Supplementary Material on

"The Asymptotic Properties of the System GMM Estimator in Dynamic Panel Data Models When Both N and T are Large"

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Abstract

This supplementary material contains (i) the explicit form of some variables (Section A), (ii) discussion on the use of redundant moment conditions (Section B), (iii) the proofs of theorems in the main part (Section C), and (iv) complete simulation results.

A On the form of $\mathbf{\Omega}_{T_1}^{-1/2}$

The explicit form of $\mathbf{\Omega}_{T_1}^{-1/2}$ is given by

$$\mathbf{\Omega}_{T_1}^{-1/2} = \text{diag}\left(\sqrt{\frac{T_1-1+\frac{1}{r}}{T_1+\frac{1}{r}}}, \sqrt{\frac{T_1-2+\frac{1}{r}}{T_1-1+\frac{1}{r}}}, ..., \sqrt{\frac{1+\frac{1}{r}}{2+\frac{1}{r}}}, \sqrt{\frac{\frac{1}{r}}{1+\frac{1}{r}}}\right) \times \begin{bmatrix} 1 & \frac{-1}{T_1-1+\frac{1}{r}} & \frac{-1}{T_1-1+\frac{1}{r}} & & \cdots & \frac{-1}{T_1-1+\frac{1}{r}} \\ 0 & 1 & \frac{-1}{T_1-2+\frac{1}{r}} & & \cdots & \frac{-1}{T_1-2+\frac{1}{r}} \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ & & & & 1 & \frac{-1}{1+\frac{1}{r}} \\ 0 & 0 & & \cdots & 0 & 1 \end{bmatrix}. \quad \text{(A.1)}$$

This form can be derived by applying a computational algorithm of Cholesky factorization to $\mathbf{\Omega}_{T_1}^{-1} = \mathbf{I}_{T_1} - \frac{1}{T_1+\frac{1}{r}}\boldsymbol{\iota}_{T_1}\boldsymbol{\iota}'_{T_1}$. A detailed derivation can be obtained upon request.

Using (A.1), we have

$$x_{it}^{+} = \begin{cases} d_t\left(x_{it} - \frac{x_{i,t+1}+\cdots+x_{iT}}{T-t+\frac{1}{r}}\right) & (i=1,...,N;\ t=2,...,T-1) \\ d_T x_{iT} & (i=1,...,N;\ t=T), \end{cases} \quad \text{(A.2)}$$

$$u_{it}^{+} = \begin{cases} d_t\left(u_{it} - \frac{u_{i,t+1}+\cdots+u_{iT}}{T-t+\frac{1}{r}}\right) = k_t\eta_i + v_{it}^{+} & (i=1,...,N;\ t=2,...,T-1) \\ d_T u_{iT} & (i=1,...,N;\ t=T), \end{cases} \quad \text{(A.3)}$$

where

$$d_t^2 = \frac{T-t+\frac{1}{r}}{T-t+1+\frac{1}{r}},$$

$$k_t = \frac{d_t}{r(T-t+\frac{1}{r})} = \frac{1}{r\sqrt{T-t+1+\frac{1}{r}}\sqrt{T-t+\frac{1}{r}}} = O\left(\frac{1}{T-t}\right), \quad \text{(A.4)}$$

$$v_{it}^{+} = d_t\left(v_{it} - \frac{v_{i,t+1}+\cdots+v_{iT}}{T-t+\frac{1}{r}}\right). \quad \text{(A.5)}$$

On comparing the transformation matrix for forward orthogonal deviations (Arellano, 2003, p. 17) and (A.1), we find that they have a somewhat similar structure. In fact, using x_{it}^* and v_{it}^*, x_{it}^+ and u_{it}^+ can be rewritten as

$$x_{it}^{+} = b_t x_{it}^* + k_t x_{it} \quad (i=1,...,N;\ t=2,...,T-1), \quad \text{(A.6)}$$

$$u_{it}^{+} = b_t v_{it}^* + k_t u_{it} \quad (i=1,...,N;\ t=2,...,T-1), \quad \text{(A.7)}$$

where

$$b_t^2 = \frac{(T-t+1)(T-t)}{(T-t+1+\frac{1}{r})(T-t+\frac{1}{r})} = 1 - \frac{2(T-t)+1+\frac{1}{r}}{r(T-t+1+\frac{1}{r})(T-t+\frac{1}{r})} < 1. \quad \text{(A.8)}$$

Note that when r is large, x_{it}^+ and u_{it}^+ coincide with x_{it}^* and v_{it}^*, respectively, for $t = 2, ..., T-1$. This corresponds to the fact that the GLS estimator is equivalent to the LSDV estimator when r and/or T are (is) large[1]. Therefore, we expect that the relationship between the FOD and optimal level GMM estimators is very similar to that between the LSDV and random effect GLS estimators.

[1] A similar observation is also made by Hahn, Kuersteiner and Cho (2004).

B On the use of redundant moment conditions

In this section, we investigate the effects of the redundant moment conditions, first, in a general framework, and then, in the system GMM estimator.

B.1 General case

Let us consider the following two sets of moment conditions:

$$E[\mathbf{g}_1(\mathbf{x}_i,\boldsymbol{\theta}_0)] = E[\mathbf{g}_{1i}(\boldsymbol{\theta}_0)] = \mathbf{0}, \tag{B.1}$$

$$E[\mathbf{g}_2(\mathbf{x}_i,\boldsymbol{\theta}_0)] = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{A} \end{bmatrix} E[\mathbf{g}_{1i}(\boldsymbol{\theta}_0)] = \mathbf{C}E[\mathbf{g}_{1i}(\boldsymbol{\theta}_0)], \tag{B.2}$$

where $\{\mathbf{x}_i\}_{i=1}^N$ are i.i.d. over i, $\boldsymbol{\theta}_0$ is a $p \times 1$ true parameter vector, $\mathbf{g}_{1i}(\boldsymbol{\theta}_0)$ and $\mathbf{g}_{2i}(\boldsymbol{\theta}_0)$ are $m \times 1$ and $(m+r)\times 1$ vectors, respectively, and $\mathbf{A}$ is a known nonrandom $r \times m$ matrix. We assume that $p < m$. Note that (B.2) includes redundant moment conditions $\mathbf{A}E[\mathbf{g}_{1i}(\boldsymbol{\theta}_0)] = \mathbf{0}$, which are linear combinations of (B.1).

The GMM estimators of these two sets of moment conditions are defined as

$$\begin{aligned} \widehat{\boldsymbol{\theta}}_1 &= \arg\min_{\theta} \widehat{\mathbf{g}}_1(\boldsymbol{\theta})'\widehat{\mathbf{W}}_{11}^{-1}\widehat{\mathbf{g}}_1(\boldsymbol{\theta}), \\ \widehat{\boldsymbol{\theta}}_2 &= \arg\min_{\theta} \widehat{\mathbf{g}}_2(\boldsymbol{\theta})'\widehat{\mathbf{W}}^{-1}\widehat{\mathbf{g}}_2(\boldsymbol{\theta}), \end{aligned}$$

respectively, where $\widehat{\mathbf{g}}_1 = N^{-1}\sum_{i=1}^N \mathbf{g}_{1i}(\boldsymbol{\theta})$, $\widehat{\mathbf{g}}_2 = N^{-1}\sum_{i=1}^N \mathbf{g}_{2i}(\boldsymbol{\theta})$, and $\widehat{\mathbf{W}}_{11}$ and $\widehat{\mathbf{W}}$ are weighting matrices that converge in probability to the positive definite matrices, $\mathbf{W}_{11}$ and $\mathbf{W}$, respectively. Under regularity conditions, the asymptotic variances are given by

$$\begin{aligned} var(\widehat{\boldsymbol{\theta}}_1) &= \left(\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1\right)^{-1}\mathbf{G}_1'\mathbf{W}_{11}^{-1}\boldsymbol{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1\left(\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1\right)^{-1}, \\ var(\widehat{\boldsymbol{\theta}}_2) &= \left(\mathbf{G}_2'\mathbf{W}^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2'\mathbf{W}^{-1}\boldsymbol{\Omega}\mathbf{W}^{-1}\mathbf{G}_2\left(\mathbf{G}_2'\mathbf{W}^{-1}\mathbf{G}_2\right)^{-1} \\ &= \left(\mathbf{G}_1'\mathbf{C}'\mathbf{W}^{-1}\mathbf{C}\mathbf{G}_1\right)^{-1}\mathbf{G}_1'\mathbf{C}'\mathbf{W}^{-1}\mathbf{C}\boldsymbol{\Omega}_{11}\mathbf{C}'\mathbf{W}^{-1}\mathbf{C}\mathbf{G}_1\left(\mathbf{G}_1'\mathbf{C}'\mathbf{W}^{-1}\mathbf{C}\mathbf{G}_1\right)^{-1} \end{aligned} \tag{B.3}$$

where

$$\mathbf{G}_1 = E\left[\frac{\partial \mathbf{g}_{1i}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}'}\right], \qquad \mathbf{G}_2 = E\left[\frac{\partial \mathbf{g}_{2i}(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}'}\right] = \mathbf{C}\mathbf{G}_1 = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{A}\mathbf{G}_1 \end{bmatrix},$$

$$E[\mathbf{g}_{1i}(\boldsymbol{\theta}_0)\mathbf{g}_{1i}(\boldsymbol{\theta}_0)'] = \boldsymbol{\Omega}_{11}, \qquad E[\mathbf{g}_{2i}(\boldsymbol{\theta}_0)\mathbf{g}_{2i}(\boldsymbol{\theta}_0)'] = \boldsymbol{\Omega} = \mathbf{C}\boldsymbol{\Omega}_{11}\mathbf{C}', \qquad \mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix}.$$

We assume that $\mathbf{W}_{11} \neq \boldsymbol{\Omega}_{11}$ since the case with $\mathbf{W}_{11} = \boldsymbol{\Omega}_{11}$ is already discussed by Breusch, Qian, Schmidt and Wyhowski (1999).

We now reformulate the form of $var(\widehat{\boldsymbol{\theta}}_2)$ to simplify the analysis. Let $\mathbf{K}$ be a non-singular matrix. Then, $var(\widehat{\boldsymbol{\theta}}_2)$ can be written as

$$\begin{aligned} var(\widehat{\boldsymbol{\theta}}_2) = {} & \Big(\mathbf{G}_1'\mathbf{C}'\mathbf{K}'\left(\mathbf{K}\mathbf{W}\mathbf{K}'\right)^{-1}\mathbf{K}\mathbf{C}\mathbf{G}_1\Big)^{-1} \\ & \times\mathbf{G}_1'\mathbf{C}'\mathbf{K}'\left(\mathbf{K}\mathbf{W}\mathbf{K}'\right)^{-1}\mathbf{K}\mathbf{C}\boldsymbol{\Omega}_{11}\mathbf{C}'\mathbf{K}'\left(\mathbf{K}\mathbf{W}\mathbf{K}'\right)^{-1}\mathbf{K}\mathbf{C}\mathbf{G}_1\Big(\mathbf{G}_1'\mathbf{C}'\mathbf{K}'\left(\mathbf{K}\mathbf{W}\mathbf{K}'\right)^{-1}\mathbf{K}\mathbf{C}\mathbf{G}_1\Big)^{-1}. \end{aligned}$$

If we set

$$\mathbf{K} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ -\mathbf{W}_{21}\mathbf{W}_{11}^{-1} & \mathbf{I}_r, \end{bmatrix},$$

we have

$$\mathbf{V} = \mathbf{K}\mathbf{W}\mathbf{K}' = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} - \mathbf{W}_{21}\mathbf{W}_{11}^{-1}\mathbf{W}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} \end{bmatrix}, \quad \mathbf{K}\mathbf{C} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{A} - \mathbf{W}_{21}\mathbf{W}_{11}^{-1} \end{bmatrix}.$$

Using these, we then have

$$\begin{aligned}\mathbf{G}_1'\mathbf{C}'\mathbf{K}'\left(\mathbf{KWK}'\right)^{-1}\mathbf{KCG}_1 &= \mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\left(\mathbf{A}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\right)'\mathbf{V}_{22}^{-1}\left(\mathbf{A}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\right)\mathbf{G}_1\\ &= \mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{G}_1,\end{aligned}$$

$$\begin{aligned}\mathbf{G}_1'\left\{\mathbf{C}'\mathbf{K}'\left(\mathbf{KWK}'\right)^{-1}\mathbf{KC}\right\}\mathbf{\Omega}_{11}\left\{\mathbf{C}'\mathbf{K}'\left(\mathbf{KWK}'\right)^{-1}\mathbf{KC}\right\}\mathbf{G}_1 &= \mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{H}\mathbf{G}_1\\ &\quad + \mathbf{G}_1'\mathbf{H}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{\Omega}_{11}\mathbf{H}\mathbf{G}_1\end{aligned}$$

where

$$\begin{aligned}\mathbf{H} &= \left(\mathbf{A}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\right)'\mathbf{V}_{22}^{-1}\left(\mathbf{A}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\right)\\ &= \left(\mathbf{A}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\right)'\left(\mathbf{W}_{22}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\mathbf{W}_{12}\right)^{-1}\left(\mathbf{A}-\mathbf{W}_{21}\mathbf{W}_{11}^{-1}\right).\end{aligned}$$

Substituting these expressions into (B.3), we have

$$\begin{aligned}var(\widehat{\boldsymbol{\theta}}_2) &= \left[\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{G}_1\right]^{-1}\\ &\quad \times\left[\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{H}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{\Omega}_{11}\mathbf{H}\mathbf{G}_1\right]\\ &\quad \times\left[\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{G}_1\right]^{-1}.\end{aligned}$$

If $\boldsymbol{\theta}$ is a scalar, we have

$$\begin{aligned}var(\widehat{\boldsymbol{\theta}}_1) &= \frac{\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1}{\left(\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1\right)^2} = \frac{a_1}{b_1^2},\\ var(\widehat{\boldsymbol{\theta}}_2) &= \frac{\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \left(\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{\Omega}_{11}\mathbf{H}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{\Omega}_{11}\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{\Omega}_{11}\mathbf{H}\mathbf{G}_1\right)}{\left(\mathbf{G}_1'\mathbf{W}_{11}^{-1}\mathbf{G}_1 + \mathbf{G}_1'\mathbf{H}\mathbf{G}_1\right)^2} = \frac{a_1+a_r}{(b_1+b_r)^2}.\end{aligned} \tag{B.4}$$

From this, we find that, depending on the relative magnitudes of a_r and b_r, or in other words, depending on the structure of redundancy, $\mathbf{A}$, and the weighting matrix, $\mathbf{W}_{21}$ and $\mathbf{W}_{22}$, which appear in $\mathbf{H}$, we could have both cases $var(\widehat{\boldsymbol{\theta}}_2) > var(\widehat{\boldsymbol{\theta}}_1)$ and $var(\widehat{\boldsymbol{\theta}}_2) < var(\widehat{\boldsymbol{\theta}}_1)$. If $\mathbf{H} = \mathbf{0}$ or $\mathbf{H} = \mathbf{W}_{11}^{-1}$, we have $var(\widehat{\boldsymbol{\theta}}_2) = var(\widehat{\boldsymbol{\theta}}_1)$. This implies that using redundant moment conditions with a non-optimal weighting matrix improves efficiency in some cases and worsens it in other cases. Which case happens depends on the structure of redundancy and weighting matrix associated with the redundant moment conditions.

B.2 Results for system model

In Section B.1, we showed that whether redundant moment conditions improve efficiency crucially depends on the structure of moment conditions and weighting matrix. In this section, we demonstrate that in the panel AR(1) cases, using redundant moment conditions improves efficiency if a suitable weighting matrix is used.

Consider the following system (the model (13) in the body):

$$\underline{\mathbf{y}}_i^\dagger = \alpha\underline{\mathbf{x}}_i^\dagger + \underline{\mathbf{u}}_i^\dagger.$$

Note that by changing the ordering, the moment conditions $E(\underline{\mathbf{Z}}_i^{s2\prime}\underline{\mathbf{u}}_i^\dagger) = \mathbf{0}$ can be reformulated as

$$\begin{bmatrix} E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^\dagger\right] \\ E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{u}}_i^\dagger\right]\end{bmatrix} = \begin{bmatrix} E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^\dagger\right] \\ \mathbf{A}E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^\dagger\right]\end{bmatrix} = \mathbf{0},$$

where

$$\underline{\mathbf{Z}}_i^r = \begin{bmatrix} 0 & & & & & & \\ \Delta y_{i1} & & & & & & \\ & \Delta y_{i1} & \Delta y_{i2} & & & & \\ & & & \ddots & & & \\ & & & & \Delta y_{i1} & \cdots & \Delta y_{i,T-2} \end{bmatrix}.$$

Note that $E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{u}}_i^{\dagger}\right] = \mathbf{A}E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^{\dagger}\right] = \mathbf{0}$ are the redundant moment conditions, and the non-redundant moment conditions for the models in levels are included in $E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^{\dagger}\right] = \mathbf{0}$. Additionally, note that the nonrandom matrix $\mathbf{A}$ is endogenously determined in the model and we cannot control it.

Using these expressions, we can set

$$\begin{aligned} E[\mathbf{g}_{1i}(\alpha_0)] &= E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^{\dagger}\right] = \mathbf{0}, \\ E[\mathbf{g}_{2i}(\alpha_0)] &= E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{u}}_i^{\dagger}\right] = \mathbf{A}E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^{\dagger}\right] = \mathbf{0}, \end{aligned}$$

where $\boldsymbol{\theta}_0 = \alpha_0$. Then, two GMM estimators are defined as

$$\begin{aligned} \widehat{\alpha}_{S1}^* &= \arg\min_{\alpha} \widehat{\mathbf{g}}_1(\alpha)'\widehat{\mathbf{W}}_{11}^{-1}\widehat{\mathbf{g}}_1(\alpha), \\ \widehat{\alpha}_{S2}^* &= \arg\min_{\alpha} \widehat{\mathbf{g}}_2(\alpha)'\widehat{\mathbf{W}}^{-1}\widehat{\mathbf{g}}_2(\alpha), \end{aligned}$$

where $\widehat{\mathbf{W}}_{11}$ and $\widehat{\mathbf{W}}$ are the consistent estimates of

$$\begin{aligned} \mathbf{W}_{11} &= E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{Z}}_i^{s1}\right] = \begin{bmatrix} E\left[\underline{\mathbf{Z}}_i^{l\prime}\underline{\mathbf{Z}}_i^{l}\right] & 0 \\ 0 & E\left[\underline{\mathbf{Z}}_i^{d1\prime}\underline{\mathbf{Z}}_i^{d1}\right] \end{bmatrix}, \\ \mathbf{W} &= \begin{bmatrix} \mathbf{W}_{11} & \begin{matrix} \mathbf{0} \\ E\left[\underline{\mathbf{Z}}_i^{d1\prime}\underline{\mathbf{Z}}_i^{r}\right] \end{matrix} \\ \mathbf{0} \quad E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{Z}}_i^{d1}\right] & E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{Z}}_i^{r}\right] \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix}. \end{aligned}$$

The explicit expressions of $\widehat{\alpha}_{S1}^*$ and $\widehat{\alpha}_{S2}^*$ are given in (15) and (17), respectively. We also consider an alternative GMM estimator using redundant moment conditions but using a more flexible weighting matrix. Specifically, we consider[2]

$$\widehat{\alpha}_{R2}^*(\kappa) = \arg\min_{\alpha} \widehat{\mathbf{g}}_2(\alpha)'\widehat{\mathbf{W}}(\kappa)^{-1}\widehat{\mathbf{g}}_2(\alpha),$$

where $\widehat{\mathbf{W}}(\kappa)$ is a consistent estimate of

$$\mathbf{W}(\kappa) = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \kappa\mathbf{W}_{22} \end{bmatrix}.$$

When $\kappa = 1$, we have $\mathbf{W}(\kappa) = \mathbf{W}$, and therefore, $\widehat{\alpha}_{R2}^*(1) = \widehat{\alpha}_{S2}^*$. Otherwise, $\widehat{\alpha}_{R2}^*(\kappa) \neq \widehat{\alpha}_{S2}^*$. To see the effect of κ, we numerically compare the variances of $\widehat{\alpha}_{S2}^*$, $\widehat{\alpha}_{S1}^*$, and $\widehat{\alpha}_{R2}^*(\kappa)$. For this, we need to derive some expressions in (B.4).

First, note that

$$y_{it} = \mu_i + w_{it} \qquad (t = 1, ..., T)$$

[2] We also considered the case where $\mathbf{W}_{12}$ also changes such that

$$\mathbf{W}(\kappa, \lambda) = \begin{bmatrix} \mathbf{W}_{11} & \lambda\mathbf{W}_{12} \\ \lambda\mathbf{W}_{21} & \kappa\mathbf{W}_{22} \end{bmatrix}.$$

However, since the result is very similar, we consider $\mathbf{W}(\kappa)$ only.

$$\begin{aligned}
x_{it}^{*} &= \psi_t w_{i,t-1} - c_t \tilde{v}_{itT} \qquad\qquad (t=1,...,T-1) \\
x_{it}^{+} &= b_t x_{it}^{*} + k_t x_{it} = k_t \mu_i + (b_t\psi_t + k_t)\, w_{i,t-1} - b_t c_t \tilde{v}_{itT}, \qquad (t=2,...,T-1) \\
x_{iT}^{+} &= d_T x_{iT} = k_T\,(\mu_i + w_{i,T-1})
\end{aligned}$$

where

$$w_{it} = \sum_{j=0}^{\infty} \alpha^j v_{i,t-j}, \qquad \psi_t = c_t\left(1 - \frac{\alpha\phi_{T-t}}{T-t}\right), \qquad \tilde{v}_{itT} = \frac{\phi_{T-t}v_{it} + ... + \phi_1 v_{i,T-1}}{T-t}, \qquad \phi_j = \frac{1-\alpha^j}{1-\alpha}.$$

Using these, we have

$$\begin{aligned}
E\left[y_{is}x_{it}^{*}\right] &= \psi_t E\left[w_{is}w_{i,t-1}\right] = \psi_t \alpha^{t-1-s}\frac{\sigma_v^2}{1-\alpha^2} \qquad\qquad (0 \le s < t \le T-1), \\
E\left[\Delta y_{is}x_{it}^{+}\right] &= (b_t\psi_t + k_t)\, E\left[\Delta w_{is}w_{i,t-1}\right] = (b_t\psi_t + k_t)\,\alpha^{t-1-s}\frac{\sigma_v^2}{1+\alpha} \qquad\qquad (1 \le s < t \le T-1), \\
E\left[\Delta y_{is}x_{iT}^{+}\right] &= k_T E\left[\Delta w_{is}w_{i,T-1}\right] = k_T \alpha^{T-1-s}\frac{\sigma_v^2}{1+\alpha} \qquad\qquad (1 \le s < T-1), \\
E\left(\Delta y_{i,t-j}\Delta y_{it}\right) &= E\left(w_{t-j} - w_{t-j-1}\right)\left(w_t - w_{t-1}\right) = \frac{-(1-\alpha)\sigma_v^2}{1+\alpha}\alpha^{j-1}.
\end{aligned}$$

Then, we have

$$\mathbf{G}_1 = -\begin{bmatrix} E\left[\underline{\mathbf{Z}}_i^{l\prime}\underline{\mathbf{u}}_i^{*}\right] \\ E\left[\underline{\mathbf{Z}}_i^{d1\prime}\underline{\mathbf{u}}_i^{+}\right] \end{bmatrix},$$

$$E\left[\underline{\mathbf{Z}}_i^{l\prime}\underline{\mathbf{u}}_i^{*}\right] = \frac{\sigma_v^2}{1-\alpha^2}\begin{bmatrix} \psi_1 a_0 \\ \psi_2 \mathbf{a}_1 \\ \vdots \\ \psi_{T-1}\mathbf{a}_{T-2} \end{bmatrix}, \qquad E\left[\underline{\mathbf{Z}}_i^{d1\prime}\underline{\mathbf{u}}_i^{+}\right] = \frac{\sigma_v^2}{1+\alpha}\begin{bmatrix} (b_2\psi_2 + k_2) \\ (b_3\psi_3 + k_3) \\ \vdots \\ (b_{T-1}\psi_{t-1} + k_{T-1}) \\ k_T \end{bmatrix},$$

$$\begin{aligned}
\mathbf{W}_{11} &= E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{Z}}_i^{s1}\right] = \mathrm{diag}\left(\Sigma_1^l, \mathbf{\Sigma}_2^l, ..., \mathbf{\Sigma}_{T-1}^l, \frac{2\sigma_v^2}{1+\alpha}\mathbf{I}_{T-1}\right), \\
\mathbf{W}_{22} &= E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{Z}}_i^{r}\right] = \mathrm{diag}(\Sigma_1^d, \mathbf{\Sigma}_2^d, ..., \mathbf{\Sigma}_{T-2}^d), \\
\mathbf{W}_{12} &= \begin{bmatrix} \mathbf{0} \\ E\left[\underline{\mathbf{Z}}_i^{d1\prime}\underline{\mathbf{Z}}_i^{r}\right] \end{bmatrix} = \frac{-(1-\alpha)\sigma_v^2}{1+\alpha}\begin{bmatrix} & & \mathbf{0} & \\ 0 & \cdots & \cdots & \mathbf{0} \\ a_0 & & & \mathbf{0} \\ & \mathbf{a}_1' & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{a}_{T-2}' \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
E\left(\underline{\mathbf{y}}_i^{t-1}x_{it}^{*}\right) &= \left(\frac{\sigma_v^2}{1-\alpha^2}\right)\psi_t\mathbf{a}_{t-1}, \qquad E\left(\Delta y_{i,t-1}x_{it}^{+}\right) = \frac{\sigma_v^2}{1+\alpha}\left(b_t\psi_t + k_t\right), \\
E\left(\Delta\underline{\mathbf{y}}_i^{t-1}\Delta y_{it}\right) &= \frac{-(1-\alpha)\sigma_v^2}{1+\alpha}\mathbf{a}_{t-2}, \qquad \mathbf{a}_t = \left(\alpha^t, \alpha^{t-1}, ..., \alpha, 1\right)', \\
\underline{\mathbf{y}}_i^t &= (y_{i0}, ..., y_{it})', \qquad \Delta\underline{\mathbf{y}}_i^t = (\Delta y_{i1}, ..., \Delta y_{it})', \\
\mathbf{\Sigma}_t^l &= E(\underline{\mathbf{y}}_i^t\underline{\mathbf{y}}_i^{t\prime}) = \sigma_\mu^2\boldsymbol{\iota}_t\boldsymbol{\iota}_t' + \left(\frac{\sigma_v^2}{1-\alpha^2}\right)\mathbf{V}_t, \qquad \mathbf{V}_t^{ij} = \{\alpha^{|i-j|}\}, \\
\mathbf{\Sigma}_t^d &= E(\Delta\underline{\mathbf{y}}_i^t\Delta\underline{\mathbf{y}}_i^{t\prime}) = \mathbf{D}_{t+1}\mathbf{\Sigma}_t^l\mathbf{D}_{t+1}',
\end{aligned}$$

with $\mathbf{D}_{t+1}$ being a $t \times (t+1)$ matrix that takes first differences.

