  $A$ ,

$$\begin{aligned} V &\doteq \int d\overline{\mathbf{x}} A(\overline{\mathbf{x}}) \mathcal{M}_2^{-1}(\overline{\mathbf{x}}, \overline{\mathbf{x}}') \cdot \int d\overline{\mathbf{x}} A(\overline{\mathbf{x}}) \left( \begin{array}{c} \langle N'(\overline{\mathbf{x}}') \widehat{\mathcal{T}}_{jk} N'(\mathbf{r}') \rangle \\ \langle E'(\overline{\mathbf{x}}') \widehat{\mathcal{T}}_{jk} N'(\mathbf{r}') \rangle \end{array} \right) \\ &= \left( \left\langle \frac{\delta}{\delta N'(\overline{\mathbf{x}})} \left[ \left( \int d\mathbf{y} [N'(\mathbf{y}) p_n(\mathbf{y}) + E'(\mathbf{y}) p_e(\mathbf{y})] \right) N'(\mathbf{r}') \right] \right\rangle \right) \\ &\quad \left( \left\langle \frac{\delta}{\delta E'(\overline{\mathbf{x}})} \left[ \left( \int d\mathbf{y} [N'(\mathbf{y}) p_n(\mathbf{y}) + E'(\mathbf{y}) p_e(\mathbf{y})] \right) N'(\mathbf{r}') \right] \right\rangle \right). \end{aligned} \quad (3.81a)$$

In the first line, the expectation vanishes if the  $\delta/\delta N'$  operates on either of the  $N'$ 's. Thus,

$$V = \int d\bar{\mathbf{x}} A(\bar{\mathbf{x}}) \left( \left\langle N'(\mathbf{r}') \int d\mathbf{y} \left( N'(\mathbf{y}) \frac{\delta p_n(\mathbf{y})}{\delta N'(\bar{\mathbf{x}})} + E'(\mathbf{y}) \frac{\delta p_e(\mathbf{y})}{\delta N'(\bar{\mathbf{x}})} \right) \right\rangle \right) \quad (3.82a)$$

$$\approx A(\mathbf{r}') \left( \frac{\partial}{\partial(\beta\mu)} \left( \frac{\partial p}{\partial n} \right) \right) \left( \frac{\partial}{\partial(\beta\mu)} \left( \frac{\partial p}{\partial e} \right) \right). \quad (3.82b)$$

Thus,

$$-P(E, \mathbf{P}, N) = (\bar{n}T^2)^{-1} \left[ K_{22} \frac{\partial}{\partial(\beta\mu)} \left( \frac{\partial p}{\partial n} \right) + K_{23} \frac{\partial}{\partial(\beta\mu)} \left( \frac{\partial p}{\partial e} \right) \right] \times (\nabla \cdot \mathbf{u})(-hT^{-1}\nabla T + \nabla P) \quad (3.83a)$$

$$= T^{-1} \left[ K_{22} \frac{\partial}{\partial p} \left( \frac{\partial p}{\partial n} \right) + K_{23} \frac{\partial}{\partial p} \left( \frac{\partial p}{\partial e} \right) \right] (\nabla \cdot \mathbf{u})(-hT^{-1}\nabla T + \nabla p). \quad (3.83b)$$

In obtaining the last line, I used the result

$$\left( \frac{\partial}{\partial(\beta\mu)} \right)_{\beta} = nT \frac{\partial}{\partial p}, \quad (3.84)$$

which was derived in (3.30). The term in  $(\nabla \cdot \mathbf{u})\nabla p$  is thus

$$\lambda_8^{\text{B}_{\text{PPN}}^{(2)}} (\nabla \cdot \mathbf{u})\nabla p, \quad (3.85)$$

where

$$\lambda_8^{\text{B}_{\text{PPN}}^{(2)}} \doteq \frac{\partial}{\partial p} \left( \frac{\partial p}{\partial n} \right) K_{22} + \frac{\partial}{\partial p} \left( \frac{\partial p}{\partial e} \right) K_{23}. \quad (3.86)$$

The term in  $\nabla T$  will cancel against a contribution from  $-P(E, \mathbf{P}, E)$ .

$(E, \mathbf{P}, E)$ :

$$(E, \mathbf{P}, E) = - \int_0^\infty ds \int d\mathbf{r}' \langle \hat{J}_i^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{T}}_{jk} E' \rangle (\mathbf{r}' - \mathbf{r})_l [\nabla_k (T^{-1}u_j)] (\nabla_l B_E) \quad (3.87a)$$

$$= -T^{-3} [K_{18}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + K_{19}\delta_{il}\delta_{jk}] (\nabla_k u_j) (\nabla_l T) \quad (3.87b)$$

$$= -T^{-3} [2K_{18} \mathbf{S} \cdot \nabla T + K_{19} (\nabla \cdot \mathbf{u}) \nabla T] \quad (3.87c)$$

$$= \lambda_4^{\text{B}_{\text{PE}}^{(2)}} (\nabla \cdot \mathbf{u}) \nabla T + \lambda_5^{\text{B}_{\text{PE}}^{(2)}} \mathbf{S} \cdot \nabla T, \quad (3.87d)$$

where

$$\lambda_4^{\text{B}_{\text{PE}}^{(2)}} \doteq -T^{-3} K_{19} \quad \lambda_5^{\text{B}_{\text{PE}}^{(2)}} \doteq -2T^{-3} K_{18}. \quad (3.88)$$

---

$-\mathbf{P}(E, \mathbf{P}, E)$ :

