Online Supplementary Material to "Estimating the Quadratic Variation Spectrum of Noisy Asset Prices using Generalized Flat-Top Realized Kernels"

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Abstract

This online supplementary material complements the main text by providing and establishing additional asymptotic results, namely Lemmas B.1-B.7, containing key marginal limit results, and Lemmas C.1-C.10, containing auxiliary technical results and stochastic bounds. Second, it establishes Propositions 1-2, Lemmas 2-3, Theorem A.1 as well as the asymptotic properties of the two-scale realized kernel estimator in the present setting, as stated in Section 3.4, by adapting the methods of Ikeda (2015). Third, it complements the proofs of Theorems 3 and 4 by: (1) Elaborating on the collection of terms for the Taylor expansion, leading to the representation in equation (B.15); and (2) it shows that higher-order Taylor expansion terms are of lower stochastic order than the first and second-order terms. Finally, it details how the endogenous MMS noise component relates to the locally stationary processes in Dahlhaus & Polonik (2009) and Dahlhaus (2009), and it explains how to adapt and use three asymptotic results in the two papers.

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1 Introduction

This online supplementary material complements the main text, Varneskov (2016), by establishing various asymptotic results and technical lemmas, and it provides additional details about the class of locally stationary processes, which is used to model the endogenous market microstructure (MMS) noise component. While each section may be read independently, most of the notation and definitions used are introduced in the main text. Hence, the respective sections below are easier to read once they have been referenced. Note, in particular, that Sections 2 and 3 contain Lemmas B.1-B.7, providing key marginal limit results for the proofs of Theorems 1-4, and Lemmas C.1-C.10, providing additional technical results and stochastic bounds, respectively. Finally, note that there is no overlap between the labelling of equations and lemmas below and the assumptions, asymptotic results as well as equations in the main text, which, as a result, are referenced without explicitly citing Varneskov (2016).

2 Key Lemmas for the Proofs of Theorems 1-4

In the following, let K, k, and ϵ denote generic constants where $K, k \in (0, \infty)$ and $\epsilon \in (0, 1)$, unless specified otherwise, and they may take different values in different places. Before proceeding, a lemma due to the results in Jacod (2009, 6.23) is stated below, and this will be used throughout, sometimes without explicit reference. Definition B.1 introduces notation for multiple summation and change of variables, some of which resembles the corresponding notation in Ikeda (2015). Definition B.2 fixes additional notation for local and average autocovariances and long run variances, similar to that provided in Definitions 1 and 2. All convergence results are for $n \to \infty$.

Lemma B.1 (Jacod (2009), 6.23)**.** *Under Assumptions 1 and 3, then for $i \geq 2$,*

(a) $\mathbb{E}[(\Delta t_i)^{-1/2}|\Delta p^*_{t_i} - \sigma_{t_{i-1}}\Delta W_{t_i}|^s|\mathcal{H}_{t_{i-1}}] \leq K_s n^{-\min(1,s/2)}$.

(b) $\mathbb{E}[(\Delta t_i)^{-1/2}|\int_{t_{i-1}}^{t_i} \Upsilon_t dt - \Upsilon_{t_{i-1}}\Delta t_i|^s|\mathcal{H}_{t_{i-1}}] \leq K_s n^{-\min(1,s/2)}$.

(c) $\mathbb{E}[|\sigma_{t+h} - \sigma_t|^s|\mathcal{H}_t] \leq K_s h^{\min(1,s/2)}$ *and* $\mathbb{E}[|\Upsilon_{t+h} - \Upsilon_t|^s|\mathcal{H}_t] \leq K_s h^{\min(1,s/2)}$.

Definition B.1. *Let $(h, g) \in \mathbb{Z}^2$, and recall four definitions from Section B in the main text:*

$$S^{(2,h)} = \{1 + S_h^+, \ldots, n-1+S_h^-\}, \quad S^{(1,h)} = S^{(2,h)} \setminus \{1\}, \quad \mathbb{Z}_k = \{-k, \ldots, -1, 0, 1, \ldots, k\},$$

for $k \in \mathbb{N}$ as well as $\mathbb{Z}_{k+1}^K = \mathbb{Z}_K \setminus \mathbb{Z}_k$ for $K - k \in \mathbb{N}$. Then, the following notation is used for various change of variables:

- *For $s = i - h \in S^{(2,h)} - h = \{1 - S_h^-, \ldots, n-1-S_h^+\} \equiv S^{(2,-h)}$.*
- *For $s = j - i$ in $\sum_{i\in S^{(2,h)}}\sum_{j\in S^{(2,g)}} = \sum_{s\in S^{(2,g)}-S^{(2,h)}}\sum_{i\in S^{(2,g)}\cap(S^{(2,h)}+s)}$; where*
- *$S^{(2,g)} - S^{(2,h)} = \{-(n-1+S_h^-), \ldots, -1, 0, 1, \ldots, n-1+S_g^-\} \equiv \mathbb{Z}_{n-1,h,g}$; and*

- $S^{(2,h)} + s = \{1 + S^{+}_{h,s}, \dots, n-1+S^{-}_{h,s}\} \equiv S^{(2,h)}_s$; *with*
- $S^{+}_{h,s} = \max(h,0) + \max(s,0)$ *and* $S^{-}_{h,s} = \min(h,0) + \min(s,0)$.
- *Last, denote* $S^{(2,g,h)}_s = S^{(2,g)} \cap S^{(2,h)}_s$.

Definition B.2. *Let $\tilde{\Omega}^{(ee)}$ and $\tilde{\Omega}^{(ep)}$ replace $\Omega^{(ee)}$ and $\Omega^{(ep)}$, respectively, in Definitions 1 and 2 for all combinations of local and average h-th autocovariance (covariance) and long run variance (covariance) terms when $\theta_t(g)$ is replaced by $\theta(t,g)$.*

The next lemma establishes marginal $\mathcal{H}_1$-stable central limit theory for the main contribution of the MMS noise to the asymptotic distribution for the flat-top realized kernels, that is, for $A(U)$.

Lemma B.2. *Under the conditions of Theorem 1 and let $\nu \in (1/3, 2/3)$, then*

$$(H^3 n^{-1})^{1/2} \left(A(U) - O_p\left(\alpha(cH) n H^{-2}\right)\right) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, 4\lambda^{(22)} \int_0^1 \Omega_t^2 dt\right).$$

Proof. First, recall that $A(U) = A_1(U) + A_2(U)$. Moreover, due to the two-component structure of the MMS noise, $U = e + u$, $A(U)$ also decomposes as

$$A(U) = A(e) + A(u) + A(e,u) + A(u,e), \tag{B.17}$$

similarly to (B.1). Now, write $\mathbb{E}[A(U)|\mathcal{H}_1] = \mathbb{E}[A_2(e) + A_2(u)|\mathcal{H}_1]$ using $a(|h|/H) = 0$ for $|h| < cH$ in conjunction with independence of e and u to eliminate the cross-products in (B.17). Next, the technical results in Lemmas C.6 (d) and C.7 (a) may be invoked to show convergence of the average h-th autocovariances in $A(e)$ and $A(u)$, i.e., the limits

$$\frac{1}{n} \sum_{i \in S^{(2,h)}} e_{t_i} e_{t_{i-h}} \overset{\mathbb{P}}{\to} \Omega^{(ee)}(h) \quad \text{and} \quad \frac{1}{n} \sum_{i \in S^{(2,h)}} u_{t_i} u_{t_{i-h}} \overset{\mathbb{P}}{\to} \bar{\Omega}^{(uu)}(h) \int_0^1 \zeta_t^2,$$

respectively. The use of these results in conjunction with $\sup_{h \in \mathbb{Z}^{n-1}_{cH}} |a(|h|/H)| \leq K$ by the regularity conditions for $\mathcal{K}^*$, $\sup_{t \in [0,1]} \zeta_t \in (0, \infty)$ by Assumption 2, and the triangle inequality gives

$$\begin{aligned}
\mathbb{E}[A_2(e) + A_2(u)|\mathcal{H}_1] &\leq n H^{-2} K \sum_{h \in \mathbb{Z}^{n-1}_{cH}} \left(|\Omega^{(ee)}(h)| + |\bar{\Omega}^{(uu)}(h)|\right)(1 + o_p(1)) \\
&\leq O_p\left(\alpha(cH) n H^{-2}\right)
\end{aligned} \tag{B.18}$$

where the last inequality for the h-th autocovariances of e and $\bar{u}$ follows by Lemma C.4.

Next, to establish the asymptotic variance and $\mathcal{H}_1$-stable central limit theorem for $A(U)$, marginal $\mathcal{H}_1$-stable central limit theorems are initially provided for each of the terms in (B.17) and subsequently combined. First, consider $A(e,u)+A(u,e)=2A(e,u)+o_p(1)$, which may be rewritten as

$$A(e,u)=\frac{n}{H^2}\sum_{h\in\mathbb{Z}_{n-1}} a\left(\frac{|h|}{H}\right)\frac{1}{n}\sum_{i\in S^{(2,h)}} e_{t_i}\zeta_{t_{i-h}}\bar{u}_{t_{i-h}}=\frac{n}{H^2}\sum_{h\in\mathbb{Z}_{n-1}} a\left(\frac{|h|}{H}\right)\frac{1}{n}\sum_{i\in S^{(2,h)}} \tilde{e}_{t_i}\bar{u}_{t_{i-h}},$$

where $\tilde{e}_{t_i}=e_{t_i}\zeta_{t_{i-h}}=e_{t_i}\zeta_{t_i}(1+O_p(|h|/n))$ using Lipschitz continuity of ζ_{t_i} in Assumption 2. Then, as $\tilde{e}_{t_i}$ is $\mathcal{H}_1$-measurable, and since Lemma C.7 (b) shows that $n^{-1}\sum_{i\in S^{(2,h)}}\tilde{e}_{t_i}\tilde{e}_{t_{i-h}}\xrightarrow{\mathbb{P}}\int_0^1\Omega_t^{(ee)}(h)\zeta_t^2dt$ for all $h\in\mathbb{Z}_{n-1}$, $\bar{u}_{t_{i-h}}$ obeys Assumption 2, and $a(|h|/H)$, when $|h|\geq cH$, satisfies the regularity conditions of $\mathcal{K}$, the general $\mathcal{H}_1$-stable central limit theorem in Lemma C.5 may be invoked to show

$$(H^3n^{-1})^{1/2}A(e,u)\overset{d_s(\mathcal{H}_1)}{\to} MN\left(0,2\lambda^{(22)}\bar{\Omega}^{(uu)}\int_0^1\Omega_t^{(ee)}\zeta_t^2dt\right),$$

such that the result for $A(e,u)+A(u,e)=2A(e,u)+o_p(1)$ is immediate by Lemma C.1 (a).

Second, to establish the corresponding marginal $\mathcal{H}_1$-stable central limit theorem for $A(e)$, define first $\beta(x)=a(x)/\lambda^{(2)}(0)$, $\mathrm{i}=\sqrt{-1}$, $\delta\in[-\pi,\pi]$, and the periodogram and autocovariance functions,

$$\mathcal{I}_n(\delta,e)=\frac{1}{2\pi n}\Big|\sum_{j\in S^{(2,0)}} e_{t_j}\exp(-\mathrm{i}\delta j)\Big|^2,\quad \mathcal{C}_n(h,e)=\frac{1}{n}\sum_{j\in S^{(2,h)}} e_{t_j}e_{t_{j-h}}=\int_{-\pi}^{\pi}\mathcal{I}_n(\delta,e)\exp(\mathrm{i}\delta h)\,d\delta,$$

where $|\cdot|^2$ denotes the complex conjugate product. These are, then, used to rewrite $A(e)$ as

$$A(e)=s_n\int_{-\pi}^{\pi}\mathcal{I}_n(\delta,e)\mathcal{K}_n(\delta)\,d\delta,\quad \mathcal{K}_n(\delta)=\frac{1}{2\pi}\sum_{h\in\mathbb{Z}_{n-1}}\beta(|h|/H)\exp(\mathrm{i}\delta h),$$

where $s_n=2\pi\lambda^{(2)}(0)nH^{-2}$ and $\mathcal{K}_n(\delta)$ is the spectral window of $\beta(x)$. Apart from s_n, this spectrum representation of $A(e)$ is equivalent to that in Dahlhaus (2009, (10)-(11)).[1] Hence, as the conditions on e in Assumption 3 satisfy Dahlhaus (2009, Assumption 2.1), and Lemmas C.2 (b) and (c) provide sufficient regularity conditions on the weight functions $\beta(x)$ and $\mathcal{K}_n(\delta)$ for Dahlhaus (2009, Theorems 2.4 and 3.2), it follows by the latter that $\mathbb{V}[A(e)]\xrightarrow{\mathbb{P}}4nH^{-3}\lambda^{(22)}\int_0^1[\Omega_t^{(ee)}]^2dt$, and

$$(H^3n^{-1})^{1/2}\left(A(e)-O\left(\alpha_e(cH)nH^{-2}\right)\right)\xrightarrow{d}N\left(0,4\lambda^{(22)}\int_0^1[\Omega_t^{(ee)}]^2dt\right).$$

Since $\theta_t(h)$ is $\mathcal{H}_1$-measurable $\forall h\in\mathbb{Z}$, Lemma C.1 (b) implies that this result is $\mathcal{H}_1$-stable.

Third, the marginal $\mathcal{H}_1$-stable central limit theorem for $A(u)$ follows by (B.18), Lemma C.7 (d), which shows that $\mathbb{V}[A(u)|\mathcal{H}_1]\xrightarrow{\mathbb{P}}4nH^{-3}\lambda^{(22)}\int_0^1[\Omega_t^{(uu)}]^2dt$, and, if additionally restricting the band-

[1]The notation in this paper differs from the notation used by Dahlhaus (2009). Hence, Section 7 in this online supplementary material clarifies exactly how the present notation and Assumption 3 should be mapped to the framework of the latter, in addition to providing a discussion of locally stationary processes.

width rate such that $\nu \in (1/3, 2/3)$, Lemma C.7 (e), which establishes

$$(H^3 n^{-1})^{1/2} \left(A(u) - \mathbb{E}[A(u)|\mathcal{H}_1]\right) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, 4\lambda^{(22)} \int_0^1 [\Omega_t^{(uu)}]^2 dt\right).$$

Finally, before collecting marginal results to establish a joint limit for $A(U)$, note that the $\mathcal{H}_1$-conditional cross-covariance terms, $\mathrm{Cov}[A(e), A(u)|\mathcal{H}_1] = 0$, $\mathrm{Cov}[A(e), 2A(e,u)|\mathcal{H}_1] = 0$, and

$$\begin{aligned}
&\mathrm{Cov}[A(u), 2A(e,u)|\mathcal{H}_1] \\
&= \frac{2n^2}{H^4} \sum_{h \in \mathbb{Z}_{n-1}} \sum_{g \in \mathbb{Z}_{n-1}} a\left(\frac{|h|}{H}\right) a\left(\frac{|g|}{H}\right) \frac{1}{n^2} \sum_{i \in S^{(2,h)}} \sum_{j \in S^{(2,g)}} e_{t_j} \zeta_{t_i} \zeta_{t_{i-h}} \zeta_{t_{j-g}} \bar{\kappa}_3(i, i-h, j-g) \\
&\leq \frac{2n}{H^4} k \sup_{h \in \mathbb{Z}_{n-1}} a\left(\frac{|h|}{H}\right)^2 \sum_{h \in \mathbb{Z}_{n-1}} \sum_{g \in \mathbb{Z}_{n-1}} \sum_{i \in S^{(2,h)}} \sup_{j \in S^{(2,g)}} |\bar{\kappa}_3(i, i-h, j-g)| \times \frac{1}{n} \sum_{j \in S^{(2,g)}} |e_{t_j}| \\
&\leq O_p(1) \times \frac{2n}{H^4} kK \sum_{h \in \mathbb{Z}_{n-1}} \sum_{g \in \mathbb{Z}_{n-1}} \sum_{i \in S^{(2,h)}} |\bar{\kappa}_3(i, i-h, -g)| = O_p\left(nH^{-4}\right) = o_p\left(nH^{-3}\right),
\end{aligned}$$

since ζ_t is bounded $\forall t \in [0,1]$,

$$\sup_{h \in \mathbb{Z}_{n-1}} a(|h|/H)^2 \leq k, \qquad \sup_{j \in S^{(2,g)}} |\bar{\kappa}_3(i, i-h, j-g)| \leq |\bar{\kappa}_3(i, i-h, -g)| K,$$

as well as $\bar{\kappa}_3(\cdot)$ being absolutely summable by Assumption 2 and $n^{-1} \sum_{j \in S^{(2,g)}} |e_{t_j}| = O_p(1)$. Hence, the final convergence result follows by combining the marginal $\mathcal{H}_1$-stable laws for $A(e)$, $A(u)$ and $A(e,u)$ with (B.18), Lemma C.1 (a) and the stable Cramér-Wold theorem in Lemma C.1 (e). □

Remark 1. *The upper bound $\nu < 2/3$ is only needed when establishing the $\mathcal{H}_1$-stable central limit theorem for $A(U)$, specifically when invoking Lemma C.7 (e), not for computing its $\mathcal{H}_1$-conditional moments. This explains why the bound is not imposed in Theorem 1(1).*

The following lemma establishes marginal $\mathcal{H}_1$-stable central limit theory for the main contribution of cross-products between the efficient log-returns and the MMS noise to the asymptotic distribution for the flat-top realized kernels, that is, for $B(r^*, U)$.