The remaining expressions we need to derive are $\mathbf{A}$ and $\mathbf{\Omega}_{11}$. However, since they are too complicated to derive analytically, their values are computed by simulation. Specifically, we use

$$\mathbf{\Omega}_{11}^{*} = \frac{1}{N^{*}} \sum_{i=1}^{N^{*}} \underline{\mathbf{Z}}_i^{s1\prime} \underline{\mathbf{u}}_i^{\dagger} \underline{\mathbf{u}}_i^{\dagger\prime} \underline{\mathbf{Z}}_i^{s1}.$$

For $\mathbf{A}$, since $E\left[\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{u}}_i^{+}\right] = \mathbf{A}E\left[\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^{\dagger}\right]$ can be seen as an expectation expression of multivariate regression $\underline{\mathbf{Z}}_i^{r\prime}\underline{\mathbf{u}}_i^{+} = \mathbf{A}\cdot\underline{\mathbf{Z}}_i^{s1\prime}\underline{\mathbf{u}}_i^{\dagger} + \mathbf{U}_i$ with $E(\mathbf{U}_i) = \mathbf{0}$, we compute as

$$\mathbf{A}^{*} = \left(\frac{1}{N^{*}} \sum_{i=1}^{N^{*}} \underline{\mathbf{Z}}_i^{r\prime} \underline{\mathbf{u}}_i^{+} \underline{\mathbf{u}}_i^{\dagger\prime} \underline{\mathbf{Z}}_i^{s1}\right) \left(\frac{1}{N^{*}} \sum_{i=1}^{N^{*}} \underline{\mathbf{Z}}_i^{s1\prime} \underline{\mathbf{u}}_i^{\dagger} \underline{\mathbf{u}}_i^{\dagger\prime} \underline{\mathbf{Z}}_i^{s1}\right)^{-1}.$$

In the numerical studies, we compute $\mathbf{\Omega}_{11}^{*}$ and $\mathbf{A}^{*}$ as the average of 10 replications with $N^{*} = 25,000$. We report the values of $\sqrt{var(\widehat{\alpha}_{S2}^{*})/N}$, $\sqrt{var(\widehat{\alpha}_{S1}^{*})/N}$, and $\sqrt{var(\widehat{\alpha}_{R2}^{*}(\kappa))/N}$ relative to $\sqrt{var(\widehat{\alpha}_{SYS}^{*})/N}$ for the cases with $\alpha = 0.3, 0.6, 0.9$, $T = 5, 10, 20$, $N = 200$, $\sigma_v^2 = 1$, and $\sigma_\eta^2 = 1$. The results are given in Figure 1.

From the figure, we first find that since the line for $var(\widehat{\alpha}_{S2}^{*})$(dotted line) is below that for $var(\widehat{\alpha}_{S1}^{*})$(dashed-dotted line) in all figures, we may conclude that $var(\widehat{\alpha}_{S2}^{*}) < var(\widehat{\alpha}_{S1}^{*})$. This implies that using redundant moment conditions as in (16) improves efficiency. We also find that κ has a significant effect on the variance. When $\kappa \geq 1$, using redundant moment conditions improves efficiency over $\widehat{\alpha}_{S1}^{*}$ where the redundant moment conditions are not used. As κ becomes larger, $var(\widehat{\alpha}_{R2}^{*}(\kappa))$ gets close to $var(\widehat{\alpha}_{S1}^{*})$. This is because the relative effect of $\kappa\mathbf{W}_{22}$ to $\mathbf{W}_{11}$ in $\mathbf{W}^{-1}$ becomes minor when κ is large. However, when $\kappa < 1$, the results are very sensitive to the value of κ. Focusing on the case of $T = 5$ and $\alpha = 0.3$, if κ is slightly smaller than 1, we have $var(\widehat{\alpha}_{R2}^{*}(\kappa)) < var(\widehat{\alpha}_{S2}^{*})$. For instance, when $\kappa = 0.7$, $var(\widehat{\alpha}_{R2}^{*}(0.7)) = 0.0564 < var(\widehat{\alpha}_{R2}^{*}(1)) = var(\widehat{\alpha}_{S2}^{*}) = 0.0572$. However, as κ gets smaller, $var(\widehat{\alpha}_{R2}(\kappa))$ becomes large, and we have $var(\widehat{\alpha}_{R2}^{*}(\kappa)) > var(\widehat{\alpha}_{S2}^{*})$. For instance, when $\kappa = 0.3$, we have $var(\widehat{\alpha}_{R2}^{*}(0.3)) = 0.0732 > var(\widehat{\alpha}_{R2}^{*}(1)) = var(\widehat{\alpha}_{S2}^{*}) = 0.0572$. This is because $\kappa\mathbf{W}_{22}$ behaves dominantly in $\mathbf{W}^{-1}$ compared with $\mathbf{W}_{11}$ when κ is close to zero.

This indicates that, in terms of the choice for $\mathbf{W}_{22}$, $\widehat{\alpha}_{S2}^{*}$ is not the best choice since $\widehat{\alpha}_{R2}^{*}(\kappa)$ can have a smaller variance for some values of κ. However, there are several reasons that motivate us to use $\widehat{\alpha}_{S2}^{*}$. First, the values of κ that lead to efficiency gain are unknown. Second, there is a danger of substantial efficiency loss when $\kappa < 1$.

From this numerical exercise, we may conclude that using redundant moment conditions as in $\widehat{\alpha}_{S2}^{*}$ improves efficiency over $\widehat{\alpha}_{S1}^{*}$ where the redundant moment conditions are not used. In Theorem 2, we show that this is also true when both N and T are large. However, it should be noted that this result is specific to the AR(1) case and inconclusive in a more general model with exogenous variables.

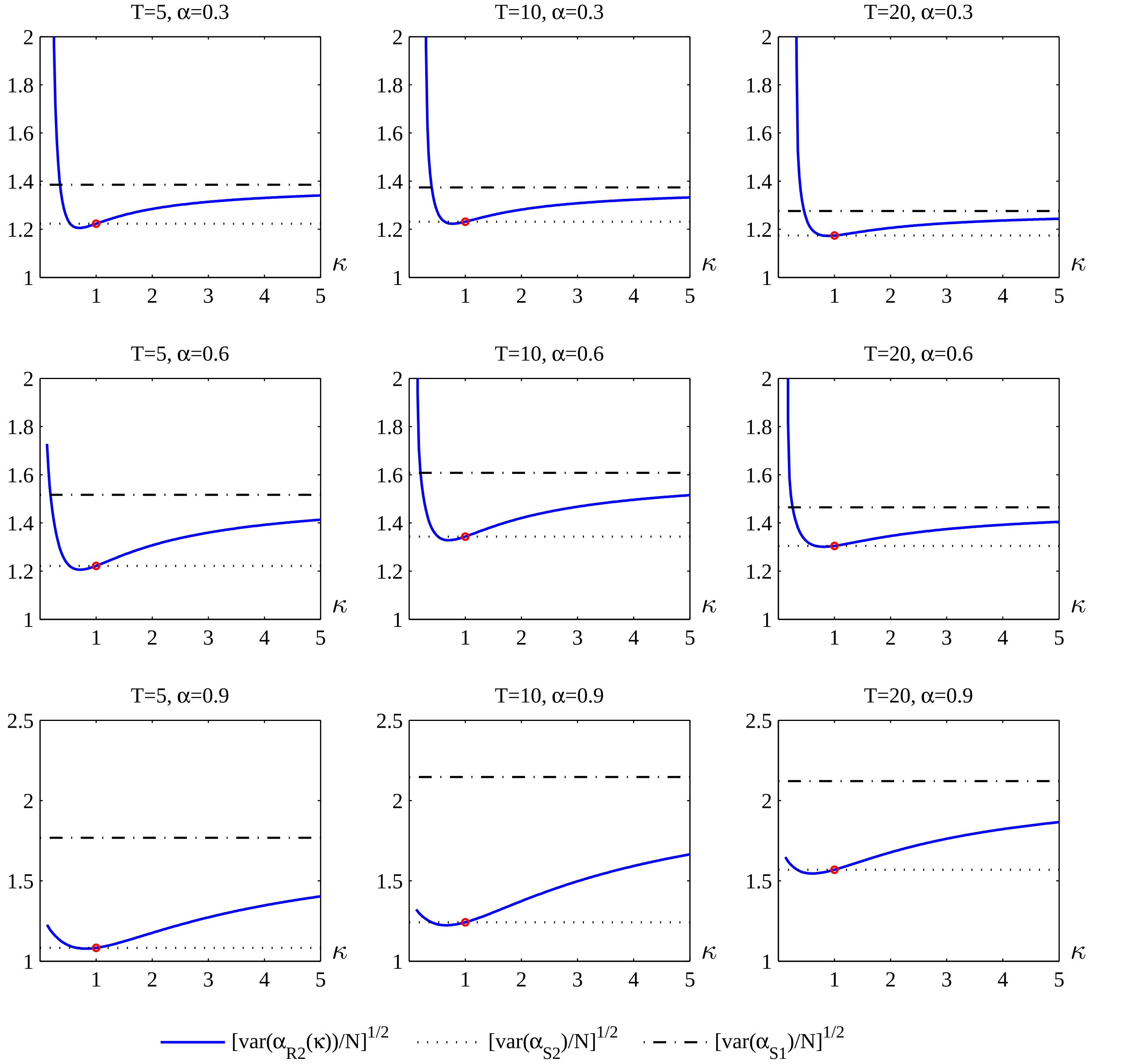

Figure 1: Values of $\frac{\sqrt{var(\widehat{\alpha}^*_{S2})/N}}{\sqrt{var(\widehat{\alpha}_{SYS})/N}}$, $\frac{\sqrt{var(\widehat{\alpha}^*_{S1})/N}}{\sqrt{var(\widehat{\alpha}_{SYS})/N}}$ and $\frac{\sqrt{var(\widehat{\alpha}^*_{R2}(\kappa))/N}}{\sqrt{var(\widehat{\alpha}_{SYS})/N}}$

C Proof of Theorems

In this section, we provide the proofs of the lemmas and theorems given in the main context. Throughout the appendix, we use C as a generic constant.

Lemma A1. *Under Assumptions 1 and 3, we obtain*

$$
\begin{aligned}
(a)\ \left[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})\right]^{-1} &= \sigma_v^{-2}\begin{bmatrix} \frac{2}{1+\alpha} & -\left(\frac{1-\alpha}{1+\alpha}\right) & \cdots & -\alpha^{t-3}\left(\frac{1-\alpha}{1+\alpha}\right) \\ -\left(\frac{1-\alpha}{1+\alpha}\right) & \frac{2}{1+\alpha} & & \\ \vdots & & \ddots & \\ -\alpha^{t-3}\left(\frac{1-\alpha}{1+\alpha}\right) & & & \frac{2}{1+\alpha} \end{bmatrix}^{-1} \\
&= \boldsymbol{\Upsilon}_{11} - \frac{\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}}{E(w_{i,t-1}^2)+E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})},
\end{aligned}
$$

$$(b)\ E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1}) = \frac{\sigma_v^2(t-1)}{(1+\alpha)^2}$$

where w_{it} is defined below (2), $\mathbf{z}_{it}^{d2} = (\Delta w_{i1},...,\Delta w_{i,t-1})'$, $E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime}) = \sigma_v^2(\alpha^{t-2},\cdots,1)/(1+\alpha)$, $E(w_{i,t-1}^2) = \sigma_v^2/(1-\alpha^2)$, and

$$
\begin{aligned}
\boldsymbol{\Upsilon}_{11} &= \sigma_v^{-2}(\alpha^2-\alpha+1)\mathbf{I}_{t-1} + \sigma_v^{-2}(1-\alpha)\begin{bmatrix} \alpha & \boldsymbol{\iota}_{t-2}' \\ \boldsymbol{\iota}_{t-2} & \boldsymbol{\Lambda} \end{bmatrix}, \qquad \text{(C.1)} \\
\boldsymbol{\Lambda} &= \begin{bmatrix} 1 & 2-\alpha & 2-\alpha & 2-\alpha & \cdots & \cdots & 2-\alpha \\ 2-\alpha & 2-\alpha & 3-2\alpha & 3-2\alpha & \cdots & \cdots & 3-2\alpha \\ 2-\alpha & 3-2\alpha & 3-2\alpha & 4-3\alpha & \cdots & \cdots & 4-3\alpha \\ 2-\alpha & 3-2\alpha & 4-3\alpha & 4-3\alpha & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & (t-2)-(t-3)\alpha \\ 2-\alpha & 3-2\alpha & 4-3\alpha & \cdots & \cdots & (t-2)-(t-3)\alpha & (t-2)-(t-3)\alpha \end{bmatrix}, \\
\boldsymbol{\iota}_{t-2} &= (1,...,1)'.
\end{aligned}
$$

Proof of Lemma A1

(a): To derive the explicit expression of $[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})]^{-1}$, let us define $\widetilde{\mathbf{z}}_{it}$ as follows:

$$
\begin{aligned}
\widetilde{\mathbf{z}}_{it} &= \begin{bmatrix} \mathbf{z}_{it}^{d2} \\ -w_{i,t-1} \end{bmatrix} = \begin{bmatrix} \Delta w_{i1} \\ \Delta w_{i2} \\ \vdots \\ \Delta w_{i,t-1} \\ -w_{i,t-1} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & & & \mathbf{0} \\ 0 & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & 0 \\ & & & & -1 & 1 \\ \mathbf{0} & & & & 0 & -1 \end{bmatrix} \begin{bmatrix} w_{i0} \\ w_{i1} \\ \vdots \\ w_{i,t-1} \end{bmatrix} \\
&= \widetilde{\mathbf{D}}\mathbf{w}_i^{(t-1)}
\end{aligned}
$$

where $\mathbf{w}_i^{(t-1)} = (w_{i0},...,w_{i,t-1})'$. Then it follows that

$$\left[E(\widetilde{\mathbf{z}}_{it}\widetilde{\mathbf{z}}_{it}')\right]^{-1} = \left[\widetilde{\mathbf{D}}E(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime})\widetilde{\mathbf{D}}'\right]^{-1} = \left(\widetilde{\mathbf{D}}'\right)^{-1}\left[E(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime})\right]^{-1}\left(\widetilde{\mathbf{D}}\right)^{-1} \qquad \text{(C.2)}$$

$$= \begin{bmatrix} E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime}) & -E(\mathbf{z}_{it}^{d2}w_{i,t-1}) \\ -E(\mathbf{z}_{it}^{d2\prime}w_{i,t-1}) & E(w_{i,t-1}^2) \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{\Upsilon}_{11} & \boldsymbol{\Upsilon}_{12} \\ \boldsymbol{\Upsilon}_{21} & \Upsilon_{22} \end{bmatrix}$$

where $\boldsymbol{\Upsilon}_{11}$, $\boldsymbol{\Upsilon}_{12}$, $\boldsymbol{\Upsilon}_{21}$, and Υ_{22} are partitioned conformably, i.e., $\boldsymbol{\Upsilon}_{11}$ is a $(t-1)\times(t-1)$ matrix, $\boldsymbol{\Upsilon}_{12}$ and $\boldsymbol{\Upsilon}_{21}'$ are $(t-1)\times 1$ vectors, and Υ_{22} is a scalar. From the partitioned inverse formula, we obtain

$$\boldsymbol{\Upsilon}_{11} = \left[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime}) - E(w_{i,t-1}\mathbf{z}_{it}^{d2})[E(w_{i,t-1}^2)]^{-1}E(\mathbf{z}_{it}^{d2}w_{i,t-1})\right]^{-1}.$$

After some algebra, we obtain

$$\left[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})\right]^{-1} = \left[\boldsymbol{\Upsilon}_{11}^{-1} + E(w_{i,t-1}\mathbf{z}_{it}^{d2})[E(w_{i,t-1}^2)]^{-1}E(\mathbf{z}_{it}^{d2\prime}w_{i,t-1})\right]^{-1}. \tag{C.3}$$

Using the fact that $(\mathbf{A}+\mathbf{BCB}')^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}[\mathbf{C}^{-1}+\mathbf{B}'\mathbf{A}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\mathbf{A}^{-1}$, $(C.3)$ can be expressed as

$$\left[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})\right]^{-1} = \boldsymbol{\Upsilon}_{11} - \frac{\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}}{E(w_{i,t-1}^2) + E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(w_{i,t-1}\mathbf{z}_{it}^{d2})}. \tag{C.4}$$

If $\boldsymbol{\Upsilon}_{11}$ can be expressed explicitly, the explicit form of $[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})]^{-1}$ is readily obtained since it is straightforward to obtain the expectations in $(C.4)$. To this end, we need to calculate $\widetilde{\mathbf{D}}^{-1}$ and $[E(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime})]^{-1}$ in $(C.2)$. From Tanaka (1996), we have

$$\widetilde{\mathbf{D}}^{-1} = \begin{bmatrix} -1 & \cdots & -1 \\ & \ddots & \vdots \\ \mathbf{0} & & -1 \end{bmatrix}$$

Also, from Hamilton (1994, p.120), we have $\left[E(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime})\right]^{-1} = \sigma_v^{-2}\mathbf{L}'\mathbf{L}$ where

$$\mathbf{L} = \begin{bmatrix} \sqrt{1-\alpha^2} & 0 & 0 & \cdots & 0 & 0 \\ -\alpha & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha & 1 \end{bmatrix}.$$

Thus, using these results, we can calculate $\boldsymbol{\Upsilon}_{11}$ as in $(C.1)$ and hence, the explicit formula of $[E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})]^{-1}$ is obtained.