The evaluation of  $\mathbf{P}(E, \mathbf{P}, E)$  proceeds similarly to that for  $\mathbf{P}(E, \mathbf{P}, N)$ , and leads to

$$-\mathbf{P}(E, \mathbf{P}, E) = T^{-3} \left[ K_{22} \frac{\partial}{\partial(-\beta)} \left( \frac{\partial p}{\partial n} \right) + K_{23} \frac{\partial}{\partial(-\beta)} \left( \frac{\partial p}{\partial e} \right) \right] (\nabla \cdot \mathbf{u}) \nabla T. \quad (3.89)$$

Now

$$\left( \frac{\partial}{\partial(-\beta)} \right)_{(\beta\mu)} = T^2 \left( \frac{\partial}{\partial T} \right)_{(\beta\mu)}. \quad (3.90)$$

We have

$$\left( \frac{\partial}{\partial T} \right)_{(\beta\mu)} = \left( \frac{\partial}{\partial T} \right)_p + \left( \frac{\partial p}{\partial T} \right)_{(\beta\mu)} \left( \frac{\partial}{\partial p} \right)_T, \quad (3.91)$$

where the result

$$\left( \frac{\partial p}{\partial T} \right)_{(\beta\mu)} = \beta h \quad (3.92)$$

was derived in (3.39). The  $\partial/\partial p$  term cancels with the  $\nabla T$  term in  $-\mathbf{P}(E, \mathbf{P}, N)$ . Thus, the  $\nabla T$  contributions from the  $E$  projections are

$$-[\mathbf{P}(E, \mathbf{P}, N) + \mathbf{P}(E, \mathbf{P}, E)]_{\nabla T} = \lambda_4^{\mathbf{B}_{\mathbf{P}\mathbf{P}(N+E)}^{(2)}} (\nabla \cdot \mathbf{u}) \nabla T, \quad (3.93)$$

where

$$\boxed{\lambda_4^{\mathbf{B}_{\mathbf{P}\mathbf{P}(N+E)}^{(2)}} \doteq T^{-1} \frac{\partial}{\partial T} \left( \frac{\partial p}{\partial n} \right) K_{22} + T^{-1} \frac{\partial}{\partial T} \left( \frac{\partial p}{\partial e} \right) K_{23}.} \quad (3.94)$$

### 3.3. Nonlinear Burnett terms — time derivatives

The nonlinear Burnett terms relating to first-order time derivatives are

$$\underbrace{\left[ \frac{\partial}{\partial t} \left( \mathbf{k}_2^\beta [\mathbf{J}^\alpha](\mu, t) \cdot \nabla B_\beta(\mathbf{r}, t) \right) \right]}_{\mathbf{B}^\alpha \partial_t \mathbf{k}_\beta + \mathbf{B}^\alpha \partial_t B_\beta}^{(1)}, \quad (3.95)$$

where

$$\mathbf{k}_2^\beta [\tilde{\mathbf{J}}^\alpha](\mu, t) \doteq \int_0^\infty ds \int d\mathbf{r}' s \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \hat{\mathbf{J}}^\beta(\mathbf{r}') \rangle = \int_0^\infty ds s \langle \hat{\mathbf{J}}^\alpha(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{J}}^\beta \rangle_0. \quad (3.96)$$

#### 3.3.1. Nonlinear Burnett — momentum ( $\partial_t$ )

Note that  $\widehat{\mathcal{N}} = 0$ , so  $(\mathbf{P}, N)_{\partial_t} = 0$ .

---



---

$(\mathbf{P}, \mathbf{P})_{\partial_t}$ :

$$(\mathbf{P}, \mathbf{P})_{\partial_t} = \frac{\partial}{\partial t} \left( \underbrace{\int_0^\infty ds s \langle \widehat{\boldsymbol{\tau}}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\mathcal{T}} \rangle_{ijkl}}_{K^{\text{IV}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K^{\text{V}}\delta_{ij}\delta_{kl}} : [\nabla(\beta \mathbf{u})]_{kl} \right)^{(1)}. \quad (3.97a)$$

One needs

$$\nabla \partial_t(\beta \mathbf{u}) = \nabla(-T^{-2} \partial_t T \mathbf{u} + T^{-1} \partial_t \mathbf{u}) \quad (3.98a)$$

$$= -T^{-2}(\partial_t T)(\nabla \mathbf{u}) - T^{-2}(\nabla T)(\partial_t \mathbf{u}) + T^{-1} \nabla(\partial_t \mathbf{u}). \quad (3.98b)$$

The momentum equation is through first order

$$\partial_t \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} - (mn)^{-1} \nabla p. \quad (3.99)$$

Hence

$$\nabla \partial_t(\beta \mathbf{u}) = -T^{-2}(\partial_t T)(\nabla \mathbf{u}) + [\dots], \quad (3.100)$$

where

$$[\dots] \doteq (mnT^2)^{-1}(\nabla T)(\nabla p) - T^{-1}[(\nabla \mathbf{u}) \cdot (\nabla \mathbf{u}) + (mn)^{-1} \nabla \nabla p - (mn)^{-1} (n^{-1} \nabla n)(\nabla p)]. \quad (3.101)$$

Define

$$\alpha \doteq -\frac{1}{n} \left( \frac{\partial n}{\partial T} \right)_p \quad (\text{expansion coefficient}), \quad (3.102a)$$

$$\kappa_T \doteq \frac{1}{n} \left( \frac{\partial n}{\partial p} \right)_T \quad (\text{isothermal compressibility}) \quad (3.102b)$$