Lemma B.3. *Under the conditions of Theorem 1,*

$$H^{1/2}\left(B(r^*, U) - O_p\left(H^{-1} n^{1/2} \alpha_e(cH)\right)\right) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, 8\lambda^{(11)} \int_0^1 \left(\left(\Omega_t^{(ep)}\right)^2 + \Omega_t \sigma_t^2\right) dt\right).$$

Proof. Before proceeding, recall that $B(r^*, U) = B_1(r^*, U) + B_2(r^*, U)$. Moreover, it is convenient to decompose $B(r^*, U) = B(r^*, e) + B(r^*, u)$ and provide marginal results for $B(r^*, e)$ and $B(r^*, u)$

separately. Now, consider first $B(r^*, u)$, whose h-th realized autocovariance may be rewritten as

$$\sum_{i \in S^{(1,h)}} r_i^* u_{t_{i-h}} = (1 + O_p(|h|/n)) \sum_{i \in S^{(1,h)}} \tilde{r}_i^* \bar{u}_{t_{i-h}}, \quad \text{where} \quad \tilde{r}_i^* = r_i^* \zeta_{t_i},$$

using Lipschitz continuity of ζ_{t_i} in Assumption 2. By applying this representation in conjunction with the technical results in Lemmas C.2 (d) and C.7 (c), which establish

$$\frac{1}{H} \sum_{h \in \mathbb{Z}_{n-1}} b\left(\frac{|h|}{H}\right)^2 \to 2\lambda^{(11)} \quad \text{and} \quad \sum_{h \in \mathbb{Z}_{n-1}} \sum_{i \in S^{(1,h)}} \tilde{r}_i^* \tilde{r}_{i-h}^* \xrightarrow{\mathbb{P}} \int_0^1 \sigma_t^2 \zeta_t^2 dt,$$

respectively, the general $\mathcal{H}_1$-stable central limit theorem in Lemma C.5 may be invoked to show

$$H^{1/2} B(r^*, u) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, 8\lambda^{(11)} \int_0^1 \Omega_t^{(uu)} \sigma_t^2 dt\right).$$

Second, consider $B(r^*, e)$. Using $b(|h|/H) = 0$ for $|h| < cH$,

$$\begin{aligned}
\mathbb{E}[B(r^*, e)|\mathcal{H}_1] = \mathbb{E}[B_2(r^*, e)|\mathcal{H}_1] &= \frac{2}{Hn^{1/2}} \sum_{h \in \mathbb{Z}_{cH}^{n-1}} b\left(\frac{|h|}{H}\right) \sum_{i \in S^{(1,h)}} \theta(t_{i-h}, h) \Upsilon_{t_{i-1}} \sigma_{t_{i-1}} (1 + o_p(1)) \\
&\leq \frac{2}{Hn^{1/2}} \sup_{t \in [0,1]} |\Upsilon_t \sigma_t| \sup_{h \in \mathbb{Z}_{cH}^{n-1}} \left| b\left(\frac{|h|}{H}\right) \right| \sum_{h \in \mathbb{Z}_{cH}^{n-1}} \sum_{i \in S^{(1,h)}} |\theta(t_{i-h}, h)| (1 + o_p(1)) \\
&= O_p\left(H^{-1} n^{1/2} \alpha_e(cH)\right)
\end{aligned}$$

since $\sup_{t \in [0,1]} |\Upsilon_t \sigma_t| \leq k$, $\sup_{h \in \mathbb{Z}_{cH}^{n-1}} |b(|h|/H)| \leq K$, and $\sum_{i \in S^{(1,h)}} |\theta(t_{i-h}, h)| \leq O(n\alpha_e(h))$. Next,

$$\mathbb{V}[B(r^*, e)|\mathcal{H}_1] = \frac{4}{H^2} \sum_{h \in \mathbb{Z}_{n-1}} \sum_{g \in \mathbb{Z}_{n-1}} b\left(\frac{|h|}{H}\right) b\left(\frac{|g|}{H}\right) \sum_{i \in S^{(1,h)}} \sum_{j \in S^{(1,g)}} \operatorname{Cov}\left[r_i^* e_{t_{i-h}}, r_j^* e_{t_{j-g}} | \mathcal{H}_1\right] \tag{B.19}$$

where, since $(r_i^*, e_{t_{i-h}}, r_j^*, e_{t_{j-g}})'$ is a 4-variate Gaussian vector, the cross-covariance term in (B.19) simplifies by invoking Brillinger (1981, Theorem 2.3.2),

$$\operatorname{Cov}\left[r_i^* e_{t_{i-h}}, r_j^* e_{t_{j-g}} | \mathcal{H}_1\right] = \operatorname{Cov}\left[r_i^*, r_j^* | \mathcal{H}_1\right] \operatorname{Cov}\left[e_{t_{i-h}}, e_{t_{j-g}} | \mathcal{H}_1\right] + \operatorname{Cov}\left[r_i^*, e_{t_{j-g}} | \mathcal{H}_1\right] \operatorname{Cov}\left[e_{t_{i-h}}, r_j^* | \mathcal{H}_1\right],$$

and let (B.19) = (B.19.1) + (B.19.2) denote the resulting decomposition of $\mathbb{V}[B(r^*, e)|\mathcal{H}_1]$. The marginal result for (B.19.1) mirrors the variance of $B(r^*, u)$ and is provided by Lemma C.6 (e),

$$H(\text{B.19.1}) \xrightarrow{\mathbb{P}} 8\lambda^{(11)} \int_0^1 \sigma_t^2 \Omega_t^{(ee)} dt.$$

For (B.19.2), since $\operatorname{Cov}\left[r_i^*, e_{t_{j-g}} | \mathcal{H}_1\right] = n^{-1/2} \tilde{\Omega}_{t_{j-g}}^{(ep)} (i - j + g)(1 + o_p(1))$, use Lemma C.6 (f) and two change of variables $s = i - h$ and $x = j - g$ to write

$$(\text{B.19.2}) = O_p\left((Hn)^{-1}\right) + \frac{4}{H^2} \sum_{h \in \mathbb{Z}_{n-1}} \sum_{g \in \mathbb{Z}_{n-1}} b\left(\frac{|h|}{H}\right) b\left(\frac{|g|}{H}\right) C_n^{(ep)}(h,g),$$

where, using another change of variables $x - s = k$, $C_n^{(ep)}(h,g)$ may be written and rewritten as

$$\begin{aligned} C_n^{(ep)}(h,g) &= \frac{1}{n} \sum_{s \in S^{(1,-h)}} \sum_{x \in S^{(1,-g)}} \Omega_{t_x}^{(ep)}(h+s-x) \Omega_{t_s}^{(ep)}(g+x-s) \\ &= \frac{1}{n} \sum_{k \in \mathbb{Z}_{n-1,-h,-g}} \sum_{s \in S_k^{(1,-g,-h)}} \Omega_{t_{s+k}}^{(ep)}(h-k) \Omega_{t_s}^{(ep)}(g+k) \\ &= O_p(n^{-1}) + \sum_{k \in \mathbb{Z}_{n-1,-h,-g}} \int_0^1 \Omega_t^{(ep)}(h-k) \Omega_t^{(ep)}(g+k) dt, \end{aligned} \tag{B.20}$$

whose final representation follows by splitting the sum $C_n^{(ep)}(h,g) = C_n^{(ep)}(h,g,1) + C_n^{(ep)}(h,g,2)$,

$$\begin{aligned} C_n^{(ep)}(h,g,1) &= \sum_{k \in \mathbb{Z}_{n-1,-h,-g}} \frac{1}{n} \sum_{s \in S_k^{(1,-g,-h)}} \Omega_{t_s}^{(ep)}(h-k) \Omega_{t_s}^{(ep)}(g+k) \\ &= \sum_{k \in \mathbb{Z}_{n-1,-h,-g}} \int_0^1 \Omega_t^{(ep)}(h-k) \Omega_t^{(ep)}(g+k) dt (1 + o_p(1)) \\ C_n^{(ep)}(h,g,2) &= \frac{1}{n} \sum_{k \in \mathbb{Z}_{n-1,-h,-g}} \sum_{s \in S_k^{(1,-g,-h)}} \Omega_{t_s}^{(ep)}(g+k) \sum_{k_1 = S_k^-}^{S_k^+} \Big(\Omega_{t_{s+k}}^{(ep)}(h-k) - \Omega_{t_{s+k-k_1}}^{(ep)}(h-k) \\ &\quad + \Omega_{t_{s+k-k_1}}^{(ep)}(h-k) - \Omega_{t_s}^{(ep)}(h-k) \Big) \leq \frac{K}{n} \sum_{k \in \mathbb{Z}_{n-1,-h,-g}} |k| \alpha_e(h-k) \alpha_e(g+k) = O(n^{-1}) \end{aligned}$$

using that, for $C_n^{(ep)}(h,g,2)$, Assumption 3 implies

$$\sup_{s \in S_k^{(1,-g,-h)}} |\Omega_{t_s}^{(ep)}(g+k)| \leq K \alpha_e(g+k), \qquad \sum_{s \in S_k^{(1,-g,-h)}} \left| \Omega_{t_{s+k}}^{(ep)}(h-k) - \Omega_{t_{s+k\pm 1}}^{(ep)}(h-k) \right| \leq K \alpha_e(h-k)$$

for $k \lessgtr 0$, $|k|$ times, and $\sum_{k \in \mathbb{Z}_{n-1,-h,-g}} |k| \alpha_e(h-k) \alpha_e(g+k) < \infty$. By inserting (B.20) in (B.19.2) and using a fourth change of variable $g - h = z$ along with a Taylor approximation of $b(|h+z|/H)$, write

$$\begin{aligned} H(\text{B.19.2}) &= O_p(n^{-1}) + (1 + O(H^{-1})) \frac{4}{H} \sum_{z \in \mathbb{Z}_{2(n-1)}} \sum_{h \in \mathbb{Z}_{n-1,z,z}} b\left(\frac{|h|}{H}\right)^2 \sum_{k \in \mathbb{Z}_{n-1,-h,-(h+z)}} \\ &\quad \int_0^1 \Omega_t^{(ep)}(h-k) \Omega_t^{(ep)}(h+z+k) dt \xrightarrow{\mathbb{P}} 8 \lambda^{(11)} \int_0^1 \left(\Omega_t^{(ep)}\right)^2 dt \end{aligned}$$

where the final convergence in probability follows by changing the order of summation with respect to z, h, and k and taking the limit. By combining results,

$$H\mathbb{V}[B(r^*,e)|\mathcal{H}_1] \xrightarrow{\mathbb{P}} 8\lambda^{(11)} \int_0^1 \left(\left(\Omega_t^{(ep)}\right)^2 + \Omega_t^{(ee)}\sigma_t^2 \right) dt.$$

Next, the individual moment results for $B(r^*,e)$ and $B(r^*,u)$ may be used, in conjunction with the observation that $\mathrm{Cov}[B(r^*,e), B(r^*,u)|\mathcal{H}_1] = 0$, to show

$$\mathbb{E}[B(r^*,U)|\mathcal{H}_1] \leq O_p\left(H^{-1}n^{1/2}\alpha_e(cH)\right), \quad H\mathbb{V}[B(r^*,U)|\mathcal{H}_1] \xrightarrow{\mathbb{P}} 8\lambda^{(11)} \int_0^1 \left(\left(\Omega_t^{(ep)}\right)^2 + \Omega_t\sigma_t^2 \right) dt.$$

Lastly, to establish the $\mathcal{H}_1$-stable central limit theorem, denote $B(z) = B(z,z)$ for some generic z, possibly a vector, and note that $B(r^*,U)$ corresponds to an off-diagonal element of

$$B\begin{pmatrix} r^*n^{1/2} \\ Un^{-1/2} \end{pmatrix} = \begin{pmatrix} B(r^*n^{1/2}) & B(r^*n^{1/2}, Un^{-1/2}) \\ B(Un^{-1/2}, r^*n^{1/2}) & B(Un^{-1/2}) \end{pmatrix} \tag{B.21}$$

since $B(r^*n^{1/2}, Un^{-1/2}) = B(r^*,U)$. Then, as the mixed normal distribution is additive, the stable Cramér-Wold results in Lemmas C.1 (d) and (e) show that it suffices to prove $\mathcal{H}_1$-stable central limit theorems for both $B(r^*n^{1/2})$ and $B(Un^{-1/2})$, and use these in conjunction with boundedness and convergence in probability of the $\mathcal{H}_1$-conditional first and second moment of $B(r^*,U)$ to get the stable limit for the latter. Hence, and as for $K(r^*)$ in Section B.1 of the main text, it follows by Barndorff-Nielsen, Hansen, Lunde & Shephard (2008, Theorem 1), Lemma C.2 (d), and $b(|h|/H) = 0$ for $|h| < cH$ that

$$H^{1/2}B(r^*n^{1/2}) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, 8\lambda^{(11)} \int_0^1 \sigma_t^4 dt\right).$$

For $B(Un^{-1/2})$, it follows using exactly the same arguments as for Lemma B.2 that

$$H^{1/2}\left(B(Un^{-1/2}) - O_p\left(H^{-1}\alpha(cH)\right)\right) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, 8\lambda^{(11)} \int_0^1 \Omega_t^2 dt\right),$$

thus providing the final $\mathcal{H}_1$-stable central limit theorem for $B(r^*,U)$, concluding the proof. □

Remark 2. *This modified Cramér-Wold argument with carefully selected scales is needed for the stable central limit theorem since the endogenous noise, e_{t_i}, is not adapted to $\mathcal{H}_{t_i}$. The latter implies that a martingale difference-type argument for $B(r^*,U)$ cannot be applied in this setting.*

The next lemma provides a bound for the realized lag structure of finite activity jumps.