(b): Using (C.2), we get

$$\begin{aligned} E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1}) &= \begin{bmatrix} E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime}) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon}_{11} & \boldsymbol{\Upsilon}_{12} \\ \boldsymbol{\Upsilon}_{21} & \Upsilon_{22} \end{bmatrix} \begin{bmatrix} E(\mathbf{z}_{it}^{d2}w_{i,t-1}) \\ 0 \end{bmatrix} \\ &= \frac{\sigma_v^2(t-1)}{(1+\alpha)^2}. \end{aligned}$$

■

Lemma A2. *Let us define* $\mathbf{w}_{t-1} = (w_{1,t-1}, ..., w_{N,t-1})'$ *where* w_{it} *is defined below (2). Then, under Assumptions 1, 2, and 3, we obtain*

$$(a) \quad \frac{1}{NT_1}\sum_{t=1}^{T-1}\mathbf{w}_{t-1}'\mathbf{M}_t^l\mathbf{w}_{t-1} \xrightarrow[N,T\to\infty]{p} \frac{\sigma_v^2}{1-\alpha^2},$$

$$(b) \quad \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1} \xrightarrow[N,T\to\infty]{p} \frac{\sigma_v^2}{1-\alpha^2},$$

$$(c) \quad \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{w}_{t-1} \xrightarrow[N(T)\to\infty]{p} \frac{\sigma_v^2}{2}\left(\frac{1}{1+\alpha}\right).$$

Proof of Lemma A2

(a): See Lemma C2 in Alvarez and Arellano (2003).
(b): Let $\mathbf{r}_t$ denote the $N \times 1$ vector of errors of the population linear projection of $\mathbf{w}_{t-1}$ on $\mathbf{Z}_t^{d2}$:

$$\mathbf{w}_{t-1} = \mathbf{Z}_t^{d2}\boldsymbol{\delta}_t + \mathbf{r}_t \tag{C.5}$$

where $\boldsymbol{\delta}_t = [E(\mathbf{z}_{it}^{d2}\mathbf{z}_{it}^{d2\prime})]^{-1}E(\mathbf{z}_{it}^{d2}w_{i,t-1})$. Using Lemma A1, $\boldsymbol{\delta}_t$ can be expressed as

$$\boldsymbol{\delta}_t = \boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})\left(\frac{E(w_{i,t-1}^2)}{E(w_{i,t-1}^2) + E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})}\right). \tag{C.6}$$

Further, note that

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1}) \\ \boldsymbol{\Upsilon}_{21}E(\mathbf{z}_{it}^{d2}w_{i,t-1}) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{\Upsilon}_{11} & \boldsymbol{\Upsilon}_{12} \\ \boldsymbol{\Upsilon}_{21} & \Upsilon_{22} \end{bmatrix} \begin{bmatrix} E(\mathbf{z}_{it}^{d2}w_{i,t-1}) \\ 0 \end{bmatrix} \\
&= \left(\widetilde{\mathbf{D}}'\right)^{-1}\left[E(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime})\right]^{-1}\left(\widetilde{\mathbf{D}}\right)^{-1}\begin{bmatrix} E(\mathbf{z}_{it}^{d2}w_{i,t-1}) \\ 0 \end{bmatrix} \\
&= \left(\frac{1}{1+\alpha}\right)\begin{bmatrix} 1 & & \mathbf{0} \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} 1 \\ (1-\alpha) \\ \vdots \\ (1-\alpha) \\ -\alpha \end{bmatrix}.
\end{aligned} \tag{C.7}$$

Hence, from $(C.5)$, $(C.6)$, and $(C.7)$, it follows that

$$\begin{aligned}
r_{it} &= w_{i,t-1} - \mathbf{z}_{it}^{d2\prime}\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})\left(\frac{E(w_{i,t-1}^2)}{E(w_{i,t-1}^2) + E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})}\right) \\
&= w_{i,t-1} - \frac{w_{i,t-1} - w_{i0} + (1-\alpha)\{(t-2)w_{i,t-1} - (w_{i1} + \cdots + w_{i,t-2})\}}{1+\alpha} \\
&\qquad \times \left(\frac{E(w_{i,t-1}^2)}{E(w_{i,t-1}^2) + E(w_{i,t-1}\mathbf{z}_{it}^{d2\prime})\boldsymbol{\Upsilon}_{11}E(\mathbf{z}_{it}^{d2}w_{i,t-1})}\right) \\
&= \frac{\left(\frac{\sigma_v^2}{1-\alpha^2}\right)\left(\frac{1}{1+\alpha}\right)[(1-\alpha)(w_{i0} + \cdots + w_{i,t-1}) + \alpha(w_{i,t-1} + w_{i0})]}{\left(\frac{\sigma_v^2}{1-\alpha^2}\right) + \frac{\sigma_v^2}{(1+\alpha)^2}(t-1)} \\
&= \frac{(1+\alpha)^{-1}(v_{i,t-1} + \cdots + v_{i1}) + w_{i0}}{1 + \left(\frac{1-\alpha}{1+\alpha}\right)(t-1)}.
\end{aligned} \tag{C.8}$$

The last equality is due to the fact that $(1-\alpha)(w_{i,t-1} + \cdots + w_{i0}) = (v_{i,t-1} + \cdots + v_{i1} + w_{i0}) - \alpha w_{i,t-1}$ where v_{it} and w_{it} are defined below (1) and (2), and $w_{i0} = \sum_{j=0}^{\infty}\alpha^j v_{i,-j}$. Since $v_{i,t-1}, ..., v_{i1}, w_{i0}$ in $(C.8)$ are mutually uncorrelated for each i under Assumption 1, we obtain

$$E(r_{it}^2) = \frac{(1+\alpha)^{-2}(t-1)\sigma_v^2 + \frac{\sigma_v^2}{1-\alpha^2}}{\left[1 + \left(\frac{1-\alpha}{1+\alpha}\right)(t-1)\right]^2} = O\left(\frac{1}{t}\right), \qquad (i = 1, ..., N). \tag{C.9}$$

Next, we consider the decomposition:

$$\begin{aligned}
\frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1} &= \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'\mathbf{w}_{t-1} - \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'(\mathbf{I}_N - \mathbf{M}_t^{d2})\mathbf{w}_{t-1} \\
&= \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'\mathbf{w}_{t-1} - \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{r}_t'(\mathbf{I}_N - \mathbf{M}_t^{d2})\mathbf{r}_t.
\end{aligned} \tag{C.10}$$

The second equality is due to the fact that $(\mathbf{I}_N - \mathbf{M}_t^{d2})\mathbf{w}_{t-1} = (\mathbf{I}_N - \mathbf{M}_t^{d2})(\mathbf{Z}_t^{d2}\boldsymbol{\delta}_t + \mathbf{r}_t)$ (see (C.5)). For the first term, we have

$$\frac{1}{NT_1}\sum_{t=2}^{T} E(\mathbf{w}_{t-1}'\mathbf{w}_{t-1}) = \frac{1}{NT_1}\sum_{i=1}^{N}\sum_{t=2}^{T} E(w_{i,t-1}^2) = E(w_{i,t-1}^2).$$

For the second term, since the maximum eigenvalue of $(\mathbf{I}_N - \mathbf{M}_t^{d2})$ is equal to 1,

$$\frac{1}{NT_1}\sum_{t=2}^{T} \mathbf{r}_t'(\mathbf{I}_N - \mathbf{M}_t^{d2})\mathbf{r}_t \quad \leq \quad \frac{1}{NT_1}\sum_{t=2}^{T} \mathbf{r}_t'\mathbf{r}_t = \frac{1}{NT_1}\sum_{i=1}^{N}\sum_{t=2}^{T} r_{it}^2.$$

Since $E(r_{it}^2) = O(1/t)$ as shown in (C.9), it follows that the second term of (C.10) is $T_1^{-1}\sum_{t=2}^{T} O(1/t) = \frac{1}{T_1}O(\log T) \to 0$. Hence, as $N, T \to \infty$, it follows that

$$\frac{1}{NT_1}\sum_{t=2}^{T} \mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1} \underset{N,T\to\infty}{\overset{p}{\longrightarrow}} \frac{\sigma_v^2}{1-\alpha^2}.$$

With regard to the proofs that the variances of $(NT_1)^{-1}\sum_{t=2}^{T}\mathbf{w}_{t-1}'\mathbf{w}_{t-1}$ and $(NT_1)^{-1}\sum_{t=2}^{T}\mathbf{r}_t'\mathbf{r}_t$ tend to zero, see Alvarez and Arellano (2003).
(c) Using the Slutsky theorem and $E[(z_{it}^{d1})^2] \underset{N(T)\to\infty}{\overset{p}{\longrightarrow}} 2\sigma_v^2/(1+\alpha)$ and $[E(w_{i,t-1}z_{it}^{d1})]^2 \underset{N(T)\to\infty}{\overset{p}{\longrightarrow}} \sigma_v^4/(1+\alpha)^2$, the result follows.

■

Lemma A3. *Let κ_3^η and κ_4^η denote the third and fourth order cumulants of η_i. Also, let $\mathbf{d}_t$ and $\mathbf{d}_s$ $(t \geq s)$ denote the diagonal elements of $\mathbf{M}_t^d$ and $\mathbf{M}_s^d$, respectively such that, for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, $\mathbf{d}_t'\mathbf{d}_s \leq (s-1)$, and for $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$, $\mathbf{d}_t'\mathbf{d}_s \leq 1$. Define $\boldsymbol{\eta} = (\eta_1, ..., \eta_N)'$. Then under Assumptions 1, 2, and 3, for $t \geq s$, we have*

$$(a) \quad cov(\boldsymbol{\eta}'\mathbf{M}_t^d\boldsymbol{\eta}, \boldsymbol{\eta}'\mathbf{M}_s^d\boldsymbol{\eta}) = E(\mathbf{d}_t'\mathbf{d}_s)\kappa_4^\eta + 2\sigma_\eta^4 E[\mathrm{tr}(\mathbf{M}_t^d\mathbf{M}_s^d)] \leq \begin{cases} (\kappa_4^\eta + 2\sigma_\eta^4)(s-1) & \text{if} \quad \mathbf{M}_t^d = \mathbf{M}_t^{d2} \\ (\kappa_4^\eta + 2\sigma_\eta^4) & \text{if} \quad \mathbf{M}_t^d = \mathbf{M}_t^{d1} \end{cases},$$

$$(b) \quad |cov(\boldsymbol{\eta}'\mathbf{M}_t^d\mathbf{w}_{t-1}, \boldsymbol{\eta}'\mathbf{M}_s^d\mathbf{w}_{s-1})| \leq N\left(\frac{\sigma_v^2\sigma_\eta^2}{1-\alpha^2}\right).$$

Proof of Lemma A3

(a): The proof is analogous to Lemma C1 in Alvarez and Arellano (2003).
(b): Since $var(\boldsymbol{\eta}'\mathbf{M}_t^d\mathbf{w}_{t-1}) = \sigma_\eta^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1})$, we obtain

$$\begin{aligned} |cov(\boldsymbol{\eta}'\mathbf{M}_t^d\mathbf{w}_{t-1}, \boldsymbol{\eta}'\mathbf{M}_s^d\mathbf{w}_{s-1})| &\leq \sqrt{var(\boldsymbol{\eta}'\mathbf{M}_t^d\mathbf{w}_{t-1})}\sqrt{var(\boldsymbol{\eta}'\mathbf{M}_s^d\mathbf{w}_{s-1})} \\ &= \sigma_\eta^2\sqrt{E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1})}\sqrt{E(\mathbf{w}_{s-1}'\mathbf{M}_s^d\mathbf{w}_{s-1})} \\ &\leq \sigma_\eta^2\sqrt{E(\mathbf{w}_{t-1}'\mathbf{w}_{t-1})}\sqrt{E(\mathbf{w}_{s-1}'\mathbf{w}_{s-1})} = N\left(\frac{\sigma_v^2\sigma_\eta^2}{1-\alpha^2}\right). \end{aligned}$$

■

The following Lemma is obtained in Alvarez and Arellano (2003).

Lemma A4. *Let Assumptions 1, 2, and 3 hold. Then, as N and T tend to infinity, we have*

$$(a)\quad \mu_{F2} = E\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{v}_t^*\right) = -T_1\left(\frac{\sigma_v^2}{1-\alpha}\right)\left[1-\frac{1}{T_1(1-\alpha)}\sum_{t=1}^{T}\frac{1-\alpha^t}{t}\right] = -T_1\left(\frac{\sigma_v^2}{1-\alpha}\right)\left[1-O\left(\frac{\log T}{T}\right)\right],$$

$$(b)\quad var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{v}_t^*\right)\underset{N,T\to\infty}{\longrightarrow}\frac{\sigma_v^4}{1-\alpha^2},\qquad \text{if}\quad (\log T)^2/N\to 0,$$

$$(c)\quad \frac{1}{NT_1}\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*\underset{N,T\to\infty}{\overset{p}{\longrightarrow}}\frac{\sigma_v^2}{1-\alpha^2},\qquad \text{if}\quad (\log T)^2/N\to 0.$$

Proof of Lemma A4

See Alvarez and Arellano (2003). ■

Preliminary results

Before providing Lemma A5, we show some preliminary results. First, since $\widehat{\sigma}_v^2=\sigma_v^2+O_p\left(\frac{1}{\sqrt{NT}}\right)$ and $\widehat{\sigma}_\eta^2=\sigma_\eta^2+O_p\left(\frac{1}{\sqrt{N}}\right)$, by expanding $f(\widehat{\sigma}_\eta^2,\widehat{\sigma}_v^2)=\widehat{\sigma}_\eta^2/\widehat{\sigma}_v^2=\widehat{r}$ around $(\sigma_\eta^2,\sigma_v^2)$, we obtain

$$\widehat{r}=r+\frac{\widehat{\sigma}_\eta^2-\sigma_\eta^2}{\sigma_v^2}+O_p\left(\frac{1}{\sqrt{NT}}\right)=r+O_p\left(\frac{1}{\sqrt{N}}\right). \tag{C.11}$$

Furthermore, let $\widehat{k}_t$ and $\widehat{b}_t$ be estimates of k_t and b_t where r is replaced with $\widehat{r}$ (see (A.4) and (A.8)). Then, applying the mean-value theorem to $\widehat{k}_t^2$, $\widehat{b}_t^2$, $\widehat{b}_t\widehat{k}_t$ and $\widehat{d}_T^2$, for $t=2,...,T-1$, we have[3]

$$\begin{aligned}\widehat{k}_t^2 &= k_t^2+O_p\left(\frac{1}{(T-t)^2}\right)(\widehat{r}-r), \qquad \widehat{b}_t^2=b_t^2+O_p\left(\frac{1}{(T-t)}\right)(\widehat{r}-r),\\ \widehat{b}_t\widehat{k}_t &= b_tk_t+O_p\left(\frac{1}{(T-t)}\right)(\widehat{r}-r), \qquad \widehat{d}_T^2=d_T^2+O_p(1)(\widehat{r}-r).\end{aligned} \tag{C.12}$$

Define the following variables:

$$\begin{aligned}\Xi_{1t}^d &= \mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{v}_t^*,\quad \Xi_{2t}^d=\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{v}_t,\quad \Xi_{3t}^d=\mathbf{w}_{t-1}'\mathbf{M}_t^d\boldsymbol{\eta},\quad \Xi_{4t}^d=\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\mathbf{v}_t,\quad \Xi_{5t}^d=\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\mathbf{v}_t^*,\\ \Xi_{6t}^d &= \widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\boldsymbol{\eta},\quad \Xi_{7t}^d=\boldsymbol{\mu}'\mathbf{M}_t^d\mathbf{v}_t,\quad \Xi_{8t}^d=\boldsymbol{\mu}'\mathbf{M}_t^d\mathbf{v}_t^*,\quad \Xi_{9t}^d=\boldsymbol{\mu}'\mathbf{M}_t^d\boldsymbol{\eta}.\end{aligned} \tag{C.13}$$

Also, note the following expressions[4]

$$\begin{aligned}\mathbf{x}_t^+ &= b_t\mathbf{x}_t^*+k_t\mathbf{x}_t=(b_t\psi_t+k_t)\mathbf{w}_{t-1}-b_tc_t\widetilde{\mathbf{v}}_{tT}+k_t\boldsymbol{\mu}, \qquad (t=2,...,T-1),\\ \mathbf{x}_T^+ &= d_T\mathbf{x}_T=d_T\boldsymbol{\mu}+d_T\mathbf{w}_{T-1},\\ \mathbf{x}_t^* &= \psi_t\mathbf{w}_{t-1}-c_t\widetilde{\mathbf{v}}_{tT}, \qquad (t=1,...,T-1),\end{aligned}$$

---

[3]These results can be derived as follows. Let $\bar{r}$ be the mean-value lying between $r_0=\sigma_\eta^2/\sigma_v^2$ and $\widehat{r}$ and let us denote $\widehat{k}_t^2$ and k_t^2 as $k_t^2(\widehat{r})$ and $k_t^2(r)$, respectively. Then, we have

$$\widehat{k}_t^2=k_t^2(\widehat{r})=k_t^2(r_0)+\left.\frac{dk_t^2(r)}{dr}\right|_{r=\bar{r}}(\widehat{r}-r_0),\quad \text{where}\quad \frac{dk_t^2(r)}{dr}=\frac{-[2r(T-t+1)(T-t)+2(T-t)+1]}{[r(T-t+1)+1]^2[r(T-t)+1]^2}.$$

The expression is obtained by noting that $\widehat{r}$ is a consistent estimator of r_0. Other results are obtained in the same way.

[4]See Alvarez and Arellano (2003, p.1143). k_t and b_t are defined in (A.4) and (A.8), respectively. c_t is defined below (3).

$$\widetilde{\mathbf{v}}_{tT} = \frac{\phi_{T-t}\mathbf{v}_t + \cdots + \phi_1 \mathbf{v}_{T-1}}{T-t} = (\widetilde{v}_{1tT}, ..., \widetilde{v}_{NtT})',$$
$$\psi_t = c_t\left(1 - \frac{\alpha\phi_{T-t}}{T-t}\right), \qquad \phi_j = \frac{1-\alpha^j}{1-\alpha},$$

where $\boldsymbol{\mu} = (\mu_1, ..., \mu_N)'$ with $\mu_i = \eta_i/(1-\alpha)$. Using (C.13), we have the followings

$$\mathbf{x}_t^{*\prime}\mathbf{M}_t^d\mathbf{v}_t^* = \psi_t\Xi_{1t} - c_t\Xi_{5t} = W_{1t}, \qquad (t=2,...,T-1), \tag{C.14}$$
$$\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{v}_t^* = \Xi_{1t} + \Xi_{8t} = W_{2t}, \qquad (t=2,...,T-1), \tag{C.15}$$
$$\mathbf{x}_t^{*\prime}\mathbf{M}_t^d\mathbf{u}_t = \psi_t\Xi_{2t} + \psi_t\Xi_{3t} - c_t\Xi_{4t} - c_t\Xi_{6t} = W_{3t}, \qquad (t=2,...,T-1), \tag{C.16}$$
$$\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{u}_t = \Xi_{2t} + \Xi_{3t} + \Xi_{7t} + \Xi_{9t} = W_{4t}, \qquad (t=2,...,T-1). \tag{C.17}$$

Then, using (A.6) and (A.7), we have

$$\begin{aligned}
\mathbf{x}_t^{+\prime}\mathbf{M}_t^d\mathbf{u}_t^+ &= b_t^2 W_{1t} + b_t k_t (W_{2t} + W_{3t}) + k_t^2 W_{4t} \\
&= b_t(b_t\psi_t + k_t)\Xi_{1t} + k_t(b_t\psi_t + k_t)\Xi_{2t} + k_t(b_t\psi_t + k_t)\Xi_{3t} - (b_t c_t k_t)\Xi_{4t} \\
&\quad -(b_t^2 c_t)\Xi_{5t} - (b_t c_t k_t)\Xi_{6t} + (k_t^2)\Xi_{7t} + (b_t k_t)\Xi_{8t} + (k_t^2)\Xi_{9t} \\
&= h_{1t}\Xi_{1t}^d + h_{2t}\Xi_{2t}^d + \cdots + h_{9t}\Xi_{9t}^d \qquad (t=2,...,T-1)
\end{aligned} \tag{C.18}$$

$$\mathbf{x}_T^{+\prime}\mathbf{M}_T^d\mathbf{u}_T^+ = d_T^2\mathbf{x}_T^{\prime}\mathbf{M}_T^d\mathbf{u}_T = d_T^2 W_{4T} \tag{C.19}$$

and

$$\begin{aligned}
\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^d\widehat{\mathbf{u}}_t^+ &= \widehat{b}_t^2 W_{1t} + \widehat{b}_t\widehat{k}_t(W_{2t} + W_{3t}) + \widehat{k}_t^2 W_{4t} \\
&= b_t^2 W_{1t} + b_t k_t (W_{2t} + W_{3t}) + k_t^2 W_{4t} \\
&\quad +(\widehat{b}_t^2 - b_t^2)W_{1t} + (\widehat{b}_t\widehat{k}_t - b_t k_t)W_{2t} + (\widehat{b}_t\widehat{k}_t - b_t k_t)W_{3t} + (\widehat{k}_t^2 - k_t^2)W_{4t} \\
&= \mathbf{x}_t^{+\prime}\mathbf{M}_t^d\mathbf{u}_t^+ + \widehat{m}_{1t}^d + \widehat{m}_{2t}^d + \widehat{m}_{3t}^d + \widehat{m}_{4t}^d, \qquad (t=2,...,T-1)
\end{aligned} \tag{C.20}$$

$$\widehat{\mathbf{x}}_T^{+\prime}\mathbf{M}_T^d\widehat{\mathbf{u}}_T^+ = \mathbf{x}_T^{+\prime}\mathbf{M}_T^d\mathbf{u}_T^+ + \widehat{m}_{5T}^d \tag{C.21}$$

where[5]

$$\widehat{m}_{1t}^d = (\widehat{b}_t^2 - b_t^2)W_{1t}, \qquad \widehat{m}_{2t}^d = (\widehat{b}_t\widehat{k}_t - b_t k_t)W_{2t}, \qquad \widehat{m}_{3t}^d = (\widehat{b}_t\widehat{k}_t - b_t k_t)W_{3t},$$
$$\widehat{m}_{4t}^d = (\widehat{k}_t^2 - k_t^2)W_{4t}, \qquad \widehat{m}_{5T}^d = (\widehat{d}_T^2 - d_T^2)W_{4T},$$

$$h_{1t} = b_t(b_t\psi_t + k_t) = 1 + O\left(\frac{1}{T-t}\right), \qquad h_{2t} = k_t(b_t\psi_t + k_t) = O\left(\frac{1}{T-t}\right),$$
$$h_{3t} = k_t(b_t\psi_t + k_t) = O\left(\frac{1}{T-t}\right), \qquad h_{4t} = -b_t c_t k_t = O\left(\frac{1}{T-t}\right),$$
$$h_{5t} = -b_t^2 c_t = O(1), \qquad h_{6t} = -b_t c_t k_t = O\left(\frac{1}{T-t}\right),$$
$$h_{7t} = k_t^2 = O\left(\frac{1}{(T-t)^2}\right), \qquad h_{8t} = b_t k_t = O\left(\frac{1}{T-t}\right),$$
$$h_{9t} = k_t^2 = O\left(\frac{1}{(T-t)^2}\right).$$

In what follows, we assume that expectations of $\widehat{m}_{jt}^d, (j=1,2,3,4)$ and $\widehat{m}_{5T}^d$ exist.

The following Lemma A5 is used to prove Lemma A6 below.

[5] Note that $b_t^2 = 1 + O(1/(T-t))$, $b_t k_t = O(1/(T-t))$ and $k_t^2 = O(1/(T-t)^2)$ (see (A.4) and (A.8)).