Then the contribution from  $\partial_t \nabla(\beta \mathbf{u})$  is

$$\begin{aligned} & [K^{\text{IV}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + K^{\text{V}}\delta_{ij}\delta_{kl}] \\ & \times \{ -T^{-2}(\partial_t T)(\nabla \mathbf{u}) \\ & + [(mnT)^{-1}(T^{-1} - \alpha)(\nabla T)(\nabla p) \\ & - T^{-1}(\nabla \mathbf{u}) \cdot (\nabla \mathbf{u}) - (mnT)^{-1} \nabla \nabla p + (mnT)^{-1} \kappa_T(\nabla p)(\nabla p)] \}, \end{aligned} \quad (3.103)$$

where the  $[\dots]$  piece is

$$\begin{aligned} & (mn)^{-1} \eta_1 (T^{-1} - \alpha)(\nabla T \nabla p + \nabla p \nabla T) + (mn)^{-1} \eta_2 (T^{-1} - \alpha) \nabla T \cdot \nabla p \\ & - 2\eta_1 [\mathbf{S} \cdot \mathbf{S} + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}] - \eta_2 (\mathbf{S} : \mathbf{S} + \boldsymbol{\Omega} : \boldsymbol{\Omega}) \\ & - 2(mn)^{-1} \eta_1 (\nabla \nabla p - \kappa_T \nabla p \nabla p) - (mn)^{-1} \eta_2 (\nabla^2 p - \kappa_T |\nabla p|^2) \end{aligned} \quad (3.104)$$

with

$$\boxed{\eta_1 \doteq T^{-1} K^{\text{IV}}}, \quad \boxed{\eta_2 \doteq T^{-1} K^{\text{V}}}. \quad (3.105)$$

The first term combines with the time derivative of the  $K$ 's to give

$$2(\partial_t \eta_1) \mathbf{S} + (\partial_t \eta_2)(\nabla \cdot \mathbf{u}) \mathbf{I}. \quad (3.106)$$

If one uses  $n$  and  $s$  as variables, the time derivatives can be worked out in terms of  $\partial_t n = -\nabla \cdot (n\mathbf{u}) \rightarrow -n \nabla \cdot \mathbf{u}$  and  $\partial_t s$ . To find the entropy equation, begin with

$$ds = \beta(de - \mu dn). \quad (3.107)$$

Thus, to first order,

$$\partial_t s = \beta(\partial_t e - \mu \partial_t n) \quad (3.108a)$$

$$= \beta(-h \nabla \cdot \mathbf{u} + n\mu \nabla \cdot \mathbf{u}) = -\beta(e + p - n\mu)(\nabla \cdot \mathbf{u}) \quad (3.108b)$$

$$= -s \nabla \cdot \mathbf{u}. \quad (3.108c)$$

In a homogeneous ensemble, one can choose a constant of integration that makes

$s(n, T) = 0$ . Assume that this can still be done to lowest order in the gradients in the reference ensemble. Then the derivatives of the  $\eta$ 's do not enter, and one obtains

$$\boxed{(\mathbf{P}, \mathbf{P})_{\partial_t} = -2n \left( \frac{\partial \eta_1}{\partial n} \right)_s (\nabla \cdot \mathbf{u}) \mathbf{S} - n \left( \frac{\partial \eta_2}{\partial n} \right)_s (\nabla \cdot \mathbf{u})^2 \mathbf{I}. \quad (3.109)}$$

$(\mathbf{P}, E)_{\partial_t}$ :

$$(\mathbf{P}, E)_{\partial_t} = \frac{\partial}{\partial t} \left( \int_0^\infty ds s \langle \widehat{\boldsymbol{\tau}}(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\boldsymbol{\mathcal{J}}}^E \rangle \cdot (T^{-2} \nabla T) \right)^{(1)}. \quad (3.110)$$

By symmetry, the expectation (a third-rank tensor) vanishes to lowest order, but in general it can be built from a gradient such as  $\nabla T$ . Since it vanishes to lowest order, the time derivatives of the  $T$  factors do not enter through second order. Thus, one only needs to evaluate  $\partial_t \langle \widehat{\boldsymbol{\tau}} e^{-i\mathcal{L}s} \widehat{\boldsymbol{\mathcal{J}}}^E \rangle$ . It can be shown that the contribution from this term cancels the  $T^{-1}$  term in the  $(T^{-1} - \alpha)$  factors in (3.103). For more detailed discussion, see the closely related calculation in §3.3.2.

### 3.3.2. Nonlinear Burnett — energy ( $\partial_t$ )

Note that  $\widehat{\mathcal{N}} = 0$ , so  $(\mathbf{P}, N)_{\partial_t} = 0$ .

$(E, \mathbf{P})_{\partial_t}$ :

$$(E, \mathbf{P})_{\partial_t} = \frac{\partial}{\partial t} \left( \int_0^\infty ds s \langle \widehat{\mathbf{J}}^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\boldsymbol{\mathcal{T}}} \rangle : [\nabla(T^{-1} \mathbf{u})] \right)^{(1)}. \quad (3.111)$$

The expectation is a third-rank tensor. By symmetry in velocity space, it vanishes to lowest order; however, in general it does not vanish, as it can be built from a vector such as  $\mathbf{u}$ .