Lemma B.4. *Under the conditions of Theorem 2, $RKL(J) = O_p\left(H^{1/2}/n\right) + O_p((mH^{1/2})/n^2)$.*

Proof. First, recall the bound in (B.10), $\mathbb{E}[\Delta N_t] \leq K\Delta t_i(1 + O(n^{-1}))$. Next, as for the proof of

Theorem 1 in the main text, write

$$RKL(J) = \underbrace{\sum_{h\in\mathbb{Z}_1^{n-1}} k\left(\frac{|h|}{H}\right) \sum_{i\in S^{(1,h)}} \Delta J_{t_i}\Delta J_{t_{i-h}}}_{=K_1(J)} + \underbrace{2\sum_{h=1}^{n-1} k(h/H)\left(\Delta J_{t_1}\Delta J_{t_{1+h}} + \Delta J_{t_n}\Delta J_{t_{n-h}}\right)}_{=Z_4(J)}$$

Then, Lemma C.8 (b) derives the uniform end-effects bound: $Z_4(J) = O_p((mH^{1/2})/n^2)$. Next, for the main component of the realized lag structure, $K_1(J)$,

$$\begin{aligned}
\mathbb{E}\left[|K_1(J)|\right] &\leq \sum_{h\in\mathbb{Z}_1^{n-1}} |k\left(|h|/H\right)| \sum_{i\in S^{(1,h)}} \mathbb{E}[|\Delta J_{t_i}|]\mathbb{E}[|\Delta J_{t_{i-h}}|] \\
&\leq \sum_{h\in\mathbb{Z}_1^{n-1}} |k\left(|h|/H\right)| \frac{1}{n^2} \sum_{i\in S^{(1,h)}} K^2\mathbb{E}[|d_{t_i}|]\mathbb{E}[|d_{t_{i-h}}|](1+O(n^{-1})) \\
&\leq \frac{1}{n} \sum_{h\in\mathbb{Z}_1^{n-1}} |k\left(|h|/H\right)| \frac{1}{n} \sum_{i\in S^{(1,h)}} K^2 k(1+O(n^{-1})) \leq \frac{K}{n} \sum_{h\in\mathbb{Z}_1^{n-1}} |k\left(|h|/H\right)|,
\end{aligned}$$

where the first inequality follows by the triangle inequality along with independent increments, ΔJ_{t_i} and $\Delta J_{t_{i-h}}$, the second by Lipschitz continuity and boundedness of η_t, and the third inequality by $\mathbb{E}[|d_t|] < \infty \; \forall t \in [0,1]$. The final $O_p(H^{1/2}/n)$ bound, thus, follows by using Lemma C.2 (a) for $k(\cdot)$. $\square$

The following lemma provides marginal $\mathcal{U}_1$-stable central limit theory for cross-products between the efficient log-returns and the finite activity jumps.

Lemma B.5. *Under the conditions of Theorem 2,*

$$\sqrt{n/H}\left(K(J,r^*) + K(r^*,J)\right) \xrightarrow{d_s(\mathcal{U}_1)} MN\Big(0, 4(\lambda^{(00)}+c)\sum_{0\leq t\leq 1} d_t^2\sigma_t^2\Big).$$

Proof. First, recall the definition $\tilde{\mathcal{U}}_{t,s} = \mathcal{J}_t \vee \mathcal{H}_s$ from the main text. Next, rewrite

$$K(J,r^*) + K(r^*,J) = 2\sum_{i\in S^{(1,0)}} \Delta J_{t_i} \sum_{h\in S^{(1,0)}} k\left(\frac{h-i}{H}\right) r_h^*.$$

Then, for each $i \in S^{(1,0)}$ such that $\Delta N_{t_i} = 1$, a $\tilde{\mathcal{U}}_{t_i,1}$-stable central limit theorem may be established and subsequently summed over all $i \in S^{(1,0)}$ since the increments of the counting process ΔN_{t_i} and ΔN_{t_j} are independent $\forall i \neq j$. That is, for some $\Delta N_{t_i} = 1$, use the Lemma B.1 (a) approximation $r_h^* = \sigma_{t_{h-1}}\Delta W_{t_h}(1+o_p(n^{-1/2}))$, for which $\sum_{h\in S^{(1,0)}} k\left(\frac{h-i}{H}\right)\mathbb{E}[d_{t_i}\sigma_{t_{h-1}}\Delta W_{t_h}|\tilde{\mathcal{U}}_{t_i,t_{h-1}}] = 0$. Next, as

$$\frac{n}{H}\Big(2d_{t_i}\sum_{h\in S^{(1,0)}} k\left(\frac{h-i}{H}\right)\sigma_{t_{h-1}}\Delta W_{t_h}\Big)^2 \xrightarrow{\mathbb{P}} 4d_{t_i}^2\frac{n}{H}\sum_{h\in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 \sigma_{t_{h-1}}^2\Delta W_{t_h}^2,$$

using independence of the Brownian increments ΔW_{t_h} and ΔW_{t_g}, for $h \neq g$, in conjunction with Lemma C.2 (a) to show $H^{-1} \sum_{h \in S^{(1,0)}} k((h-i)/H)^2 \to \lambda^{(00)} + c$ for fixed i. Then,

$$\frac{4n}{H} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 \mathbb{E}\left[d_{t_i}^2 \sigma_{t_{h-1}}^2 \Delta W_{t_h}^2 | \tilde{\mathcal{U}}_{t_i,t_{h-1}}\right] \xrightarrow{\mathbb{P}} 4d_{t_i}^2(\lambda^{(00)} + c)\sigma_{t_i}^2$$

establishes the asymptotic variance for each jump, $\Delta N_{t_i} = 1$, where, in particular, the final convergence result follows by addition and subtraction as

$$\frac{4d_{t_i}^2}{H} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 \sigma_{t_{h-1}}^2 = \frac{4d_{t_i}^2}{H} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 \left(\sigma_{t_i}^2 + \sigma_{t_{h-1}}^2 - \sigma_{t_i}^2\right)$$

such that $\sigma_{t_i}^2 H^{-1} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 \to \sigma_{t_i}^2(\lambda^{(00)} + c)$, using, again, Lemma C.2 (a) for $k(\cdot)^2$, and

$$\begin{aligned}
\frac{1}{H} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 (\sigma_{t_{h-1}}^2 - \sigma_{t_i}^2) &\leq \frac{2\Lambda_2}{H} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 |\sigma_{t_{h-1}} - \sigma_{t_i}| \\
&\leq \frac{2\Lambda_2 K_1}{H} \sum_{h \in S^{(1,0)}} k\left(\frac{h-i}{H}\right)^2 \left(\frac{|h-1-i|}{n}\right)^{1/2} (1 + o_p(1)) \xrightarrow{\mathbb{P}} 0
\end{aligned}$$

using Assumption 1 for the first inequality, Lemma B.1 (c) for the second, and Lemma C.2 (a) for the final convergence in probability. Having established the first two $\tilde{\mathcal{U}}_{t_i,1}$-conditional moments for each jump, $\Delta N_{t_i} = 1$, its accompanying $\tilde{\mathcal{U}}_{t_i,1}$-stable central limit theory follows by invoking a central limit theorem for martingale difference arrays and using it in conjunction with Lemma C.1 (a).[2] The final result follows by sequentially summing over the N_1 independent jumps. □

The next lemma establishes marginal $\mathcal{U}_1$-stable central limit theory for cross-products between the MMS noise and the finite activity jumps.

Lemma B.6. *Under the conditions of Theorem 2,*

$$H^{1/2} B(J,U) \overset{d_s(\mathcal{U}_1)}{\to} MN\Big(0, 4\lambda^{(11)} \sum_{0 \leq t \leq 1} d_t^2 \Omega_t\Big).$$

Proof. Similar to the strategy behind Lemmas B.2 and B.5 above, rewrite

$$B(J,U) = 2 \sum_{i \in S^{(1,0)}} \Delta J_{t_i} \frac{1}{H} \sum_{h \in S^{(1,0)}} b\left(\frac{|h-i|}{H}\right) U_{t_h}.$$

Then, for each $i \in S^{(1,0)}$ such that $\Delta N_{t_i} = 1$, a $\tilde{\mathcal{U}}_{t_i,1}$-stable central limit theorem may be established

[2] A similar martingale difference array argument is applied for Barndorff-Nielsen, Hansen, Lunde & Shephard (2011, Proposition A.5). In their Proposition A.5, however, the argument is used to establish stable central limit theory for a term that is equivalent to $K(r^*)$ in Section B.1 of the main text, albeit without a flat-top kernel function.

and subsequently summed over all $i \in S^{(1,0)}$. Before deducing the joint $\tilde{\mathcal{U}}_{t_i,1}$-stable law, however, the marginal $\mathcal{J}_{t_i}$-conditional law is established for e and the corresponding marginal $\tilde{\mathcal{U}}_{t_i,1}$-stable law is provided for u. First, for some $\Delta N_{t_i} = 1$, write $\mathbb{E}[2d_{t_i} H^{-1} \sum_{h \in S^{(1,0)}} b\left(|h-i|/H\right) U_{t_h} | \mathcal{J}_{t_i}] = 0$,

$$\mathbb{E}\Big[\Big(2d_{t_i} H^{-1} \sum_{h \in S^{(1,0)}} b\left(\frac{|h-i|}{H}\right) U_{t_h}\Big)^2 |\mathcal{J}_{t_i}\Big] = 4d_{t_i}^2 \mathbb{E}\Big[\Big(H^{-1} \sum_{h \in S^{(1,0)}} b\left(\frac{|h-i|}{H}\right)(e_{t_h} + u_{t_h})\Big)^2\Big],$$

and define $E_i = \mathbb{E}[(H^{-1} \sum_{h \in S^{(1,0)}} b\left(|h-i|/H\right) e_{t_h})^2]$. Then,

$$\begin{aligned}
HE_i &= \frac{1}{H} \sum_{h \in S^{(1,0)}} \sum_{g \in S^{(1,0)}} b\left(\frac{|h-i|}{H}\right) b\left(\frac{|g-i|}{H}\right) \mathrm{Cov}[e_{t_h}, e_{t_g}] \\
&= \frac{1}{H}(1 + O(H^{-1})) \sum_{z \in \mathbb{Z}_{n-1}} \Omega_{t_i}^{(ee)}(z) \sum_{h \in S^{(1,0)} \cap S_z^{(1,0)}} b\left(\frac{|h-i|}{H}\right)^2 \\
&\quad + \frac{1}{H}(1 + O(H^{-1})) \sum_{z \in \mathbb{Z}_{n-1}} \sum_{h \in S^{(1,0)} \cap S_z^{(1,0)}} b\left(\frac{|h-i|}{H}\right)^2 \left(\Omega_{t_h}^{(ee)}(z) - \Omega_{t_i}^{(ee)}(z)\right),
\end{aligned}$$

whose representation follows by first using a change of variable $g = h+z$, switching the summation with respect to h and z, then a Taylor approximation of $b(|h+z-i|/H)$, Lemma C.6 (a) for $\mathrm{Cov}[e_{t_h}, e_{t_g}]$, and, finally, by adding and subtracting sums involving $\Omega_{t_i}^{(ee)}(z)$. For the second term in the decomposition of HE_i, Lemma C.6 (g) shows that $|\Omega_{t_h}^{(ee)}(z) - \Omega_{t_i}^{(ee)}(z)| \leq |h-i|\alpha_e(z)K/n$. Hence, it readily follows that $HE_i \to \Omega_{t_i}^{(ee)} \lambda^{(11)}$, since

$$\frac{1}{H} \sum_{z \in \mathbb{Z}_{n-1}} \alpha_e(z) \sum_{h \in S^{(1,0)} \cap S_z^{(1,0)}} b\left(\frac{|h-i|}{H}\right)^2 \frac{|h-i|}{n} \leq \frac{K}{H} \sum_{h \in S^{(1,0)} \cap S^{(1,0)}} b\left(\frac{|h-i|}{H}\right)^2 \frac{|h-i|}{n} \to 0,$$

using Lemma C.2 (d) for $b(\cdot)^2$. This implies

$$H\mathbb{E}\left[\Big(2d_{t_i} H^{-1} \sum_{h \in S^{(1,0)}} b\left(\frac{|h-i|}{H}\right) e_{t_h}\Big)^2 |\mathcal{J}_{t_i}\right] \to 4\lambda^{(11)} d_{t_i}^2 \Omega_{t_i}^{(ee)}.$$

These derivations show that the e_{t_h} variables in $H^{-1} \sum_{h \in S^{(1,0)}} b(|h-i|/H) e_{t_h}$ are locally stationary (close to t_i) with long-run variance $\Omega_{t_i}^{(ee)}$, up to an $o(H^{-1})$ error. Hence, a marginal $\mathcal{J}_{t_i}$-conditional law follows directly by applying standard central limit theory for linear processes with Gaussian errors. Moreover, since e is $\tilde{\mathcal{U}}_{t_i,1}$-measurable, this limit result is $\tilde{\mathcal{U}}_{t_i,1}$-stable by Lemma C.1 (b).

Next, for u, $\mathbb{E}[2d_{t_i} H^{-1} \sum_{h \in S^{(1,0)}} b\left(|h-i|/H\right) u_{t_h} | \tilde{\mathcal{U}}_{t_i,1}] = 0$ and

$$H\mathbb{E}\left[\Big(2d_{t_i} H^{-1} \sum_{h \in S^{(1,0)}} b\left(\frac{|h-i|}{H}\right) u_{t_h}\Big)^2 |\tilde{\mathcal{U}}_{t_i,1}\right] \xrightarrow{\mathbb{P}} 4\lambda^{(11)} d_{t_i}^2 \zeta_{t_i}^2 \bar{\Omega}^{(uu)} = 4\lambda^{(11)} d_{t_i}^2 \Omega_{t_i}^{(uu)}$$

using arguments similar to those for HE_i above in conjunction with Lemma C.2 (d) for $b(\cdot)^2$ and Lemma C.7 (a) to show convergence of the $\tilde{\mathcal{U}}_{t_i,1}$-conditional autocovariances for u_{t_i}. Moreover, since the σ-fields, $\mathcal{J}_s \perp\!\!\!\perp \mathcal{H}_t$, $\forall s,t$, convergence of the asymptotic variance for u and the regularity conditions on $b(\cdot)$ by Definitions 4-5 suffice to invoke the stable central limit theorem for orthogonal variables in Lemma C.5 (b), providing the corresponding marginal $\tilde{\mathcal{U}}_{t_i,1}$-stable central limit theory for u.

Finally, since the $\tilde{\mathcal{U}}_{t_i,1}$-conditional covariance between e and u is zero, the joint $\mathcal{U}_1$-stable central limit theorem is established by summing up the respective marginal limit theorems over the N_1 independent jumps, together with Lemma C.1 (c) and the stable Cramér-Wold theorem in Lemma C.1 (e). $\square$

The next lemma provides a useful representation result for the blocks of flat-top realized kernel estimates, $RK_i^T(p)$, $i = 1, \ldots, n_L$. This is used to establish Theorems 3 and 4.