Lemma A5. *Let Assumptions 1, 2 and 3 hold. Then, we have*

$$(a)\quad E(W_{1t}^2) = O(N) + O\left(\frac{\mathrm{tr}(\mathbf{M}_t^d)}{T-t}\right) + O\left(\frac{[\mathrm{tr}(\mathbf{M}_t^d)]^2}{(T-t)^2}\right), \qquad (t=2,...,T-1),$$

$$(b)\quad E(W_{2t}^2) = O(N) + O\left(\mathrm{tr}(\mathbf{M}_t^d)\right), \qquad (t=2,...,T-1),$$

$$(c)\quad E(W_{3t}^2) = O(N) + O\left(\frac{\mathrm{tr}(\mathbf{M}_t^d)}{T-t}\right) + O\left(\frac{[\mathrm{tr}(\mathbf{M}_t^d)]^2}{(T-t)^2}\right), \qquad (t=2,...,T-1),$$

$$(d)\quad E(W_{4t}^2) = O(N) + O\left(\mathrm{tr}(\mathbf{M}_t^d)\right) + O\left(\{\mathrm{tr}(\mathbf{M}_t^d)\}^2\right), \qquad (t=2,...,T-1),$$

$$(e)\quad E(W_{kt}^4) = O(N^4), \qquad (k=1,2,3,4).$$

Proof of Lemma A5

(a)-(d): Let us denote W_{kt} as $W_{kt} = \sum_m c_m \Xi_{mt}$ for $k = 1,2,3,4$ where Ξ_{mt}^d is denoted as Ξ_{mt} for notational simplicity. Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} E(W_{kt}^2) &= \sum_{m_1,m_2} c_{m_1} c_{m_2} E(\Xi_{m_1 t}\Xi_{m_2 t}) \leq \sum_{m_1,m_2} |c_{m_1}||c_{m_2}| \sqrt{E(\Xi_{m_1 t}^2)}\sqrt{E(\Xi_{m_2 t}^2)} \\ &= \left[\sum_m |c_m| \left[E(\Xi_{mt}^2)\right]^{1/2}\right]^2. \end{aligned}$$

Hence, to derive the order of magnitude, we need to assess that of $E(\Xi_{mt}^2)$. Let $E_t(\cdot) = E(\cdot|\mathbf{v}_{t-1}, \mathbf{v}_{t-2}, ..., \mathbf{w}_0, \boldsymbol{\eta})$ be an expectation conditional on $\boldsymbol{\eta}$ and $\{\mathbf{v}_{t-j}\}_{j=1}^{\infty}$. Then, we obtain

$$E(\Xi_{1t}^2) = E[\mathbf{w}_{t-1}'\mathbf{M}_t^d E_t(\mathbf{v}_t^*\mathbf{v}_t^{*\prime})\mathbf{M}_t^d\mathbf{w}_{t-1}] = \sigma_v^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}) = O(N), \tag{C.22}$$

$$E(\Xi_{2t}^2) = E[\mathbf{w}_{t-1}'\mathbf{M}_t^d E_t(\mathbf{v}_t\mathbf{v}_t')\mathbf{M}_t^d\mathbf{w}_{t-1}] = \sigma_v^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}) = O(N), \tag{C.23}$$

$$E(\Xi_{3t}^2) = E[\mathbf{w}_{t-1}'\mathbf{M}_t^d E(\boldsymbol{\eta}\boldsymbol{\eta}')\mathbf{M}_t^d\mathbf{w}_{t-1}] = \sigma_\eta^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}) = O(N), \tag{C.24}$$

$$E(\Xi_{4t}^2) = O\left(\frac{\mathrm{tr}(\mathbf{M}_t^d)}{T-t}\right) + O\left(\frac{[\mathrm{tr}(\mathbf{M}_t^d)]^2}{(T-t)^2}\right), \tag{C.25}$$

$$E(\Xi_{5t}^2) = O\left(\frac{\mathrm{tr}(\mathbf{M}_t^d)}{T-t}\right) + O\left(\frac{[\mathrm{tr}(\mathbf{M}_t^d)]^2}{(T-t)^2}\right), \tag{C.26}$$

$$E(\Xi_{6t}^2) = \sigma_\eta^2 E(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\widetilde{\mathbf{v}}_{tT}) = \sigma_\eta^2 \,\mathrm{tr}(\mathbf{M}_t^d)E(\widetilde{v}_{itT}^2) = \frac{\sigma_\eta^2\,\mathrm{tr}(\mathbf{M}_t^d)(\phi_1^2 + \cdots + \phi_{T-t}^2)}{(T-t)^2} = O\left(\frac{\mathrm{tr}(\mathbf{M}_t^d)}{T-t}\right), \tag{C.27}$$

$$E(\Xi_{7t}^2) = \sigma_v^2 E(\boldsymbol{\mu}'\mathbf{M}_t^d\boldsymbol{\mu}) = \sigma_v^2\sigma_\mu^2\,\mathrm{tr}(\mathbf{M}_t^d) = O\left(\mathrm{tr}(\mathbf{M}_t^d)\right), \tag{C.28}$$

$$E(\Xi_{8t}^2) = \sigma_v^2 E(\boldsymbol{\mu}'\mathbf{M}_t^d\boldsymbol{\mu}) = \sigma_v^2\sigma_\mu^2\,\mathrm{tr}(\mathbf{M}_t^d) = O\left(\mathrm{tr}(\mathbf{M}_t^d)\right), \tag{C.29}$$

$$E(\Xi_{9t}^2) = \frac{E[(\boldsymbol{\eta}'\mathbf{M}_t^d\boldsymbol{\eta})^2]}{(1-\alpha)^2} \leq \frac{(2\sigma_\eta^2 + \kappa_4^\eta)\,\mathrm{tr}(\mathbf{M}_t^d) + \sigma_\eta^4[\mathrm{tr}(\mathbf{M}_t^d)^2]}{(1-\alpha)^2} = O\left(\{\mathrm{tr}(\mathbf{M}_t^d)\}^2\right). \tag{C.30}$$

Note that $\mathrm{tr}(\mathbf{M}_t^d) = t-1$ for $d = d2$ and $\mathrm{tr}(\mathbf{M}_t^d) = 1$ for $d = d1$. The results for $E(\Xi_{5t}^2)$ can be obtained as follows. Let $m_{t,ij}$ be the (i,j)th element of $\mathbf{M}_t^d$. Then, the order of magnitude of $E(\Xi_{5t}^2)$ becomes

$$\begin{aligned} E(\Xi_{5t}^2) &= \sum_i\sum_j\sum_k\sum_\ell E\left[m_{t,ij}m_{t,k\ell}E_t(\widetilde{v}_{itT}v_{jt}^*\widetilde{v}_{ktT}v_{\ell t}^*)\right] \\ &= \sum_i E\left[m_{t,ii}^2 E_t(\widetilde{v}_{itT}^2 v_{it}^{*2})\right] + \sum_{i=k\neq j=\ell}\sum E\left[m_{t,ij}^2 E_t(\widetilde{v}_{itT}^2)E_t(v_{jt}^{*2})\right] \end{aligned}$$

$$+\sum_{i,\ j\neq\ell}\sum E\left[(m_{t,ii}m_{t,jj}+m_{t,ij}^2)[E_t(\widetilde{v}_{itT}v_{it}^*)]^2\right]$$

$$\leq \operatorname{tr}(\mathbf{M}_t^d)E(\widetilde{v}_{itT}^2v_{it}^{*2})+\operatorname{tr}[(\mathbf{M}_t^d)^2]E(\widetilde{v}_{itT}^2)E(v_{it}^{*2})+\left\{[\operatorname{tr}(\mathbf{M}_t^d)]^2+\operatorname{tr}[(\mathbf{M}_t^d)^2]\right\}[E(\widetilde{v}_{itT}v_{it}^*)]^2$$

where we used the law of iterated expectation and the fact that $\operatorname{tr}(\mathbf{M}_t^d)$ and $\operatorname{tr}[(\mathbf{M}_t^d)^2]$ are non-stochastic. Since $E(\widetilde{v}_{itT}^2v_{it}^{*2})$, $E(\widetilde{v}_{itT}^2)$, $E(\widetilde{v}_{itT}v_{it}^*)$ are $O(1/(T-t))$ and $E(v_{it}^{*2})=O(1)$, the result follows. Similarly, we can prove for the case of $E(\Xi_{4t}^2)$. Combining the results (C.22) to (C.30) and definitions of (C.14) to (C.17), we obtain the results.

(e): Using $W_{kt}=\sum_m c_m\Xi_{mt}$, we have

$$\begin{aligned}
E(W_{kt}^4) &= \sum_{m_1,m_2,m_3,m_4} c_{m_1}c_{m_2}c_{m_3}c_{m_4}E(\Xi_{m_1t}\Xi_{m_2t}\Xi_{m_3t}\Xi_{m_4t}) \\
&\leq \sum_{m_1,m_2,m_3,m_4} c_{m_1}c_{m_2}c_{m_3}c_{m_4}\sqrt{E(\Xi_{m_1t}^2\Xi_{m_2t}^2)}\sqrt{E(\Xi_{m_3t}^2\Xi_{m_4t}^2)} \\
&\leq \sum_{m_1,m_2,m_3,m_4} c_{m_1}c_{m_2}c_{m_3}c_{m_4}[E(\Xi_{m_1t}^4)]^{1/4}[E(\Xi_{m_2t}^4)]^{1/4}[E(\Xi_{m_3t}^4)]^{1/4}[E(\Xi_{m_4t}^4)]^{1/4} \\
&= \left[\sum_m c_m[E(\Xi_{mt}^4)]^{1/4}\right]^4.
\end{aligned}$$

Let us write Ξ_{mt} as $\Xi_{mt}=\mathbf{f}'\mathbf{M}_t^d\mathbf{g}$ where $\mathbf{f}$ and $\mathbf{g}$ are $\mathbf{w}_{t-1}$, $\mathbf{v}_t$, $\mathbf{v}_t^*$, $\boldsymbol{\eta}$, or $\widetilde{\mathbf{v}}_{tT}$. Then, from the Cauchy-Schwarz inequality and $\mathbf{x}'\mathbf{A}\mathbf{x}\leq\lambda_{max}(\mathbf{A})\mathbf{x}'\mathbf{x}$ with $\lambda_{max}(\mathbf{A})$ being the largest eigenvalue of $\mathbf{A}$, we obtain

$$\begin{aligned}
E(\Xi_{mt}^4) &= E\left[(\mathbf{f}'\mathbf{M}_t^d\mathbf{g})^4\right]\leq E\left[(\mathbf{f}'\mathbf{M}_t^d\mathbf{f})^2(\mathbf{g}'\mathbf{M}_t^d\mathbf{g})^2\right]\leq E\left[\left(\sum_{i=1}^N f_i^2\right)^2\left(\sum_{j=1}^N g_j^2\right)^2\right] \\
&\leq \sum_{i_1=1}^N\sum_{i_2=1}^N\sum_{j_1=1}^N\sum_{j_2=1}^N E(f_{i_1}^2f_{i_2}^2g_{j_1}^2g_{j_2}^2)\leq\sum_{i_1=1}^N\sum_{i_2=1}^N\sqrt{E\left(f_{i_1}^4f_{i_2}^4\right)}\sum_{j_1=1}^N\sum_{j_2=1}^N\sqrt{E\left(g_{j_1}^4g_{j_2}^4\right)} \\
&\leq \sum_{i_1=1}^N[E\left(f_{i_1}^8\right)]^{1/4}\sum_{i_2=1}^N[E\left(f_{i_2}^8\right)]^{1/4}\sum_{j_1=1}^N[E\left(g_{j_1}^8\right)]^{1/4}\sum_{j_2=1}^N[E\left(g_{j_2}^8\right)]^{1/4}.
\end{aligned}$$

Since $E(f_i^8)$ and $E(g_j^8)$ are $O(1)$ under Assumption 2, it follows that $E(\Xi_{mt}^4)=O(N^4)$ and therefore $E(W_{mt}^4)=O(N^4)$. ■

The following lemma is used to derive the asymptotic properties of the level and system GMM estimators.

Lemma A6. *Let Assumptions 1, 2, and 3 hold. Then, as N and T tend to infinity, we have*

$$(a)\quad \mu_{L2}=E\left(\sum_{t=2}^T\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{u}}_t^+\right)=-T_1\left(\frac{\sigma_v^2}{1-\alpha}\right)\left[\left(\frac{\alpha}{r+1}\right)+O\left(\frac{\log T}{T}\right)+O\left(\frac{\log T}{\sqrt{NT}}\right)+O\left(\frac{1}{\sqrt{N}}\right)\right],$$

$$(b)\quad \mu_{L1}=E\left(\sum_{t=2}^T\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{u}}_t^+\right)=O\left(\log T\right),$$

$$(c)\quad var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^T\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{u}}_t^+\right)\underset{N,T\to\infty}{\longrightarrow}\frac{\sigma_v^4}{1-\alpha^2}\qquad\text{if}\quad(\log T)^2/N\to 0,$$

$$(d)\quad var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^T\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{u}}_t^+\right)\underset{N,T\to\infty}{\longrightarrow}\frac{\sigma_v^4}{2(1+\alpha)},$$

$$(e)\quad \frac{1}{NT_1}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^{+}\ \underset{N,T\to\infty}{\overset{p}{\longrightarrow}}\ \frac{\sigma_v^2}{1-\alpha^2}\qquad \text{if}\quad (\log T)^2/N\to 0,$$

$$(f)\quad \frac{1}{NT_1}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{x}}_t^{+}\ \underset{N,T\to\infty}{\overset{p}{\longrightarrow}}\ \frac{\sigma_v^2}{2(1+\alpha)}.$$

Proof of Lemma A6

(a), (b): From (C.20), we have

$$\begin{aligned}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d}\widehat{\mathbf{u}}_t^{+} &= \sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d}\widehat{\mathbf{u}}_t^{+}+\widehat{\mathbf{x}}_T^{+\prime}\mathbf{M}_T^{d}\widehat{\mathbf{u}}_T^{+}\\ &= \sum_{t=2}^{T-1}\mathbf{x}_t^{+\prime}\mathbf{M}_t^{d}\mathbf{u}_t^{+}+\sum_{t=2}^{T-1}\left[\widehat{m}_{1t}^{d}+\widehat{m}_{2t}^{d}+\widehat{m}_{3t}^{d}+\widehat{m}_{4t}^{d}\right]+\mathbf{x}_T^{+\prime}\mathbf{M}_T^{d}\mathbf{u}_T^{+}+\widehat{m}_{5T}^{d}. \qquad \text{(C.31)}\end{aligned}$$

For the first term in (C.31), using (A.6) and (A.7), for $t=2,...,T-1$, after some algebra, we obtain

$$\begin{aligned}E[\mathbf{x}_t^{+\prime}\mathbf{M}_t^{d}\mathbf{u}_t^{+}] &= E\left[\operatorname{tr}\left(\mathbf{M}_t^{d}E_t(\mathbf{u}_t^{+}\mathbf{x}_t^{+\prime})\right)\right]=\operatorname{tr}(\mathbf{M}_t^{d})E(u_{it}^{+}x_{it}^{+})\\ &= \frac{-\sigma_v^2\operatorname{tr}(\mathbf{M}_t^{d})\left[(r+1-\alpha)\phi_{T-t}-r(T-t)\alpha^{T-t}-1+\alpha\right]}{r(1-\alpha)\left(T-t+1+\frac{1}{r}\right)\left(T-t+\frac{1}{r}\right)}. \qquad \text{(C.32)}\end{aligned}$$

Then, for $\mathbf{M}_t^{d}=\mathbf{M}_t^{d2}$ with $\operatorname{tr}(\mathbf{M}_t^{d2})=t-1$, by changing the index to $s=T-t$, we obtain

$$\begin{aligned}\sum_{t=2}^{T-1}E[\mathbf{x}_t^{+\prime}\mathbf{M}_t^{d2}\mathbf{u}_t^{+}] &= \frac{-\sigma_v^2}{r(1-\alpha)}\left[T_1\sum_{s=1}^{T-2}\frac{(r+1-\alpha)\phi_s-rs\alpha^s-1+\alpha}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)}\right.\\ &\left.\quad -\sum_{s=1}^{T-2}\frac{(r+1-\alpha)s\phi_s-rs^2\alpha^s-s(1-\alpha)}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)}\right].\end{aligned}$$

Consequently, using (C.32) and

$$\begin{aligned}\sum_{s=1}^{T-2}\frac{\phi_s}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)} &= \frac{r}{r+1}-\frac{\phi_{T-2}}{T-1+\frac{1}{r}}+\sum_{s=2}^{T-2}\frac{\alpha^{s-1}}{s+\frac{1}{r}},\\ \sum_{s=1}^{T-2}\frac{s\alpha^s}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)} &= \frac{\alpha r}{r+1}-\frac{(T-2)\alpha^{T-2}}{T-1+\frac{1}{r}}+\sum_{s=2}^{T-2}\frac{\alpha^{s-1}(s(\alpha-1)+1)}{s+\frac{1}{r}},\\ \sum_{s=1}^{T-2}\frac{1}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)} &= \frac{r}{r+1}-\frac{1}{T-1+\frac{1}{r}},\\ E(\mathbf{x}_T^{+\prime}\mathbf{M}_T^{d2}\mathbf{u}_T^{+}) &= T_1\left(\frac{\sigma_v^2}{1-\alpha}\right)\frac{r}{r+1},\end{aligned}$$

we obtain

$$\begin{aligned}\bar{\mu}_{L2} &= \sum_{t=2}^{T}E(\mathbf{x}_t^{+\prime}\mathbf{M}_t^{d2}\mathbf{u}_t^{+})=\sum_{t=2}^{T-1}E(\mathbf{x}_t^{+\prime}\mathbf{M}_t^{d2}\mathbf{u}_t^{+})+E(\mathbf{x}_T^{+\prime}\mathbf{M}_T^{d2}\mathbf{u}_T^{+})\\ &= \frac{-\sigma_v^2}{r(1-\alpha)}\left\{T_1\left[r-\frac{(r+1-\alpha)(1-\alpha^{T-2})-r(1-\alpha)(T-2)\alpha^{T-2}}{(1-\alpha)(T-1+\frac{1}{r})}-r\alpha^{T-2}\right.\right.\\ &\left.\left.\quad -(1-\alpha)\left(\frac{r}{r+1}-\frac{1}{T-1+\frac{1}{r}}\right)\right]-\sum_{s=1}^{T-2}\frac{(r+1-\alpha)s\phi_s-rs^2\alpha^s-s(1-\alpha)}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)}\right\}\end{aligned}$$

$$
\begin{aligned}
& +T_1 \left(\frac{\sigma_v^2}{1-\alpha}\right) \frac{r}{r+1} \\
= \; & \frac{-\sigma_v^2}{1-\alpha} \left[T_1 \left(\frac{r+\alpha}{r+1} + O\left(\frac{1}{T}\right)\right) + O(\log T)\right] + T_1 \left(\frac{\sigma_v^2}{1-\alpha}\right) \frac{r}{r+1} \\
= \; & -T_1 \left(\frac{\sigma_v^2}{1-\alpha}\right) \frac{\alpha}{r+1} + O(\log T). \qquad \text{(C.33)}
\end{aligned}
$$

Similarly, for $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$, we have

$$
\begin{aligned}
\bar{\mu}_{L1} &= \sum_{t=2}^{T} E(\mathbf{x}_t^{+\prime} \mathbf{M}_t^{d1} \mathbf{u}_t^{+}) = \sum_{t=2}^{T-1} E(\mathbf{x}_t^{+\prime} \mathbf{M}_t^{d1} \mathbf{u}_t^{+}) + E(\mathbf{x}_T^{+\prime} \mathbf{M}_T^{d1} \mathbf{u}_T^{+}) \\
&= \frac{-\sigma_v^2}{r(1-\alpha)} \sum_{s=1}^{T-2} \frac{(r+1-\alpha)\phi_s - rs\alpha^s - 1 + \alpha}{\left(s+1+\frac{1}{r}\right)\left(s+\frac{1}{r}\right)} + \left(\frac{\sigma_v^2}{1-\alpha}\right) \frac{r}{r+1} = O(1). \qquad \text{(C.34)}
\end{aligned}
$$

Next, we evaluate the remaining terms in (C.31). For the case of $d = d2$, using (C.11), (C.12) and Lemma A5, we have

$$
\begin{aligned}
\sum_{t=2}^{T-1} E\left(\widehat{m}_{1t}^{d2}\right) &= \sum_{t=2}^{T} O\left(\frac{1}{T-t}\right) E\left[(\widehat{r}-r) W_{1t}\right] \le \sum_{t=2}^{T} O\left(\frac{1}{T-t}\right) \sqrt{E(\widehat{r}-r)^2} \sqrt{E(W_{1t}^2)} \\
&= O\left(\frac{1}{\sqrt{N}}\right) \sum_t \left[O\left(\frac{\sqrt{N}}{T-t}\right) + O\left(\frac{\sqrt{t}}{(T-t)^{3/2}}\right) + O\left(\frac{t}{(T-t)^2}\right)\right] \\
&= O(\log T) + O\left(\frac{\sqrt{T}\log T}{\sqrt{N}}\right) + O\left(\frac{T}{\sqrt{N}}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{t=2}^{T-1} E\left(\widehat{m}_{2t}^{d2}\right) &\le \sum_{t=2}^{T} O\left(\frac{1}{T-t}\right) \sqrt{E(\widehat{r}-r)^2} \sqrt{E(W_{2t}^2)} = O\left(\frac{1}{\sqrt{N}}\right) \sum_t \left[O\left(\frac{\sqrt{N}}{T-t}\right) + O\left(\frac{\sqrt{t}}{T-t}\right)\right] \\
&= O(\log T) + O\left(\frac{\sqrt{T}\log T}{\sqrt{N}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{t=2}^{T-1} E\left(\widehat{m}_{3t}\right) &\le \sum_{t=2}^{T} O\left(\frac{1}{T-t}\right) \sqrt{E(\widehat{r}-r)^2} \sqrt{E(W_{3t}^2)} = O\left(\frac{1}{\sqrt{N}}\right) \sum_t \left[O\left(\frac{\sqrt{t}}{(T-t)^{3/2}}\right) + O\left(\frac{t}{(T-t)^2}\right)\right] \\
&= O(\log T) + O\left(\frac{\sqrt{T}\log T}{\sqrt{N}}\right) + O\left(\frac{T}{\sqrt{N}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{t=2}^{T-1} E\left(\widehat{m}_{4t}^{d2}\right) &\le \sum_{t=2}^{T} O\left(\frac{1}{(T-t)^2}\right) \sqrt{E(\widehat{r}-r)^2} \sqrt{E(W_{4t}^2)} \\
&= O\left(\frac{1}{\sqrt{N}}\right) \sum_t O\left(\frac{1}{(T-t)^2}\right) \left[O\left(\sqrt{N}\right) + O\left(\sqrt{t}\right) + O(t)\right] \\
&= O(1) + O\left(\frac{\sqrt{T}\log T}{\sqrt{N}}\right) + O\left(\frac{T}{\sqrt{N}}\right).
\end{aligned}
$$

and

$$
E\left(\widehat{m}_{5T}^{d2}\right) \le \sqrt{E(\widehat{r}-r)^2} \sqrt{E(W_{4T}^2)} = O\left(\frac{1}{\sqrt{N}}\right) \left[O\left(\sqrt{N}\right) + O\left(\sqrt{T}\right) + O(T)\right]
$$

$$= O(1) + O\left(\frac{\sqrt{T}}{\sqrt{N}}\right) + O\left(\frac{T}{\sqrt{N}}\right).$$

Hence, we have

$$\sum_{t=2}^{T-1}\left[E(\widehat{m}_{1t}^{d2}) + E(\widehat{m}_{2t}^{d2}) + E(\widehat{m}_{3t}^{d2}) + E(\widehat{m}_{4t}^{d2})\right] + E\left(\widehat{m}_{5T}^{d2}\right) = O(\log T) + O\left(\frac{\sqrt{T}\log T}{\sqrt{N}}\right) + O\left(\frac{T}{\sqrt{N}}\right) \quad \text{(C.35)}$$

Therefore, from (C.31), (C.33) and (C.35), it follows that for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$,

$$E\left(\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{u}}_t^{+}\right) = -T_1\left(\frac{\sigma_v^2}{1-\alpha}\right)\left[\left(\frac{\alpha}{r+1}\right) + O\left(\frac{\log T}{T}\right) + O\left(\frac{\log T}{\sqrt{NT}}\right) + O\left(\frac{1}{\sqrt{N}}\right)\right].$$

Similarly, for the case of $d = d1$, we have

$$\begin{aligned}
\sum_{t=2}^{T-1}E\left(\widehat{m}_{1t}^{d1}\right) &= O\left(\frac{1}{\sqrt{N}}\right)\sum_{t}\left[O\left(\frac{\sqrt{N}}{T-t}\right) + O\left(\frac{1}{(T-t)^{3/2}}\right) + O\left(\frac{1}{(T-t)^2}\right)\right] \\
&= O(\log T) + O\left(\frac{\log T}{\sqrt{N}}\right) + O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}$$

$$\sum_{t=2}^{T-1}E\left(\widehat{m}_{2t}^{d1}\right) = O\left(\frac{1}{\sqrt{N}}\right)\sum_{t}\left[O\left(\frac{\sqrt{N}}{T-t}\right) + O\left(\frac{1}{T-t}\right)\right] = O(\log T) + O\left(\frac{\log T}{\sqrt{N}}\right),$$

$$\sum_{t=2}^{T-1}E\left(\widehat{m}_{3t}^{d1}\right) = O\left(\frac{1}{\sqrt{N}}\right)\sum_{t}\left[O\left(\frac{1}{(T-t)^{3/2}}\right) + O\left(\frac{1}{(T-t)^2}\right)\right] = O\left(\frac{\log T}{\sqrt{N}}\right) + O\left(\frac{1}{\sqrt{N}}\right),$$

$$\sum_{t=2}^{T-1}E\left(\widehat{m}_{4t}^{d1}\right) = O\left(\frac{1}{\sqrt{N}}\right)\sum_{t}O\left(\frac{1}{(T-t)^2}\right)\left[O\left(\sqrt{N}\right) + O(1)\right] = O(1),$$

$$E\left(\widehat{m}_{5T}^{d1}\right) = O\left(\frac{1}{\sqrt{N}}\right)\left[O\left(\sqrt{N}\right) + O(1)\right] = O(1).$$

Hence, we have

$$\sum_{t=2}^{T}\left[E(\widehat{m}_{1t}^{d1}) + E(\widehat{m}_{2t}^{d1}) + E(\widehat{m}_{3t}^{d1}) + E(\widehat{m}_{4t}^{d1})\right] + E\left(\widehat{m}_{5T}^{d1}\right) = O(\log T). \quad \text{(C.36)}$$

Therefore, from (C.31), (C.34) and (C.36), it follows that for $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$, it follows that

$$E\left(\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{u}}_t^{+}\right) = O(\log T). \quad \text{(C.37)}$$

(c), (d): From (C.20) and (C.21), we obtain

$$\begin{aligned}
\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d}\widehat{\mathbf{u}}_t^{+} &= \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{x}_t^{+\prime}\mathbf{M}_t^{d}\mathbf{u}_t^{+} + \frac{1}{\sqrt{NT_1}}\mathbf{x}_T^{+\prime}\mathbf{M}_T^{d}u_T^{+} \\
&+ \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\left[\widehat{m}_{1t}^{d} + \widehat{m}_{2t}^{d} + \widehat{m}_{3t}^{d} + \widehat{m}_{4t}^{d}\right] + \frac{1}{\sqrt{NT_1}}\widehat{m}_{5T}^{d}. \quad \text{(C.38)}
\end{aligned}$$

To begin with, we consider the first term in (C.38). Using (C.13), we can decompose

$$\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{x}_t^{+\prime}\mathbf{M}_t^d\mathbf{u}_t^+ = \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}(h_{1t}\Xi_{1t}^d + h_{2t}\Xi_{2t}^d + \cdots + h_{9t}\Xi_{9t}^d) = \Xi_1^d + \Xi_2^d + \cdots + \Xi_9^d$$

where $\Xi_j^d = \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1} h_{jt}\Xi_{jt}^d$. We show that only Ξ_1^d has a nonzero variance and others have zero variances as $N, T \to \infty$, i.e., $o_p(1)$.