Since the expectation vanishes to lowest order, the term in  $\partial_t(\nabla \mathbf{u})$  will not contribute through second order in the gradients. Thus, one need to calculate

$$\boldsymbol{\tau} \doteq \frac{\partial}{\partial t} \left( \int_0^\infty ds s \langle \widehat{\mathbf{J}}^E(\mathbf{r}) e^{-i\mathcal{L}s} \widehat{\boldsymbol{\mathcal{T}}} \rangle \right). \quad (3.112)$$

One has

$$\widehat{\mathbf{J}}^E = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{x}_i)} \sum_{i=1}^{\mathcal{N}} \left[ \widetilde{E}_i \mathbf{v}_i + \Delta \widetilde{\boldsymbol{\tau}}_i(\mathbf{k}) \cdot \mathbf{v}_i - \left( \frac{h}{n} \right) \mathbf{v}_i \right], \quad (3.113a)$$

$$\widehat{\boldsymbol{\mathcal{T}}} = \sum_{i=1}^{\mathcal{N}} m[\mathbf{v}_i \mathbf{v}_i + \Delta \widetilde{\boldsymbol{\tau}}_i(0)] - \mathbf{I} \int d\mathbf{y} [p + N' p_n + E' p_e](\mathbf{y}). \quad (3.113b)$$

Contributions to the time derivative come from the projected parts of  $\widehat{\mathbf{J}}^E$  and  $\widehat{\boldsymbol{\mathcal{T}}}$ , as well as from the time derivative of  $f_0$ . The integrals of the projections leave expectations that are odd in velocity, so those will be proportional to  $\mathbf{u}$ , which one can take to vanish. Thus, we focus on  $\partial_t f_0$ . One has

$$f_0 = e^{-\beta \widetilde{\mathcal{E}}_0} / Z_0, \quad (3.114)$$

where

$$\tilde{\mathcal{E}}_0 \doteq \sum_{i=1}^{\mathcal{N}} \left( \frac{1}{2} m(\mathbf{v}_i - \mathbf{u})^2 + \tilde{U}_i \right), \quad Z_0 \doteq \int d\Gamma e^{-\beta \tilde{\mathcal{E}}}. \quad (3.115)$$

Thus,

$$\begin{aligned} \partial_t f_0 &= (-\beta)(-\partial_t \mathbf{u}) \cdot \sum_{i=1}^{\mathcal{N}} m(\mathbf{v}_i - \mathbf{u}) f_0 - \partial_t \beta \tilde{\mathcal{E}}_0 f_0 \\ &\quad - f_0 \left\langle \left( (-\beta)(-\partial_t \mathbf{u}) \cdot \sum_{i=1}^{\mathcal{N}} m(\mathbf{v}_i - \mathbf{u}) - \partial_t \beta \tilde{\mathcal{E}}_0 \right) \right\rangle \end{aligned} \quad (3.116a)$$

$$= \beta(\partial_t \mathbf{u}) \cdot \left( \sum_{i=1}^{\mathcal{N}} m \mathbf{w}_i \right) f_0 - (\partial_t \beta) \tilde{\mathcal{E}}_0' f_0. \quad (3.116b)$$

The  $\partial_t \beta$  term again leads to an expectation that is odd in velocity, which one can ignore. Thus, one finds that

$$\begin{aligned} \mathbf{T} &= \beta(\partial_t \mathbf{u}) \cdot \int_0^\infty ds s \int \frac{d\mathbf{k}}{(2\pi)^3} \left\langle e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{x}_i)} \sum_{i=1}^{\mathcal{N}} \right. \\ &\quad \times \left[ \tilde{E}_i m \mathbf{w}_i \mathbf{w}_i + m \mathbf{w}_i \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k}) \cdot \mathbf{w}_i - \left( \frac{\hbar}{n} \right) m \mathbf{w}_i \mathbf{w}_i \right] e^{-i\mathcal{L}s} \widehat{\boldsymbol{\mathcal{T}}} \rangle. \end{aligned} \quad (3.117)$$

From the Euler equation for  $\mathbf{u}$ , one can replace  $\partial_t \mathbf{u} \rightarrow -(mn)^{-1} \nabla p$ . Now the integral  $\beta \int_0^\infty ds s \dots$  has the same dimensions as  $\int_0^\infty ds s \langle \widehat{\boldsymbol{\tau}} e^{-i\mathcal{L}s} \widehat{\boldsymbol{\mathcal{T}}} \rangle$ , so it can be written as that term plus a correction:

$$\begin{aligned} [\dots] &= (\tilde{E}_i - T) m \mathbf{w}_i \mathbf{w}_i + T m \mathbf{w}_i \mathbf{w}_i + m \left( \mathbf{w}_i \mathbf{w}_i - \frac{1}{3} \langle w^2 \rangle \mathbf{I} \right) : \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k}) + T \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k}) \\ &\quad - \left( \frac{\hbar}{n} \right) m \left( \mathbf{w}_i \mathbf{w}_i - \frac{1}{3} \langle w^2 \rangle \mathbf{I} \right) - \left( \frac{\hbar}{n} \right) \mathbf{I} T \end{aligned} \quad (3.118a)$$

$$\begin{aligned} &= T [m \mathbf{w}_i \mathbf{w}_i + \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k}) - n^{-1} \mathbf{I} (p + N' p_n + E' p_e)] \\ &\quad + (\tilde{E}_i - T) m \left( \mathbf{w}_i \mathbf{w}_i - \frac{1}{3} \langle w^2 \rangle \mathbf{I} \right) + (\tilde{E}_i - T) T \mathbf{I} + m \left( \mathbf{w}_i \mathbf{w}_i - \frac{1}{3} \langle w^2 \rangle \mathbf{I} \right) : \Delta \tilde{\boldsymbol{\tau}}_i(\mathbf{k}) \\ &\quad - \left( \frac{\hbar}{n} \right) m \left( \mathbf{w}_i \mathbf{w}_i - \frac{1}{3} \langle w^2 \rangle \mathbf{I} \right) - n^{-1} e T \mathbf{I} + n^{-1} \mathbf{I} (N' p_n + E' p_e). \end{aligned} \quad (3.118b)$$