Lemma B.7. *Under the conditions of Theorem 2, then for $y = p$, $RK_i^T(p)$ has representation $RK_i^T(p) = \int_{\tau_{i-1}}^{\tau_i} \sigma_t^2 dt + \Delta\tilde{M}_{\tau_i}(1 + o_p(1))$, $i = 1, \ldots, n_L$ where $\tilde{M}_t$ is an $\mathcal{U}_t$-measurable sequence of continuous local martingales on $t \in [0,1]$, which satisfies $n^{1/2}[\tilde{M}, \tilde{M}] \xrightarrow{\mathbb{P}} \mathcal{V}(\lambda, a)$ and $n^{1/4}[\tilde{M}, W] \xrightarrow{\mathbb{P}} 0$. The properties of $\tilde{M}_t$ also hold under the statistical risk neutral distribution, $\mathbb{Q}$.*[3]

Proof. Theorem 1 and Lemma 3 provide the representation of $RK_i^T(p)$ where for fixed τ_i, $i = 1, \ldots, n_L$, $(iL)^{-1/4}\tilde{M}_{\tau_i} \overset{d_s(\mathcal{U}_1)}{\rightarrow} MN(0, \text{plim}_{n\to\infty} \tau_i \int_0^{\tau_i} \mathcal{V}(\lambda, a, t)dt)$. Interchangeability of limits and quadratic variation follows by Mykland & Zhang (2012, Proposition 4). Finally, existence of equivalent results under the risk neutral distribution, $\mathbb{Q}$, follows by Girsanov's Theorem due to the absence of drift in $\tilde{M}_t$. $\square$

3 Technical Results and Definitions

This section contains ten lemmas to complement the proofs in the previous section as well as those in Section B of the appendix in the main text. Since these generally rely on prior definitions or pertain to specific terms, they are easier to read after having been referred to. However, to ease their readability, each lemma is briefly introduced before stated. Before proceeding, two definitions are given. These are referred to in Lemmas C.1 and C.2.

Definition C.1. *(Stable Convergence, A General Class of Kernels)*

(a) *Jacod & Shiryaev (2003, pp. 512-513): Suppose $\mathcal{X}$ is a σ-field on $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ such that $\mathcal{X} \subseteq \mathcal{F}$, then Y_n converges $\mathcal{X}$-stably in law to Y, $Y_n \overset{d_s(\mathcal{X})}{\rightarrow} Y$, if and only if the pair (Y_n, W) converges in law to (Y, W) for any $\mathcal{X}$-measurable random variable W.*

(b) *$\mathcal{K}_1^A$ of Andrews (1991, p. 812) is defined as the set of functions $k : \mathbb{R} \to [-1,1]$, which satisfy (a) $k(0) = 1$, (b) $k(x) = k(-x)$, (c) $k^{(00)} < \infty$, and (d) $k(\cdot)$ is continuous at 0 and at all but a finite number of points.*

[3]The risk neutral distribution is defined in, e.g., Mykland & Zhang (2009, Section 2.2).

The first lemma provides five results for stable convergence. The first three of these are due to Barndorff-Nielsen et al. (2008), Podolskij & Vetter (2010), and Ikeda (2015). These three are supplemented with a stable Cramér-Wold theorem and a variant of the latter.

Lemma C.1. *(Stable Convergence Results) Let $\mathcal{L}\{\cdot|\mathcal{X}\}$ denote the $\mathcal{X}$-conditional law. Then,*

(a) *If $Y_n \overset{d_s(\mathcal{X})}{\to} Y$ and $\{W_n\}$ is a sequence of positive random variables on $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ tending in probability to a positive $\mathcal{X}$-measurable random variable W, that is $W_n \overset{\mathbb{P}}{\to} W$, then $W_n Y_n \overset{d_s(\mathcal{X})}{\to} WY$ and $Y_n + W_n \overset{d_s(\mathcal{X})}{\to} Y + W$.*

(b) *For some $\mathcal{X}$-measurable variable W, the following three assertions are equivalent: (i) $Y_n \overset{d_s(\mathcal{X})}{\to} Y$, (ii) $(Y_n, W) \overset{d}{\to} (Y, W)$, and (iii) $(Y_n, W) \overset{d_s(\mathcal{X})}{\to} (Y, W)$.*

(c) *Let $\{Y_n\}$ and $\{W_n\}$ be sequences of random vectors. Suppose $Y_n \overset{d_s(\mathcal{X})}{\to} Y$ and $\mathcal{L}\{W_n|\mathcal{X}\} \overset{\mathbb{P}}{\to} \mathcal{L}\{W|\mathcal{X}\}$. Then $(Y_n, W_n) \overset{d_s(\mathcal{X})}{\to} (Y, W)$.*

(d) *Let Y_n be a p-dimensional sequence of random vectors satisfying $(\alpha' Y_n, \alpha' W \alpha) \overset{d}{\to} (\alpha' Y, \alpha' W \alpha)$ for every α satisfying $\alpha'\alpha = 1$ and some $\mathcal{X}$-measurable $p \times p$ positive definite matrix W, then $(\alpha' Y_n, \alpha' W \alpha) \overset{d_s(\mathcal{X})}{\to} (\alpha' Y, \alpha' W \alpha)$ and $(Y_n, W) \overset{d_s(\mathcal{X})}{\to} (Y, W)$.*

(e) *Let Y_n be p-dimensional random vectors satisfying $Y_n \overset{d_s(\mathcal{X})}{\to} Y$ and W_n be a sequence of $p \times p$ random matrices satisfying $W_n \overset{\mathbb{P}}{\to} W$ for some $\mathcal{X}$-measurable positive definite matrix W, then $(\alpha' Y_n, \alpha' W_n \alpha) \overset{d_s(\mathcal{X})}{\to} (\alpha' Y, \alpha' W \alpha)$ for every α satisfying $\alpha'\alpha = 1$ and $W_n Y_n \overset{d_s(\mathcal{X})}{\to} WY$.*

Proof. **(a)** Ikeda (2015, Lemma 1). **(b)** Podolskij & Vetter (2010, Proposition 2.2). **(c)** Barndorff-Nielsen et al. (2008, Proposition 5). **(d)** follows by **(b)** in conjunction with the Cramér-Wold theorem, cf. Davidson (2002, Theorem 25.6). **(e)** follows by combining **(a)** and **(d)**. □

The second lemma collects convergence and regularity results for the kernel function, its first and second derivative, and different transformations thereof.

Lemma C.2. *(Kernel functions) Let $k(x) \in \mathcal{K}^*$, then for large H,*

(a) *Define the $n \times n$ matrix $A = \operatorname{diag}(1, 2, \ldots, 2)$, the $n \times 1$ vector $w = \left(1, \ldots, 1, \lambda\left(\frac{h-c}{H}\right), \ldots, \lambda\left(\frac{n-1-c}{H}\right)\right)'$ for $h/H \geq c$, then $w' A w = 2H\lambda^{(00)} + 2cH + O(1)$.*

(b) *$\beta(x) = 0$ for $|x| < c$ and $\beta(|x|) \in \mathcal{K}_1^A$ for $|x| \geq c$. For $|x| \geq c$, $\beta(|x|)$ is differentiable at all but a finite number of points, and $\int_0^\infty [\beta^{(j)}(|x|)]^2 < \infty$ for $j = 1$ almost everywhere.*

(c) $\sup_\delta \left\{\sum_{j=1}^g |\mathcal{K}(\delta_j) - \mathcal{K}(\delta_{j-1})| : -\pi \leq \delta_0 < \cdots < \delta_g \leq \pi;\ g \in \mathbb{N}\right\} < \infty.$

(d) *Define the $n \times 1$ vector $w_z = \left(0, \ldots, 0, z\left(\frac{h-c}{H}\right), \ldots, z\left(\frac{n-1-c}{H}\right)\right)'$ for $h/H \geq c$ and $z = \{a, b\}$, then $w_b' A w_b = 2H\lambda^{(11)} + O(1)$ and $w_a' A w_a = 2H\lambda^{(22)} + O(1)$.*

Proof. **(a)** follows from summing over cH elements in A and then using Barndorff-Nielsen et al. (2008, Theorem 2). **(b)** For $|x| < c$, the results follow immediately. For $|x| \geq c$, it follows that $\beta(|x|) : \mathbb{R} \to [-1, 1]$ because $|\lambda^{(2)}(|x| - c)|$ achieves its maximum at $|x| = c$, and $\beta(c) = 1$ by construction. Continuity, symmetry and square integrability follow from the properties of $\lambda(x)$. Differentiability of $\beta(|x|)$ and square summability of $\beta^{(1)}(|x|)$ follow from Definitions 4 (a) and (d). **(c)** is provided by the properties of $\beta(x)$ in **(b)**. **(d)** is similar to **(a)**. □

The third lemma establishes asymptotic bounds on end-effects for the proof of Theorem 1.

Lemma C.3. *(Jittered variables) Under the conditions of Theorem 1, four uniform bounds hold:*

(a) $r_1^* + r_n^* = O_p((m/n)^{1/2})$.

(b) $Z_1(r^*) = O_p(mn^{-1}) + O_p((Hm)^{1/2}n^{-1})$.

(c) $Z_2(U) = O_p(m^{-1})$.

(d) $Z_3(r^*, U) = O_p(H^{1/2}(nm)^{-1/2}) + O_p(m(Hn)^{-1/2}) + O_p(n^{-1/2})$.

Proof. First, recall the definition of the jittered end-point returns,

$$r_1^* = p_{t'_m}^* - \frac{1}{m}\sum_{i=1}^{m} p_{t'_{i-1}} \quad \text{and} \quad r_n^* = \frac{1}{m}\sum_{i=1}^{m} p_{t'_{N-m+i}}^* - p_{t'_{N-m}}^*.$$

The results below are derived for either r_1^* or r_n^* since the corresponding result for the other term follows by symmetry. **(a)** is provided by Ikeda (2015, Lemma 9). **(b)** The third component of

$$Z_1(r^*) = (r_1^*)^2 + (r_n^*)^2 + 2\sum_{h=1}^{n-1} k\left(\frac{h}{n}\right)\left(r_{h+1}^* r_1^* + r_n^* r_{n-h}^*\right)$$

is $O_p((Hm)^{1/2}n^{-1})$ by calculating the mean and variance, given $h > 0$, of a sum of conditionally independent Gaussian variables, and using **(a)** in conjunction with Lemma C.2 (a) for $k(\cdot)$. Hence, the stated asymptotic order readily follows by **(a)** for the first two terms, since the boundary terms for $h = n - 1$ are of order $O_p(H^{-1/2})$ smaller than these. **(c)** is established by using Barndorff-Nielsen et al. (2011, Proposition A.2), since Assumptions 2 and 3 provide sufficient summability conditions on the MMS noise components u and e, respectively, to ensure $\sum_{h\in\mathbb{Z}} |\Omega(h)| < \infty$.[4]

For **(d)**, since $Z_3(r^*, U)$ decomposes as

$$\begin{aligned} Z_3(r^*, U) = {} & 2\sum_{h=1}^{n-2} k\left(\frac{h}{H}\right)\left(U_{t_n} r_{n-h}^* - U_{t_0} r_{h+1}^*\right) + \frac{2}{H}\sum_{h=1}^{n-1} b\left(\frac{h}{H}\right)\left(r_n^* U_{t_{n-h}} - r_1^* U_{t_h}\right) \\ & + k(0)(U_{t_n} r_n^* - U_{t_0} r_1^*) + k((n-1)/H)(U_{t_n} r_1^* - U_{t_0} r_n^*), \end{aligned}$$

[4] In their notation, see also Section B.1 for definitions, this follows since $\sum_{h\in\mathbb{Z}} |\Omega(h)| < \infty$, together with the remaining conditions in Assumptions 2 and 3, suffices to show $mZ_h = O_p(1)$ for all $h \in \mathbb{Z}$, see also Lemmas C.6 (d) and C.7 (a).

it suffices to characterize probabilistic orders of $U_{t_n} r^*_{n-h}$ and $r^*_n U_{t_{n-h}}$in the first two terms, as the last two terms are $O_p(n^{-1/2})$ by the Cauchy-Schwarz inequality, **(a)** and **(c)**. Next, for the first term in $Z_3(r^*, U)$, make the decomposition $U_{t_n} r^*_{n-h} = e_{t_n} r^*_{n-h} + u_{t_n} r^*_{n-h}$. Then, for the exogenous noise contribution, it follows that $\mathbb{E}[u_{t_n} r^*_{n-h} | \mathcal{H}_1] = 0$ and

$$\mathbb{V}\left[u_{t_n} r^*_{n-h} | \mathcal{H}_1\right] = \frac{(r^*_{n-h})^2}{m^2} \sum_{i,j=1}^{m} \zeta_{t_{N-m+i}} \zeta_{t_{N-m+j}} \bar{\Omega}^{(uu)}(i-j) = O_p((mn)^{-1}),$$

since ζ_t is bounded $\forall t \in [0,1]$. Hence, $|u_{t_n} r^*_{n-h}| = O_p((mn)^{-1/2})$. Next, for the contribution of the endogenous noise component,

$$\begin{aligned} e_{t_n} r^*_{n-h} &= \frac{1}{mn^{1/2}} \sum_{j=1}^{m} \theta(t'_{N-m+j}, -(h+j)) \Upsilon_{t_{n-h-1}} \sigma_{t_{n-h-1}} (1 + o_p(1)) \\ &\leq \sup_{t \in [0,1]} |\Upsilon_t \sigma_t| \frac{1}{mn^{1/2}} (1 + o_p(1)) \sum_{j=1}^{m} \alpha_e(h+j) = O_p((mn^{1/2})^{-1}), \end{aligned}$$

since $\sup_{t \in [0,1]} |\Upsilon_t \sigma_t| \leq K$ and $\sum_{j=1}^{m} \alpha_e(h+j) = O(1)$, $\forall h \in \mathbb{Z}$. By combining these results with Lemma C.2 (a) for $k(\cdot)$, the first term in $Z_3(r^*, U)$ is uniformly $O_p((H/(mn))^{1/2})$.

Next, for the second term in $Z_3(r^*, U)$, decompose $r^*_n U_{t_{n-h}} = r^*_n e_{t_{n-h}} + r^*_n u_{t_{n-h}}$. Then, for the exogenous noise contribution, it follows that $\mathbb{E}[r^*_n u_{t_{n-h}} | \mathcal{H}_1] = 0$ and

$$\mathbb{V}[r^*_n u_{t_{n-h}} | \mathcal{H}_1] = \bar{\Omega}^{(uu)}(0) \zeta^2_{t_{n-h}} (r^*_n)^2 = O_p(m/n),$$

using **(a)**. Hence, $|r^*_n u_{t_{n-h}}| = O_p((m/n)^{1/2})$. Moreover, noting that $r^*_n = m^{-1} \sum_{i=1}^{m} \sum_{j=1}^{i} \Delta p^*_{t'_{N-m+j}}$, the contribution of the endogenous noise component is determined by writing

$$\begin{aligned} r^*_n e_{t_{n-h}} &= \frac{1}{mn^{1/2}} \sum_{i=1}^{m} \sum_{j=1}^{i} \theta(t_{n-h}, h+j) \Upsilon_{t'_{N-m+j-1}} \sigma_{t'_{N-m+j-1}} (1 + o_p(1)) \\ &\leq \frac{1}{mn^{1/2}} \sup_{t \in [0,1]} |\Upsilon_t \sigma_t| (1 + o_p(1)) \sum_{i=1}^{m} \sum_{j=1}^{i} \alpha_e(h+j) \leq O_p(mn^{-1/2}), \end{aligned}$$

since $\sum_{i=1}^{m} \sum_{j=1}^{i} \alpha_e(h+j) \leq O(m^2)$, $\forall h \in \mathbb{Z}$. Hence, by combining results, the stochastic order of second term in $Z_3(r^*, U)$ is uniformly $O_p(m(Hn)^{-1/2})$, thus providing the final result. □

Remark 3. *Lemma C.3 shows that slightly stronger conditions on m are required to avoid end-effects influencing the asymptotic distribution than those in Barndorff-Nielsen et al. (2008, 2011a) and Ikeda (2015). This is due to cross-products involving the endogenous noise component.*

The fourth lemma provides maximal inequalities to bound partial sums of polynomially decaying autocovariances for the MMS noise.

