Supplementary Material on "Testing for changes in Kendall's tau"

Herold Dehling
University of Bochum

Daniel Vogel*
University of Aberdeen

Martin Wendler
University of Greifswald

Dominik Wied
University of Cologne

Abstract

This document contains proofs and further technical results for the article "Testing for changes in Kendall's tau". It consists of one section, labeled Appendix D. The results are labeled D.1, D.2, ... Results from the main document are referred to as in the main document. The labels contain references to the respective section, e.g., Corollary 3.1 can be found in Section 3, Lemma A.2 in Appendix A of the main document.

D. Proofs of lemmas

We first prove Lemmas A.1 and A.2 from Appendix A.

Proof of Lemma A.1. Part (i) is straightforward.
Part (ii): There are positive constants C_1, C_2 such that

$$a_{p,k}^p = E\left|\boldsymbol{X}_0 - E\left(\boldsymbol{X}_0\middle|\mathscr{F}_{-k}^k\right)\right|_p^p \leq C_1 E\left|\boldsymbol{X}_0 - E\left(\boldsymbol{X}_0\middle|\mathscr{F}_{-k}^k\right)\right|_1 \leq C_2 E\left|\boldsymbol{X}_0 - f_k\left(\boldsymbol{Z}_{-k},\ldots,\boldsymbol{Z}_k\right)\right|_1. \qquad (1)$$

The first inequality is due to the boundedness of $\boldsymbol{X}_0$. The second inequality can be shown by applying Jensen's inequality for the conditional expectation $E(\cdot|\mathscr{F}_{-k}^k)$ to the convex function $|\cdot|_1$. Further, for any $\varepsilon > 0$ we have

$$E\left|\boldsymbol{X}_0 - f_k\left(\boldsymbol{Z}_{-k},\ldots,\boldsymbol{Z}_k\right)\right|_1 \leq C_3 P\left(\left|\boldsymbol{X}_0 - f_k\left(\boldsymbol{Z}_{-k},\ldots,\boldsymbol{Z}_k\right)\right|_1 > \varepsilon\right) + \varepsilon \leq C_3\Phi(\varepsilon)a_k + \varepsilon$$

Combining this with (1) we arrive at $a_{p,k}^p \leq C_2C_3\Phi(\varepsilon)a_k + C_2\varepsilon$. By first choosing ε sufficiently small and then k sufficiently large we can make the left-hand side arbitrarily small, and hence

*Corresponding author: Daniel Vogel, E-Mail: daniel.vogel@abdn.ac.uk.

$(\boldsymbol{X}_n)_{n\in\mathbb{Z}}$ is L_p-NED, $p \geq 1$, on $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$. In particular, if condition (13) holds, we get $a_{p,k}^p \leq C_2C_3\Phi(s_k)a_k + C_2s_k = O(s_k)$. *Part (iii):* Let $f_k(\boldsymbol{Z}_{-k},\ldots,\boldsymbol{Z}_k) = E(\boldsymbol{X}_0|\mathscr{F}_{-k}^k)$. By means of the Hölder and the Markov inequalities we have for every $\varepsilon > 0$:

$$P\left(|\boldsymbol{X}_0 - f_k\left(\boldsymbol{Z}_{-k},\ldots,\boldsymbol{Z}_k\right)|_1 > \varepsilon\right) \leq q^{p-1}\varepsilon^{-p}\, E\left|\boldsymbol{X}_0 - E(\boldsymbol{X}_0|\mathscr{F}_{-k}^k)\right|_p^p$$

Choosing $\Phi(\varepsilon) = q^{p-1}\varepsilon^{-p}$ and $a_k = a_{p,k}^p$ we have $a_k \to 0$ as $k \to \infty$, and $(\boldsymbol{X}_n)_{n\in\mathbb{Z}}$ is hence P-NED on $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$. □

Proof of Lemma A.2. By the definition of g_1, we have that, for any $\boldsymbol{x},\boldsymbol{x}' \in \mathbb{R}^d$,

$$\left|g_1(\boldsymbol{x}) - g_1(\boldsymbol{x}')\right| = \left|Eg(\boldsymbol{x},\boldsymbol{X}_0) - Eg(\boldsymbol{x}',\boldsymbol{X}_0)\right| \leq \int \left|g(\boldsymbol{x},\boldsymbol{y}) - g(\boldsymbol{x}',\boldsymbol{y})\right| dF(\boldsymbol{y})$$

and consequently, for independent copies $\boldsymbol{X},\boldsymbol{Y}$ of $\boldsymbol{X}_0$,

$$\begin{aligned}
&E\left(\sup_{|\boldsymbol{x}-\boldsymbol{X}|\leq\epsilon} |g_1(\boldsymbol{x}) - g_1(\boldsymbol{X})|\right)^2 \\
&\leq E\left(\sup_{|\boldsymbol{x}-\boldsymbol{X}|\leq\epsilon}\int |g(\boldsymbol{x},\boldsymbol{y}) - g(\boldsymbol{X},\boldsymbol{y})|\, dF(\boldsymbol{y})\right)^2 \leq E\left(\int \sup_{|\boldsymbol{x}-\boldsymbol{X}|\leq\epsilon} |g(\boldsymbol{x},\boldsymbol{y}) - g(\boldsymbol{X},\boldsymbol{y})|\, dF(\boldsymbol{y})\right)^2 \\
&\quad\leq E\left(\sup_{|\boldsymbol{x}-\boldsymbol{X}|\leq\epsilon} |g(\boldsymbol{x},\boldsymbol{Y}) - g(\boldsymbol{X},\boldsymbol{Y})|\right)^2 \leq E\left(\sup_{|\boldsymbol{x}-\boldsymbol{X}|\leq\epsilon,|\boldsymbol{y}-\boldsymbol{Y}|\leq\epsilon} |g(\boldsymbol{x},\boldsymbol{y}) - g(\boldsymbol{X},\boldsymbol{Y})|\right)^2 \leq L\epsilon.
\end{aligned}$$