The first line leads to contributions

$$\begin{aligned} &- (mn)^{-1} \nabla p \cdot [K^{IV} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + K^V \delta_{ij} \delta_{kl}] (T^{-1} \nabla_l u_k) \\ &= \lambda_7^{\partial B_P^E} \mathbf{S} \cdot \nabla p + \lambda_8^{\partial B_P^E} (\nabla \cdot \mathbf{u}) \nabla p, \end{aligned} \quad (3.119a)$$

where

$$\boxed{\lambda_7^{\partial B_P^E} \doteq -2(mn)^{-1} \eta_1, \quad \lambda_8^{\partial B_P^E} \doteq -(mn)^{-1} \eta_2,} \quad (3.120)$$

$\eta_1$  and  $\eta_2$  having already been defined in (3.105).

$(E, E)_{\partial_t}$ :

$$(E, E)_{\partial_t} = \frac{\partial}{\partial t} \left( \underbrace{\int_0^\infty ds s \langle \widehat{\mathcal{J}}^E e^{-i\mathcal{L}s} \widehat{\mathcal{J}}^E \rangle \cdot \nabla B_E}_{K^{\text{VI}} \mathbf{I}} \right)^{(1)}. \quad (3.121a)$$

First let us calculate  $\partial_t \nabla B_E$ . One has

$$\nabla \partial_t B_E = \nabla \partial_t (-T^{-1}) \quad (3.122a)$$

$$= \nabla (T^{-2} \partial_t T) \quad (3.122b)$$

$$= -2T^{-3} (\partial_t T) \nabla T + T^{-2} \nabla (\partial_t T). \quad (3.122c)$$

It is convenient to express  $T$  in terms of  $n$  and  $e$ , which have simple Euler equations:

$$\partial_t T(n, e) = T_n (\partial_t n) + T_e (\partial_t e) \quad (3.123a)$$

$$= -[T_n (n \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla n) + T_e (h \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla e)]. \quad (3.123b)$$

For  $\partial_t T$  itself, after setting  $\mathbf{u} = \mathbf{0}$ ,

$$\partial_t T \rightarrow -(n T_n + h T_e) (\nabla \cdot \mathbf{u}). \quad (3.124)$$

Now one can show that

$$n T_n + h T_e = T p_e, \quad (3.125)$$

so the Euler temperature equation is<sup>2</sup>

$$\partial_t T = -T p_e \nabla \cdot \mathbf{u}. \quad (3.126)$$

To prove (3.125), note that from

$$de = T ds + \mu dn \quad (3.127)$$

one has

$$0 = T ds + \mu dn \quad (e = \text{const}), \quad (3.128a)$$

$$de = T ds \quad (n = \text{const}). \quad (3.128b)$$

Thus, upon expressing  $p = p(n, s)$ ,

$$\left( \frac{\partial p}{\partial e} \right)_n = \left( \frac{\partial s}{\partial e} \right)_n \left( \frac{\partial p}{\partial s} \right)_n = \frac{1}{T} \left( \frac{\partial p}{\partial s} \right)_n, \quad (3.129a)$$

$$\left( \frac{\partial T}{\partial n} \right)_e = \left( \frac{\partial T}{\partial n} \right)_s + \left( \frac{\partial s}{\partial n} \right)_e \left( \frac{\partial T}{\partial s} \right)_n = \left( \frac{\partial T}{\partial n} \right)_s - \left( \frac{\mu}{T} \right) \left( \frac{\partial T}{\partial s} \right)_n, \quad (3.129b)$$

$$\left( \frac{\partial T}{\partial e} \right)_n = \left( \frac{\partial s}{\partial e} \right)_n \left( \frac{\partial T}{\partial s} \right)_n = \frac{1}{T} \left( \frac{\partial T}{\partial s} \right)_n. \quad (3.129c)$$

Then

$$n (T_n)|_e + h (T_e)|_n = n (T_n)|_s + T^{-1} \underbrace{(-n\mu + e + p)}_s (T_s)|_n. \quad (3.130)$$

From the Gibbs–Duhem relation,

$$0 = s dT - dp + n d\mu, \quad (3.131)$$

---

<sup>2</sup>When the ideal-gas values of  $p$  and  $e$  are used, (3.126) reduces to  $\frac{3}{2}n \partial_t T = -nT \nabla \cdot \mathbf{u}$ , which is the Euler part of the familiar Braginskii temperature equation.

so

$$(p_s)|_n = s (T_s)|_n + n (\mu_s)|_n. \quad (3.132)$$

The relationship is then proven if

$$(\mu_s)|_n = (T_n)|_s. \quad (3.133)$$

But this is the Maxwell relation that follows from

$$de = T ds + \mu dn. \quad (3.134)$$

Thus,

$$\nabla \partial_t B_E = 2T^{-2} \left( \frac{\partial p}{\partial e} \right)_n (\nabla \cdot \mathbf{u}) \nabla T - T^{-2} \nabla [T_n (n \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla n) + T_e (h \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla e)]. \quad (3.135)$$