Lemma C.4. *(Maximal inequalities) Under Assumptions 2 and 3, denote* $\Omega^{(zz)} = \{\Omega^{(ee)}, \bar{\Omega}^{(uu)}\}$ *as well as* $\alpha_z(n) = \{\alpha_e(n), \alpha_u(n)\}$ *for* $z = \{e, \bar{u}\}$*, then, for some* $n, p \in \mathbb{N}^+$*, it follows that*

$$\sum_{j=n}^{\infty} |\Omega^{(zz)}(j)|^p \leq \sum_{j=n}^{\infty} \alpha_z(j)^p \leq K\alpha_z(n)^p.$$

Proof. First, write $f(n) = \alpha_z(n)^{-p}$, $h(n) = n^{(r_z+\epsilon)p}$, $F(n) = \sum_{j=n}^{\infty} f(j)^{-1}$ and $H(n) = \sum_{j=n}^{\infty} h(j)^{-1}$. Then, by applying the mixing inequality for the exogenous noise component, $z = \bar{u}$, e.g. Davidson (2002, Corollary 14.3), and Dahlhaus & Polonik (2009, Proposition 5.4 (48)) for the endogenous noise, $z = e$, it follows that $|\bar{\Omega}^{(uu)}(n)|^p \leq \alpha_u(n)^p$ and $|\Omega^{(ee)}(n)|^p \leq \alpha_e(n)^p$, respectively.[5] This establishes the first inequality. For the second, use the first inequality to write

$$\sum_{j=n}^{\infty} |\Omega^{(zz)}(j)|^p \leq F(n) = \sum_{j=n}^{\infty} j^{-(r_z+\epsilon)p} \left(\alpha_z(j)^p j^{(r_z+\epsilon)p}\right) \leq \sup_{j\geq n} \left(\frac{h(j)}{f(j)}\right) H(n) \leq K\frac{h(n)}{f(n)} H(n).$$

Then, since $h(n+1)H(n+1) \leq kh(n)H(n)$ by $h(n+1)/h(n) \leq k$ and $H(n+1)/H(n) \leq 1$, this descending sequence may be bounded as $h(n)H(n) \leq kH(1)$. Finally, since $H(1)$ is a p-series with exponent $p(r_z + \epsilon) > 1$, $kH(1) \leq K$, providing the second inequality. □

The fifth lemma derives a general stable central limit theorem for weighted cross-products between orthogonal variables that may individually exhibit temporal dependence, however, with one of the variables being the exogenous α-mixing part of the MMS noise, $\bar{u}$.

Lemma C.5. *(Cross-product CLT) Under the conditions of Theorem 1, suppose that* $\{x_t\}_{t\in[0,1]}$ *is an* $\mathcal{X}_1$*-measurable, bounded random variable where* $\mathcal{X}_t \subset \mathcal{F}_t$ *is a* σ*-algebra on* $\left(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P}\right)$ *satisfying* $\mathcal{X}_t \perp\!\!\!\perp \mathcal{G}_s$ $\forall (t,s) \in [0,1]^2$*. Furthermore, let* $(b_1, b_2) = \{(1,0),(0,1)\}$ *such that*

$$\sum_{h\in\mathbb{Z}_{n-1}} \frac{1}{n^{b_2}} \sum_{i\in S^{(1,h)}} x_{t_i} x_{t_{i-h}} \xrightarrow{\mathbb{P}} \sum_{h\in\mathbb{Z}} \int_0^1 c_t(h)dt = \Omega^{(xx)} \tag{C.1}$$

where $c_t(h)$ *is* $\mathcal{X}_1$*-measurable* $\forall h \in \mathbb{Z}$*,* $\mathbb{P}$*-uniformly bounded* $\forall (h,t) \in \mathbb{Z} \times [0,1]$ *and* $\Omega^{(xx)} \in (0,\infty)$ $\mathbb{P}$*-almost surely. Lastly, define the cross-product realized kernel estimator*

$$RK(f, x, \bar{u}) = \frac{1}{H^{b_1}} \sum_{h\in\mathbb{Z}_{n-1}} f\left(\frac{h}{H}\right) \frac{1}{n^{b_2}} \sum_{i\in S^{(1,h)}} x_{t_i} \bar{u}_{t_{i-h}},$$

where $f(x)$ *is a weight function, which is differentiable at all but a finite number of points and, moreover,* $f^{(jj)} = \int_{-\infty}^{\infty} [f^{(j)}(x)]^2 dx < \infty$ *for* $j = 0$ *and* $j = 1$ *almost everywhere. Then,*

(a) $\mathbb{E}[RK(f, x, \bar{u})|\mathcal{X}_1] = 0$ *and* $n^{b_2} H^{2b_1-1} \mathbb{V}[RK(f, x, \bar{u})|\mathcal{X}_1] \xrightarrow{\mathbb{P}} f^{(00)} \bar{\Omega}^{(uu)} \Omega^{(xx)}$.

[5] Details on how to adapt the notation in Dahlhaus & Polonik (2009, Proposition 5.4) to the present setting are provided in Section 7 of this online supplementary material, along with details on locally stationary processes.

(b) $n^{b_2/2}H^{(2b_1-1)/2}RK(f,x,\bar{u}) \overset{d_s(\mathcal{X}_1)}{\to} MN\left(0, f^{(00)}\bar{\Omega}^{(uu)}\Omega^{(xx)}\right)$.

Proof. **(a)** $\mathbb{E}[RK(f,x,\bar{u})|\mathcal{X}_1] = 0$ is trivial. Next, write

$$\mathbb{V}[RK(f,x,\bar{u})|\mathcal{X}_1] = \frac{1}{H^{2b_1}}\frac{1}{n^{b_2}}\sum_{h\in\mathbb{Z}_{n-1}}\sum_{g\in\mathbb{Z}_{n-1}} f\left(\frac{h}{H}\right) f\left(\frac{g}{H}\right) C_n(h,g)$$

where $C_n(h,g)$ may be written, using a change of variables $j-i=l$, as

$$C_n(h,g) = \frac{1}{n^{b_2}}\sum_{i\in S^{(1,h)}}\sum_{l\in S^{(1,g)}-i} x_{t_i}x_{t_{i+l}}\bar{\Omega}^{(uu)}(l+g-h) = \sum_{l\in\mathbb{Z}_{n-1,h,g}}\bar{\Omega}^{(uu)}(l+g-h)\frac{1}{n^{b_2}}\sum_{i\in S_l^{(1,g,h)}} x_{t_i}x_{t_{i+l}}$$

By another change of variables $g-h=z$,

$$\begin{aligned} n^{b_2}H^{2b_1-1}\mathbb{V}[RK(f,x,\bar{u})|\mathcal{X}_1] &= \frac{1}{H}\sum_{h\in\mathbb{Z}_{n-1}}\sum_{z\in\mathbb{Z}_{n-1}-h} f\left(\frac{h}{H}\right) f\left(\frac{h+z}{H}\right)\sum_{l\in\mathbb{Z}_{n-1,h,h+z}}\bar{\Omega}^{(uu)}(l+z) \\ &\quad\times\frac{1}{n^{b_2}}\sum_{i\in S_u^{(1,h+z,h)}} x_{t_i}x_{t_{i+l}} \\ &= \left(1+O(H^{-1})\right)\sum_{z\in\mathbb{Z}_{2(n-1)}}\frac{1}{H}\sum_{h\in\mathbb{Z}_{n-1,z,z}} f\left(\frac{h}{H}\right)^2\sum_{l\in\mathbb{Z}_{n-1,h,h+z}}\bar{\Omega}^{(uu)}(l+z) \\ &\quad\times\frac{1}{n^{b_2}}\sum_{i\in S_l^{(1,h+z,h)}} x_{t_i}x_{t_{i+l}} \overset{\mathbb{P}}{\to} f^{(00)}\bar{\Omega}^{(uu)}\Omega^{(xx)}, \end{aligned}$$

where the second equality follows by Taylor's theorem since $f(x)$ is differentiable at all but a finite number of points and $f^{(11)} < \infty$ almost everywhere. The final convergence to the probability limit follows by switching the order of summation with respect to h and (l,i) and using (C.1). **(b)** First, rewrite $RK(f,x,\bar{u})$ as

$$RK(f,x,\bar{u}) = \sum_{i\in S^{(1,0)}}\bar{u}_{t_i}\bar{w}_{n,i}, \qquad \bar{w}_{n,i} = \frac{1}{H^{b_1}}\frac{1}{n^{b_2}}\sum_{h\in S^{(1,0)}} f\left(\frac{h-i}{H}\right)x_{t_h}$$

and define the sequences $(\bar{K}_n,\bar{L}_n)\in\mathbb{R}^+\times\mathbb{R}^+$ where $\bar{K}_n = O(n^{\bar{k}})$ and $\bar{L}_n = O(n^{\bar{l}})$ for $0<\bar{k}<\bar{l}<1$. The stable central limit theorem, then, follows by the central limit theorem for weighted α-mixing processes from Yang (2007, Theorem 3.1), $\mathcal{X}_1$-conditionally, in conjunction with the moment results in **(a)** and Lemmas C.1 (a)-(b), since the following four conditions are shown below to hold:

(1) $\bar{w}_n = \max_{i\in S^{(1,0)}}|\bar{w}_{n,i}| \le O_p(\sum_{i\in S^{(1,0)}}\bar{w}_{n,i}^2) = O_p(\mathbb{V}[RK(f,x,\bar{u})|\mathcal{X}_1])$,

(2) $n\bar{L}_n^{-1}\alpha_u(\bar{K}_n) = o(1)$,

(3) $n\bar{K}_n\bar{L}_n^{-1}\bar{w}_n^2\mathbb{V}[RK(f,x,\bar{u})|\mathcal{X}_1]^{-1} = o_p(1)$, and

(4) $\bar{L}_n \sum_{i\in S^{(1,0)}} \bar{w}_{n,i}^2 = o_p(1)$.

First, for (1), the last equality of orders is immediate from the derivations in **(a)**. The first inequality follows by observing that $\max_{i\in S^{(1,0)}} |\bar{w}_{n,i}| = O_p(H^{(1-2b_1)/2} n^{-(1+b_2)/2}) \leq O_p(H^{1-2b_1} n^{-b_2})$ since $O(1) \leq O(H^{(1-2b_1)/2} n^{(1-b_2)/2})$ for both combinations of (b_1, b_2) using (C.1) as well as Lemmas C.2 (a) and (d) for the kernel function to construct the bounds. Since $r_u \in \mathbb{N}^+$, (2) is satisfied by having $0 < (1-\bar{l})/(1+r_u+\epsilon) < \bar{k} < \bar{l} < 1$. For (3), $n\bar{K}_n \bar{L}_n^{-1} \bar{w}_n^2 \mathbb{V}[RK(f,x,\bar{u})|\mathcal{X}_1]^{-1} = n\bar{K}_n \bar{L}_n^{-1} O_p(n^{-1}) \xrightarrow{\mathbb{P}} 0$ trivially by condition (1). Last, (4) follows by noting that the conditions for $\bar{L}_n O_p(H^{(1-2b_1)} n^{-b_2}) \xrightarrow{\mathbb{P}} 0$ are $\bar{l} - \nu < 0$ for $(b_1, b_2) = (1, 0)$ and $\bar{l} - (1-\nu) < 0$ for $(b_1, b_2) = (0, 1)$. Setting $\nu = 1/2$ as required for the central limit theorem in Theorem 1, such conditions are easily satisfied. □

Remark 4. *Lemma C.5 generalizes Ikeda (2015, Lemma 4) by applying more generally to series of weighted products of orthogonal variables, which individually may exhibit temporal dependence. Note that condition (3) is slightly different from Yang (2007, (3.4)) and that Yang (2007, Assumption 2 (i)) is omitted. However, careful inspection of the proof on pp. 1022-1023 shows that condition (3) suffices for line 7, and that condition (4) is sufficient for Yang (2007, (3.14)), as $\tilde{r} = r + 2$, in their notation, may replace r in the last five lines of their proof since $\exists v > 4 : \sup_{i=0,\ldots,N} \mathbb{E}[|\bar{u}_{t_i}|^v] < \infty$.*

The sixth lemma provides several convergence results for moments of the endogenous noise component, e, as well as for its covariation with increments of the efficient price process, r^*. Some of these results rely on Dahlhaus & Polonik (2009, Proposition 5.4). As previously mentioned, details about the latter are deferred to Section 7 below.

Lemma C.6. *(Moment Results for e) Under the conditions of Theorem 1,*

(a) $\sum_{i\in S_0^{(1,h,-s)}} \left|\mathrm{Cov}[e_{t_{i-h}}, e_{t_{i+s}}] - \Omega_{t_i}^{(ee)}(h+s)\right| \leq K\left(1 + \min(|h|, n)\alpha_e(h+s)\right)$.

(b) $n^{-1} \sum_{i\in S_0^{(1,h,-s)}} \mathrm{Cov}[e_{t_{i-h}}, e_{t_{i+s}}] \to \int_0^1 \Omega_t^{(ee)}(h+s)dt$.

(c) $\sum_{i\in S_0^{(1,h,-s)}} \left|\mathrm{Cov}[e_{t_{i-h}}, e_{t_{i+s}}|\mathcal{H}_1] - \Omega_{t_i}^{(ee)}(h+s)\right| \leq K\left(1 + \min(|h|, n)\alpha_e(h+s)\right)(1 + o_p(1))$.

(d) $n^{-1} \sum_{i\in S_0^{(1,h,-s)}} \mathrm{Cov}[e_{t_{i-h}}, e_{t_{i+s}}|\mathcal{H}_1] \xrightarrow{\mathbb{P}} \int_0^1 \Omega_t^{(ee)}(h+s)dt$.

(e) $H(\mathrm{B.19.1}) \xrightarrow{\mathbb{P}} 8\lambda^{(11)} \int_0^1 \sigma_t^2 \Omega_t^{(ee)} dt$.

(f) $\sum_{i\in S^{(1,h)}} \sum_{j\in S^{(1,g)}} \left|\tilde{\Omega}_{t_{j-g}}^{(ep)}(i-j+g)\tilde{\Omega}_{t_{i-h}}^{(ep)}(j-i+h) - \Omega_{t_{j-g}}^{(ep)}(i-j+g)\Omega_{t_{i-h}}^{(ep)}(j-i+h)\right| \leq O_p(1)$.

(g) $\left|\Omega_{t_h}^{(ee)}(z) - \Omega_{t_i}^{(ee)}(z)\right| \leq |h-i|\alpha_e(z)K/n$.