Recall that the conditional expectation minimizes the L_2-distance, so

$$E\left\{g_1(\boldsymbol{X}_0) - E(g_1(\boldsymbol{X}_0)|\mathscr{F}_{-k}^k)\right\}^2 \leq E\left\{g_1(\boldsymbol{X}_0) - g_1(\boldsymbol{X}_{0,k})\right\}^2,$$

where $\boldsymbol{X}_{0,k} = f_k(\boldsymbol{Z}_{-k},\ldots,\boldsymbol{Z}_k)$. Now we will make use of the P-near epoch dependence and the Hölder inequality and obtain

$$\begin{aligned}
&E\left(g_1(\boldsymbol{X}_0) - g_1(\boldsymbol{X}_{0,k})\right)^2 \\
&\quad= E\left(g_1(\boldsymbol{X}_0) - g_1(\boldsymbol{X}_{0,k})\right)^2 \mathbf{1}_{\{|\boldsymbol{X}_0-\boldsymbol{X}_{0,k}|>s_l\}} + E\left(g_1(\boldsymbol{X}_0) - g_1(\boldsymbol{X}_{0,k})\right)^2 \mathbf{1}_{\{|\boldsymbol{X}_0-\boldsymbol{X}_{0,k}|\leq s_l\}} \\
&\qquad\leq \left(2\left\|g_1^2(\boldsymbol{X}_0)\right\|_{\frac{2+\delta}{2}} + 2\left\|g_1^2(\boldsymbol{X}_{0,k})\right\|_{\frac{2+\delta}{2}}\right)\left(P(|\boldsymbol{X}_0 - \boldsymbol{X}_{0,k}| > s_l)\right)^{\frac{\delta}{2+\delta}} \\
&\qquad\qquad + E\left(g_1(\boldsymbol{X}_0) - g_1(\boldsymbol{X}_{0,k})\right)^2 \mathbf{1}_{\{|\boldsymbol{X}_0-\boldsymbol{X}_{0,k}|\leq s_l\}} \leq C(s_l^{\frac{2+\delta}{\delta}})^{\frac{\delta}{2+\delta}} + Ls_l \leq Cs_l
\end{aligned}$$

and finally

$$\left\|g_1(\boldsymbol{X}_0) - E(g_1(\boldsymbol{X}_0)|\mathscr{F}_{-k}^k)\right\|_2 \leq \left(E\left(g_1(\boldsymbol{X}_0) - g_1(\boldsymbol{X}_{0,k})\right)^2\right)^{1/2} \leq Cs_l^{1/2},$$

which completes the proof. □

The following lemmas D.1 to D.6 are required for the proof of Theorem 2.5 (Invariance principle for the sequential U-process).

Lemma D.1. *Let $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$ be a stationary and absolutely regular stochastic process with mixing coefficients $(\beta_k)_{k\in\mathbb{N}}$. Then there exist processes $(\boldsymbol{Z}_n')_{n\in\mathbb{Z}}$ and $(\boldsymbol{Z}_n'')_{n\in\mathbb{Z}}$, independent of each* *other and both with the same distribution as* $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$ *such that* $P\{(\boldsymbol{Z}'_n)_{n\geq k} = (\boldsymbol{Z}_n)_{n\geq k}\} = 1-\beta_k$ *and* $P\{(\boldsymbol{Z}''_n)_{n\leq 0} = (\boldsymbol{Z}_n)_{n\leq 0}\} = 1-\beta_k$.

This can be proved in a similar way to Lemma 2.5 of Borovkova, Burton, and Dehling (2001).

Lemma D.2. *Let* $(\boldsymbol{X}_n)_{n\in\mathbb{Z}}$ *be* P*-NED of an absolutely regular sequence* $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$ *and* g *be a kernel, such that Assumptions 2.4 and 2.3 hold. Then there is a constant* $C>0$*, such that for* $i,k,l\in\mathbb{N}$, $\epsilon>0$*:*

$$\|g_2(\boldsymbol{X}_i, \boldsymbol{X}_{i+k+2l}) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{i+k+2l,l})\|_2 \leq C(\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}}\Phi^{\frac{\delta}{2+\delta}}(\epsilon) + \beta_k^{\frac{\delta}{2+\delta}}),$$

where $\boldsymbol{X}_{i,l} = f_l(\boldsymbol{Z}_{i-l}, \ldots, \boldsymbol{Z}_{i+l})$.