The last term is  $-T^{-2}[\dots]$ , where

$$\begin{aligned} [\dots] &= (\nabla T_n) n (\nabla \cdot \mathbf{u}) + (\nabla T_e) h (\nabla \cdot \mathbf{u}) \\ &\quad + T_n [(\nabla n) (\nabla \cdot \mathbf{u}) + n \nabla (\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u}) \cdot \nabla n] \\ &\quad + T_e [(\nabla h) (\nabla \cdot \mathbf{u}) + h \nabla (\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u}) \cdot \nabla e]. \end{aligned} \quad (3.136a)$$

$$\begin{aligned} &= (\nabla T_n) n (\nabla \cdot \mathbf{u}) + (\nabla T_e) h (\nabla \cdot \mathbf{u}) \\ &\quad + (n T_n + h T_e) \nabla (\nabla \cdot \mathbf{u}) + (\nabla \cdot \mathbf{u}) \mathbf{I} (T_n \nabla n + T_e \nabla h) \\ &\quad + (\nabla \mathbf{u}) \cdot (T_n \nabla n + T_e \nabla e) \end{aligned} \quad (3.136b)$$

$$= [\nabla (n T_n + h T_e)] (\nabla \cdot \mathbf{u}) + (n T_n + h T_e) \nabla (\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u}) \cdot \nabla T \quad (3.136c)$$

$$= \left[ \frac{\partial (T p_e)}{\partial T} \nabla T + \frac{\partial (T p_e)}{\partial p} \nabla p \right] (\nabla \cdot \mathbf{u}) + T p_e \nabla (\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u}) \cdot \nabla T. \quad (3.136d)$$

The full set of terms one needs to calculate is

$$(\partial_t K^{\text{VI}}) T^{-2} \nabla T + K^{\text{VI}} \nabla \partial_t B_E. \quad (3.137)$$

Now  $K^{\text{VI}}$  depends on time only through its dependence on the state variables. It is convenient to view it as a function of  $n$  and  $s$ , since the Euler equation for  $s$  is trivial. Thus,

$$\partial_t K^{\text{VI}} = \left( \frac{\partial K^{\text{VI}}}{\partial n} \right)_s (\partial_t n) = - \left[ n \left( \frac{\partial K^{\text{VI}}}{\partial n} \right)_s + s \left( \frac{\partial K^{\text{VI}}}{\partial s} \right)_n \right] (\nabla \cdot \mathbf{u}). \quad (3.138)$$

One therefore has

$$(\partial_t K^{\text{VI}}) T^{-2} \nabla T = - \left[ n \left( \frac{\partial K^{\text{VI}}}{\partial n} \right)_s + s \left( \frac{\partial K^{\text{VI}}}{\partial s} \right)_n \right] (\nabla \cdot \mathbf{u}) T^{-2} \nabla T \quad (3.139a)$$

$$= \left[ -n \frac{\partial}{\partial n} (T^{-2} K^{\text{VI}}) - 2T^{-3} n \left( \frac{\partial T}{\partial n} \right)_s K^{\text{VI}} - s \left( \frac{\partial K^{\text{VI}}}{\partial s} \right)_n \right] (\nabla \cdot \mathbf{u}) \nabla T. \quad (3.139b)$$

The second term adds to the first term of (3.135) to give

$$2T^{-3} (\nabla \cdot \mathbf{u}) \nabla T \left[ T \left( \frac{\partial p}{\partial e} \right)_n - n \left( \frac{\partial T}{\partial n} \right)_s \right] K^{\text{VI}} = 2T^{-3} (\nabla \cdot \mathbf{u}) \nabla T \left[ s \left( \frac{\partial T}{\partial s} \right)_n \right] K^{\text{VI}}; \quad (3.140)$$

adding that to the third term gives a correction

$$- s \frac{\partial}{\partial s} (T^{-2} K^{\text{VI}}) (\nabla \cdot \mathbf{u}) \nabla T \quad (3.141)$$

which vanishes with  $s = 0$ . (Compare the analogous situation with  $\eta_1$ .)

If one defines

$$\boxed{\lambda_1 \doteq T^{-2} K^{\text{VI}}}, \quad (3.142)$$

then

$$\begin{aligned} (E, E)_{\partial_t} = & -n \left( \frac{\partial \lambda_1}{\partial n} \right)_s - \lambda_1 \frac{\partial}{\partial T} \left[ \left( \frac{\partial p}{\partial e} \right)_n \right] (\nabla \cdot \mathbf{u}) \nabla T - \lambda_1 T \frac{\partial}{\partial p} \left[ \left( \frac{\partial p}{\partial e} \right)_n \right] (\nabla \cdot \mathbf{u}) \nabla P \\ & - \lambda_1 T \left( \frac{\partial p}{\partial e} \right)_n \nabla (\nabla \cdot \mathbf{u}) - \lambda_1 (\mathbf{S} - \boldsymbol{\Omega}) \cdot \nabla T, \end{aligned} \quad (3.143)$$

which generates all of the  $\lambda_1$  terms in Eq. (A2) of [Brey \(1983\)](#).

#### 4. Summary of the dissipative fluxes

Here I summarize the dissipative fluxes that have been derived. The results agree with those of [Brey \(1983\)](#), although I have slightly reordered some of the terms.