Proof. First, **(a)** follows directly by Dahlhaus & Polonik (2009, Proposition 5.4 (49)). **(b)** follows by applying **(a)** and Riemann integration. **(c)** follows similarly to the proof of Dahlhaus & Polonik (2009, Proposition 5.4) after using the Markov inequality to ensure convergence in probability of squares and cross-products of standard Brownian motions. **(d)** follows by applying **(c)**. For **(e)**, using the same change of variables as in the proof of Lemma C.5 (a), the last part of H(B.19.1) may be rewritten as

$$\sum_{i\in S^{(1,h)}}\sum_{j\in S^{(1,g)}}\mathrm{Cov}\left[r_i^*,r_j^*|\mathcal{H}_1\right]\mathrm{Cov}\left[e_{t_{i-h}},e_{t_{j-g}}|\mathcal{H}_1\right]$$
$$=\sum_{l\in\mathbb{Z}_{n-1,h,h+z}}\sum_{i\in S_l^{(1,h+z,h)}}r_i^*r_{i+l}^*\,\mathrm{Cov}\left[e_{t_{i-h}},e_{t_{i+l-(h+z)}}|\mathcal{H}_1\right]=(\text{C.19.1})+(\text{C.19.2}),$$

where, using the bound in **(c)**, it follows that

$$(\text{C.19.1})=\sum_{i\in S_0^{(1,h+z,h)}}(r_i^*)^2\,\mathrm{Cov}\left[e_{t_{i-h}},e_{t_{i-(h+z)}}|\mathcal{H}_1\right]=\sum_{i\in S_0^{(1,h+z,h)}}(r_i^*)^2\Omega_{t_i}^{(ee)}(-z)(1+o_p(1)),$$
$$(\text{C.19.2})=\sum_{l\in\mathbb{Z}_{n-1,h,h+z}\setminus\{0\}}\sum_{i\in S_l^{(1,h+z,h)}}r_i^*r_{i+l}^*\,\mathrm{Cov}\left[e_{t_{i-h}},e_{t_{i+l-(h+z)}}|\mathcal{H}_1\right]$$
$$\le\sum_{l\in\mathbb{Z}_{n-1,h,h+z}\setminus\{0\}}\sup_{i\in S_l^{(1,h+z,h)}}\left|\Omega_{t_i}^{(ee)}(l-z)(1+o_p(1))\right|\sum_{i\in S_l^{(1,h+z,h)}}|r_ir_{i+l}|=O_p(n^{-1/2}),$$

since, uniformly, $\sum_{i\in S_l^{(1,h+z,h)}}r_ir_{i+l}=O_p(n^{-1/2})$ for $l\neq 0$, and

$$\sum_{l\in\mathbb{Z}_{n-1,h,h+z}\setminus\{0\}}\sup_{i\in S_l^{(1,h+z,h)}}\left|\Omega_{t_i}^{(ee)}(l-z)\right|\le\sum_{l\in\mathbb{Z}_{n-1,h,h+z}\setminus\{0\}}K\alpha(l-z)<\infty.$$

Hence, using a Taylor approximation for $b(|h+z|/H)$, (C.19.1) and (C.19.2) are used to write

$$H(\text{B.19.1})=4(1+O(H^{-1}))\sum_{z\in\mathbb{Z}_{2(n-1)}}\frac{1}{H}\sum_{h\in\mathbb{Z}_{n-1,z,z}}b\left(\frac{|h|}{H}\right)^2\left(O_p(n^{-1/2})+\sum_{i\in S_0^{(1,h+z,h)}}(r_i^*)^2\Omega_{t_i}^{(ee)}(-z)\right)$$
$$\xrightarrow{\mathbb{P}}8\lambda^{(11)}\int_0^1\sigma_t^2\Omega_t^{(ee)}dt,$$

where the final convergence to the probability limit follows by switching the order of summation with respect to h and i. For **(f)**, denote

$$(\text{C.19.3})=\sum_{i\in S^{(1,h)}}\sum_{j\in S^{(1,g)}}\left|\tilde{\Omega}_{t_{j-g}}^{(ep)}(i-j+g)\tilde{\Omega}_{t_{i-h}}^{(ep)}(j-i+h)-\Omega_{t_{j-g}}^{(ep)}(i-j+g)\Omega_{t_{i-h}}^{(ep)}(j-i+h)\right|,$$

then

$$(\text{C.19.3})\le\sum_{i\in S^{(1,h)}}\sum_{j\in S^{(1,g)}}\left|\tilde{\Omega}_{t_{i-h}}^{(ep)}(j-i+h)\right|\left|\tilde{\Omega}_{t_{j-g}}^{(ep)}(i-j+g)-\Omega_{t_{j-g}}^{(ep)}(i-j+g)\right|$$
$$+\sum_{i\in S^{(1,h)}}\sum_{j\in S^{(1,g)}}\left|\Omega_{t_{j-g}}^{(ep)}(i-j+g)\right|\left|\tilde{\Omega}_{t_{i-h}}^{(ep)}(j-i+h)-\Omega_{t_{i-h}}^{(ep)}(j-i+h)\right|$$

$$\leq \frac{K}{n} \sum_{i \in S^{(1,h)}} \sum_{j \in S^{(1,g)}} \left(\left| \tilde{\Omega}^{(ep)}_{t_{i-h}}(j-i+h) \right| + \left| \Omega^{(ep)}_{t_{j-g}}(i-j+g) \right| \right) (1+o_p(1)) \leq O_p(1),$$

using first the triangle inequality, and then $\sup_g \sum_{i=1}^{n} |\theta(t_i, g) - \theta_{t_i}(g)| \leq K$ and

$$\left| \tilde{\Omega}^{(ep)}_{t_{j-g}}(i-j+g) - \Omega^{(ep)}_{t_{j-g}}(i-j+g) \right| \leq \sup_{t \in [0,1]} |\Upsilon_t \sigma_t| \left| \theta(t_{j-g}, i-j+g) - \theta_{t_{j-g}}(i-j+g) \right|$$

to establish the second inequality. Finally, for **(g)**, the triangle inequality is used repeatedly to make the decomposition $|\Omega^{(ee)}_{t_h}(z) - \Omega^{(ee)}_{t_i}(z)| \leq |h-i| K |\Omega^{(ee)}_{t_h}(z) - \Omega^{(ee)}_{t_{h-1}}(z)|$. Then,

$$\begin{aligned}
\left| \Omega^{(ee)}_{t_h}(z) - \Omega^{(ee)}_{t_{h-1}}(z) \right| &\leq \sum_{j=-\infty}^{\infty} \left(|\theta_{t_h}(z+j)||\theta_{t_h}(j) - \theta_{t_{h-1}}(j)| + |\theta_{t_{h-1}}(j)||\theta_{t_{h-1}}(z+j) - \theta_{t_h}(z+j)| \right) \\
&\leq \frac{k}{n} \sum_{j=-\infty}^{\infty} \left(\alpha_e(z+j)\alpha_e(j) + \alpha_e(j)\alpha_e(z+j) \right) \leq \frac{K}{n} \alpha_e(z)
\end{aligned}$$

where the first inequality follows by adding and subtracting $\theta_{t_h}(z+j)\theta_{t_{h-1}}(j)$ along with the triangle inequality, and the second by Assumptions 3(3) and 3(5). This provides the final result. □

The seventh lemma not only establishes a convergence result for autocovariances of an α-mixing process multiplied with a diurnally heteroskedastic component, ζ, it also provides the asymptotic variance and stable central limit theory for a long-run variance estimator applied to such variables. Finally, it studies conditional autocovariances for the transformed variables $\tilde{e}$ and $\tilde{r}^*$.

Lemma C.7. *(Heteroskedasticity Robust Results for u) Under the conditions of Theorem 1,*

(a) $n^{-1} \sum_{i \in S^{(2,h)}} \text{Cov}\left[u_{t_i} u_{t_{i-h}} | \mathcal{H}_1 \right] \xrightarrow{\mathbb{P}} \bar{\Omega}^{(uu)}(h) \int_0^1 \zeta_t^2 dt.$

(b) $n^{-1} \sum_{i \in S_0^{(1,h,-s)}} \text{Cov}[\tilde{e}_{t_{i-h}}, \tilde{e}_{t_{i+s}} | \mathcal{H}_1] \xrightarrow{\mathbb{P}} \int_0^1 \Omega_t^{(ee)}(h+s) \zeta_t^2 dt.$

(c) $\sum_{h \in \mathbb{Z}_{n-1}} \sum_{i \in S^{(1,h)}} \tilde{r}_i^* \tilde{r}_{i-h}^* \xrightarrow{\mathbb{P}} \int_0^1 \sigma_t^2 \zeta_t^2 dt.$

(d) $\mathbb{V}[A(u)|\mathcal{H}_1] = 4nH^{-3}\lambda^{(22)}[\bar{\Omega}^{(uu)}]^2 \int_0^1 \zeta_t^4 dt(1+o_p(1)) = 4nH^{-3}\lambda^{(22)} \int_0^1 [\Omega_t^{(uu)}]^2 dt(1+o_p(1)).$

(e) $(H^3 n^{-1})^{1/2} \left(A(u) - \mathbb{E}[A(u)|\mathcal{H}_1] \right) \xrightarrow{d_s(\mathcal{H}_1)} MN\left(0, 4\lambda^{(22)} \int_0^1 [\Omega_t^{(uu)}]^2 dt\right)$, *when* $\nu \in (1/3, 2/3)$.

Proof. First, for **(a)**, write

$$\frac{1}{n} \sum_{i \in S^{(2,h)}} \text{Cov}\left[u_{t_i} u_{t_{i-h}} | \mathcal{H}_1 \right] = \bar{\Omega}^{(uu)}(h) \frac{1}{n} \sum_{i \in S^{(2,h)}} \zeta_{t_i} \zeta_{t_{i-h}} = \bar{\Omega}^{(uu)}(h) \frac{1}{n} \sum_{i \in S^{(2,h)}} \zeta_{t_i}^2 (1 + O_p(|h|/n))$$

by Lipschitz continuity of ζ_{t_i} in Assumption 2. The result, then, follows by Riemann integration.

Next, **(b)** follows, similarly to **(a)**, by writing

$$\frac{1}{n}\sum_{i\in S_0^{(1,h,-s)}}\mathrm{Cov}[\tilde{e}_{t_{i-h}},\tilde{e}_{t_{i+s}}|\mathcal{H}_1]=\frac{1}{n}\sum_{i\in S_0^{(1,h,-s)}}\mathrm{Cov}[e_{t_{i-h}},e_{t_{i+s}}|\mathcal{H}_1]\zeta_{t_i}^2(1+O_p((|h|+|s|)/n))$$

and using Riemann integration, independence of ζ and e along with Lemma C.6 (d). **(c)** follows by applying the approximation $r_i^*=\sigma_{t_{i-1}}\Delta W_{t_i}(1+o_p(n^{-1/2}))$ from Lemma B.1 in conjunction with independence between the Brownian increments in the sum, independence between r and ζ, and Riemann integration. **(d)** Let $|h|,|s|\leq k$, then $\zeta_{t_i}\zeta_{t_{i-h}}\zeta_{t_{i-g}}\zeta_{t_{i-h-s}}=\zeta_{t_i}^4(1+O_p((|g|+|k|)/n))$, a change of variables $s=i-j$, and Riemann integration provide the representation

$$\frac{1}{n}\sum_{i\in S^{(2,h)}}\sum_{j\in S^{(2,g)}}\mathrm{Cov}\left[\bar{u}_{t_i}\bar{u}_{t_{i-h}},\bar{u}_{t_j}\bar{u}_{t_{j-g}}|\mathcal{H}_1\right]\zeta_{t_i}\zeta_{t_{i-h}}\zeta_{t_j}\zeta_{t_{j-g}}=$$

$$\sum_{s\in Z}\left(\bar{\Omega}_s\bar{\Omega}_{s-h+g}+\bar{\Omega}_{s-h}\bar{\Omega}_{s+g}+\bar{\kappa}_4(0,s,s-h,-g)\right)\int_0^1\zeta_t^4dt(1+o_p(1)),$$

as in Ikeda (2015, Section A.3.4). This conditional fourth moment result together with a spectral representation,

$$A(\bar{u})=s_n\int_{-\pi}^{\pi}\mathcal{I}_n(\delta,\bar{u})\mathcal{K}_n(\delta)\,d\delta,$$

using the definitions from the proof of Lemma B.2, imply that Rosenblatt (1984, Theorem 2) may be invoked to establish the asymptotic variance, since the re-scaled kernel function, $\beta(x)$, and its associated spectral window, $\mathcal{K}_n(\delta)$, satisfy the regularity conditions in Lemmas C.2 (b) and (c), and since the scalar contribution from the diurnally heteroskedastic component, $\int_0^1\zeta_t^4dt>0$, may easily be accommodated in the scale, s_n, when conditioning on $\mathcal{H}_1$.[6]

Finally, for **(e)**, and as in the proof of Lemma C.5 (b), write

$$A(u)=\sum_{i\in S^{(1,0)}}\bar{u}_{t_i}\bar{w}_{n,i},\qquad \bar{w}_{n,i}=\frac{n}{H^2}\frac{1}{n}\sum_{h\in S^{(1,0)}}a\left(\frac{|h|-i}{H}\right)\bar{u}_{t_h}\zeta_{t_h}\zeta_{t_i}$$

and use the same argument as in **(a)** to show

$$\bar{w}_{n,i}=(1+O_p(n^{-1}))\frac{n}{H^2}\frac{1}{n}\sum_{h\in S^{(1,0)}}a\left(\frac{|h|-i}{H}\right)\bar{u}_{t_h}\zeta_{t_h}^2.$$

The stable central limit theorem, then, follows by the central limit theorem for weighted α-mixing processes from Yang (2007, Theorem 3.1), $\mathcal{H}_1$-conditionally, in conjunction with **(d)** and Lemmas C.1 (a)-(b), if the *same* four conditions, as used to establish Lemma C.5 (b) above, can be shown to hold. First, for (1), the last equality of asymptotic orders is immediate since ζ_t is bounded $\forall t\in[0,1]$. The

[6] Section 7 of this online supplementary material explains how this representation relates to spectral analysis in the context of locally stationary processes.

order of $\bar{w}_n = \max_{i\in S^{(1,0)}} \leq O_p(H^{-3/2})$ follows since

$$\sup_{h\in S^{(1,0)}} |\bar{u}_{t_h}| = O_p(1), \quad \text{and} \quad \sup_{i\in S^{(1,0)}} \Big(\sum_{h\in S^{(1,0)}} a\left((|h|-i)/H\right)\Big)^2 = O(H)$$

by Lemma C.2 (d). Moreover, $\sum_{i\in S^{(1,0)}} \bar{w}_{n,i}^2 = O_p(nH^{-3})$ follows as in **(d)**. Hence, the bound $O_p(H^{-3/2}) \leq O_p(nH^{-3})$ is satisfied when $H \propto n^{\nu}$, $\nu < 2/3$. (2) is immediate from the corresponding proof of Lemma C.5 (b). (3) $n\bar{K}_n\bar{L}_n^{-1}\bar{w}_n^2\mathbb{V}[A(u)|\mathcal{H}_1]^{-1} = n\bar{K}_n\bar{L}_n^{-1}O_p(n^{-1}) \xrightarrow{\mathbb{P}} 0$ trivially by $0 < \bar{k} < \bar{l} < 1$. Finally, for (4), $\bar{L}_n\sum_{i\in S^{(1,0)}} \bar{w}_{n,i}^2 = \bar{L}_nO_p(n^{1-3\nu}) \xrightarrow{\mathbb{P}} 0$ by setting $0 < \bar{k} < \bar{l} < \min(1, 3\nu-1)$. □

Remark 5. *Lemma C.7 (e) generalizes the central limit theory result for spectral density estimates (at frequency zero) of α-mixing random variables in Rosenblatt (1984, Theorem 2) by allowing the variables to exhibit diurnal heteroskedasticity through the multiplicative component, ζ_t.*

The eighth lemma establishes additional bounds on end-effects for the proof of Theorem 2.