Proof of Lemma D.2. First note that we can rewrite any $\boldsymbol{X}_i$ such that

$$\boldsymbol{X}_i = f_\infty\left((\boldsymbol{Z}_{i+n})_{n\in\mathbb{Z}}\right).$$

By Lemma D.1 there exist independent copies $(\boldsymbol{Z}'_n)_{n\in\mathbb{Z}}$ and $(\boldsymbol{Z}''_n)_{n\in\mathbb{Z}}$ of $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}}$ satisfying $P\{(\boldsymbol{Z}'_n)_{n\geq i+l+k} = (Z_n)_{n\geq i+l+k}\} = 1-\beta_k$ and $P\{(\boldsymbol{Z}''_n)_{n\leq i+l} = (Z_n)_{n\leq i+l}\} = 1-\beta_k$. We define $\boldsymbol{X}'_i = f_\infty((\boldsymbol{Z}'_{n+i})_{n\in\mathbb{Z}})$ and $\boldsymbol{X}''_i = f_\infty((\boldsymbol{Z}''_{n+i})_{n\in\mathbb{Z}})$. Analogously, $\boldsymbol{X}'_{i,l} = f_l(\boldsymbol{Z}'_{i-l}, \ldots, \boldsymbol{Z}'_{i+l})$ and $\boldsymbol{X}''_{i,l} = f_l(\boldsymbol{Z}'_{i-l}, \ldots, \boldsymbol{Z}'_{i+l})$. Now we can conclude that

$$\begin{aligned}\|g_2(\boldsymbol{X}_i, \boldsymbol{X}_{i+k+2l}) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{i+k+2l,l})\|_2 &\leq \left\|g_2(\boldsymbol{X}_i, \boldsymbol{X}_{i+k+2l}) - g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l})\right\|_2 \\ + \left\|g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l}) - g_2(\boldsymbol{X}''_{i,l}, \boldsymbol{X}'_{i+k+2l,l})\right\|_2 &+ \left\|g_2(\boldsymbol{X}''_{i,l}, \boldsymbol{X}'_{i+k+2l,l}) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{i+k+2l,l})\right\|_2\end{aligned}$$

We will treat the three summands on the right-hand side separately. For the first summand, we use the Hölder inequality to obtain

$$\begin{aligned}&\left\|g_2(\boldsymbol{X}_i, \boldsymbol{X}_{i+k+2l}) - g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l})\right\|_2 \\ &= \left\|\{g_2(\boldsymbol{X}_i, \boldsymbol{X}_{i+k+2l}) - g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l})\}\mathbf{1}_{\{(\boldsymbol{Z}'_n)_{n\geq i+l+k}\neq(\boldsymbol{Z}_n)_{n\geq i+l+k} \text{ or } (\boldsymbol{Z}''_n)_{n\leq i+l}\neq(\boldsymbol{Z}_n)_{n\leq i+l}\}}\right\|_2 \\ &\qquad\qquad \leq \left\|(g_2(\boldsymbol{X}_i, \boldsymbol{X}_{i+k+2l}) - g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l})\right\|_{\frac{2+\delta}{2}} \\ &\times \left(P\{(\boldsymbol{Z}'_n)_{n\geq i+l+k} \neq (\boldsymbol{Z}_n)_{n\geq i+l+k} \text{ or } (\boldsymbol{Z}''_n)_{n\leq i+l} \neq (\boldsymbol{Z}_n)_{n\leq i+l}\}\right)^{\frac{\delta}{2+\delta}} \leq 2M^{2/(2+\delta)}2\beta_k^{\delta/(2+\delta)}\end{aligned}$$

with Assumption 2.3 and Lemma D.1. With the same arguments, the third summand is also bounded by $C\beta_k^{\delta/(2+\delta)}$. For the second summand, we make use of the variation condition and the P-NED property

$$\begin{aligned}&\left\|g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l}) - g_2(\boldsymbol{X}''_{i,l}, \boldsymbol{X}'_{i+k+2l,l})\right\|_2 \\ &\quad \leq \left\|\{g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l}) - g_2(\boldsymbol{X}''_{i,l}, \boldsymbol{X}'_{i+k+2l,l})\}\mathbf{1}_{\{|\boldsymbol{X}''_i - \boldsymbol{X}''_{i,l}|\leq\epsilon, |\boldsymbol{X}'_{i+k+2l} - \boldsymbol{X}'_{i+k+2l,l}|\leq\epsilon\}}\right\|_2 \\ &\quad + \left\|\{(g_2(\boldsymbol{X}''_i, \boldsymbol{X}'_{i+k+2l}) - g_2(\boldsymbol{X}''_{i,l}, \boldsymbol{X}'_{i+k+2l,l})\}\mathbf{1}_{\{|\boldsymbol{X}''_i - \boldsymbol{X}''_{i,l}|>\epsilon \text{ or } |\boldsymbol{X}'_{i+k+2l} - \boldsymbol{X}'_{i+k+2l,l}|>\epsilon\}}\right\|_2 \\ &\leq \sqrt{L\epsilon} + 2M^{2/(2+\delta)}2P\left(\left|\boldsymbol{X}_{i+k+2l} - f_l(\boldsymbol{Z}'_{i+k+l}, \ldots, \boldsymbol{Z}'_{i+k+3l})\right| > \epsilon\right)^{\frac{\delta}{2+\delta}} \leq C(\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}}\Phi^{\frac{\delta}{2+\delta}}(\epsilon)).\end{aligned}$$

The proof is complete. □

Lemma D.3. *Under Assumptions 2.2, 2.3 and 2.4, we have for* $n_1 < n_2 \leq n$*:*

$$\Bigg\| \sum_{\substack{1\leq i<j \\ n_1<j\leq n_2}} \left(g_2(\boldsymbol{X}_i, \boldsymbol{X}_j) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l})\right)\Bigg\|_2 \leq C(n_2 - n_1)n^{1/4}.$$