Since $E(\mathbf{v}_t^*\mathbf{v}_s^{*\prime}|\boldsymbol{\eta}, \mathbf{v}_{t-1}, \mathbf{v}_{t-2}, \cdots) = \mathbf{0}$ for $t > s$ and $h_{1t}^2 = 1 - O(1/(T-t))$, using Lemma A2, for the case $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, we obtain

$$var\left(\Xi_1^{d2}\right) = \frac{\sigma_v^2}{NT_1}\sum_{t=2}^{T-1} h_{1t}^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}) \underset{N,T\to\infty}{\longrightarrow} \frac{\sigma_v^4}{1-\alpha^2}.$$

Similarly for $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$, we obtain

$$var\left(\Xi_1^{d1}\right) = \frac{\sigma_v^2}{NT_1}\sum_{t=2}^{T-1} h_{1t}^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{w}_{t-1}) \underset{N(T)\to\infty}{\longrightarrow} \frac{\sigma_v^4}{2(1+\alpha)}.$$

To complete the proof, we need to show that the variances of other terms tend to zero. We consider in sequence from Ξ_2^d. Using $h_{2t} = O(1/(T-t))$ and Lemma A2, for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}, \mathbf{M}_t^{d1}$, we obtain

$$var\left(\Xi_2^d\right) = \frac{\sigma_v^2}{NT_1}\sum_{t=2}^{T-1} h_{2t}^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}) \underset{N,T\to\infty}{\longrightarrow} 0.$$

Since $h_{3t} = O(1/(T-t))$, for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}, \mathbf{M}_t^{d1}$, using Lemma A3(b),

$$\begin{aligned}
\left|var\left(\Xi_3^d\right)\right| &\leq \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} h_{3t}^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{3t}h_{3s}|cov(\mathbf{w}_{t-1}'\mathbf{M}_t^d\boldsymbol{\eta}, \mathbf{w}_{s-1}'\mathbf{M}_s^d\boldsymbol{\eta})| \\
&\leq \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} h_{3t}^2 E(\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}) + \frac{2C}{T}\sum_t\sum_{t>s} O\left(\frac{1}{T-t}\right) O\left(\frac{1}{T-s}\right) \\
&= O\left(\frac{(\log T)^2}{T}\right) \underset{T\to\infty}{\longrightarrow} 0.
\end{aligned}$$

To show that $var(\Xi_4^d) \to 0$ and $var(\Xi_5^d) \to 0$, we use an alternative expression. Using $c_t\mathbf{v}_t^* = \mathbf{v}_t - \bar{\mathbf{v}}_{tT}$, $h_{4t}\Xi_{4t}^d + h_{5t}\Xi_{5t}^d$ can be written as[6],

$$h_{4t}\Xi_{4t}^d + h_{5t}\Xi_{5t}^d = \bar{h}_{4t}\Xi_{4t}^d + \bar{h}_{5t}\bar{\Xi}_{5t}^d$$

where $\bar{h}_{4t} = -(b_t c_t k_t + b_t^2)$, $\bar{h}_{5t} = b_t^2$ and $\bar{\Xi}_{5t}^d = \tilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\bar{\mathbf{v}}_{tT}$. To show $var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\bar{h}_{4t}\Xi_{4t}^d\right) \to 0$ and $var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\bar{h}_{5t}\bar{\Xi}_{5t}^d\right) \to 0$, we follow Alvarez and Arellano (2003). Let κ_3^v and κ_4^v denote the third and fourth order cumulants of v_{it}. We have

$$\begin{aligned}
var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\bar{h}_{4t}\Xi_{4t}^d\right) &= \frac{1}{NT_1}var\left[\sum_{t=2}^{T-1}\frac{\bar{h}_{4t}}{T-t}\mathbf{v}_t'\mathbf{M}_t^d(\phi_{T-t}\mathbf{v}_t + \cdots + \phi_1\mathbf{v}_{T-1})\right] \\
&= a_{0NT}^d + a_{1NT}^d
\end{aligned}$$

where

$$a_{0NT}^d = \frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{\bar{h}_{4t}^2\left[\phi_{T-t}^2 var(\mathbf{v}_t'\mathbf{M}_t^d\mathbf{v}_t) + \phi_{T-t-1}^2 var(\mathbf{v}_t'\mathbf{M}_t^d\mathbf{v}_{t+1}) + \cdots + \phi_1^2 var(\mathbf{v}_t'\mathbf{M}_t^d\mathbf{v}_{T-1})\right]}{(T-t)^2}$$

[6] See Alvarez and Arellano (2003, p.1144).

$$= \frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{\bar{h}_{4t}^2\phi_{T-t}^2[2\sigma_v^4\operatorname{tr}(\mathbf{M}_t^d)+\kappa_4^v E(\mathbf{d}_t'\mathbf{d}_t)]+\bar{h}_{4t}^2(\phi_{T-t-1}^2+\cdots+\phi_1^2)\operatorname{tr}(\mathbf{M}_t^d\mathbf{M}_s^d)\sigma_v^4}{(T-t)^2}$$

and

$$\begin{aligned} a_{1NT}^d &= \frac{2}{NT_1}\sum_{t=2}^{T-2}\left[\frac{\bar{h}_{4t}^2\phi_{T-t-1}^2 cov(\mathbf{v}_t'\mathbf{M}_t^d\mathbf{v}_{t+1},\mathbf{v}_{t+1}'\mathbf{M}_{t+1}^d\mathbf{v}_{t+1})}{(T-t)(T-t-1)}+\cdots\right.\\ &\qquad\left.\cdots+\frac{\bar{h}_{4t}^2\phi_1^2 cov(\mathbf{v}_t'\mathbf{M}_t^d\mathbf{v}_{T-1},\mathbf{v}_{T-1}'M_{T-1}^d\mathbf{v}_{T-1})}{(T-t)}\right]\\ &= \frac{2}{NT_1}\sum_{t=2}^{T-2}\left[\frac{\bar{h}_{4t}^2\phi_{T-t-1}^2\kappa_3^v E(\mathbf{d}_{t+1}'\mathbf{M}_t^d\mathbf{v}_t)}{(T-t)(T-t-1)}+\cdots+\frac{\bar{h}_{4t}^2\phi_1^2\kappa_3^v E(\mathbf{d}_{T-1}'\mathbf{M}_t^d\mathbf{v}_t)}{(T-t)}\right]. \end{aligned}$$

For the case of $\mathbf{M}_t^d=\mathbf{M}_t^{d2}$, by noting that

$$a_{0NT}^{d2} < \frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{\phi_{T-t}^2[2\sigma_v^4\operatorname{tr}(\mathbf{M}_t^d)+\kappa_4^v E(\mathbf{d}_t'\mathbf{d}_t)]+(\phi_{T-t-1}^2+\cdots+\phi_1^2)\operatorname{tr}(\mathbf{M}_t^d\mathbf{M}_s^d)\sigma_v^4}{(T-t)^2}$$

where $\bar{h}_{4t}^2<1$ is used, and from (A64) and (A66) in Alvarez and Arellano (2003), we have $a_{0NT}^{d2}=O(\log T/N)\to 0$. For the case of $\mathbf{M}_t^d=\mathbf{M}_t^{d1}$, using the fact that $\phi_j^2<1/(1-\alpha)^2$ for all j, we have

$$\begin{aligned} a_{0NT}^{d1} &\le \frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{\phi_{T-t}^2|2\sigma_v^4+\kappa_4^v|+(\phi_{T-t-1}^2+\cdots+\phi_1^2)\sigma_v^4}{(T-t)^2}\\ &\le \frac{1}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{|2\sigma_v^4+\kappa_4^v|+(T-t-1)\sigma_v^4}{(T-t)^2}\\ &= \frac{|2\sigma_v^4+\kappa_4^v|}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{1}{(T-t)^2}+\frac{\sigma_v^4}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-1}\frac{T-t-1}{(T-t)^2}\underset{N(T)\to\infty}{\longrightarrow}0. \end{aligned}$$

For a_{1NT}^d, from the triangle inequality, the facts that $|E(\mathbf{d}_{t+j}'\mathbf{M}_t^{d2}\mathbf{v}_t)|\le(t+j)\sigma_v$ and $|E(\mathbf{d}_{t+j}'\mathbf{M}_t^{d1}\mathbf{v}_t)|\le\sigma_v$ and $\bar{h}_{4t}^2<1$,

$$\begin{aligned} |a_{1NT}^{d2}| &< \frac{2|\kappa_3^v|\sigma_v}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-2}\frac{1}{T-t}\left[\frac{t+1}{(T-t-1)}+\cdots+\frac{T-1}{1}\right]\\ &< \frac{2|\kappa_3^v|\sigma_v}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-2}\frac{1}{T-t}\left[\sum_{s=1}^{T-1}\frac{T_1}{s}\right]=O\left(\frac{(\log T)^2}{N}\right)\underset{N,T\to\infty}{\longrightarrow}0 \end{aligned}$$

under the assumption of $(\log T)^2/N\to 0$ and

$$\begin{aligned} |a_{1NT}^{d1}| &< \frac{2|\kappa_3^v|\sigma_v}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-2}\frac{1}{T-t}\left[\frac{1}{(T-t-1)}+\cdots+\frac{1}{1}\right]\\ &< \frac{2|\kappa_3^v|\sigma_v}{(1-\alpha)^2}\frac{1}{NT_1}\sum_{t=2}^{T-2}\frac{1}{T-t}\left[\sum_{s=1}^{T-1}\frac{1}{s}\right]=O\left(\frac{(\log T)^2}{NT}\right)\underset{N(T)\to\infty}{\longrightarrow}0. \end{aligned}$$

Thus, the variance of $\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\bar{h}_{4t}\Xi_{4t}^d$ is shown to tend to zero.

With regard to $\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\bar{h}_{5t}\bar{\Xi}_{5t}^d$, following Alvarez and Arellano (2003), we decompose as follows:

$$var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\bar{h}_{5t}\bar{\Xi}_{5t}^d\right)=\frac{1}{NT_1}var\left(\sum_{t=2}^{T-1}\bar{h}_{5t}\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^d\tilde{\mathbf{v}}_{tT}\right)=b_{0NT}^d+b_{1NT}^d$$

where

$$b_{0NT}^{d} = \frac{1}{NT_1}\sum_{t=2}^{T-1}\bar{h}_{5t}^{2} var(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^d\widetilde{\mathbf{v}}_{tT}),$$

$$b_{1NT}^{d} = \frac{2}{NT_1}\sum_{s}\sum_{s>t}\bar{h}_{5t}\bar{h}_{5s} cov(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^d\widetilde{\mathbf{v}}_{tT}, \bar{\mathbf{v}}_{sT}'\mathbf{M}_s^d\widetilde{\mathbf{v}}_{sT}).$$

For the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, by noting that

$$b_{0NT}^{d2} < \frac{1}{NT_1}\sum_{t=2}^{T-1} var(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^{d2}\widetilde{\mathbf{v}}_{tT})$$

where $\bar{h}_{5t}^2 < 1$ is used, and (A75) in Alvarez and Arellano (2003), we have $b_{0NT}^{d2} = O(1/N) \to 0$. For the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$, using[7]

$$var(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^{d1}\widetilde{\mathbf{v}}_{tT}) = O\left(\frac{1}{(T-t)^2}\right),$$

we have $b_{0NT}^{d1} = O(1/NT) \to 0$. Next, we consider the term b_{1NT}^{d}. For the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, using $|\bar{h}_{5t}||\bar{h}_{5s}| < 1$ and (A77) in Alvarez and Arellano (2003), we obtain

$$|b_{1NT}^{d2}| < \frac{2}{NT_1}\sum_{s}\sum_{s>t}|cov(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^{d1}\widetilde{\mathbf{v}}_{tT}, \bar{\mathbf{v}}_{sT}'\mathbf{M}_s^{d2}\widetilde{\mathbf{v}}_{sT})| = O\left(\frac{(\log T)^2}{N}\right) \to 0$$

under the assumption of $(\log T)^2/N \to 0$. Similarly, for the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$, we have

$$\begin{aligned} |b_{1NT}^{d1}| &\le \frac{2}{NT_1}\sum_{s}\sum_{s>t}|cov(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^{d1}\widetilde{\mathbf{v}}_{tT}, \bar{\mathbf{v}}_{sT}'\mathbf{M}_s^{d1}\widetilde{\mathbf{v}}_{sT})| \\ &\le \frac{2}{NT_1}\sum_{s}\sum_{s>t}\sqrt{var(\bar{\mathbf{v}}_{tT}'\mathbf{M}_t^{d1}\widetilde{\mathbf{v}}_{tT})}\sqrt{var(\bar{\mathbf{v}}_{sT}'\mathbf{M}_s^{d1}\widetilde{\mathbf{v}}_{sT})} \\ &\le \frac{2}{NT_1}\sum_{s} O\left(\frac{1}{T-t}\right)\sum_{t} O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{NT}\right) \underset{N(T)\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus, the variance of $\frac{1}{\sqrt{NT}}\sum_{t=2}^{T-1}\bar{h}_{5t}\Xi_{5t}^d$ is shown to tend to zero.

Next, we turn to consider Ξ_6^d. Since $\phi_j^2 < 1/(1-\alpha)^2$ for all j, we obtain

$$E(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\widetilde{\mathbf{v}}_{tT}) = \sigma_v^2\,\mathrm{tr}(\mathbf{M}_t^d)\frac{\phi_1^2 + \cdots + \phi_{T-t}^2}{(T-t)^2} < \frac{\sigma_v^2}{(1-\alpha)^2}\frac{\mathrm{tr}(\mathbf{M}_t^d)}{(T-t)}, \tag{C.39}$$

$$|cov(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\boldsymbol{\eta}, \widetilde{\mathbf{v}}_{sT}'\mathbf{M}_s^d\boldsymbol{\eta})| \le \sigma_\eta^2\sqrt{E(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^d\widetilde{\mathbf{v}}_{tT})}\sqrt{E(\widetilde{\mathbf{v}}_{sT}'\mathbf{M}_s^d\widetilde{\mathbf{v}}_{sT})}. \tag{C.40}$$

Using (C.39), (C.40), $h_{6t} = O(1/(T-t))$, and triangular inequality, for the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, we obtain

$$\begin{aligned} \left|var\left(\Xi_6^d\right)\right| &\le \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} h_{6t}^2 E(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^{d2}\widetilde{\mathbf{v}}_{tT}) + \frac{2}{NT_1}\sum_{t}\sum_{t>s}\left|h_{6t}h_{6s} cov(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^{d2}\boldsymbol{\eta}, \widetilde{\mathbf{v}}_{sT}'\mathbf{M}_s^{d2}\boldsymbol{\eta})\right| \\ &< \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} h_{6t}^2\frac{\sigma_v^2\,\mathrm{tr}(\mathbf{M}_t^{d2})}{(T-t)(1-\alpha)^2} + \frac{C}{NT_1}\sum_{t}\sum_{t>s}|h_{6t}h_{6s}|\sqrt{\frac{\mathrm{tr}(\mathbf{M}_t^{d2})}{T-t}}\sqrt{\frac{\mathrm{tr}(\mathbf{M}_s^{d2})}{T-s}} \end{aligned}$$

[7] This can be proved in the same way as (A73) in Alvarez and Arellano (2003).

$$
\begin{aligned}
&< \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{t}{(T-t)^3}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{\sqrt{t}}{(T-t)^{3/2}}\right) O\left(\frac{\sqrt{s}}{(T-s)^{3/2}}\right) \\
&< O\left(\frac{1}{N}\right) + \frac{2C}{NT_1}\sum_t O\left(\frac{t}{T-t}\right)\sum_t O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{N}\right).
\end{aligned}
$$

Thus if we assume that $(\log T)^2/N \to 0$, $var(\Xi_6^{d2}) \underset{N,T\to\infty}{\longrightarrow} 0$. Similarly, for the case of $\mathbf{M}_t^{d1}$, we obtain

$$
\begin{aligned}
\left|var\left(\Xi_6^{d1}\right)\right| &\leq \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} h_{6t}^2 E(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^{d1}\widetilde{\mathbf{v}}_{tT}) + \frac{2}{NT_1}\sum_t\sum_{t>s}\left|h_{6t}h_{6s}cov(\widetilde{\mathbf{v}}_{tT}'\mathbf{M}_t^{d1}\boldsymbol{\eta}, \widetilde{\mathbf{v}}_{sT}'\mathbf{M}_s^{d1}\boldsymbol{\eta})\right| \\
&< \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} h_{6t}^2 \frac{\sigma_v^2\,\mathrm{tr}(\mathbf{M}_t^{d1})}{(T-t)(1-\alpha)^2} + \frac{C}{NT_1}\sum_t\sum_{t>s}|h_{6t}h_{6s}|\sqrt{\frac{\mathrm{tr}(\mathbf{M}_t^{d1})}{T-t}}\sqrt{\frac{\mathrm{tr}(\mathbf{M}_s^{d1})}{T-s}} \\
&< \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{1}{(T-t)^3}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{1}{(T-t)^{3/2}}\right) O\left(\frac{1}{(T-s)^{3/2}}\right) \\
&< O\left(\frac{1}{NT}\right) + \frac{2C}{NT_1}\sum_t O\left(\frac{1}{T-t}\right)\sum_t O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{NT}\right).
\end{aligned}
$$

Thus, $var(\Xi_6^{d1}) \underset{N(T)\to\infty}{\longrightarrow} 0$ for the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$.

Using $h_{7t} = O(1/(T-t)^2)$, we obtain

$$
\begin{aligned}
var\left(\Xi_7^d\right) &= \frac{1}{NT_1}\sum_{t=2}^{T-1} h_{7t}^2 var(\boldsymbol{\mu}'\mathbf{M}_t^{d2}\mathbf{v}_t) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{7t}h_{7s}cov(\boldsymbol{\mu}'\mathbf{M}_t^{d2}\mathbf{v}_t, \boldsymbol{\mu}'\mathbf{M}_s^{d2}\mathbf{v}_s) \\
&\leq \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} h_{7t}^2\,\mathrm{tr}(\mathbf{M}_t^{d2}) + \frac{C}{NT_1}\sum_t\sum_{t>s} h_{7t}h_{7s}\sqrt{\mathrm{tr}(\mathbf{M}_t^{d2})}\sqrt{\mathrm{tr}(\mathbf{M}_s^{d2})} \\
&\leq \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{t}{(T-t)^4}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{\sqrt{s}}{(T-s)^2}\right) O\left(\frac{\sqrt{t}}{(T-t)^2}\right) \\
&< O\left(\frac{1}{N}\right) + \frac{C}{NT_1}\sum_t O\left(\frac{t}{T-t}\right)\sum_t O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{N}\right)
\end{aligned}
$$

where $\sigma_\mu^2 = var(\mu_i)$. If we assume that $(\log T)^2/N \to 0$, $var(\Xi_7^{d2}) \underset{N,T\to\infty}{\longrightarrow} 0$. Similarly, for the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$,

$$
\begin{aligned}
var\left(\Xi_7^{d1}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T-1} h_{7t}^2 var(\boldsymbol{\mu}'\mathbf{M}_t^{d1}\mathbf{v}_t) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{7t}h_{7s}cov(\boldsymbol{\mu}'\mathbf{M}_t^{d1}\mathbf{v}_t, \boldsymbol{\mu}'\mathbf{M}_s^{d1}\mathbf{v}_s) \\
&\leq \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} h_{7t}^2\,\mathrm{tr}(\mathbf{M}_t^{d1}) + \frac{C}{NT_1}\sum_t\sum_{t>s} h_{7t}h_{7s}\sqrt{\mathrm{tr}(\mathbf{M}_t^{d1})}\sqrt{\mathrm{tr}(\mathbf{M}_s^{d1})} \\
&\leq \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{1}{(T-t)^4}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{1}{(T-s)^2}\right) O\left(\frac{1}{(T-t)^2}\right) \\
&< O\left(\frac{1}{NT}\right) + \frac{C}{NT_1}\sum_t O\left(\frac{1}{T-t}\right)\sum_t O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{NT}\right) \underset{N(T)\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Using $h_{8t} = O(1/(T-t))$, for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, we obtain

$$
var\left(\Xi_8^{d2}\right) = \frac{1}{NT_1}\sum_{t=2}^{T-1} h_{8t}^2 var(\boldsymbol{\mu}'\mathbf{M}_t^{d2}\mathbf{v}_t^*) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{8t}h_{8s}cov(\boldsymbol{\mu}'\mathbf{M}_t^{d2}\mathbf{v}_t^*, \boldsymbol{\mu}'\mathbf{M}_s^{d2}\mathbf{v}_s^*)
$$

$$\begin{aligned}
&\leq \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} h_{8t}^2\,\mathrm{tr}(\mathbf{M}_t^{d2}) + \frac{C}{NT_1}\sum_t\sum_{t>s} h_{8t}h_{8s}\sqrt{\mathrm{tr}(\mathbf{M}_t^{d2})}\sqrt{\mathrm{tr}(\mathbf{M}_s^{d2})}\\
&< \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{t}{(T-t)^2}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{\sqrt{t}}{T-t}\right)O\left(\frac{\sqrt{s}}{T-s}\right)\\
&< O\left(\frac{1}{N}\right) + \frac{C}{NT_1}\sum_t O\left(\frac{t}{T-t}\right)\sum_{t>s} O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{N}\right).
\end{aligned}$$

Thus under the condition $(\log T)^2/N \to 0$, $var(\Xi_9^{d2}) \underset{N,T\to\infty}{\longrightarrow} 0$. Similarly, for the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$,

$$\begin{aligned}
var\left(\Xi_8^{d1}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T-1} h_{8t}^2 var(\boldsymbol{\mu}'\mathbf{M}_t^{d1}\mathbf{v}_t^*) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{8t}h_{8s}cov(\boldsymbol{\mu}'\mathbf{M}_t^{d1}\mathbf{v}_t^*, \boldsymbol{\mu}'\mathbf{M}_s^{d1}\mathbf{v}_s^*)\\
&\leq \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} h_{8t}^2\,\mathrm{tr}(\mathbf{M}_t^{d1}) + \frac{C}{NT_1}\sum_t\sum_{t>s} h_{8t}h_{8s}\sqrt{\mathrm{tr}(\mathbf{M}_t^{d1})}\sqrt{\mathrm{tr}(\mathbf{M}_s^{d1})}\\
&< \frac{\sigma_v^2\sigma_\mu^2}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{1}{(T-t)^2}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{1}{T-t}\right)O\left(\frac{1}{T-s}\right)\\
&< O\left(\frac{1}{N}\right) + \frac{C}{NT_1}\sum_t O\left(\frac{1}{T-t}\right)\sum_{t>s} O\left(\frac{1}{T-s}\right) = O\left(\frac{(\log T)^2}{NT}\right) \underset{N(T)\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus, $var(\Xi_8^{d2}) \underset{N,T\to\infty}{\longrightarrow} 0$ for the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$.