Lemma C.8. *(Jittered Variables for Jump-diffusions) Under the conditions of Theorem 2, four uniform bounds hold:*

(a) $\Delta J_{t_1} + \Delta J_{t_n} = O_p(m/n)$.

(b) $Z_4(J) = O_p((mH^{1/2})/n^2)$.

(c) $Z_1(r^*, J) = O_p((m/n)^{3/2}) + O_p((mH)^{1/2}/n)$.

(d) $Z_3(J,U) = O_p(n^{-1}) + O_p(m/(H^{1/2}n)) + O_p(H^{1/2}/(mn)^{1/2})$.

Proof. For (**a**), recall for $\Delta J_{t_1} = m^{-1}\sum_{i=1}^m (J_{t'_m} - J_{t'_{i-1}}) = \sum_{i=1}^m (1-(i-1)/m)\Delta J_{t'_{i-1}}$. Then,

$$\mathbb{E}\left[|\Delta J_{t_1}|\right] \leq \sum_{i=1}^m (1-(i-1)/m)\mathbb{E}[|\Delta J_{t'_{i-1}}|] \leq \sum_{i=1}^m (1-(i-1)/m)\frac{K}{n}(1+O(n^{-1})) \leq O(m/n).$$

using the triangle inequality, $\forall t \in [0,1]$ $\mathbb{E}[N_t] < \infty$, and $\forall s = 1,\ldots,N_t$ $\mathbb{E}[|d_s|] < \infty$. This implies that $|\Delta J_{t_1}| \leq O_p(m/n)$, and the analogous result for ΔJ_{t_n} follows by symmetry. Next, the results in (**b**), (**c**), and (**d**) follow using arguments that are almost identical to those provided for Lemma C.3 above, noting that for $i \in S^{(1,0)}$, it follows that $\mathbb{V}[\Delta J_{t_i}] = \mathbb{V}[\Delta N_{t_i}]\mathbb{V}[\Delta d_{t_i}] = O(n^{-1})$. □

Finally, Lemmas C.9 and C.10 below provide first and second-order approximations, respectively, to establish consistency results for the (medium) blocked realized kernels.

Lemma C.9. *(First-order Approximation) Under the conditions of Theorem 3, then*

(a) $|\sum_{i=1}^{n_L}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - t)d\sigma_t^2| \leq O_p(n^{-\beta})$.

(b) $|\sum_{i=1}^{n_L}(\sigma_{\tau_i}^2 - \sigma_{\tau_{i-1}}^2)\Delta\tau| \leq O_p(n^{-\beta})$.

(c) *For $y = p$ and some fixed $j \in \mathbb{N}^+$,* $\big|\sum_{i=1+j}^{n_L}\big(RK_i^T(p) - RK_{i-j}^T(p)\big)\big| \leq O_p(n^{-\beta}) + o_p(n^{-1/4})$.

Proof. First, for (**a**), define $\mathcal{T}_1 = \sum_{i=1}^{n_L}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - t)d\sigma_t^2$. Then, as

$$
\begin{aligned}
[\mathcal{T}_1, \mathcal{T}_1] &= \sum_{i=1}^{n_L}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - t)^2 d[\sigma^2,\sigma^2]_t(1 + o_p(1)) = \frac{\Delta\tau^2}{3}\sum_{i=1}^{n_L}([\sigma^2,\sigma^2]_{\tau_i} - [\sigma^2,\sigma^2]_{\tau_{i-1}})(1 + o_p(1)) \\
&= \frac{\Delta\tau^2}{3}\int_0^1 d[\sigma^2,\sigma^2]_t(1 + o_p(1)) = O_p(n^{-2\beta}),
\end{aligned}
$$

the result follows by the Burkholder-Davis-Gundy (BDG) inequality.

Next, for (**b**), let $\mathcal{T}_2 = \sum_{i=1}^{n_L}(\sigma_{\tau_i}^2 - \sigma_{\tau_{i-1}}^2)\Delta\tau$. Then, as for (**a**),

$$
[\mathcal{T}_2, \mathcal{T}_2] = \Delta\tau^2\sum_{i=1}^{n_L} d[\sigma^2,\sigma^2]_{\tau_i}(1 + o_p(1)) = O_p(n^{-2\beta}),
$$

which, using again the BDG inequality, provides the result.

Lastly, for (**c**), use Lemma B.7 to write (up to an $o_p(1)$ error),

$$
\sum_{i=1+j}^{n_L}\big(RK_i^T(p) - RK_{i-j}^T(p)\big) = \sum_{i=1+j}^{n_L}\Big(\int_{\tau_{i-1}}^{\tau_i}\sigma_t^2 dt - \int_{\tau_{i-j-1}}^{\tau_{i-j}}\sigma_t^2 dt + \Delta\tilde{M}_{\tau_i} - \Delta\tilde{M}_{\tau_{i-j}}\Big).
$$

Then, it readily follows by Itô's lemma in conjunction with (**a**) and (**b**) that

$$
\Big|\sum_{i=1+j}^{n_L}\Big(\int_{\tau_{i-1}}^{\tau_i}\sigma_t^2 dt - \int_{\tau_{i-j-1}}^{\tau_{i-j}}\sigma_t^2 dt\Big)\Big| \leq O_p(n^{-\beta}).
$$

Finally, since $n^{1/4}|\sum_{i=1+j}^{n_L}\Delta\tilde{M}_{\tau_i} - \sum_{i=1+j}^{n_L}\Delta\tilde{M}_{\tau_{i-j}}| \leq o_p(1)$ follows by the same arguments used to establish equation (B.13), this concludes the proof. ☐

Lemma C.10. *(Second-order Approximation) Under the conditions of Theorem 3, then for the case without jumps $y = p$ and $j - j_1 \in [1, \ldots, B-1]$, it holds that*

$$
\frac{1}{\Delta\tau}\sum_{i=B}^{n_L}\sigma_{\tau_{i-B}}^{-2}\left(RK_{i-j_1}^T(p) - RK_{i-j}^T(p)\right)^2 \leq O_p(n^{-1/2+\beta}) + O_p(n^{-\beta}).
$$

Proof. Before proceeding, note that the initial steps of this proof follow the corresponding for Mykland, Shephard & Sheppard (2012, Theorem 10) closely. The later steps, however, differ, and, in fact, simplify, since only the stochastic orders of the approximation errors are of interest for Theorem 3, not the asymptotic central limit theory. First, write

$$
RK_{i-j_1}^T(p) - RK_{i-j}^T(p) = \int_{\tau_{i-j_1-1}}^{\tau_{i-j_1}}\sigma_t^2 dt + \Delta\tilde{M}_{\tau_{i-j_1}} - \int_{\tau_{i-j-1}}^{\tau_{i-j}}\sigma_t^2 dt - \Delta\tilde{M}_{\tau_{i-j}}
$$

$$= \left[\sigma^2_{\tau_{i-j_1-1}} - \sigma^2_{\tau_{i-j}}\right] \Delta\tau + L_{\tau_{i-j_1}} - L^-_{\tau_{i-j}}, \tag{C.2}$$

where $L_{\tau_{i-j_1}}$ and $L^-_{\tau_{i-j}}$ are defined as,

$$L_{\tau_{i-j_1}} = \int_{\tau_{i-j_1-1}}^{\tau_{i-j_1}} (\tau_{i-j_1} - t) d\sigma_t^2 + \Delta\tilde{M}_{\tau_{i-j_1}} \quad \text{and} \quad L^-_{\tau_{i-j}} = \int_{\tau_{i-j-1}}^{\tau_{i-j}} (t - \tau_{i-j-1}) d\sigma_t^2 + \Delta\tilde{M}_{\tau_{i-j}},$$

respectively. The second equality in (C.2) follows by Itô's lemma. Note, however, that unlike the setting of Mykland et al. (2012), the $[\sigma^2_{\tau_{i-j_1-1}} - \sigma^2_{\tau_{i-j}}]\Delta\tau$ term does not cancel out. This implies that the infinitesimal changes in the quadratic variation of $L_{\tau_{i-j_1}}$ for $t \in (\tau_{i-j_1-1}, \tau_{i-j_1}]$ expand as

$$d[L, L]_{\tau_{i-j_1},t} = d[\tilde{M}, \tilde{M}]_{\tau_{i-j_1},t} + (\tau_{i-j_1} - t)^2 d[\sigma^2\sigma^2]_t + 2(\tau_{i-j_1} - t) d[\tilde{M}, \sigma^2]_{\tau_{i-j_1},t},$$

whose cross-product term is seen to be of lower stochastic order by Lemma B.7. In particular, as the latter shows that $n^{1/4}[\tilde{M}, \sigma^2] = o_p(1)$, this implies $\left|(\tau_{i-j_1} - t) d[\tilde{M}, \sigma^2]_{\tau_{i-j_1},t}\right| \leq o_p(n^{-1/4-2\beta})$. A similar decomposition applies to $d[L^-, L^-]_{\tau_{i-j},t}$.

Now, since there is no overlap between $[\sigma^2_{\tau_{i-j_1-1}} - \sigma^2_{\tau_{i-j}}]$ in (C.2) and the stochastic volatility components of $L_{\tau_{i-j_1}}$ and $L^-_{\tau_{i-j}}$, respectively, it follows that

$$\begin{aligned}
\left(RK^T_{i-j_1}(p) - RK^T_{i-j}(p)\right)^2 = \Delta\tau^2 &\left([\sigma^2, \sigma^2]_{\tau_{i-j_1-1}} - [\sigma^2, \sigma^2]_{\tau_{i-j}}\right) \\
&+ \left(\int_{\tau_{i-j_1-1}}^{\tau_{i-j_1}} (\tau_{i-j_1} - t)^2 d[\sigma^2, \sigma^2]_t + \int_{\tau_{i-j-1}}^{\tau_{i-j}} (t - \tau_{i-j-1})^2 d[\sigma^2, \sigma^2]_t\right) \\
&+ \left(d[\tilde{M}, \tilde{M}]_{\tau_{i-j_1}} + d[\tilde{M}, \tilde{M}]_{\tau_{i-j}}\right) + \mathcal{R}_{i,j_1,j},
\end{aligned}$$

as in Mykland et al. (2012), where $|\mathcal{R}_{i,j_1,j}| = o_p(n^{-1/4-2\beta})$ collects lower order approximation errors.

Next, for the *first* term in the decomposition, define $\tilde{k} = \inf_{i=1,\dots,n_L} \sigma^2_{\tau_{i-B}}$, then $\tilde{k} > 0$, and

$$\begin{aligned}
\frac{1}{\Delta\tau} \sum_{i=B}^{n_L} \sigma^{-2}_{\tau_{i-B}} \Delta\tau^2 \left([\sigma^2, \sigma^2]_{\tau_{i-j_1-1}} - [\sigma^2, \sigma^2]_{\tau_{i-j}}\right) &\leq \frac{\Delta\tau}{\tilde{k}} \sum_{i=B}^{n_L} \sum_{g=0}^{j-j_1-1} d[\sigma^2, \sigma^2]_{\tau_{i-j+g}} \\
&\leq K \frac{\Delta\tau}{\tilde{k}} \sum_{i=B}^{n_L} d[\sigma^2, \sigma^2]_{\tau_{i-j_1-1}} (1 + o_p(1)) = O_p(n^{-\beta}),
\end{aligned}$$

since $\sum_{i=B}^{n_L} d[\sigma^2, \sigma^2]_{\tau_{i-j_1-1}} \xrightarrow{\mathbb{P}} \int_0^1 d[\sigma^2, \sigma^2]_t$ by Riemann integration, similarly to Lemma C.9 (b).

Now, for the *second* term,

$$\frac{1}{\Delta\tau} \sum_{i=B}^{n_L} \sigma^{-2}_{\tau_{i-B}} \int_{\tau_{i-j_1-1}}^{\tau_{i-j_1}} (\tau_{i-j_1} - t)^2 d[\sigma^2, \sigma^2]_t \leq \frac{\Delta\tau}{3\tilde{k}} \sum_{i=B}^{n_L} d[\sigma^2, \sigma^2]_{\tau_{i-j_1}} (1 + o_p(1)) = O_p(n^{-\beta}),$$

as above, and analogously for $\int_{\tau_{i-j-1}}^{\tau_{i-j}} (t - \tau_{i-j-1})^2 d[\sigma^2, \sigma^2]_t$. This is similar to Lemma C.9 (a).

Finally, for the *third* term,

$$\begin{aligned}\frac{1}{\Delta\tau}\sum_{i=B}^{n_L}\sigma_{\tau_{i-B}}^{-2}\left(d[\tilde{M},\tilde{M}]_{\tau_{i-j_1}}+d[\tilde{M},\tilde{M}]_{\tau_{i-j}}\right) &\leq \frac{1}{\tilde{k}}\frac{n^{-1/2}}{\Delta\tau}\sum_{i=B}^{n_L}n^{1/2}\left(d[\tilde{M},\tilde{M}]_{\tau_{i-j_1}}+d[\tilde{M},\tilde{M}]_{\tau_{i-j}}\right)\\ &=\frac{2}{\tilde{k}}\frac{n^{-1/2}}{\Delta\tau}\mathcal{V}(\lambda,a)(1+o_p(1))=O_p(n^{-1/2+\beta}),\end{aligned}$$

using Lemma B.7, thus establishing the approximation bounds. □

Remark 6. *Lemma C.10 generalizes the result in Mykland et al. (2012, Theorem 10) by providing stochastic bounds for blocks that are allowed to be separated by $j-j_1\in[1,\ldots,B-1]$ time increments, thus nesting the $j-j_1=1$ case. Note, however, that the proof of the former is highly similar to that of the latter. In addition, Mykland et al. (2012) also provide central limit theory for the $B=2$ case, albeit with end-averaged blocking and with jumps assumed absent.*

4 Asymptotic Results for the TSRK

This section establishes the central limit theory result stated for the TSRK in Section 3.4, including the properties of the characteristic parameters for its implied jack-knife kernel window.