Proof of Lemma D.3. We set $l = \lfloor n^{1/4} \rfloor$ and $\epsilon = l^{-6}$. If $k < 0$, we set $\beta_k = 1$. With Lemma D.2 and some straightforward calculus, we obtain

$$\left\| \sum_{\substack{1 \le i < j \\ n_1 < j \le n_2}} (g_2(\boldsymbol{X}_i, \boldsymbol{X}_j) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l})) \right\|_2 \le C(n_2 - n_1) \sum_{k=1}^{n} (\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}} \Phi^{\frac{\delta}{2+\delta}}(\epsilon) + \beta_{k-l}^{\frac{\delta}{2+\delta}})$$
$$\le C(n_2 - n_1) \sum_{k=1}^{n} \left(\lfloor n^{1/4} \rfloor^{-3} + a_l^{\frac{\delta}{2+\delta}} \Phi^{\frac{\delta}{2+\delta}}(\lfloor n^{1/4} \rfloor^{-6}) + \beta_{k-\lfloor n^{1/4} \rfloor}^{\frac{\delta}{2+\delta}} \right) \le C(n_2 - n_1) n^{1/4}.$$

□

As we approximate the random variables $\boldsymbol{X}_i$ by $\boldsymbol{X}_{i,l} = f_l(\boldsymbol{Z}_{i-l}, \ldots, \boldsymbol{Z}_{i+l})$, we introduce the Hoeffding decomposition of the kernel g with respect to these approximating random variables. Let $(\tilde{\boldsymbol{Z}}_n)_{n \in \mathbb{Z}}$ be an independent copy of the sequence $(\boldsymbol{Z}_n)_{n \in \mathbb{Z}}$ and $\tilde{\boldsymbol{X}}_{i,l} = f_l(\tilde{\boldsymbol{Z}}_{i-l}, \ldots, \tilde{\boldsymbol{Z}}_{i+l})$. We define

$$U_l = Eg(\boldsymbol{X}_{0,l}, \tilde{\boldsymbol{X}}_{0,l}), \quad g_{1,l}(x) = Eg(\boldsymbol{x}, \tilde{\boldsymbol{X}}_{0,l}) - U_l, \quad g_{2,l}(\boldsymbol{x}, \boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y}) - g_{1,l}(\boldsymbol{x}) - g_{1,l}(\boldsymbol{y}) - U_l.$$

Lemma D.4. *Under Assumptions 2.2, 2.3 and 2.4, we have for $n_1 < n_2 \le n$:*

$$\left\| \sum_{\substack{1 \le i < j \\ n_1 < j \le n_2}} (g_{2,l}(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l}) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l})) \right\|_2 \le C(n_2 - n_1) n^{1/4}.$$

Proof of Lemma D.4. Let $(\tilde{\boldsymbol{Z}}_n)_{n \in \mathbb{Z}}$ be an independent copy of the sequence $(\boldsymbol{Z}_n)_{n \in \mathbb{Z}}$ and $\tilde{\boldsymbol{X}}_i = f_\infty((\tilde{\boldsymbol{Z}}_{n+l})_{l \in \mathbb{Z}})$. Then $g_2(\boldsymbol{x}, \boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y}) - Eg(\boldsymbol{x}, \tilde{\boldsymbol{X}}_j) - Eg(\tilde{\boldsymbol{X}}_i, \boldsymbol{y}) + Eg(\tilde{\boldsymbol{X}}_i, \boldsymbol{X}_j)$ and $g_{2,l}(\boldsymbol{x}, \boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y}) - Eg(\boldsymbol{x}, \tilde{\boldsymbol{X}}_{j,l}) - Eg(\tilde{\boldsymbol{X}}_{i,l}, \boldsymbol{y}) + Eg(\tilde{\boldsymbol{X}}_{i,l}, \boldsymbol{X}_{j,l})$. for every $i, j, l \in \mathbb{N}$. So we can conclude that

$$\|g_{2,l}(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l}) - g_2(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l})\|_2 \le \left\| g(\boldsymbol{X}_{i,l}, \tilde{\boldsymbol{X}}_{j,l}) - g(\boldsymbol{X}_{i,l}, \tilde{\boldsymbol{X}}_j) \right\|_2$$
$$+ \left\| g(\tilde{\boldsymbol{X}}_{i,l}, \boldsymbol{X}_{j,l}) - g(\tilde{\boldsymbol{X}}_i, \boldsymbol{X}_{j,l}) \right\|_2 + \left\| g(\tilde{\boldsymbol{X}}_{i,l}, \boldsymbol{X}_{j,l}) - g(\tilde{\boldsymbol{X}}_i, \boldsymbol{X}_j) \right\|_2$$
$$\le C(\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}} \Phi^{\frac{\delta}{2+\delta}}(\epsilon) + \beta_k^{\frac{\delta}{2+\delta}}),$$

where the bound in the last line can be proved along the lines of Lemma D.2. The assertion of Lemma D.4 then follows analogously to Lemma D.3. □

Lemma D.5. *Under Assumptions 2.2, 2.3 and 2.4, we have for $n_1 < n_2 \le n$*

$$E \left(\sum_{\substack{1 \le i < j \\ n_1 < j \le n_2}} g_{2,l}(\boldsymbol{X}_{i,l}, \boldsymbol{X}_{j,l}) \right)^2 \le C(n_2 - n_1)\, n\, l^2.$$