Using $h_{9t} = O(1/(T-t)^2)$ and Lemma A3, for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$, we obtain

$$\begin{aligned}
var\left(\Xi_9^{d2}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T-1} h_{9t}^2 var(\boldsymbol{\mu}'\mathbf{M}_t^{d2}\boldsymbol{\eta}) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{9t}h_{9s}cov(\boldsymbol{\mu}'\mathbf{M}_t^{d2}\boldsymbol{\eta}, \boldsymbol{\mu}'\mathbf{M}_s^{d2}\boldsymbol{\eta})\\
&< \frac{C}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{t}{(T-t)^4}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{1}{(T-t)^2}\right)O\left(\frac{s}{(T-s)^2}\right)\\
&= O\left(\frac{1}{N}\right) \underset{N(T)\to\infty}{\longrightarrow} 0.
\end{aligned}$$

For the case of $\mathbf{M}_t^d = \mathbf{M}_t^{d1}$,

$$\begin{aligned}
var\left(\Xi_9^{d1}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T-1} h_{9t}^2 var(\boldsymbol{\mu}'\mathbf{M}_t^{d1}\boldsymbol{\eta}) + \frac{2}{NT_1}\sum_t\sum_{t>s} h_{9t}h_{9s}cov(\boldsymbol{\mu}'\mathbf{M}_t^{d1}\boldsymbol{\eta}, \boldsymbol{\mu}'\mathbf{M}_s^{d1}\boldsymbol{\eta})\\
&< \frac{C}{NT_1}\sum_{t=2}^{T-1} O\left(\frac{1}{(T-t)^4}\right) + \frac{C}{NT_1}\sum_t\sum_{t>s} O\left(\frac{1}{(T-t)^2}\right)O\left(\frac{1}{(T-s)^2}\right)\\
&= O\left(\frac{1}{NT}\right) \underset{N(T)\to\infty}{\longrightarrow} 0.
\end{aligned}$$

For the second term in (C.38), by using (A.2), (A.3) and Lemma A2, A3, we obtain

$$\frac{1}{NT_1} var\left(\mathbf{x}_T^{+\prime}\mathbf{M}_T^d\mathbf{u}_T^+\right) \underset{N(T)\to\infty}{\longrightarrow} 0.$$

To complete the proof, we show that $var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\widehat{m}_{kt}^d\right) \to 0$, $(k = 1, ..., 4)$ and $var\left(\frac{1}{\sqrt{NT_1}}\widehat{m}_{5T}^d\right) \to 0$. For $k = 1, 2, 3, 4$,

$$\frac{1}{NT_1} var\left(\sum_{t=2}^{T-1}\widehat{m}_{kt}\right) = \frac{1}{NT_1} var\left((\widehat{r} - r)\sum_{t=2}^{T-1} O_p\left(\frac{1}{T-t}\right)W_{kt}\right)$$

$$< \frac{1}{NT_1}E\left[(\widehat{r}-r)^2\left\{\sum_{t=2}^{T-1}O_p\left(\frac{1}{T-t}\right)W_{kt}\right\}^2\right]$$
$$\leq \frac{1}{NT_1}\sqrt{E(\widehat{r}-r)^4}\sqrt{E\left\{\sum_{t=2}^{T-1}O_p\left(\frac{1}{T-t}\right)W_{kt}\right\}^4}.$$

From the Cauchy-Schwarz inequality,

$$E\left\{\sum_{t=2}^{T-1}O_p\left(\frac{1}{T-t}\right)W_{kt}\right\}^4 = \sum_{t_1,t_2,t_3,t_4}O\left(\frac{1}{(T-t_1)(T-t_2)(T-t_3)(T-t_4)}\right)E(W_{kt_1}W_{kt_2}W_{kt_3}W_{kt_4})$$
$$\leq \left\{\sum_{t_1,t_2}O\left(\frac{1}{(T-t_1)(T-t_2)}\right)\sqrt{E(W_{kt_1}^2W_{kt_2}^2)}\right\}\left\{\sum_{t_3,t_4}O\left(\frac{1}{(T-t_3)(T-t_4)}\right)\sqrt{E(W_{kt_3}^2W_{kt_4}^2)}\right\}$$
$$\leq \left\{\sum_{t_1}O\left(\frac{1}{(T-t_1)}\right)[E(W_{kt_1}^4)]^{1/4}\right\}\times\cdots\times\left\{\sum_{t_4}O\left(\frac{1}{(T-t_4)}\right)[E(W_{kt_4}^4)]^{1/4}\right\}$$
$$= \left\{\sum_{t}O\left(\frac{1}{(T-t)}\right)[E(W_{kt}^4)]^{1/4}\right\}^4.$$

Since $E(W_{kt}^4)=O(N^4)$ as in Lemma A5, we have $E\left\{\sum_{t=2}^{T}O_p\left(\frac{1}{T-t}\right)W_{kt}\right\}^4=O(N^4(\log T)^4)$. Hence, for $k=1,2,3,4$

$$\frac{1}{NT_1}var\left(\sum_{t=2}^{T-1}\widehat{m}_{kt}\right)=\frac{1}{NT_1}O\left(\frac{1}{N}\right)O(N^2(\log T)^2)=O\left(\frac{(\log T)^2}{T}\right)\underset{T\to\infty}{\longrightarrow}0.$$

Finally, we have

$$var\left(\frac{1}{\sqrt{NT_1}}\widehat{m}_{5T}^d\right) = \frac{1}{NT_1}var\left(O_p(1)(\widehat{r}-r)W_{4T}\right)\leq\frac{C}{NT_1}E\left[(\widehat{r}-r)^2W_{4T}^2\right]\leq\frac{C}{NT_1}\sqrt{E(\widehat{r}-r)^4}\sqrt{E(W_{4T}^4)}$$
$$= O\left(\frac{1}{NT}\right)O\left(\frac{1}{N}\right)O(N^2)=O\left(\frac{1}{T}\right)\to 0.$$

Thus, we obtain the results.

(e), (f): Using (A.6), we have

$$\begin{aligned}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^d\widehat{\mathbf{x}}_t^{+} &= \sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^d\widehat{\mathbf{x}}_t^{+}+\widehat{\mathbf{x}}_T^{+\prime}\mathbf{M}_T^d\widehat{\mathbf{x}}_T^{+}\\ &= \sum_{t=2}^{T-1}\widehat{b}_t^2\mathbf{x}_t^{*\prime}\mathbf{M}_t^d\mathbf{x}_t^{*}+2\sum_{t=2}^{T-1}\widehat{b}_t\widehat{k}_t\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t^{*}+\sum_{t=2}^{T-1}\widehat{k}_t^2\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t+\widehat{d}_T^2\mathbf{x}_T^{\prime}\mathbf{M}_T^d\mathbf{x}_T\\ &= \sum_{t=2}^{T-1}b_t^2\mathbf{x}_t^{*\prime}\mathbf{M}_t^d\mathbf{x}_t^{*}+2\sum_{t=2}^{T-1}b_tk_t\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t^{*}+\sum_{t=2}^{T-1}k_t^2\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t+d_T^2\mathbf{x}_T^{\prime}\mathbf{M}_T^d\mathbf{x}_T\\ &\quad+\sum_{t=2}^{T-1}(\widehat{b}_t^2-b_t^2)\mathbf{x}_t^{*\prime}\mathbf{M}_t^d\mathbf{x}_t^{*}+2\sum_{t=2}^{T-1}(\widehat{b}_t\widehat{k}_t-b_tk_t)\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t^{*}\\ &\qquad+\sum_{t=2}^{T-1}(\widehat{k}_t^2-k_t^2)\mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t+(\widehat{d}_T^2-d_T^2)\mathbf{x}_T^{\prime}\mathbf{M}_T^d\mathbf{x}_T.\end{aligned} \tag{C.41}$$

For the first term in (C.41), since $b_t^2 = 1 - O(1/(T-t))$, it follows that

$$\frac{1}{NT_1}\sum_{t=2}^{T-1} b_t^2 \mathbf{x}_t^{*\prime}\mathbf{M}_t^{d2}\mathbf{x}_t^* \underset{N,T\to\infty}{\xrightarrow{p}} \frac{\sigma_v^2}{1-\alpha^2},$$
$$\frac{1}{NT_1}\sum_{t=2}^{T-1} b_t^2 \mathbf{x}_t^{*\prime}\mathbf{M}_t^{d1}\mathbf{x}_t^* \underset{N,T\to\infty}{\xrightarrow{p}} \frac{\sigma_v^2}{2(1+\alpha)}.$$

With regard to the second term, we obtain

$$\frac{2}{NT_1}\sum_{t=2}^{T-1} b_t k_t E(\mathbf{x}_t^{*\prime}\mathbf{M}_t^{d}\mathbf{x}_t) = \frac{2}{NT_1}\sum_{t=2}^{T-1} b_t k_t \psi_t E(\mathbf{w}_{t-1}^{\prime}\mathbf{M}_t^{d}\mathbf{w}_{t-1}) \underset{N,T\to\infty}{\xrightarrow{p}} 0$$

from Lemma A2 and $b_t k_t \psi_t = O(1/(T-t))$. For the third term, from Lemma A6 and $k_t^2 = O(1/(T-t)^2)$, it is shown to converge in probability to zero. For the fourth term, since $E(\mathbf{w}_{T-1}^{\prime}\mathbf{M}_T^{d}\mathbf{w}_{T-1})/N = O(1)$, for $\mathbf{M}_T^d = \mathbf{M}_T^{d2}, \mathbf{M}_T^{d1}$, we obtain

$$\frac{d_T^2}{NT_1}E(\mathbf{x}_T^{\prime}\mathbf{M}_T^{d}\mathbf{x}_T) = \frac{d_T^2}{NT_1}\left(\sigma_\mu^2 \operatorname{tr}(\mathbf{M}_T^d) + E(\mathbf{w}_{T-1}^{\prime}\mathbf{M}_T^{d}\mathbf{w}_{T-1})\right) \underset{N,T\to\infty}{\longrightarrow} 0.$$

For the fifth term, using (C.12), we have

$$\frac{1}{NT_1}\sum_{t=2}^{T-1}(\widehat{b}_t^2 - b_t^2)\mathbf{x}_t^{\prime}\mathbf{M}_t^{d}\mathbf{x}_t = O_p\left(\frac{\log T}{T\sqrt{N}}\right) \to 0.$$

The remaining terms can be proved in the same way as the fifth term.

■

Lemma A7. *Let Assumptions 1, 2, and 3 hold. Then, as both N and T tend to infinity, provided that $T/N \to c$, $0 \le c < \infty$,*

$$(a)\quad \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d2}\mathbf{u}_t \underset{N,T\to\infty}{\xrightarrow{p}} \frac{c\sigma_\eta^2}{2}\left(\frac{1}{1-\alpha}\right),$$
$$(b)\quad E\left(\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{u}_t\right) = T_1\left(\frac{\sigma_\eta^2}{1-\alpha}\right) = \mu_{L1}^{non},$$
$$(c)\quad var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{u}_t\right) \underset{N,T\to\infty}{\longrightarrow} \frac{\sigma_v^4}{2(1+\alpha)},$$
$$(d)\quad \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d2}\mathbf{x}_t \underset{N,T\to\infty}{\xrightarrow{p}} \frac{c\sigma_\eta^2}{2}\left(\frac{1}{1-\alpha}\right)^2 + \frac{\sigma_v^2}{1-\alpha^2},$$
$$(e)\quad \frac{1}{NT_1}\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{x}_t \underset{N,T\to\infty}{\xrightarrow{p}} \frac{\sigma_v^2}{2(1+\alpha)}.$$

Proof of Lemma A7

(a): Remember that $\mathbf{x}_t^{\prime}\mathbf{M}_t^{d}\mathbf{u}_t$ can be decomposed as (C.17). In these terms, only Ξ_{9t}^d has nonzero mean which is given by

$$E(\Xi_{9t}^d) = E\left(\operatorname{tr}(\mathbf{M}_t^d\boldsymbol{\eta}\boldsymbol{\eta}^{\prime})\right) = \sigma_\eta^2 \operatorname{tr}(\mathbf{M}_t^d). \tag{C.42}$$

Hence, for $\mathbf{M}_t^d = \mathbf{M}_t^{d2}$,

$$\frac{1}{NT_1}\sum_{t=2}^{T} E(\mathbf{x}_t'\mathbf{M}_t^{d2}\mathbf{u}_t) = \frac{1}{NT_1}\sum_{t=2}^{T} E(\Xi_{9t}^{d2}) \underset{N,T\to\infty}{\longrightarrow} \frac{c\sigma_\eta^2}{2}\left(\frac{1}{1-\alpha}\right).$$

Using Lemmas A2 and A3, the variances of $\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{2t}$, $\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{3t}$, $\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{7t}$, and $\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{9t}$ are shown to tend to zero as follows:

$$\begin{aligned}
var\left(\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{2t}^d\right) &= \frac{1}{N^2T_1^2}\sum_{t=2}^{T} var(\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{v}_t) + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s} cov(\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{v}_t, \mathbf{w}_{s-1}'\mathbf{M}_s^{d2}\mathbf{v}_t) \\
&= \frac{\sigma_v^2}{N^2T_1^2}\sum_{t=2}^{T} E(\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}) \underset{N(T)\to\infty}{\longrightarrow} 0,
\end{aligned}$$

$$\begin{aligned}
\left|var\left(\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{3t}^d\right)\right| &\leq \frac{1}{N^2T_1^2}\sum_{t=2}^{T} var(\boldsymbol{\eta}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}) + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s} |cov(\boldsymbol{\eta}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}, \boldsymbol{\eta}'\mathbf{M}_s^{d2}\mathbf{w}_{s-1})| \\
&= \frac{\sigma_\eta^2}{N^2T_1^2}\sum_{t=2}^{T} E(\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}) + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s} |cov(\boldsymbol{\eta}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}, \boldsymbol{\eta}'\mathbf{M}_s^{d2}\mathbf{w}_{s-1})| \\
&\leq \frac{\sigma_\eta^2}{N^2T_1^2}\sum_{t=2}^{T} E(\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{w}_{t-1}) + \frac{2}{NT_1^2}\sum_s\sum_{t>s}\left(\frac{\sigma_v^2\sigma_\eta^2}{1-\alpha^2}\right) \underset{N(T)\to\infty}{\longrightarrow} 0,
\end{aligned}$$

$$\begin{aligned}
var\left(\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{7t}^d\right) &= \frac{1}{N^2T_1^2}\sum_{t=2}^{T} var(\mathbf{v}_t'\mathbf{M}_t^{d2}\boldsymbol{\eta}) + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s} cov(\mathbf{v}_t'\mathbf{M}_t^{d2}\boldsymbol{\eta}, \mathbf{v}_s'\mathbf{M}_s^{d2}\boldsymbol{\eta}) \\
&= \frac{\sigma_\eta^2}{N^2T_1^2}\sum_{t=2}^{T} E(\mathbf{v}_t'\mathbf{M}_t^{d2}\mathbf{v}_t) = \frac{\sigma_v^2\sigma_\eta^2}{N^2T_1^2}\sum_{t=2}^{T}\mathrm{tr}(\mathbf{M}_t^{d2}) \underset{N(T)\to\infty}{\longrightarrow} 0,
\end{aligned}$$

$$\begin{aligned}
var\left(\frac{1}{NT_1}\sum_{t=2}^{T}\Xi_{9t}^d\right) &= \frac{1}{N^2T_1^2}\sum_{t=2}^{T} var(\boldsymbol{\eta}'\mathbf{M}_t^{d2}\boldsymbol{\eta}) + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s} cov(\boldsymbol{\eta}'\mathbf{M}_t^{d2}\boldsymbol{\eta}, \boldsymbol{\eta}'\mathbf{M}_s^{d2}\boldsymbol{\eta}) \\
&= \frac{1}{N^2T_1^2}\sum_{t=2}^{T}\left[E(\mathbf{d}_t'\mathbf{d}_t)\kappa_4^\eta + 2\sigma_\eta^4\,\mathrm{tr}(\mathbf{M}_t^d)\right] \\
&\quad + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s}\left[E(\mathbf{d}_t'\mathbf{d}_s)\kappa_4^\eta + 2\sigma_\eta^4 E(\mathrm{tr}(\mathbf{M}_t^{d2}\mathbf{M}_s^{d2}))\right] \\
&\leq \frac{1}{N^2T_1^2}\sum_{t=2}^{T}(\kappa_4^\eta + 2\sigma_\eta^4)(t-1) + \frac{2}{N^2T_1^2}\sum_s\sum_{t>s}(t-1)(\kappa_4^\eta + 2\sigma_\eta^2) \underset{N(T)\to\infty}{\longrightarrow} 0.
\end{aligned}$$

(b): Since $\mathrm{tr}(\mathbf{M}_t^{d1}) = 1$, using (C.42), we obtain

$$E\left(\sum_{t=2}^{T}\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{u}_t\right) = T_1\left(\frac{\sigma_\eta^2}{1-\alpha}\right).$$

(c): We use the decomposition (C.17). Only Ξ_{2t}^d has non-zero variance:

$$var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\Xi_{2t}^d\right) = \frac{1}{NT_1}\sum_{t=2}^{T} var(\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{v}_t) + \frac{2}{NT_1}\sum_s\sum_{t>s} cov(\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{v}_t, \mathbf{w}_{s-1}'\mathbf{M}_s^{d1}\mathbf{v}_t)$$

$$= \frac{\sigma_v^2}{NT_1}\sum_{t=2}^{T} E(\mathbf{w}'_{t-1}\mathbf{M}_t^{d1}\mathbf{w}_{t-1}) \underset{N(T)\to\infty}{\longrightarrow} \frac{\sigma_v^4}{2(1+\alpha)}.$$

The variances of Ξ_{3t}^{d1}, Ξ_{7t}^{d1} and Ξ_{9t}^{d1} tend to zero as follows:

$$\begin{aligned} var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\Xi_{3t}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T} var(\boldsymbol{\eta}'\mathbf{M}_t^{d1}\mathbf{w}_{t-1}) + \frac{2}{NT_1}\sum_s\sum_{t>s} cov(\boldsymbol{\eta}'\mathbf{M}_t^{d1}\mathbf{w}_{t-1}, \boldsymbol{\eta}'\mathbf{M}_s^{d1}\mathbf{w}_{s-1}) \\ &= \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T} E(\mathbf{w}'_{t-1}\mathbf{M}_t^{d1}\mathbf{w}_{t-1}) + \frac{2\sigma_\eta^2}{NT_1}\sum_s\sum_{t>s} E(\mathbf{w}'_{t-1}\mathbf{M}_t^{d1}\mathbf{M}_s^{d1}\mathbf{w}_{s-1}) \\ &= \frac{\sigma_\eta^2\sigma_v^2}{2(1+\alpha)} - \frac{\sigma_\eta^2\sigma_v^2}{2(1+\alpha)} + o_p(1) \underset{N,T\to\infty}{\longrightarrow} 0, \end{aligned}$$

where last convergence is obtained from Lemma A3(c) and

$$\begin{aligned} \frac{\mathbf{w}'_{t-1}\mathbf{M}_t^{d1}\mathbf{M}_s^{d1}\mathbf{w}_{s-1}}{N} &= \frac{(\mathbf{w}'_{t-1}\mathbf{Z}_t^{d1}/N)(\mathbf{Z}_t^{d1\prime}\mathbf{Z}_s^{d1}/N)(\mathbf{Z}_s^{d1\prime}\mathbf{w}_{s-1}/N)}{(\mathbf{Z}_t^{d1\prime}\mathbf{Z}_t^{d1}/N)(\mathbf{Z}_s^{d1\prime}\mathbf{Z}_s^{d1}/N)} \\ &= \frac{E(w_{i,t-1}z_{it}^{d1})E(z_{it}^{d1}z_{is}^{d1})E(w_{i,s-1}z_{is}^{d1}) + O_p(1/\sqrt{N})}{[E(z_{it}^{d1})^2][E(z_{is}^{d1})^2] + O_p(1/\sqrt{N})} \\ &= -\alpha^{t-s-1}\frac{\sigma_v^2(1-\alpha)}{4(1+\alpha)} + O_p\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

$$\begin{aligned} var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\Xi_{7t}^{d}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T} var(\mathbf{v}'_t\mathbf{M}_t^{d1}\boldsymbol{\eta}) + \frac{2}{NT_1}\sum_s\sum_{t>s} cov(\mathbf{v}'_t\mathbf{M}_t^{d1}\boldsymbol{\eta}, \mathbf{v}'_s\mathbf{M}_s^{d1}\boldsymbol{\eta}) \\ &= \frac{\sigma_\eta^2}{NT_1}\sum_{t=2}^{T} E(\mathbf{v}'_t\mathbf{M}_t^{d1}\mathbf{v}_t) = \frac{\sigma_v^2\sigma_\eta^2}{NT_1}\sum_{t=2}^{T}\mathrm{tr}(\mathbf{M}_t^{d1}) \underset{N\to\infty}{\longrightarrow} 0, \end{aligned}$$

$$\begin{aligned} var\left(\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\Xi_{9t}^{d}\right) &= \frac{1}{NT_1}\sum_{t=2}^{T} var(\boldsymbol{\eta}'\mathbf{M}_t^{d1}\boldsymbol{\eta}) + \frac{2}{NT_1}\sum_s\sum_{t>s} cov(\boldsymbol{\eta}'\mathbf{M}_t^{d1}\boldsymbol{\eta}, \boldsymbol{\eta}'\mathbf{M}_s^{d1}\boldsymbol{\eta}) \\ &= \frac{1}{NT_1}\sum_{t=2}^{T}\left[E(\mathbf{d}'_t\mathbf{d}_t)\kappa_4^\eta + 2\sigma_\eta^4\,\mathrm{tr}(\mathbf{M}_t^{d1})\right] \\ &\quad + \frac{2}{NT_1}\sum_s\sum_{t>s}\left[E(\mathbf{d}'_t\mathbf{d}_s)\kappa_4^\eta + 2\sigma_\eta^4 E(\mathrm{tr}(\mathbf{M}_t^{d1}\mathbf{M}_s^{d1}))\right] \underset{N,T\to\infty}{\longrightarrow} 0 \end{aligned}$$

where we used Lemma A3 and the fact that

$$\begin{aligned} \mathrm{tr}\left(\mathbf{M}_t^{d1}\mathbf{M}_s^{d1}\right) &= \frac{\left[\mathbf{Z}_t^{d1\prime}\mathbf{Z}_s^{d1}/N\right]^2}{(\mathbf{Z}_t^{d1\prime}\mathbf{Z}_t^{d1}/N)(\mathbf{Z}_s^{d1\prime}\mathbf{Z}_s^{d1}/N)} = \frac{\left[E(z_{it}^{d1}z_{is}^{d1}) + O_p\left(1/\sqrt{N}\right)\right]^2}{E[(z_{it}^{d1})^2]E[(z_{is}^{d1})^2] + O_p\left(1/\sqrt{N}\right)} \\ &= \frac{\alpha^{2(t-s-1)}(1-\alpha)^2}{4} + O_p\left(\frac{1}{\sqrt{N}}\right) \end{aligned}$$

and

$$\mathbf{d}'_t\mathbf{d}_s = \frac{1}{N}\frac{\left(N^{-1}\sum_{i=1}^{N}(z_{it}^{d1}z_{is}^{d1})^2\right)}{(\mathbf{Z}_t^{d1\prime}\mathbf{Z}_t^{d1}/N)(\mathbf{Z}_s^{d1\prime}\mathbf{Z}_s^{d1}/N)} = \frac{1}{N}\frac{E[(z_{it}^{d1}z_{is}^{d1})^2] + O_p\left(1/\sqrt{N}\right)}{E[(z_{it}^{d1})^2]E[(z_{is}^{d1})^2] + O_p\left(1/\sqrt{N}\right)} = O_p\left(\frac{1}{N}\right).$$

Thus, we get

$$var\left(\frac{1}{\sqrt{NT}}\sum_{t=2}^{T}\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{u}_t\right)\underset{N,T\to\infty}{\longrightarrow}\frac{\sigma_v^4}{2(1+\alpha)}.$$

(d), (e): We decompose as follows:

$$\mathbf{x}_t'\mathbf{M}_t^d\mathbf{x}_t=\left(\frac{2}{1-\alpha}\right)\Xi_{3t}^d+\left(\frac{1}{1-\alpha}\right)\Xi_{9t}^d+\Xi_{10t}^d$$

where Ξ_{3t}^d and Ξ_{9t}^d are defined in (C.13) and $\Xi_{10t}^d=\mathbf{w}_{t-1}'\mathbf{M}_t^d\mathbf{w}_{t-1}$. The result is obtained from $E(\Xi_{3t}^d)=0$, $E(\Xi_{9t}^d)=\sigma_\eta^2\,\mathrm{tr}(\mathbf{M}_t^d)/(1-\alpha)$ and Lemma A2.