The dissipative part of the momentum flux is

$$\begin{aligned} \boldsymbol{\tau}_{\text{diss}} = & -\eta \left( (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T - \frac{2}{d} (\nabla \cdot \mathbf{u}) \mathbf{I} \right) - \zeta (\nabla \cdot \mathbf{u}) \mathbf{I} \\ & + \eta_3 \nabla \nabla T + \eta_5 \nabla T \nabla T \\ & + [\eta_7 - (mn)^{-1} \alpha \eta_1] (\nabla T \nabla p + \nabla p \nabla T) - 2(mn)^{-1} \eta_1 (\nabla \nabla p - \kappa_T \nabla p \nabla p) \\ & + \underbrace{\left[ \eta_{11} - 2n \left( \frac{\partial \eta_1}{\partial n} \right)_s \right]}_{\partial \text{B}_P^P} (\nabla \cdot \mathbf{u}) \mathbf{S} + (\eta_{12} - 2\eta_1) \mathbf{S} \cdot \mathbf{S}^T + 2\eta_1 \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}^T \\ & + \eta_{13} (\mathbf{S}^T \cdot \boldsymbol{\Omega} + \boldsymbol{\Omega}^T \cdot \mathbf{S}) \\ & + \mathbf{I} \left\{ \eta_4 \nabla^2 T + \eta_6 |\nabla T|^2 \right. \\ & + [\eta_8 - (mn)^{-2} \alpha \eta_2] \nabla T \cdot \nabla p - (mn)^{-1} \eta_2 (\nabla^2 p - \kappa_T |\nabla p|^2) \\ & \left. + \underbrace{\left[ \eta_9 - n \left( \frac{\partial \eta_2}{\partial n} \right)_s \right]}_{\partial \text{B}_P^P} (\nabla \cdot \mathbf{u})^2 + (\eta_{10} - \eta_2) \mathbf{S} : \mathbf{S}^T + \eta_2 \boldsymbol{\Omega} : \boldsymbol{\Omega}^T \right\}. \end{aligned} \quad (4.1)$$

Note that  $\mathbf{S}^T = \mathbf{S}$  and  $\boldsymbol{\Omega}^T = -\boldsymbol{\Omega}$ . A consequence is that the trace of the  $\eta_{13}$  term vanishes.

The dissipative part of the heat flux is

$$\begin{aligned}
\mathbf{j}_{\text{diss}}^E &= \mathbf{u} \cdot \boldsymbol{\tau}_{\text{diss}} - \lambda \nabla T \\
&+ \left[ \lambda_2 + \underbrace{(-\lambda_1)T \left( \frac{\partial p}{\partial e} \right)_n}_{\partial B_E^E} \right] \nabla (\nabla \cdot \mathbf{u}) + \lambda_3 \nabla^2 \mathbf{u} \\
&+ \left( \lambda_4 - \underbrace{\left\{ \lambda_1 \left( \frac{\partial p}{\partial e} \right)_n + \lambda_1 T \left[ \frac{\partial}{\partial T} \left( \frac{\partial p}{\partial e} \right)_n \right]_p + n \left( \frac{\partial \lambda_1}{\partial n} \right)_s \right\}}_{\partial B_E^E} \right) (\nabla \cdot \mathbf{u}) \nabla T \\
&+ (\lambda_5 - \lambda_1) \mathbf{S} \cdot \nabla T + (\lambda_6 + \lambda_1) \boldsymbol{\Omega} \cdot \nabla T + \lambda_7 \mathbf{S} \cdot \nabla p \\
&+ \left\{ \lambda_8 - \underbrace{\lambda_1 T \left[ \frac{\partial}{\partial p} \left( \frac{\partial p}{\partial e} \right)_n \right]_T}_{\partial B_E^E} \right\} (\nabla \cdot \mathbf{u}) \nabla p.
\end{aligned} \tag{4.2}$$

In the above expressions,

$$\eta \doteq \underbrace{T^{-1} K^{\text{I}}}_{\text{NS}_P^P}, \quad \zeta \doteq \underbrace{T^{-2} (K^{\text{II}} + 2d^{-1} K^{\text{I}})}_{\text{NS}_P^P}, \tag{4.3a}$$

$$\eta_1 \doteq \underbrace{T^{-1} K^{\text{IV}}}_{\partial B_P^P}, \tag{4.3b}$$

$$\eta_2 \doteq \underbrace{T^{-1} K^{\text{V}}}_{\partial B_P^P}, \tag{4.3c}$$

$$\eta_3 \doteq - \underbrace{2T^{-2} K_1}_{B_E^P}, \tag{4.3d}$$

$$\eta_4 \doteq - \underbrace{T^{-2} K_2}_{B_E^P}, \tag{4.3e}$$

$$\eta_5 \doteq \underbrace{4T^{-3} K_1}_{B_E^P} + \underbrace{2(nT^4)^{-1} h K_3}_{B_{EN}^P} - \underbrace{2T^{-4} K_5}_{B_{EE}^P} + \underbrace{2T^{-2} \left[ \frac{\partial}{\partial T} \left( \frac{h}{mn} \right) \right]_p K_{20}}_{B_{-PEE}^P}, \tag{4.3f}$$

$$\eta_6 \doteq \underbrace{2T^{-3} K_2}_{B_E^P} + \underbrace{(nT^4)^{-2} h K_4}_{B_{EN}^P} - \underbrace{T^{-4} K_6}_{B_{EE}^P} + \underbrace{T^{-2} \left[ \frac{\partial}{\partial T} \left( \frac{h}{mn} \right) \right]_p K_{21}}_{B_{-PEE}^P}, \tag{4.3g}$$

$$\eta_7 \doteq - \underbrace{(nT^3)^{-1} K_3}_{B_{EN}^P} + \underbrace{T^{-2} \left[ \frac{\partial}{\partial p} \left( \frac{h}{mn} \right) \right]_T K_{20}}_{B_{-PEN}^P}, \tag{4.3h}$$

$$\eta_8 \doteq - \underbrace{(nT^3)^{-1} K_4}_{B_{EN}^P} + \underbrace{T^{-2} \left[ \frac{\partial}{\partial p} \left( \frac{h}{mn} \right) \right]_T K_{21}}_{B_{-PEN}^P}, \tag{4.3i}$$