Proof of Lemma D.5. By Lemma 1 of Yoshihara (1976) we obtain

$$\left| E\{g_{2,l}(\boldsymbol{X}_{i_{(1)},l}, \boldsymbol{X}_{i_{(2)},l}) g_{2,l}(\boldsymbol{X}_{i_{(3)},l}, \boldsymbol{X}_{i_{(4)},l})\} \right| \le C \beta_{m-l}^{\delta/(2+\delta)}$$

with $m = \max\{i_{(2)} - i_{(1)}, i_{(4)} - i_{(3)}\}$, where $i_{(1)} \le i_{(2)} \le i_{(3)} \le i_{(4)}$ are the ordered indices i_1, i_2, i_3, i_4. To simplify the notation, we define $\beta_{m-l} = 1$ for $m - l < 0$. Note that, for given m, we have less than n choices for $i_{(1)}$ and less than $n_2 - n_1$ choices for $i_{(4)}$, which has to be either i_2 or i_4. If $m = i_{(2)} - i_{(1)}$, there are m possibilities for $i_{(3)}$, and if $m = i_{(4)} - i_{(3)}$, there are m possibilities for $i_{(2)}$. We conclude that

$$E\Bigg(\sum_{\substack{1\le i<j\\ n_1<j\le n_2}} g_{2,l}(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l})\Bigg)^2 = \sum_{\substack{1\le i_1<j_1\\ n_1<j\le n_2}}\sum_{\substack{1\le i<j\\ n_1<j\le n_2}} C\beta_{m-l}^{\delta/(2+\delta)} \le C(n_2-n_1)n\sum_{m=1}^{n} m\beta_{m-l}^{\delta/(2+\delta)}$$

$$\le C(n_2-n_1)n\left(\sum_{m=1}^{l} m + l^2\sum_{m=l+1}^{n}(m-l)\beta_{m-l}^{\delta/(2+\delta)}\right) \le C(n_2-n_1)nl^2.$$

The proof is complete. □

Lemma D.6. *Under Assumptions 2.2, 2.3 and 2.4, we have*

(i) $\Big\|\max_{n\le 2^k}\sum_{1\le i<j\le n} g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)\Big\|_2 \le C2^{\frac{5}{4}k}k$ *and*

(ii) $\sum_{1\le i<j\le n} g_2(\boldsymbol{X}_i,\boldsymbol{X}_j) = O\left(n^{\frac{5}{4}}\log^2(n)\right)$ *almost surely.*

Proof of Lemma D.6 (i): We will use Lemma D.3 and Lemma D.5 with $l = l_k = \lfloor 2^{\frac{1}{4}k}\rfloor$ and split the expectation into three parts:

$$\Big\|\max_{n\le 2^k}\Big|\sum_{1\le i<j\le n} g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)\Big|\Big\|_2 \le \Big\|\max_{n\le 2^k}\Big|\sum_{1\le i<j\le n}(g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)-g_2(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l}))\Big|\Big\|_2$$

$$+\Big\|\max_{n\le 2^k}\Big|\sum_{1\le i<j\le n}(g_{2,l}(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l})-g_2(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l}))\Big|\Big\|_2$$

$$+\Big\|\max_{n\le 2^k}\Big|\sum_{1\le i<j\le n} g_{2,l}(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l})\Big|\Big\|_2 = I_k + \mathit{II}_k + \mathit{III}_k.$$

Now by Lemma D.3

$$I_k \le \Big\|\sum_{j=1}^{2^k}\Big|\sum_{1\le i<j}(g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)-g_2(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l}))\Big|\Big\|_2$$

$$\le \sum_{j=1}^{2^k}\Big\|\sum_{1\le i<j}(g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)-g_2(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l}))\Big\|_2 \le C2^{\frac{5}{4}k}.$$

Similarly, we get by Lemma D.4 that $\mathit{II}_k \le C2^{\frac{5}{4}}$. To deal with III_k, we define the random variables $Y_{j,l} = \sum_{1\le i<j} g_{2,l}(\boldsymbol{X}_{i,l},\boldsymbol{X}_{j,l})$ and rewrite III_k as $\big\|\max_{n\le 2^k}|\sum_{j=1}^{n} Y_{j,l}|\big\|_2$. As we have $E(\sum_{j=n_1}^{n_2} Y_{j,l})^2 \le C(n_2-n_1)n^{3/2}$ by Lemma D.5, we can use Theorem 1 of Móricz (1976) to obtain $E(\max_{n\le 2^k}\sum_{j=1}^{n} Y_{j,l})^2 \le C2^{5k/2}k^2$, which completes the proof of part (1).
Part (ii): It suffices to prove that

$$\max_{n\le 2^k}\sum_{1\le i<j\le n} g_2(\boldsymbol{X}_i,\boldsymbol{X}_j) = O\left(2^{\frac{5}{4}k}k^2\right)$$

almost surely. By means of the Chebychev inequality, we have for any $\epsilon > 0$

$$\sum_{k=1}^{\infty} P\left(\max_{n\leq 2^k} \sum_{1\leq i<j\leq n} g_2(\boldsymbol{X}_i,\boldsymbol{X}_j) > \epsilon 2^{5k/4}k^2\right)$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{5k/2}k^4} E\left(\max_{n\leq 2^k} \sum_{1\leq i<j\leq n} g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)\right)^2 \leq C\sum_{k=1}^{\infty}\frac{2^{5k/2}k^2}{2^{5k/2}k^4} = C\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty,$$

and the almost sure convergence follows by the Borel–Cantelli lemma. □

Towards the proof of Theorem 2.7, we further state and prove Lemmas D.7 to D.9.