■

Proof of Theorem 1

(a), (b), (c): (a) is proved in Alvarez and Arellano (2003). (b) and (c) can be proved by noting that $(NT)^{-1}\sum_t\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^d\widehat{\mathbf{u}}_t^+\underset{N,T\to\infty}{\overset{p}{\longrightarrow}}0$ while $(NT)^{-1}\sum_t\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^d\widehat{\mathbf{x}}_t^+$ is bounded in probability as $N,T\to\infty$ (see Lemma A6).
(d): See Alvarez and Arellano (2003).
(e): Using (C.33), (C.35) and Lemma A6, (C.38) can be rewritten as

$$\begin{aligned}\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{u}}_t^+-\frac{\mu_{L2}}{\sqrt{NT_1}}&=\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{w}_{t-1}'\mathbf{M}_t^{d2}\mathbf{v}_t^*+o_p(1)\\&=\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{w}_{t-1}'\mathbf{v}_t^*-\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{w}_{t-1}'(\mathbf{I}_N-\mathbf{M}_t^{d2})\mathbf{v}_t^*+o_p(1).\end{aligned}$$

Since the second term is shown to be $o_p(1)$, using the same argument as that of Alvarez and Arellano (2003), we obtain

$$\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{u}}_t^+-\frac{\mu_{L2}}{\sqrt{NT_1}}\underset{N,T\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}\left(0,\frac{\sigma_v^4}{1-\alpha^2}\right).$$

Hence, using Lemma A6, it follows that

$$\left(\frac{1}{NT_1}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^+\right)^{-1}\left[\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{u}}_t^+-\frac{\mu_{L2}}{\sqrt{NT_1}}\right]\underset{N,T\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}\left(0,1-\alpha^2\right)$$

or

$$\sqrt{NT_1}(\widehat{\alpha}_{L2}-\alpha)-\left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^+\right)^{-1}\frac{\mu_{L2}}{\sqrt{NT_1}}\underset{N,T\to\infty}{\longrightarrow}\mathcal{N}\left(0,1-\alpha^2\right).$$

The result is obtained by approximating the second term as follows:

$$\left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^+\right)^{-1}\frac{\mu_{L2}}{\sqrt{NT_1}}=-\sqrt{\frac{T}{N}}(1+\alpha)\frac{\alpha}{r+1}+O_p\left(\frac{\log T}{\sqrt{NT}}\right)+O_p\left(\frac{\log T}{N}\right)+O_p\left(\frac{\sqrt{T}}{N}\right).$$

where Lemma A6(a) and (e) are used.
(f): Using (C.34), (C.36), Lemma A6 and (C.38), we have the following convergence result:

$$\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{u}}_t^+-\frac{\mu_{L1}}{\sqrt{NT_1}}=\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{v}_t^*+o_p(1)$$

$$
\begin{aligned}
&= \frac{E(w_{i,t-1}z_{it}^{d1})}{E[(z_{it}^{d1})^2]}\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\sum_{i=1}^{N} z_{it}^{d1}v_{it}^{*} + o_p(1) \\
&\xrightarrow[N,T\to\infty]{d} \mathcal{N}\left(0, \frac{\sigma_v^4}{2(1+\alpha)}\right).
\end{aligned}
$$

The first equality comes from

$$
\begin{aligned}
\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T} h_{1t}\Xi_{1t}^{d1} &= \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\left[1+O\left(\frac{1}{T-t}\right)\right]\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{v}_t^{*} + o_p(1) \\
&= \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{w}_{t-1}'\mathbf{M}_t^{d1}\mathbf{v}_t^{*} + o_p(1).
\end{aligned}
$$

Thus, using Lemma A6(f), we obtain

$$
\left(\frac{1}{NT_1}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{x}}_t^{+}\right)^{-1}\left[\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{u}}_t^{+} - \frac{\mu_{L1}}{\sqrt{NT_1}}\right] \xrightarrow[N,T\to\infty]{d} \mathcal{N}\left(0, 2(1+\alpha)\right)
$$

or

$$
\sqrt{NT_1}(\widehat{\alpha}_{L1}-\alpha) - \left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{x}}_t^{+}\right)^{-1}\frac{\mu_{L1}}{\sqrt{NT_1}} \xrightarrow[N,T\to\infty]{} \mathcal{N}\left(0, 2(1+\alpha)\right).
$$

The result is obtained by noting that the second term is $o_p(1)$ as follows:

$$
\left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{x}}_t^{+}\right)^{-1}\frac{\mu_{L1}}{\sqrt{NT_1}} = O_p\left(\frac{\log T}{\sqrt{NT}}\right) \xrightarrow[N,T\to\infty]{} 0
$$

where Lemma A6 (b) and (f) are used.

Proof of Theorem 2

(a), (b): Using the Lemma 2 in Alvarez and Arellano (2003) and Lemma A6, consistency is obtained by noting that $(NT)^{-1}\sum_t(\mathbf{x}_t^{*\prime}\mathbf{M}_t^{l}\mathbf{v}_t^{*} + \widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d}\widehat{\mathbf{u}}_t^{+}) \xrightarrow[N,T\to\infty]{p} 0$ while $(NT)^{-1}\sum_t(\mathbf{x}_t^{*\prime}\mathbf{M}_t^{l}\mathbf{x}_t^{*} + \widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d}\widehat{\mathbf{x}}_t^{+})$ is bounded in probability as $N,T\to\infty$.
(c): From the proof of Theorem 1, we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{NT_1}}\sum_{t=1}^{T-1}\left(\mathbf{x}_t^{*\prime}\mathbf{M}_t^{l}\mathbf{v}_t^{*} + \widehat{\mathbf{x}}_{t+1}^{+\prime}\mathbf{M}_{t+1}^{d2}\widehat{\mathbf{u}}_{t+1}^{+}\right) - \frac{\mu_{F2}+\mu_{L2}}{\sqrt{NT_1}} &= \frac{2}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\mathbf{w}_{t-1}'\mathbf{v}_t^{*} + o_p(1) \\
&\xrightarrow[N,T\to\infty]{d} \mathcal{N}\left(0, \frac{4\sigma_v^4}{1-\alpha^2}\right).
\end{aligned}
$$

Also, using Lemma A6 and Lemma C1 of Alvarez and Arellano (2003), we obtain

$$
\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^{l}\mathbf{x}_t^{*} + \sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^{+}\right) \xrightarrow[N,T\to\infty]{p} \frac{2\sigma_v^2}{1-\alpha^2}.
$$

Therefore,

$$
\sqrt{NT_1}\left(\widehat{\alpha}_{S2}-\alpha\right) - \frac{\frac{1}{\sqrt{NT_1}}(\mu_{F2}+\mu_{L2})}{\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^{l}\mathbf{x}_t^{*} + \sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^{+}\right)} \xrightarrow[N,T\to\infty]{d} \mathcal{N}\left(0, 1-\alpha^2\right).
$$

Consequently, the result follows from

$$\frac{\frac{1}{\sqrt{NT_1}}(\mu_{F2}+\mu_{L2})}{\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^+\right)} = \left(\frac{1-\alpha^2}{2\sigma_v^2}\right)\left(\frac{\mu_{F2}+\mu_{L2}}{\sqrt{NT_1}}\right)+o_p(1)$$

$$= -\sqrt{\frac{T}{N}}\left(\frac{1+\alpha}{2}\right)\left(\frac{r+1+\alpha}{r+1}\right)+o_p(1)$$

where Lemma A6(a) and Lemma 2 of Alvarez and Arellano (2003) are used.
(d): We have the following convergence result:

$$\frac{1}{\sqrt{NT_1}}\sum_{t=1}^{T-1}\left(\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{v}_t^*+\widehat{\mathbf{x}}_{t+1}^{+\prime}\mathbf{M}_{t+1}^{d1}\widehat{\mathbf{u}}_{t+1}^+\right)-\frac{\mu_{F2}+\mu_{L1}}{\sqrt{NT_1}}$$

$$= \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\left(\mathbf{w}_{t-1}+\frac{E(w_{i,t-1}\mathbf{z}_{it}^{d2})}{E[(\mathbf{z}_{it}^{d2})^2]}\mathbf{z}_t^{d2}\right)^{\prime}\mathbf{v}_t^*+o_p(1)\xrightarrow[N,T\to\infty]{d}\mathcal{N}\left(0,\frac{\sigma_v^4}{1-\alpha^2}\left(\frac{5-3\alpha}{2}\right)\right).$$

Also, using Lemma A6(f) and Lemma C1 of Alvarez and Arellano (2003), we have

$$\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{x}}_t^+\right)\xrightarrow[N,T\to\infty]{p}\left(\frac{\sigma_v^2}{1-\alpha^2}\right)\left(\frac{3-\alpha}{2}\right).$$

Therefore,

$$\sqrt{NT_1}\left(\widehat{\alpha}_{S1}-\alpha\right)-\frac{\frac{1}{\sqrt{NT_1}}(\mu_{F2}+\mu_{L1})}{\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d2}\widehat{\mathbf{x}}_t^+\right)}\xrightarrow[N,T\to\infty]{d}\mathcal{N}\left(0,(1-\alpha^2)\frac{2(5-3\alpha)}{(3-\alpha)^2}\right).$$

Then the result follows from

$$\frac{\frac{1}{\sqrt{NT_1}}(\mu_{F2}+\mu_{L1})}{\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\widehat{\mathbf{x}}_t^{+\prime}\mathbf{M}_t^{d1}\widehat{\mathbf{x}}_t^+\right)} = \left(\frac{2(1-\alpha^2)}{\sigma_v^2(3-\alpha)}\right)\left(\frac{\mu_{F2}+\mu_{L1}}{\sqrt{NT_1}}\right)+o_p(1)$$

$$= -\sqrt{\frac{T}{N}}\left(\frac{2(1+\alpha)}{3-\alpha}\right)+O\left(\frac{\log T}{\sqrt{NT}}\right)+o_p(1).$$

■

Proof of Theorem 3

(a): The result is obtained from Lemma A7 and Slutsky theorem.
(b): Consistency is obtained by noting that $(NT)^{-1}\sum_t\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{u}_t\xrightarrow[N,T\to\infty]{p}0$ while $(NT)^{-1}\sum_t\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{x}_t$ is bounded in probability as $N,T\to\infty$(Lemma A7).
(c): Using Lemma A7 and applying the central limit theorem, we have

$$\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{u}_t-\frac{\mu_{L1}^{non}}{\sqrt{NT_1}} = \frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\mathbf{w}_{t-1}^{\prime}\mathbf{M}_t^{d1}\mathbf{v}_t+o_p(1)$$

$$= \frac{E(w_{i,t-1}z_{it}^{d1})}{E[(z_{it}^{d1})^2]}\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\sum_{i=1}^{N}z_{it}^{d1}v_{it}+o_p(1)\xrightarrow[N,T\to\infty]{}\mathcal{N}\left(0,\frac{\sigma_v^4}{2(1+\alpha)}\right).$$

Then, we obtain

$$\left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{x}_t\right)^{-1}\left[\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T}\mathbf{x}_t^{\prime}\mathbf{M}_t^{d1}\mathbf{u}_t-\frac{\mu_{L1}^{non}}{\sqrt{NT_1}}\right]\xrightarrow[N,T\to\infty]{}\mathcal{N}\left(0,2(1+\alpha)\right)$$

or

$$\sqrt{NT_1}(\widehat{\alpha}_{L1}^{non}-\alpha)-\left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{x}_t\right)^{-1}\frac{\mu_{L1}^{non}}{\sqrt{NT_1}}\underset{N,T\to\infty}{\longrightarrow}\mathcal{N}\left(0,2(1+\alpha)\right).$$

Under the assumption $T/N\to c$, the result is obtained from

$$\left(\frac{1}{NT_1}\sum_{t=2}^{T-1}\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{x}_t\right)^{-1}\frac{\mu_{L1}^{non}}{\sqrt{NT_1}}=\left(\frac{2(1+\alpha)}{\sigma_v^2}\right)\frac{\mu_{L1}^{non}}{\sqrt{NT_1}}+o_p(1)=\sqrt{\frac{T}{N}}\left(\frac{2(1+\alpha)}{1-\alpha}\right)r.$$

where Lemma A7(b) and (e) are used.

■

Proof of Theorem 4

(a): The result is obtained from Lemma 2 in Alvarez and Arellano (2003), Lemma A7 and Slutsky theorem.
(b): Consistency is obtained by noting that $(NT)^{-1}\sum_t(\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{v}_t^*+\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{u}_t)\underset{N,T\to\infty}{\overset{p}{\longrightarrow}}0$ while $(NT)^{-1}\sum_t(\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{x}_t)$ is bounded in probability as $N,T\to\infty$(Lemma 2 in Alvarez and Arellano (2003) and Lemma A7).
(c): We have

$$\frac{1}{\sqrt{NT_1}}\sum_{t=1}^{T-1}\left(\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{v}_t^*+\mathbf{x}_{t+1}'\mathbf{M}_{t+1}^{d1}\mathbf{u}_{t+1}\right)-\frac{\mu_{F2}+\mu_{L1}^{non}}{\sqrt{NT_1}}$$
$$=\frac{1}{\sqrt{NT_1}}\sum_{t=2}^{T-1}\left(\mathbf{w}_{t-1}+\frac{E(w_{i,t-1}\mathbf{z}_{it}^{d2})}{E[(\mathbf{z}_{it}^{d2})^2]}\mathbf{z}_t^{d1}\right)'\mathbf{v}_t+o_p(1)\underset{N,T\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}\left(0,\frac{\sigma_v^4}{1-\alpha^2}\left(\frac{5-3\alpha}{2}\right)\right).$$

Also,

$$\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{x}_t\right)\underset{N,T\to\infty}{\overset{p}{\longrightarrow}}\left(\frac{\sigma_v^2}{1-\alpha^2}\right)\left(\frac{3-\alpha}{2}\right).$$

Therefore

$$\sqrt{NT_1}\left(\widehat{\alpha}_{S1}^{non}-\alpha\right)-\frac{\frac{1}{\sqrt{NT_1}}(\mu_{F2}+\mu_{L1}^{non})}{\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\mathbf{x}_t'\mathbf{M}_t^{d2}\mathbf{x}_t\right)}\underset{N,T\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}\left(0,(1-\alpha^2)\frac{2(5-3\alpha)}{(3-\alpha)^2}\right).$$

Then, the result follows from

$$\frac{\frac{1}{\sqrt{NT_1}}(\mu_{F2}+\mu_{L1}^{non})}{\frac{1}{NT_1}\left(\sum_{t=1}^{T-1}\mathbf{x}_t^{*\prime}\mathbf{M}_t^l\mathbf{x}_t^*+\sum_{t=2}^{T}\mathbf{x}_t'\mathbf{M}_t^{d1}\mathbf{x}_t\right)}=-\sqrt{\frac{T}{N}}\left(\frac{2(1+\alpha)}{3-\alpha}\right)(1-r)+o_p(1)$$

where Lemma A7 is used.

■

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Table A1: Mean

estimator	T	N	$\alpha=0.3$ $r=0.2$	$\alpha=0.3$ $r=1$	$\alpha=0.3$ $r=5$	$\alpha=0.6$ $r=0.2$	$\alpha=0.6$ $r=1$	$\alpha=0.6$ $r=5$	$\alpha=0.9$ $r=0.2$	$\alpha=0.9$ $r=1$	$\alpha=0.9$ $r=5$
$F2$	5	200	0.290	0.283	0.278	0.578	0.562	0.534	0.699	0.490	0.399
$L2^*$	5	200	0.301	0.306	0.319	0.598	0.612	0.655	0.910	0.951	0.976
$L2$	5	200	0.300	0.307	0.332	0.597	0.612	0.677	0.904	0.946	0.979
$L2^{non}$	5	200	0.302	0.314	0.367	0.599	0.621	0.706	0.912	0.960	0.988
$L1^*$	5	200	0.301	0.304	0.307	0.601	0.609	0.626	0.911	0.947	0.956
$L1$	5	200	0.300	0.304	0.316	0.598	0.609	0.651	0.895	0.932	0.970
$L1^{non}$	5	200	0.303	0.313	0.354	0.602	0.622	0.686	0.914	0.961	0.988
$S2^*$	5	200	0.297	0.299	0.305	0.593	0.603	0.622	0.897	0.933	0.959
$S2$	5	200	0.296	0.299	0.312	0.591	0.601	0.644	0.889	0.933	0.973
$S2^{non}$	5	200	0.298	0.309	0.359	0.595	0.615	0.696	0.902	0.952	0.986
$S1^*$	5	200	0.295	0.295	0.294	0.592	0.596	0.593	0.890	0.908	0.898
$S1$	5	200	0.295	0.294	0.299	0.590	0.595	0.616	0.878	0.914	0.956
$S1^{non}$	5	200	0.297	0.305	0.342	0.595	0.613	0.677	0.898	0.946	0.984
SYS^*	5	200	0.300	0.302	0.308	0.595	0.603	0.611	0.886	0.905	0.899
SYS	5	200	0.300	0.303	0.310	0.595	0.603	0.626	0.882	0.914	0.957
$F2$	10	200	0.290	0.289	0.288	0.583	0.575	0.569	0.794	0.724	0.695
$L2^*$	10	200	0.301	0.304	0.308	0.603	0.611	0.622	0.917	0.947	0.965
$L2$	10	200	0.301	0.305	0.318	0.603	0.615	0.661	0.917	0.959	0.988
$L2^{non}$	10	200	0.307	0.338	0.454	0.610	0.651	0.776	0.924	0.966	0.991
$L1^*$	10	200	0.300	0.302	0.303	0.601	0.605	0.605	0.910	0.926	0.926
$L1$	10	200	0.299	0.303	0.308	0.600	0.606	0.625	0.903	0.943	0.980
$L1^{non}$	10	200	0.303	0.318	0.378	0.606	0.633	0.722	0.920	0.961	0.989
$S2^*$	10	200	0.296	0.297	0.299	0.596	0.597	0.599	0.904	0.927	0.940
$S2$	10	200	0.296	0.297	0.301	0.595	0.598	0.617	0.899	0.942	0.983
$S2^{non}$	10	200	0.300	0.321	0.411	0.603	0.636	0.749	0.916	0.960	0.989
$S1^*$	10	200	0.294	0.294	0.294	0.590	0.587	0.581	0.873	0.858	0.832
$S1$	10	200	0.294	0.294	0.295	0.590	0.587	0.588	0.870	0.898	0.959
$S1^{non}$	10	200	0.295	0.303	0.338	0.594	0.610	0.676	0.896	0.943	0.982
SYS^*	10	200	0.298	0.299	0.298	0.598	0.596	0.592	0.883	0.865	0.836
SYS	10	200	0.298	0.299	0.299	0.598	0.597	0.596	0.880	0.899	0.958
$F2$	20	200	0.292	0.291	0.290	0.587	0.584	0.583	0.848	0.831	0.826
$L2^*$	20	200	0.301	0.302	0.302	0.603	0.605	0.607	0.916	0.932	0.942
$L2$	20	200	0.301	0.304	0.313	0.604	0.611	0.639	0.922	0.961	0.990
$L2^{non}$	20	200	0.315	0.368	0.542	0.621	0.686	0.829	0.933	0.971	0.993
$L1^*$	20	200	0.301	0.300	0.300	0.601	0.600	0.600	0.907	0.910	0.909
$L1$	20	200	0.301	0.301	0.306	0.601	0.603	0.615	0.910	0.941	0.981
$L1^{non}$	20	200	0.305	0.317	0.379	0.608	0.635	0.727	0.925	0.962	0.989
$S2^*$	20	200	0.297	0.297	0.296	0.596	0.595	0.595	0.901	0.909	0.914
$S2$	20	200	0.296	0.297	0.299	0.596	0.596	0.603	0.896	0.927	0.979
$S2^{non}$	20	200	0.305	0.338	0.465	0.609	0.657	0.786	0.923	0.965	0.991
$S1^*$	20	200	0.295	0.294	0.293	0.591	0.588	0.587	0.867	0.854	0.848
$S1$	20	200	0.295	0.294	0.294	0.591	0.588	0.590	0.866	0.872	0.932
$S1^{non}$	20	200	0.296	0.301	0.328	0.594	0.604	0.652	0.890	0.929	0.976
SYS^*	20	200	0.295	0.294	0.294	0.592	0.588	0.588	0.868	0.854	0.848
SYS	20	200	0.295	0.294	0.295	0.592	0.589	0.590	0.867	0.872	0.932

Note: $F2$, $L2$, $L1$, $S2$, $S1$, SYS^* and SYS denote $\widehat{\alpha}_{F2}$, $\widehat{\alpha}_{L2}$, $\widehat{\alpha}_{L1}$, $\widehat{\alpha}_{S2}$, $\widehat{\alpha}_{S1}$, $\widehat{\alpha}^*_{SYS}$, and $\widehat{\alpha}_{SYS}$, respectively. $L2^*$, $L1^*$, $S2^*$, $S1^*$ denote $\widehat{\alpha}^*_{L2}$, $\widehat{\alpha}^*_{L1}$, $\widehat{\alpha}^*_{S2}$, $\widehat{\alpha}^*_{S1}$, respectively. $L2^{non}$, $L1^{non}$, $S2^{non}$, $S1^{non}$ denote $\widehat{\alpha}^{non}_{L2}$, $\widehat{\alpha}^{non}_{L1}$, $\widehat{\alpha}^{non}_{S2}$, $\widehat{\alpha}^{non}_{S1}$, respectively. $r=\sigma^2_\eta/\sigma^2_v$.

