Online Supplementary Material for 'Uniform Convergence Rates over Maximal Domains in Structural Nonparametric Cointegrating Regression'

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Proof of Lemma 3.1. Let $g_k(z) := g(z)\mathbf{1}\{|g(z)| \leq k\}$. g_k is bounded, and a straightforward extension of the argument used to verify (9.1) in Duffy (2016) gives that

$$\mathbb{E}f(Y)g_k(Z) = \frac{1}{2\pi}\int \hat{f}(\lambda)\mathbb{E}\Big[\mathrm{e}^{-\mathrm{i}\lambda' Y} g_k(Z)\Big]\,\mathrm{d}\lambda$$

for every $k \in \mathbb{N}$. Now let $k \to \infty$; the left side converges to $\mathbb{E}f(Y)g(Z)$ by dominated convergence. For the right side, using that Y_1 and (Y_2, Z) are independent, we have

$$\begin{aligned}\left|\int \hat{f}(\lambda)\mathbb{E}\Big[\mathrm{e}^{-\mathrm{i}\lambda' Y}\{g_k(Z) - g(Z)\}\Big]\,\mathrm{d}\lambda\right| &\leq \left(\int |\hat{f}(\lambda)\psi_{Y_1}(-\lambda)|\,\mathrm{d}\lambda\right)\mathbb{E}|g_k(Z) - g(Z)| \\ &\leq \|f\|_1\|\psi_{Y_1}\|_1\mathbb{E}|g(Z)|\mathbf{1}\{|g(Z)| > k\} \\ &\to 0\end{aligned}$$

using the fact that $|\hat{f}(\lambda)| \leq \|f\|_1$. $\square$

Proof of Lemma 3.2. We shall give only the proof of (3.6) here; the proof of (3.5) follows by similar arguments, and is somewhat simpler. Recall from (3.3) the decomposition

$$x'_{t+1,t+k,t+k} = a_m\epsilon_{t+k-m} + \sum_{\substack{l=0 \\ l\neq m}}^{k-1} a_l\epsilon_{t+k-l}.$$

Let $\mathcal{K} := \{\lfloor k/2\rfloor + 1, \ldots, k-1\}\backslash\{m\}$. Since the second term on the right is independent of η_{t+k-m},

$$\begin{aligned}|\mathbb{E}\eta_{t+k-m}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| &\leq |\mathbb{E}\eta_{t+k-m}\mathrm{e}^{-\mathrm{i}\lambda a_m\epsilon_{t+k-m}}|\prod_{l\in\mathcal{K}}|\psi(-\lambda a_l)| \\ &\leq [|a_m||\lambda|\mathbb{E}|\eta_0\epsilon_0| \wedge \mathbb{E}|\eta_0|]\prod_{l\in\mathcal{K}}|\psi(-\lambda a_l)|\end{aligned}$$

$$\lesssim (c_m|\lambda| \wedge 1) \prod_{l\in\mathcal{K}} |\psi(-\lambda a_l)|$$

using $\mathbb{E}|\mathrm{e}^{\mathrm{i}x} - 1| \leq |x|$, (3.4) and the Cauchy-Schwarz inequality. Hence

$$|\mathbb{E}\eta_{t+k-m}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}|^q \lesssim (c_m^q|\lambda|^q \wedge 1) \prod_{l\in\mathcal{K}} |\psi(-\lambda a_l)|.$$

Thus the left side of (3.6) may be bounded above by a constant times

$$\int_{\mathbb{R}} (z_1 c_m^q |a_k|^p |\lambda|^{p+q} F(a_k\lambda) \wedge z_2) \prod_{l\in\mathcal{K}} |\psi(-\lambda a_l)| \,\mathrm{d}\lambda.$$

The result now follows by Lemma F.2 in the Supplement to Duffy (2016). □

Proof of Lemma 3.3. (i) follows by arguments analogous to those used to prove Lemma 9.3(i) in Duffy (2016). For (ii), we recall from (3.2) the decomposition

$$x_{t+k} = x^*_{t,t+k} + x'_{t+1,t+k,t+k}.$$

Thence by Fourier inversion (Lemma 3.1) and Lemma 3.2(i),

$$\begin{aligned} |\mathbb{E}_t f(x_{t+k})\eta_{t+k-m}| &= \left| \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\lambda) \mathrm{e}^{-\mathrm{i}\lambda x^*_{t,t+k}} \mathbb{E}[\eta_{t+k-m}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}] \,\mathrm{d}\lambda \right| \\ &\lesssim \|f\|_1 \int_{\mathbb{R}} |\mathbb{E}\eta_{t+k-m}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda, \end{aligned}$$

using the fact that $|\hat{f}(\lambda)| \leq \|f\|_1$. The result now follows by Lemma 3.2(i). □

Proof of Lemma 3.4. For (i), note that $\{d_t^{-2}\}$ is regularly varying with index $-2H$, whence by Karamata's theorem and Proposition 1.5.9a in Bingham, Goldie, and Teugels (1987), $\{\sum_{t=1}^n d_t^{-2}\}$ is either slowly varying (when $H \leq 1/2$), or regularly varying with index $1 - 2H$. In comparison, $\{e_n^{1/2}\}$ is regularly varying with index

$$\frac{1}{2}(1 - H) > 1 - 2H$$

for all $H \in (\frac{1}{3}, 1)$; thus (i) holds. (ii) follows from the fact that $\{k^{-1/2} d_k^{-3/2}\}$ is regularly varying with index

$$-\frac{1}{2} - \frac{3}{2}H < -\frac{1}{2} - \frac{3}{2} \cdot \frac{1}{3} = -1$$

For (iii), note that $\{c_m\}$ and $\{m^{1/2}e_m\}$ are regularly varying with indices $H - 1/\alpha < 1$ and

$$\frac{1}{2} + 1 - H < \frac{3}{2} - \frac{1}{3} = \frac{7}{6}$$

respectively. Thus the result follows from Assumption 1(ii). □

References

Bingham, N. H., C. M. Goldie, and J. L. Teugels (1987): *Regular Variation.* C.U.P., Cambridge (UK).

Duffy, J. A. (2016): "A uniform law for convergence to the local times of linear fractional stable motions," *Annals of Applied Probability*, 25(1), 45–72.