Lemma D.7. *Under Assumptions 2.2, 2.3 and 2.4, we have for any constants* $(c_i)_{i\in\mathbb{N}}$

$$E\left(\sum_{i=1}^{n} g_1(\boldsymbol{X}_i)c_i\right)^2 \leq Cn\left(\max_{i=1,\dots,n}|c_i|\right)^2,$$

Proof of Lemma D.7. By Assumption 2.4 and Lemma A.2, the process $(g_1(\boldsymbol{X}_n))_{n\in\mathbb{Z}}$ is L_2-NED with approximation constants $a_{l,2} = O(l^{-3})$. We have the following bound for the autocovariance:

$$|Eg_1(\boldsymbol{X}_i)g_1(\boldsymbol{X}_{i+k})|$$

$$\leq \left|E\{E(g_1(\boldsymbol{X}_i)|\mathscr{F}_{i-l}^{i+l})E(g_1(\boldsymbol{X}_{i+k})|\mathscr{F}_{i+k-l}^{i+k+l})\}\right| + \left|E\{g_1(\boldsymbol{X}_i)\left(g_1(\boldsymbol{X}_{i+k}) - E(g_1(\boldsymbol{X}_{i+k})|\mathscr{F}_{i+k-l}^{i+k+l})\right)\}\right|$$

$$+\left|E\left(E(g_1(\boldsymbol{X}_{i+k})|\mathscr{F}_{i+k-l}^{i+k+l})\left(g_1(\boldsymbol{X}_i) - E(g_1(\boldsymbol{X}_i)|\mathscr{F}_{i-l}^{i+l})\right)\right)\right|$$

$$\leq 10\left\|E(g_1(\boldsymbol{X}_{i+k})|\mathscr{F}_{i+k-l}^{i+k+l})\right\|_{2+\delta}^2 \beta_{k-2l}^{\frac{\delta}{2+\delta}} + 2\left\|g_1(\boldsymbol{X}_i)\right\|_2\left\|g_1(\boldsymbol{X}_{i+k}) - E(g_1(\boldsymbol{X}_{i+k})|\mathscr{F}_{i+k-l}^{i+k+l})\right\|_2$$

$$\leq C(a_{l,2} + \beta_l^{\frac{\delta}{2+\delta}}),$$

where we used the inequality by Davydov (1970) and set $l = \lfloor \frac{k}{3}\rfloor$. Recall that $\mathscr{F}_i^j$ is defined as the σ-algebra generated by $\boldsymbol{Z}_i,\dots,\boldsymbol{Z}_j$. Now it follows from the stationarity of the process that

$$E\left(\sum_{i=1}^{n} g_1(\boldsymbol{X}_i)c_i\right)^2 \leq \sum_{i,j=1}^{n}|Eg_1(\boldsymbol{X}_i)g_1(\boldsymbol{X}_j)|\,|c_i|\,|c_j|$$

$$\leq \left(\max_{i=1,\dots,n}|c_i|\right)^2 \sum_{i,j=1}^{n}|Eg_1(\boldsymbol{X}_i)g_1(\boldsymbol{X}_j)| \leq C\left(\max_{i=1,\dots,n}|c_i|\right)^2 n\sum_{k=0}^{n}|Eg_1(\boldsymbol{X}_0)g_1(\boldsymbol{X}_k)|$$

$$\leq C\left(\max_{i=1,\dots,n}|c_i|\right)^2 n\sum_{l=0}^{\lfloor\frac{n}{3}\rfloor}(a_{l,2} + \beta_l^{\frac{\delta}{2+\delta}}) \leq Cn\left(\max_{i=1,\dots,n}|c_i|\right)^2,$$

and the Lemma is proved. □

Lemma D.8. *Under the Assumptions 2.2, 2.3 and 2.4 for any constants* $(c_{ij})_{i,j\in\mathbb{N}}$

$$\left|E\sum_{j_1,j_2,j_3,j_4=1}^{n} g_2(\boldsymbol{X}_{j_1},\boldsymbol{X}_{j_2})g_2(\boldsymbol{X}_{j_3},\boldsymbol{X}_{j_4})c_{j_1j_3}\right| \leq C\max_{i,j\in\{1,\dots,n\}}|c_{ij}|\,n^{5/2}.$$

Proof of Lemma D.8. Recall that we abbreviate $f_l(\boldsymbol{Z}_{j-l}, \dots, \boldsymbol{Z}_{j+l})$ by $\boldsymbol{X}_{j,l}$. We use the triangle inequality to obtain

$$\left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} g_2(\boldsymbol{X}_{j_1}, \boldsymbol{X}_{j_2}) g_2(\boldsymbol{X}_{j_3}, \boldsymbol{X}_{j_4}) c_{j_1 j_3} \right|$$

$$\leq \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} \left(g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l})\right) \left(g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l})\right) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} \left(g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1}, \boldsymbol{X}_{j_2})\right) \left(g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3}, \boldsymbol{X}_{j_4})\right) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) \left(g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l})\right) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} \left(g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l})\right) g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l})) \left(g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3}, \boldsymbol{X}_{j_4})\right) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} \left(g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1}, \boldsymbol{X}_{j_2})\right) g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} \left(g_{2,l}(\boldsymbol{X}_{j_1 l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l})\right) \left(g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3}, \boldsymbol{X}_{j_4})\right) c_{j_1 j_3} \right|$$