Table A2: Standard deviation

estimator	T	N	$\alpha = 0.3$, $r = 0.2$	$r = 1$	$r = 5$	$\alpha = 0.6$, $r = 0.2$	$r = 1$	$r = 5$	$\alpha = 0.9$, $r = 0.2$	$r = 1$	$r = 5$
$F2$	5	200	0.069	0.090	0.110	0.100	0.148	0.186	0.306	0.414	0.439
$L2^*$	5	200	0.061	0.077	0.101	0.066	0.086	0.118	0.073	0.067	0.051
$L2$	5	200	0.062	0.079	0.107	0.067	0.090	0.126	0.097	0.097	0.073
$L2^{non}$	5	200	0.062	0.085	0.139	0.067	0.088	0.138	0.072	0.065	0.042
$L1^*$	5	200	0.073	0.091	0.117	0.083	0.113	0.165	0.104	0.131	0.177
$L1$	5	200	0.074	0.091	0.117	0.084	0.113	0.155	0.142	0.182	0.148
$L1^{non}$	5	200	0.068	0.080	0.123	0.076	0.090	0.144	0.090	0.088	0.051
$S2^*$	5	200	0.056	0.070	0.095	0.064	0.084	0.118	0.069	0.068	0.061
$S2$	5	200	0.056	0.072	0.098	0.065	0.088	0.125	0.086	0.097	0.078
$S2^{non}$	5	200	0.055	0.073	0.115	0.063	0.082	0.117	0.067	0.061	0.042
$S1^*$	5	200	0.062	0.078	0.103	0.076	0.105	0.150	0.094	0.117	0.156
$S1$	5	200	0.062	0.079	0.103	0.077	0.106	0.147	0.115	0.131	0.122
$S1^{non}$	5	200	0.060	0.072	0.102	0.072	0.087	0.113	0.082	0.072	0.049
SYS^*	5	200	0.056	0.063	0.072	0.065	0.078	0.108	0.085	0.107	0.149
SYS	5	200	0.056	0.064	0.073	0.065	0.079	0.111	0.093	0.116	0.107
$F2$	10	200	0.033	0.038	0.042	0.041	0.049	0.055	0.086	0.117	0.125
$L2^*$	10	200	0.033	0.037	0.041	0.035	0.041	0.049	0.028	0.026	0.024
$L2$	10	200	0.034	0.038	0.046	0.037	0.045	0.065	0.035	0.029	0.018
$L2^{non}$	10	200	0.034	0.047	0.082	0.036	0.049	0.062	0.027	0.023	0.014
$L1^*$	10	200	0.045	0.049	0.055	0.053	0.063	0.076	0.065	0.085	0.105
$L1$	10	200	0.045	0.050	0.054	0.055	0.063	0.077	0.071	0.065	0.046
$L1^{non}$	10	200	0.041	0.044	0.061	0.046	0.048	0.062	0.046	0.034	0.020
$S2^*$	10	200	0.030	0.034	0.040	0.033	0.038	0.048	0.030	0.030	0.033
$S2$	10	200	0.030	0.035	0.042	0.034	0.041	0.056	0.040	0.040	0.025
$S2^{non}$	10	200	0.030	0.038	0.068	0.033	0.043	0.059	0.027	0.022	0.014
$S1^*$	10	200	0.033	0.039	0.043	0.039	0.047	0.056	0.053	0.074	0.096
$S1$	10	200	0.034	0.039	0.044	0.040	0.048	0.057	0.066	0.082	0.062
$S1^{non}$	10	200	0.033	0.037	0.046	0.038	0.041	0.054	0.043	0.035	0.021
SYS^*	10	200	0.031	0.033	0.036	0.034	0.037	0.046	0.044	0.066	0.092
SYS	10	200	0.032	0.033	0.036	0.034	0.037	0.047	0.051	0.072	0.060
$F2$	20	200	0.020	0.021	0.021	0.021	0.023	0.023	0.030	0.036	0.038
$L2^*$	20	200	0.020	0.020	0.021	0.019	0.021	0.023	0.015	0.017	0.017
$L2$	20	200	0.021	0.022	0.024	0.020	0.024	0.032	0.020	0.019	0.009
$L2^{non}$	20	200	0.022	0.034	0.062	0.021	0.032	0.039	0.014	0.011	0.006
$L1^*$	20	200	0.029	0.030	0.032	0.034	0.038	0.039	0.045	0.056	0.063
$L1$	20	200	0.030	0.030	0.031	0.035	0.038	0.037	0.045	0.043	0.026
$L1^{non}$	20	200	0.027	0.028	0.039	0.030	0.032	0.041	0.029	0.021	0.012
$S2^*$	20	200	0.019	0.020	0.021	0.018	0.021	0.022	0.016	0.019	0.020
$S2$	20	200	0.019	0.020	0.021	0.019	0.022	0.024	0.023	0.028	0.017
$S2^{non}$	20	200	0.019	0.025	0.051	0.018	0.027	0.039	0.015	0.012	0.006
$S1^*$	20	200	0.021	0.022	0.022	0.021	0.024	0.024	0.027	0.032	0.037
$S1$	20	200	0.021	0.022	0.022	0.022	0.024	0.024	0.031	0.040	0.048
$S1^{non}$	20	200	0.020	0.021	0.025	0.021	0.023	0.028	0.023	0.021	0.013
SYS^*	20	200	0.021	0.022	0.022	0.021	0.023	0.024	0.027	0.032	0.036
SYS	20	200	0.021	0.022	0.022	0.022	0.024	0.024	0.031	0.040	0.048

See note to Table A.1.

Table A3: RMSE

			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	5	200	0.069	0.091	0.112	0.102	0.153	0.197	0.366	0.583	0.666
$L2^*$	5	200	0.061	0.078	0.103	0.066	0.087	0.131	0.074	0.085	0.091
$L2$	5	200	0.062	0.080	0.112	0.067	0.091	0.148	0.098	0.108	0.108
$L2^{non}$	5	200	0.062	0.086	0.154	0.067	0.091	0.174	0.073	0.088	0.098
$L1^*$	5	200	0.073	0.091	0.117	0.083	0.113	0.167	0.105	0.140	0.186
$L1$	5	200	0.074	0.091	0.119	0.084	0.113	0.163	0.142	0.184	0.163
$L1^{non}$	5	200	0.068	0.081	0.134	0.076	0.093	0.168	0.091	0.107	0.101
$S2^*$	5	200	0.056	0.070	0.095	0.064	0.084	0.120	0.069	0.075	0.085
$S2$	5	200	0.056	0.072	0.099	0.066	0.088	0.132	0.086	0.102	0.107
$S2^{non}$	5	200	0.055	0.074	0.129	0.063	0.083	0.151	0.067	0.080	0.096
$S1^*$	5	200	0.062	0.078	0.103	0.076	0.105	0.151	0.094	0.117	0.156
$S1$	5	200	0.062	0.079	0.103	0.077	0.106	0.148	0.117	0.132	0.135
$S1^{non}$	5	200	0.060	0.072	0.110	0.072	0.088	0.137	0.082	0.086	0.097
SYS^*	5	200	0.056	0.063	0.072	0.065	0.078	0.109	0.086	0.107	0.149
SYS	5	200	0.056	0.064	0.074	0.065	0.079	0.114	0.095	0.117	0.121
$F2$	10	200	0.034	0.040	0.044	0.045	0.055	0.063	0.137	0.212	0.240
$L2^*$	10	200	0.033	0.037	0.042	0.035	0.042	0.054	0.033	0.053	0.069
$L2$	10	200	0.034	0.039	0.049	0.037	0.047	0.089	0.039	0.066	0.090
$L2^{non}$	10	200	0.035	0.060	0.175	0.038	0.071	0.187	0.036	0.069	0.092
$L1^*$	10	200	0.045	0.049	0.055	0.053	0.063	0.076	0.065	0.089	0.108
$L1$	10	200	0.045	0.050	0.055	0.055	0.063	0.081	0.071	0.078	0.092
$L1^{non}$	10	200	0.041	0.047	0.099	0.046	0.059	0.137	0.050	0.070	0.091
$S2^*$	10	200	0.030	0.035	0.040	0.033	0.038	0.048	0.030	0.040	0.052
$S2$	10	200	0.030	0.035	0.042	0.034	0.041	0.059	0.040	0.058	0.086
$S2^{non}$	10	200	0.030	0.043	0.130	0.033	0.056	0.160	0.032	0.064	0.090
$S1^*$	10	200	0.034	0.039	0.044	0.041	0.049	0.059	0.059	0.086	0.118
$S1$	10	200	0.034	0.039	0.044	0.042	0.050	0.058	0.073	0.082	0.085
$S1^{non}$	10	200	0.033	0.037	0.060	0.038	0.042	0.093	0.043	0.055	0.085
SYS^*	10	200	0.032	0.033	0.036	0.034	0.037	0.047	0.047	0.075	0.112
SYS	10	200	0.032	0.033	0.036	0.035	0.037	0.047	0.054	0.072	0.083
$F2$	20	200	0.022	0.023	0.023	0.025	0.028	0.029	0.060	0.078	0.084
$L2^*$	20	200	0.020	0.020	0.021	0.019	0.022	0.024	0.022	0.036	0.045
$L2$	20	200	0.021	0.022	0.027	0.020	0.026	0.050	0.030	0.064	0.091
$L2^{non}$	20	200	0.026	0.076	0.250	0.030	0.092	0.233	0.036	0.072	0.093
$L1^*$	20	200	0.029	0.030	0.032	0.034	0.038	0.039	0.045	0.057	0.063
$L1$	20	200	0.030	0.030	0.032	0.035	0.038	0.040	0.046	0.060	0.085
$L1^{non}$	20	200	0.027	0.033	0.088	0.031	0.047	0.133	0.038	0.065	0.090
$S2^*$	20	200	0.019	0.020	0.021	0.018	0.021	0.023	0.016	0.021	0.025
$S2$	20	200	0.019	0.021	0.021	0.019	0.022	0.024	0.023	0.039	0.081
$S2^{non}$	20	200	0.019	0.045	0.172	0.020	0.063	0.190	0.027	0.066	0.091
$S1^*$	20	200	0.021	0.022	0.023	0.023	0.027	0.027	0.042	0.057	0.064
$S1$	20	200	0.021	0.023	0.023	0.023	0.027	0.027	0.046	0.049	0.057
$S1^{non}$	20	200	0.021	0.021	0.037	0.022	0.023	0.059	0.025	0.036	0.077
SYS^*	20	200	0.021	0.022	0.023	0.023	0.026	0.027	0.042	0.056	0.064
SYS	20	200	0.021	0.022	0.023	0.023	0.026	0.026	0.045	0.049	0.057

See note to Table A.1.

Table A4: Empirical size (standard errors obtained under large N and fixed T asymptotics)

			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	5	200	0.052	0.053	0.055	0.067	0.070	0.077	0.151	0.217	0.222
$L2^*$	5	200	0.050	0.047	0.064	0.050	0.055	0.082	0.047	0.168	0.293
$L2$	5	200	0.055	0.069	0.116	0.058	0.099	0.252	0.077	0.335	0.726
$L2^{non}$	5	200	0.058	0.073	0.173	0.054	0.107	0.328	0.084	0.363	0.754
$L1^*$	5	200	0.049	0.049	0.047	0.048	0.043	0.042	0.035	0.060	0.085
$L1$	5	200	0.052	0.054	0.069	0.054	0.077	0.149	0.054	0.220	0.580
$L1^{non}$	5	200	0.057	0.060	0.139	0.060	0.082	0.273	0.058	0.237	0.669
$S2^*$	5	200	0.054	0.057	0.055	0.060	0.066	0.083	0.062	0.265	0.573
$S2$	5	200	0.060	0.065	0.092	0.072	0.101	0.199	0.087	0.326	0.715
$S2^{non}$	5	200	0.053	0.067	0.148	0.060	0.092	0.296	0.068	0.341	0.749
$S1^*$	5	200	0.056	0.059	0.054	0.061	0.061	0.065	0.039	0.067	0.167
$S1$	5	200	0.058	0.063	0.067	0.068	0.089	0.130	0.090	0.215	0.566
$S1^{non}$	5	200	0.058	0.058	0.111	0.059	0.068	0.227	0.047	0.210	0.655
SYS^*	5	200	0.079	0.091	0.125	0.096	0.132	0.279	0.174	0.386	0.704
SYS	5	200	0.078	0.093	0.132	0.097	0.140	0.298	0.199	0.434	0.774
$F2$	10	200	0.056	0.062	0.071	0.081	0.081	0.094	0.258	0.407	0.460
$L2^*$	10	200	0.052	0.054	0.065	0.061	0.052	0.090	0.108	0.344	0.541
$L2$	10	200	0.060	0.064	0.116	0.084	0.115	0.356	0.202	0.691	0.965
$L2^{non}$	10	200	0.065	0.143	0.543	0.081	0.261	0.772	0.255	0.819	0.988
$L1^*$	10	200	0.055	0.047	0.056	0.049	0.041	0.054	0.048	0.053	0.048
$L1$	10	200	0.062	0.054	0.061	0.060	0.056	0.104	0.093	0.337	0.764
$L1^{non}$	10	200	0.055	0.076	0.315	0.057	0.128	0.564	0.110	0.523	0.950
$S2^*$	10	200	0.055	0.052	0.057	0.059	0.047	0.063	0.082	0.301	0.489
$S2$	10	200	0.061	0.060	0.071	0.075	0.070	0.148	0.161	0.579	0.935
$S2^{non}$	10	200	0.051	0.098	0.453	0.069	0.180	0.704	0.179	0.776	0.985
$S1^*$	10	200	0.057	0.057	0.058	0.064	0.056	0.064	0.077	0.076	0.082
$S1$	10	200	0.061	0.062	0.063	0.066	0.057	0.075	0.145	0.289	0.681
$S1^{non}$	10	200	0.054	0.057	0.174	0.059	0.059	0.373	0.053	0.355	0.918
SYS^*	10	200	0.156	0.169	0.240	0.186	0.205	0.405	0.350	0.675	0.872
SYS	10	200	0.159	0.173	0.245	0.191	0.214	0.402	0.417	0.733	0.942
$F2$	20	200	0.071	0.068	0.069	0.096	0.114	0.115	0.428	0.545	0.562
$L2^*$	20	200	0.059	0.048	0.050	0.046	0.057	0.063	0.192	0.437	0.630
$L2$	20	200	0.079	0.080	0.133	0.073	0.136	0.472	0.381	0.895	1.000
$L2^{non}$	20	200	0.118	0.589	0.985	0.197	0.772	0.997	0.654	0.994	1.000
$L1^*$	20	200	0.052	0.050	0.051	0.049	0.038	0.052	0.049	0.048	0.046
$L1$	20	200	0.066	0.056	0.064	0.066	0.050	0.075	0.064	0.255	0.815
$L1^{non}$	20	200	0.055	0.098	0.602	0.062	0.213	0.879	0.183	0.798	1.000
$S2^*$	20	200	0.056	0.052	0.051	0.054	0.061	0.060	0.067	0.132	0.189
$S2$	20	200	0.068	0.060	0.062	0.072	0.080	0.092	0.192	0.539	0.974
$S2^{non}$	20	200	0.070	0.374	0.951	0.082	0.627	0.992	0.450	0.985	1.000
$S1^*$	20	200	0.064	0.061	0.056	0.071	0.081	0.077	0.213	0.260	0.265
$S1$	20	200	0.068	0.064	0.061	0.078	0.088	0.071	0.243	0.230	0.509
$S1^{non}$	20	200	0.060	0.054	0.244	0.060	0.057	0.543	0.068	0.352	0.988
SYS^*	20	200	0.752	0.764	0.799	0.804	0.823	0.865	0.922	0.968	0.988
SYS	20	200	0.748	0.768	0.796	0.801	0.828	0.864	0.926	0.944	0.985

See note to Table A.1.

Table A5: Empirical size (standard errors obtained under large N and large T asymptotics)

			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	5	200	0.268	0.399	0.481	0.525	0.673	0.732	0.783	0.865	0.894
$L2^*$	5	200	0.213	0.340	0.453	0.327	0.474	0.663	0.551	0.599	0.680
$L2$	5	200	0.219	0.344	0.503	0.333	0.486	0.735	0.577	0.606	0.629
$L1^*$	5	200	0.078	0.154	0.262	0.087	0.201	0.380	0.105	0.171	0.210
$L1$	5	200	0.083	0.160	0.268	0.089	0.196	0.407	0.133	0.124	0.089
$L1^{non}$	5	200	0.060	0.112	0.334	0.058	0.110	0.386	0.071	0.073	0.045
$S2^*$	5	200	0.166	0.282	0.406	0.313	0.449	0.626	0.554	0.633	0.737
$S2$	5	200	0.170	0.289	0.436	0.322	0.465	0.689	0.591	0.626	0.646
$S1^*$	5	200	0.188	0.292	0.428	0.381	0.518	0.655	0.567	0.539	0.605
$S1$	5	200	0.198	0.304	0.432	0.392	0.532	0.680	0.624	0.622	0.598
$S1^{non}$	5	200	0.173	0.265	0.461	0.348	0.455	0.692	0.552	0.570	0.605
$F2$	10	200	0.169	0.235	0.281	0.355	0.468	0.520	0.883	0.946	0.961
$L2^*$	10	200	0.157	0.208	0.273	0.270	0.373	0.483	0.592	0.875	0.937
$L2$	10	200	0.163	0.227	0.347	0.306	0.426	0.693	0.634	0.885	0.790
$L1^*$	10	200	0.079	0.106	0.150	0.095	0.162	0.255	0.142	0.272	0.316
$L1$	10	200	0.083	0.110	0.147	0.105	0.163	0.281	0.123	0.172	0.315
$L1^{non}$	10	200	0.061	0.086	0.485	0.051	0.132	0.717	0.046	0.131	0.299
$S2^*$	10	200	0.118	0.179	0.242	0.240	0.308	0.404	0.503	0.720	0.839
$S2$	10	200	0.125	0.188	0.263	0.258	0.339	0.501	0.600	0.839	0.821
$S1^*$	10	200	0.148	0.209	0.261	0.308	0.381	0.479	0.705	0.765	0.814
$S1$	10	200	0.150	0.209	0.264	0.311	0.384	0.474	0.748	0.786	0.752
$S1^{non}$	10	200	0.138	0.176	0.440	0.271	0.333	0.752	0.610	0.765	0.828
$F2$	20	200	0.145	0.161	0.167	0.280	0.363	0.356	0.904	0.948	0.955
$L2^*$	20	200	0.106	0.127	0.139	0.169	0.237	0.282	0.627	0.882	0.941
$L2$	20	200	0.134	0.158	0.258	0.214	0.337	0.671	0.743	0.977	0.903
$L1^*$	20	200	0.068	0.078	0.097	0.092	0.118	0.135	0.159	0.251	0.306
$L1$	20	200	0.077	0.087	0.094	0.101	0.122	0.140	0.152	0.363	0.806
$L1^{non}$	20	200	0.052	0.101	0.732	0.057	0.213	0.949	0.084	0.452	0.981
$S2^*$	20	200	0.097	0.120	0.132	0.146	0.211	0.251	0.403	0.534	0.645
$S2$	20	200	0.108	0.126	0.139	0.172	0.243	0.286	0.548	0.807	0.971
$S1^*$	20	200	0.112	0.137	0.149	0.227	0.299	0.317	0.778	0.847	0.864
$S1$	20	200	0.118	0.138	0.140	0.234	0.302	0.296	0.769	0.783	0.818
$S1^{non}$	20	200	0.110	0.118	0.425	0.194	0.227	0.811	0.554	0.807	0.989

See note to Table A.1.

Table A6: Theoretical values of GMM estimators

(a) Approximated μ and asymptotic q

mean			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
Estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	20	200	0.294	0.294	0.294	0.592	0.592	0.592	0.891	0.891	0.891
$L2^{*}$	20	200	0.298	0.299	0.300	0.596	0.598	0.599	0.893	0.896	0.899
$L2^{non}$	20	200	0.312	0.357	0.514	0.615	0.664	0.795	0.915	0.947	0.982
$L1^{*}$	20	200	0.300	0.300	0.300	0.599	0.599	0.599	0.891	0.891	0.891
$L1^{non}$	20	200	0.304	0.319	0.393	0.608	0.640	0.800	0.938	1.090	1.850
$S2^{*}$	20	200	0.304	0.304	0.303	0.606	0.605	0.604	0.908	0.907	0.906
$S2^{non}$	20	200	0.306	0.330	0.427	0.608	0.635	0.729	0.908	0.931	0.969
$S1^{*}$	20	200	0.295	0.295	0.295	0.593	0.593	0.593	0.891	0.891	0.891
$S1^{non}$	20	200	0.296	0.300	0.319	0.595	0.600	0.627	0.893	0.900	0.936

(b) Exact μ and asymptotic q

mean			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
Estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	20	200	0.295	0.295	0.295	0.595	0.595	0.595	0.897	0.897	0.897
$L2^{*}$	20	200	0.300	0.301	0.301	0.599	0.600	0.602	0.900	0.902	0.905
$L2^{non}$	20	200	0.312	0.357	0.514	0.615	0.664	0.795	0.915	0.947	0.982
$L1^{*}$	20	200	0.300	0.300	0.300	0.599	0.600	0.600	0.898	0.900	0.903
$L1^{non}$	20	200	0.304	0.319	0.393	0.608	0.640	0.800	0.938	1.090	1.850
$S2^{*}$	20	200	0.298	0.298	0.298	0.597	0.598	0.598	0.899	0.900	0.901
$S2^{non}$	20	200	0.306	0.330	0.427	0.608	0.635	0.729	0.908	0.931	0.969
$S1^{*}$	20	200	0.296	0.296	0.296	0.596	0.596	0.596	0.897	0.897	0.898
$S1^{non}$	20	200	0.297	0.301	0.320	0.597	0.602	0.629	0.899	0.906	0.943

(c) Approximate μ and simulated q

mean			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
Estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	20	200	0.288	0.287	0.286	0.578	0.573	0.571	0.730	0.670	0.648
$L2^{*}$	20	200	0.297	0.298	0.299	0.591	0.593	0.597	0.861	0.869	0.889
$L2^{non}$	20	200	0.312	0.357	0.514	0.615	0.664	0.795	0.915	0.947	0.982
$L1^{*}$	20	200	0.300	0.300	0.300	0.598	0.598	0.599	0.878	0.880	0.892
$L1^{non}$	20	200	0.306	0.329	0.432	0.613	0.658	0.813	0.942	1.002	1.044
$S2^{*}$	20	200	0.293	0.293	0.293	0.585	0.584	0.585	0.830	0.824	0.833
$S2^{non}$	20	200	0.306	0.330	0.427	0.608	0.635	0.729	0.908	0.931	0.969
$S1^{*}$	20	200	0.291	0.290	0.290	0.582	0.579	0.577	0.765	0.715	0.696
$S1^{non}$	20	200	0.293	0.300	0.337	0.587	0.600	0.669	0.825	0.900	1.004

(d) Exact μ and simulated q

mean			$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.9$		
Estimator	T	N	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$	$r = 0.2$	$r = 1$	$r = 5$
$F2$	20	200	0.291	0.290	0.289	0.586	0.583	0.581	0.849	0.832	0.825
$L2^{*}$	20	200	0.300	0.301	0.303	0.598	0.601	0.607	0.899	0.917	0.941
$L2^{non}$	20	200	0.312	0.357	0.514	0.615	0.664	0.795	0.915	0.947	0.982
$L1^{*}$	20	200	0.300	0.300	0.300	0.599	0.599	0.600	0.894	0.901	0.917
$L1^{non}$	20	200	0.306	0.329	0.432	0.613	0.658	0.813	0.942	1.002	1.044
$S2^{*}$	20	200	0.296	0.296	0.296	0.592	0.593	0.595	0.888	0.897	0.914
$S2^{non}$	20	200	0.306	0.330	0.427	0.608	0.635	0.729	0.908	0.931	0.969
$S1^{*}$	20	200	0.293	0.293	0.292	0.588	0.586	0.585	0.860	0.847	0.843
$S1^{non}$	20	200	0.295	0.302	0.339	0.593	0.606	0.675	0.891	0.950	1.022

Note: μ denotes μ_{F2}, μ_{L2}, μ_{L1} or μ_{L1}^{non}. q denotes the probability limit of $\frac{1}{NT}\sum_t \mathbf{x}_t^{*\prime}\mathbf{M}_t^d\mathbf{x}_t^*$, $\frac{1}{NT}\sum_t \mathbf{x}_t^{\prime}\mathbf{M}_t^d\mathbf{x}_t$, or $\frac{1}{NT}\sum_t \mathbf{x}_t^{+\prime}\mathbf{M}_t^d\mathbf{x}_t^+$. For the definition of estimators see note to Table A.1.