$$\eta_9 \doteq - \underbrace{T^{-2}K_7}_{B_{PP}^P} - \underbrace{T^{-1}\left(\frac{\partial p}{\partial e}\right)_n K_{21}}_{B_{-PP}^P}, \quad (4.3j)$$

$$\eta_{10} \doteq - \underbrace{2T^{-2}K_8}_{B_{PP}^P} + \underbrace{2T^{-1}K_{21}}_{B_{-PP}^P}, \quad (4.3k)$$

$$\eta_{11} \doteq - \underbrace{2T^{-2}(K_9 + K_{10})}_{B_{PP}^P} - \underbrace{2T^{-1}\left(\frac{\partial p}{\partial e}\right)_n K_{20}}_{B_{-PP}^P}, \quad (4.3l)$$

$$\eta_{12} \doteq - \underbrace{4T^{-2}(K_{11} + K_{12})}_{B_{PP}^P} + \underbrace{4T^{-1}K_{20}}_{B_{-PP}^P}, \quad (4.3m)$$

$$\eta_{13} \doteq - \underbrace{2T^{-2}(K_{11} - K_{12})}_{B_{PP}^P} + \underbrace{2T^{-1}K_{20}}_{B_{-PP}^P}, \quad (4.3n)$$

and

$$\lambda \doteq T^{-2}K^{\text{III}}, \quad (4.4a)$$

$$\lambda_1 \doteq \underbrace{T^{-2}K^{\text{VI}}}_{\partial B_E^E}, \quad (4.4b)$$

$$\lambda_2 \doteq \underbrace{-T^{-1}(K_1 + K_2)}_{B_P^E}, \quad (4.4c)$$

$$\lambda_3 \doteq \underbrace{-T^{-1}K_1}_{B_P^E}, \quad (4.4d)$$

$$\begin{aligned} \lambda_4 \doteq & \underbrace{T^{-2}(K_1 + K_2)}_{B_P^E} - \underbrace{T^{-3}K_{19}}_{B_{PE}^E} - \underbrace{T^{-3}K_{13}}_{B_{EP}^E} \\ & + \underbrace{(nT^3)^{-1}h K_{17}}_{B_{PN}^E} + \underbrace{T^{-1}\left\{\left[\frac{\partial}{\partial T}\left(\frac{\partial p}{\partial n}\right)_e\right]_p K_{22} + \left[\frac{\partial}{\partial T}\left(\frac{\partial p}{\partial e}\right)_n\right]_p K_{23}\right\}}_{B_{-PPN}^E}, \end{aligned} \quad (4.4e)$$

$$\begin{aligned} \lambda_5 \doteq & \underbrace{T^{-2}(3K_1 + K_2)}_{B_P^E} + \underbrace{2(nT^3)^{-1}h K_{16}}_{B_{PN}^E} - \underbrace{2T^{-3}K_{18}}_{B_{PE}^E} \\ & - \underbrace{T^{-3}(K_{14} + K_{15})}_{B_{EP}^E} + \underbrace{T^{-2}\left\{\left[\left(\frac{\partial p}{\partial n}\right)_e - \frac{h}{n}\right] K_{22} + \left[1 + \left(\frac{\partial p}{\partial e}\right)_n\right] K_{23}\right\}}_{B_{-PEP}^E}, \end{aligned} \quad (4.4f)$$

$$\begin{aligned} \lambda_6 \doteq & \underbrace{T^{-2}(K_1 - K_2)}_{B_P^E} - \underbrace{T^{-3}(K_{14} - K_{15})}_{B_{EP}^E} \\ & - \underbrace{T^{-2}\left\{\left[\left(\frac{\partial p}{\partial n}\right)_e - \frac{h}{n}\right] K_{22} + \left[1 + \left(\frac{\partial p}{\partial e}\right)_n\right] K_{23}\right\}}_{B_{-PEP}^E}, \end{aligned} \quad (4.4g)$$

$$\begin{aligned}
\lambda_7 &\doteq -\underbrace{2(nT^2)^{-1}K_{16}}_{B_{PN}^E} - \underbrace{2(mn)^{-1}\eta_1}_{\partial B_P^E}, \\
\lambda_8 &\doteq -\underbrace{(nT^2)^{-1}K_{17}}_{B_{PN}^E} + \underbrace{T^{-1}\left\{\left[\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial n}\right)_e\right]_T K_{22} + \left[\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial \epsilon}\right)_n\right]_T K_{23}\right\}}_{B_{-PN}^E} - \underbrace{(mn)^{-1}\eta_2}_{\partial B_P^E}.
\end{aligned} \tag{4.4h}$$

$$\tag{4.4i}$$

This work was supported by the U. S. Department of Energy Contract DE-AC02-09CH11466.

#### REFERENCES

- BREY, J. J. 1983 Long time behavior of the Burnett transport coefficients. *J. Chem. Phys.* **79**, 4585–4598.
- BREY, J. J., ZWANZIG, R. & DORFMAN, J. R. 1981 Nonlinear transport equations in statistical mechanics. *Physica* **109A**, 425–444.
- KROMMES, J. A. 2015 A tutorial introduction to the statistical theory of turbulent plasmas, a half century after Kadomtsev’s *Plasma Turbulence* and the resonance-broadening theory of Dupree and Weinstock. *J. Plasma Phys.* **81**, 205810601 (80 pages).
- KROMMES, J. A. 2018 Projection-operator methods for classical transport in magnetized plasmas. Part 2. Nonlinear response and the Burnett equations. *J. Plasma Phys.* **84(6)**, <https://doi.org/10.1017/S0022377818000892> (109 pages).