$$+ \left| E \sum_{j_1,j_2,j_3,j_4=1}^{n} \left(g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1}, \boldsymbol{X}_{j_2})\right) \left(g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l})\right) c_{j_1 j_3} \right|$$

$$= I_n + II_n + III_n + IV_n + V_n + VI_n + VII_n + VIII_n + IX_n.$$

We will establish the bound only for some of the summands in order to keep this proof short. The bound for I_n can be shown in the same way as Lemma D.5. For II_n, III_n, $VIII_n$ and IX_n, we use the Hölder inequality and can then proceed similar to the proof of Lemma D.3 and D.4. For example by Lemmas D.2 and D.4, we have

$$VIII_n \leq \sum_{j_1,j_2,j_3,j_4=1}^{n} \left\|g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) - g_2(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l})\right\|_2 \left\|g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3}, \boldsymbol{X}_{j_4})\right\|_2 c_{j_1 j_3}$$

$$\leq C \max_{i,j\in\{1,\dots,n\}} |c_{ij}| \sum_{j_1,j_2,j_3,j_4=1}^{n} (\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}} \Phi^{\frac{\delta}{2+\delta}}(\epsilon) + \beta_{|j_2-j_1|}^{\frac{\delta}{2+\delta}})(\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}} \Phi^{\frac{\delta}{2+\delta}}(\epsilon) + \beta_{|j_4-j_3|}^{\frac{\delta}{2+\delta}})$$

$$\leq C \max_{i,j\in\{1,\dots,n\}} |c_{ij}|\, n^{5/2}$$

with $l = \lfloor n^{1/4} \rfloor$ and $\epsilon = l^{-6}$. For the summand IV_n, the Hölder inequality is used in a slightly different way

$$
\begin{aligned}
\mathit{IV}_n &\le \sum_{j_3,j_4=1}^{n} \Big| E\Big\{ \sum_{j_1,j_2=1}^{n} g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) \left(g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l})\right) c_{j_1 j_3} \Big\} \Big| \\
&\le \sum_{j_3,j_4=1}^{n} \Big\| \sum_{j_1,j_2=1}^{n} g_{2,l}(\boldsymbol{X}_{j_1,l}, \boldsymbol{X}_{j_2,l}) c_{j_1 j_3} \Big\|_2 \big\| g_{2,l}(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) - g_2(\boldsymbol{X}_{j_3,l}, \boldsymbol{X}_{j_4,l}) \big\|_2 \\
&\le C \sum_{j_3,j_4=1}^{n} n^{5/4} \max_{i,j\in\{1,\dots,n\}} |c_{ij}| \left(\sqrt{\epsilon} + a_l^{\frac{\delta}{2+\delta}} \Phi^{\frac{\delta}{2+\delta}}(\epsilon) + \beta_{|j_4-j_3|}^{\frac{\delta}{2+\delta}}\right) \le C \max_{i,j\in\{1,\dots,n\}} |c_{ij}|\, n^{5/2}.
\end{aligned}
$$

A similar treatment of the summands V_n, VI_n and VII_n completes the proof. □

Lemma D.9. *Under Assumptions 2.2, 2.3, 2.4 and 2.6, we have, for $n \to \infty$,*

$$
E\Big| \sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \left(g_1(\boldsymbol{X}_i) g_1(\boldsymbol{X}_{i+|r|}) - \hat{g}_1(\boldsymbol{X}_i) \hat{g}_1(\boldsymbol{X}_{i+|r|})\right) \kappa(|r|/b_n) \Big| \to 0.
$$

Proof of Lemma D.9. We first expand the difference of g_1 and its estimator $\hat{g}_1$ as

$$
g_1(\boldsymbol{x}) - \hat{g}_1(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} g_1(\boldsymbol{X}_i) - \frac{1}{n} \sum_{i=1}^{n} g_2(\boldsymbol{x}, \boldsymbol{X}_i) + \frac{1}{n^2} \sum_{i,j=1}^{n} g_2(\boldsymbol{X}_i, \boldsymbol{X}_j).
$$

With the help of this, we split the expectation into six parts and apply the triangle inequality:

$$
\begin{aligned}
&E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \left(g_1(\boldsymbol{X}_i) g_1(\boldsymbol{X}_{i+|r|}) - \hat{g}_1(\boldsymbol{X}_i) \hat{g}_1(\boldsymbol{X}_{i+|r|})\right) \kappa(|r|/b_n) \Big| \\
&\le E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \left(g_1(\boldsymbol{X}_i) - \hat{g}_1(\boldsymbol{X}_i)\right) g_1(\boldsymbol{X}_{i+|r|}) \kappa(|r|/b_n) \Big| \\
&+E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \left(g_1(\boldsymbol{X}_{i+|r|}) - \hat{g}_1(\boldsymbol{X}_{i+|r|})\right) \hat{g}_1(\boldsymbol{X}_i) \kappa(|r|/b_n) \Big| \\
&\le E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \frac{1}{n} \sum_{j=1}^{n} g_1(\boldsymbol{X}_j) g_1(\boldsymbol{X}_{i+|r|}) \kappa(|r|/b_n) \Big| \\
&+E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \frac{1}{n} \sum_{j=1}^{n} g_2(\boldsymbol{X}_i, \boldsymbol{X}_j) g_1(\boldsymbol{X}_{i+|r|}) \kappa(|r|/b_n) \Big| \\
&+E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} g_2(\boldsymbol{X}_{j_1}, \boldsymbol{X}_{j_2}) g_1(\boldsymbol{X}_{i+|r|}) \kappa(|r|/b_n) \Big| \\
&+E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \frac{1}{n} \sum_{j=1}^{n} g_1(\boldsymbol{X}_j) \hat{g}_1(\boldsymbol{X}_i) \kappa(|r|/b_n) \Big| \\
&+E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \frac{1}{n} \sum_{j=1}^{n} g_2(\boldsymbol{X}_{i+|r|}, \boldsymbol{X}_j) \hat{g}_1(\boldsymbol{X}_i) \kappa(|r|/b_n) \Big| \\
&+E\Big| \textstyle\sum_{r=-n}^{n} \frac{1}{n} \sum_{i=1}^{n-|r|} \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} g_2(\boldsymbol{X}_{j_1}, \boldsymbol{X}_{j_2}) \hat{g}_1(\boldsymbol{X}_i) \kappa(|r|/b_n) \Big| \\
&= I_n + \mathit{II}_n + \mathit{III}_n + \mathit{IV}_n + V_n + \mathit{VI}_n.
\end{aligned}
$$

For the first summand I_n, we use the Hölder inequality and Lemma D.7 with constants $c_i = \sum_{i_2=1}^{n} \kappa(|i-i_2|/b_n) = O(b_n)$ and obtain

$$
I_n = E\Big| \textstyle\frac{1}{n} \sum_{j=1}^{n} g_1(\boldsymbol{X}_j) \Big| \cdot \Big| \frac{1}{n} \sum_{i=1}^{n} g_1(\boldsymbol{X}_i) \sum_{i_2=1}^{n} \kappa(|i-i_2|/b_n) \Big|
$$

$$\leq \Big\|\tfrac{1}{n}\textstyle\sum_{j=1}^n g_1(\boldsymbol{X}_j)\Big\|_2 \Big\|\tfrac{1}{n}\textstyle\sum_{i=1}^n g_1(\boldsymbol{X}_i)\sum_{i_2=1}^n \kappa(|i-i_2|/b_n)\Big\|_2 \leq C\tfrac{1}{\sqrt{n}}\tfrac{1}{\sqrt{n}}b_n \to 0$$

as $n \to \infty$ due to the assumptions on b_n. For $I\!I_n$, we use Lemma D.8 to obtain

$$I\!I_n = E\Big|\tfrac{1}{n}\textstyle\sum_{i_1=1}^n \tfrac{1}{n}\sum_{i,j=1}^n g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)g_1(\boldsymbol{X}_{i_1})\kappa(|i-i_1|/b_n)\Big|$$

$$\leq E\left[\left\{\tfrac{1}{n}\textstyle\sum_{i=1}^n g_1(\boldsymbol{X}_{i_1})^2\right\}^{1/2}\left\{\tfrac{1}{n}\textstyle\sum_{i_1=1}^n \left(\tfrac{1}{n}\sum_{i,j=1}^n g_2(\boldsymbol{X}_i,\boldsymbol{X}_j)\kappa(|i-i_1|/b_n)\right)^2\right\}^{1/2}\right]$$

$$\leq \left[E\left\{\tfrac{1}{n}\textstyle\sum_{i=1}^n g_1(\boldsymbol{X}_{i_1})^2\right\}\right]^{1/2}$$

$$\left[E\left\{\frac{1}{n^3}\sum_{j_1,j_2,j_3,j_4=1}^n g_2(\boldsymbol{X}_{j_1},\boldsymbol{X}_{j_2})g_2(\boldsymbol{X}_{j_3},\boldsymbol{X}_{j_4})\sum_{i_1=1}^n \kappa(|j_1-i_1|/b_n)\kappa(|j_3-i_1|/b_n)\right\}\right]^{1/2}$$

$$\leq C\sqrt{\frac{n^{5/2}b_n}{n^3}} \to 0,$$

since $c_{j_1j_2} = \sum_{i_1=1}^n \kappa(|j_1-i_1|/b_n)\kappa(|j_3-i_1|/b_n) = O(b_n) = o(\sqrt{n})$ and $E\{\frac{1}{n}\sum_{i=1}^n g_1(\boldsymbol{X}_{i_1})^2\}^2 \leq E\{g_1(\boldsymbol{X}_0)^2\}^2 < \infty$ due to Assumption 2.3. For the third summand $I\!I\!I_n$, we use again the Hölder inequality and Lemma D.7 to get

$$I\!I\!I_n = E\Big|\textstyle\sum_{r=-n}^n \tfrac{1}{n}\sum_{i=1}^{n-|r|}\tfrac{1}{n^2}\sum_{j_1,j_2=1}^n g_2(\boldsymbol{X}_{j_1},\boldsymbol{X}_{j_2})g_1(\boldsymbol{X}_{i+|r|})\kappa(|r|/b_n)\Big|$$

$$\leq \left\|\frac{1}{n^2}\sum_{j_1,j_2=1}^n g_2(\boldsymbol{X}_{j_1},\boldsymbol{X}_{j_2})\right\|_2 \left\|\frac{1}{n}\sum_{i=1}^{n-|r|} g_1(\boldsymbol{X}_{i+|r|})\sum_{i_1=1}^n \kappa(|i-i_1|/b_n)\right\|_2 \leq C\frac{1}{n^{3/4}}\frac{1}{\sqrt{n}}b_n \to 0.$$

The convergence of the remaining parts $I\!V_n$, V_n and $V\!I_n$ can be shown in the same way. □

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