First, rewrite the generalized jack-knife kernel representation of the TSRK as a convex combination of realized kernels

$$TSRK(p)=(1-\tau^2)^{-1}\left(RK(p,H)-\tau^2RK(p,G)\right),$$

whose asymptotic properties may, then, be deduced using Lemma 1, similarly to the corresponding proof in Ikeda (2015, Section A.3.2). Hence, under the conditions for Lemma 1, the TSRK has a bias

$$\mathbb{B}[TSRK(p)|\mathcal{H}_1]=\frac{nH^{-2}}{1-\tau^2}\left(H^{-q}-G^{-q}\right)\lambda_q^{(2)}\sum_{h\in\mathbb{Z}}|h|^q\Omega(h)+o_p(1)=O_p(nH^{-2}G^{-q}),$$

as the remaining bias terms disappear. Next, define $\mathcal{Z}(H)$, $\mathcal{E}_n(H)$ and $\mathcal{V}_n(\lambda,H)$ as the asymptotic distribution, end-point errors and variance, respectively, for $RK(p,H)$ and let analogous definitions be written with G in place of H for $RK(p,G)$. Then, it readily follows that $O_p(\tau^2\mathcal{E}_n(G))\leq O_p(\mathcal{E}_n(H))$ and, hence, both terms are asymptotically negligible when $\xi\in(1/4,1/2)$, as for Theorem 1(2). Moreover, since the mixed normal distribution is additive, Lemma C.1 (d) gives

$$\mathcal{Z}_{\mathcal{T}}\equiv(1-\tau^2)^{-1}\left(\mathcal{Z}(H)-\tau^2\mathcal{Z}(G)\right)\overset{d_s(\mathcal{H}_1)}{\rightarrow}MN\left(0,\lim_{n\to\infty}\mathcal{V}_{\mathcal{T},n}(\lambda)\right),$$

whose variance

$$\mathcal{V}_{\mathcal{T},n}(\lambda)=\left(\mathcal{V}_n(\lambda,H)+\tau^4\mathcal{V}_n(\lambda,G)-2\tau^2\operatorname{Cov}[RK(p,H),RK(p,G)|\mathcal{H}_1]\right)/(1-\tau^2)^2$$

may be rewritten using addition and subtraction of $\tau^4\mathcal{V}_n(\lambda,H)$ and $2\tau^2\mathcal{V}_n(\lambda,H)$ as

$$\begin{aligned}\mathcal{V}_{\mathcal{T},n}(\lambda) &= \mathcal{V}_n(\lambda,H) + \frac{\tau^4\left(\mathcal{V}_n(\lambda,G) - \mathcal{V}_n(\lambda,H)\right) + 2\tau^2\left(\mathcal{V}_n(\lambda,H) - \mathrm{Cov}[RK(p,H),RK(p,G)|\mathcal{H}_1]\right)}{(1-\tau^2)^2} \\ &\equiv \mathcal{V}_n(\lambda,H) + \mathcal{V}_{\mathcal{R},n}(\lambda).\end{aligned}$$

Next, the first term in $\mathcal{V}_{\mathcal{R},n}(\lambda)$ may readily be expanded as

$$\begin{aligned}\mathcal{V}_n(\lambda,G) - \mathcal{V}_n(\lambda,H) = (\tau-1)\times 4Hn^{-1}\lambda^{(00)}\int_0^1\sigma_t^4dt + (1/\tau^3-1)\times 4nH^{-3}\lambda^{(22)}\int_0^1\Omega_t^2dt \\ + (1/\tau-1)\times 8H^{-1}\lambda^{(11)}\int_0^1\left(\Omega_t\sigma_t^2 + 2\left(\Omega_t^{(ep)}\right)^2\right).\end{aligned}$$

Moreover, by defining $\lambda^{(jj)}(\tau) \equiv \int_0^\infty \lambda^{(j)}(x)\lambda^{(j)}(x/\tau)dx$ for $j=0,1,2$ and rewriting $RK(p,G) = RK(p,H/\tau)$, it immediately follows by the same derivations as for Lemma 1 and Theorem 1 that the second term of the residual asymptotic variance, $\mathcal{V}_{\mathcal{R},n}(\lambda)$, may be expanded as

$$\begin{aligned}\mathcal{V}_n(\lambda,H) - \mathrm{Cov}[RK(p,H),RK(p,G)|\mathcal{H}_1] = \left(\lambda^{(00)} - \lambda^{(00)}(\tau)\right)\times 4Hn^{-1}\int_0^1\sigma_t^4dt \\ + \left(\lambda^{(22)} - \lambda^{(22)}(\tau)\right)\times 4nH^{-3}\int_0^1\Omega_t^2dt \\ + \left(\lambda^{(11)} - \lambda^{(11)}(\tau)\right)\times 8H^{-1}\int_0^1\left(\Omega_t\sigma_t^2 + 2\left(\Omega_t^{(ep)}\right)^2\right).\end{aligned}$$

This suggests to make the decomposition $\mathcal{V}_{\mathcal{R},n}(\lambda) = \mathcal{V}_{\mathcal{R},n,1}(\lambda) + \mathcal{V}_{\mathcal{R},n,2}(\lambda) + \mathcal{V}_{\mathcal{R},n,3}(\lambda)$, where

$$\begin{aligned}\mathcal{V}_{\mathcal{R},n,1}(\lambda) &= \frac{\tau^4(\tau-1)\lambda^{(00)} + 2\tau^2\left(\lambda^{(00)} - \lambda^{(00)}(\tau)\right)}{(1-\tau^2)^2}\times 4Hn^{-1}\int_0^1\sigma_t^4dt, \\ \mathcal{V}_{\mathcal{R},n,2}(\lambda) &= \frac{\tau^4(1/\tau^3-1)\lambda^{(22)} + 2\tau^2\left(\lambda^{(22)} - \lambda^{(22)}(\tau)\right)}{(1-\tau^2)^2}\times 4nH^{-3}\int_0^1\Omega_t^2dt, \\ \mathcal{V}_{\mathcal{R},n,3}(\lambda) &= \frac{\tau^4(1/\tau-1)\lambda^{(11)} + 2\tau^2\left(\lambda^{(11)} - \lambda^{(11)}(\tau)\right)}{(1-\tau^2)^2}\times 8H^{-1}\int_0^1\left(\Omega_t\sigma_t^2 + 2\left(\Omega_t^{(ep)}\right)^2\right).\end{aligned}$$

Hence, $\mathcal{V}_{\mathcal{R},n}(\lambda)$ has the same decomposition as $\mathcal{V}_n(\lambda,H)$, which shows that the asymptotic variance $\mathcal{V}_{\mathcal{T},n}(\lambda)$ has the conjectured form $\mathcal{V}(\Phi,a)$ where $\Phi^{(jj)}(\tau) = \lambda^{(jj)} + f_j(\tau)$ $j=0,1,2$ with the $f_j(\tau)$ functions corresponding the highlighted scales in $\mathcal{V}_{\mathcal{R},n,1}(\lambda)$, $\mathcal{V}_{\mathcal{R},n,3}(\lambda)$ and $\mathcal{V}_{\mathcal{R},n,2}(\lambda)$ for $j=0,1,2$, respectively. However, it remains to be shown that $f_j(\tau)\in\mathbb{R}_+$ for all $j=0,1,2$ as well as $f_j(\tau)=O(\tau^2)$ for $j=0,1$ and $f_2(\tau)=O(\tau)$. The asymptotic orders readily follow since $\lambda^{(jj)}$ and $\lambda^{(jj)}(\tau)$ are bounded for all $j=0,1,2$, ensuring that the dominant components in the scales of $\mathcal{V}_{\mathcal{R},n,1}(\lambda)$, $\mathcal{V}_{\mathcal{R},n,2}(\lambda)$ and $\mathcal{V}_{\mathcal{R},n,3}(\lambda)$ are of orders $O(\tau^2)$, $O(\tau)$ and $O(\tau^2)$, respectively. It is more elaborate to show that $f_j(\tau)\in\mathbb{R}_+$ for $j=0,1,2$. First, consider the case $j=0$, for which showing $f_0(\tau)\in\mathbb{R}_+$ amounts to proving that

$$\tau^4(\tau-1)\lambda^{(00)} + 2\tau^2\left(\lambda^{(00)} - \lambda^{(00)}(\tau)\right) > 0. \tag{D.3}$$

Now, by a change of variables $y = x/\tau$ along with addition and subtraction, write

$$\int_0^\infty \lambda^{(j)}(x)\lambda^{(j)}(x/\tau)dx = \tau\int_0^\infty \lambda^{(j)}(y)\lambda^{(j)}(\tau y)dy = \tau\lambda^{(jj)} - \tau\int_0^\infty \lambda^{(j)}(y)\left(\lambda^{(j)}(y) - \lambda^{(j)}(\tau y)\right)dy,$$

which may be inserted into (D.3) and rewritten as

$$\begin{aligned}\tau^4(\tau-1)\lambda^{(00)} + \tau^2\left(\lambda^{(00)} - \lambda^{(00)}(\tau)\right) &= \tau^2\lambda^{(00)} + \tau^2\lambda^{(00)}(1-\tau-\tau^2+\tau^3)\\ &\quad + \tau^3\int_0^\infty \lambda^{(0)}(y)\left(\lambda^{(0)}(y) - \lambda^{(0)}(\tau y)\right)dy.\end{aligned}$$

Since $\tau \in (0,1)$ for all finite n, the polynomial $(1-\tau-\tau^2+\tau^3) > 0$. For the last term,

$$\begin{aligned}\int_0^\infty \lambda^{(0)}(y)\left(\lambda^{(0)}(y) - \lambda^{(0)}(\tau y)\right)dy &= \lambda^{(00)} - \int_0^\infty \lambda^{(0)}(y)\lambda^{(0)}(\tau y)dy \\ &\geq \lambda^{(00)} - \Big|\int_0^\infty \lambda^{(0)}(y)\lambda^{(0)}(\tau y)dy\Big| \geq \lambda^{(00)} - \sqrt{\lambda^{(00)}\times\lambda^{(00)}} = 0\end{aligned} \tag{D.4}$$

by the Cauchy-Schwarz inequality. This gives (D.3). The arguments to show that $f_j(\tau) \in \mathbb{R}_+$ for $j = 1, 2$ are similar since it suffices to prove $\lambda^{(jj)} - \lambda^{(jj)}(\tau) \geq 0$. As the latter follows by the same arguments given in (D.4), this shows that the asymptotic variance for the TSRK, $\mathcal{V}_{\mathcal{T},n}(\lambda)$, has the conjectured form $\mathcal{V}(\Phi, a)$, concluding the proof. □

5 Proofs of Asymptotic Results in the Main Text

This section provides proofs of Propositions 1-2, Lemmas 2-3, and Theorem A.1.

5.1 Proof of Proposition 1

(1) follows as $\mathbb{B}[TSRK(p)|\mathcal{H}_1] = O_p\left(n^{-1/2q}\right)$ and $\mathbb{B}[RK^*(p)|\mathcal{H}_1] \leq O_p\left(n^{-1/2(1-\gamma)(1+r)}\right)$ where the asymptotic order of the latter is strictly smaller when $q < (1-\gamma)(1+r)$ or, equivalently, when $\gamma < (1+r-q)/(1+r)$. (2) is trivial as $\lim_{n\to\infty}\Phi^{(jj)}(\tau) > \lambda^{(jj)}$ for $j = 0, 1, 2$. □

5.2 Proof of Proposition 2

First, (1) follows by considering

$$\mathbb{B}[RK^*(p)|\mathcal{H}_1] \leq O_p\left(n^{-1/2(1-(1/2+q)/(3/2+q))(1+r)}\right) \quad \text{and} \quad \mathbb{B}[TSRK(p)|\mathcal{H}_1] = O_p\left(n^{-q/(2q+1)}\right)$$

The asymptotic order of the former is, then, strictly smaller than that of the latter, which may be seen by algebraic manipulation of the powers to show $q/(1+r) < (2q+1)/(2q+3)$ as $q/(1+r) \leq 1/2$ and $(2q+1)/(2q+3) \in [3/5, 1)$. Next, to show (2), write the respective variances

$$n^{1/2}\mathbb{V}[RK^*(p)|\mathcal{H}_1] = \mathcal{V}(\lambda, a) + 4ac\int_0^1 \sigma_t^4 dt + o_p(1)$$

and

$$\begin{aligned} n^{1/2}\mathbb{V}[TSRK(p)|\mathcal{H}_1] = \mathcal{V}(\lambda, a) + f_0(\tau)4a\int_0^1 \sigma_t^4 dt + f_2(\tau)4a^{-3}\int_0^1 \Omega_t^2 dt \\ + f_1(\tau)8a^{-1}\int_0^1 \left(\Omega_t\sigma_t^2 + 2\left(\Omega_t^{(ep)}\right)^2\right) dt + o_p(1) \end{aligned}$$

such that the ratio $\mathbb{V}[RK^*(p)|\mathcal{H}_1]/\mathbb{V}[TSRK(p)|\mathcal{H}_1] = v_n(q, \psi^2, \rho)$ may be defined as

$$v_n(q, \psi^2, \rho) = \frac{\mathcal{V}(\lambda, a)(4a\int_0^1 \sigma_t^4 dt)^{-1} + c + o_p(1)}{\mathcal{V}(\lambda, a)(4a\int_0^1 \sigma_t^4 dt)^{-1} + [f_0(\tau) + f_2(\tau)a^{-4}\psi^4 \jmath_1 + f_1(\tau)2a^{-2}\psi^2\rho(\jmath_2 + 2\jmath_3)]}.$$

Since $f_j(\tau) \in \mathbb{R}_+$ for $j = 0, 1, 2$, $v_n(q, \psi^2, \rho)$ decreases in both ψ^2 and ρ. Finally, it follows that $\text{plim}_{n\to\infty} v_n(q, \psi^2, \rho) = 1$ since $c = o(1)$ and $\tau = o(1)$ such that $f_j(\tau) = o(1)$ for all $j = 0, 1, 2$. $\square$

5.3 Proof of Lemma 2

First, convergence in law of $\bar{U}_{t_i}$ follows using Yang (2007, Theorem 3.1) as

$$M^{1/2}\bar{U}_{t_i} \xrightarrow{d} N(0, \psi_1\Omega), \quad 0 \leq i \leq n - M.$$

This is immediately seen by writing $\bar{U}_{t_i} = -\sum_{j=1}^{M} \Delta g\left(\frac{j+1}{M}\right) U_{t_{i+j}}$ and using the arguments from the proof of Lemma C.5 (b). In this case, the blocks $\bar{U}_{t_i}$ become asymptotically serially dependent. However, $\exists \varpi \in (\kappa/(1+r_u), \kappa)$ such that $\text{Cov}\left(\bar{U}_{t_i}, \bar{U}_{t_j}\right) = o_p\left(M^{-1}\right)$ for $|i - j| = M + n^{\varpi}$. To see this, write $M\,\text{Cov}\left(\bar{U}_{t_i}, \bar{U}_{t_j}\right) \leq KM\alpha_u(n^{\varpi}) \xrightarrow{\mathbb{P}} 0$ whenever $\varpi > \kappa(1+r_u)^{-1}$ by applying Lemma C.4, and where $\kappa(1+r_u)^{-1} < \kappa$ trivially. Thus, the additional distance between blocks to make them asymptotically independent, compared with the i.i.d. noise case, increases with n at a slower rate than M, i.e., $n^{\varpi}/M = o(1)$. This implies that the big block-small block technique, see Jacod, Podolskij & Vetter (2010, p. 1494), may be used without asymptotic implications as the size of the asymptotically dominant big blocks is strictly larger than M, the size of the smaller, asymptotically dominant blocks. Using this in conjunction with Jacod et al. (2010, Theorem 3.3) gives the desired result. $\square$

Remark 7. *Hautsch & Podolskij (2013) use a similar argument to establish a corresponding lemma for modulated realized volatility when the MMS noise exhibits finite dependence.*

5.4 Proof of Lemma 3

The proof is only provided for $RK^T(p)$ since the result for $RK^T(y)$ follows using the same arguments. Let $z^T = RK^T(p) - RK^*(p) = \max(0, -RK^*(p))$. Then, it suffices to show $z^T = o_p(n^{-1/4})$. From Theorem 1(2), $RK^*(p) = \int_0^1 \sigma_t^2 dt + n^{-1/4}\mathcal{Z} + o_p(n^{-1/4})$ where $\mathcal{Z} \overset{d_s(\mathcal{U}_1)}{\to} MN\left(0, \mathcal{V}(\lambda, a)\right)$ and $\mathcal{V}(\lambda, a) = \lim_{n\to\infty} n^{1/2}\mathcal{V}_n(\lambda)$. Now, similarly to the argument for Ikeda (2015, Proposition 1), decompose

$$\left|z^T\right| = \left|-RK^*(p)\mathbf{1}_{\{RK^*(p)\le 0\}}\right| = \left|RK^*(p)\right| \mathbf{1}_{\{RK^*(p)\le 0\}}$$

and define $\bar{\eta}_n = kn^{-\epsilon} \in (0, k]$ for $\epsilon \in (0, 1/8)$ such that $\bar{\eta}_n \le \int_0^1 \sigma_t^2 dt$. As $|RK^*(p)| = O_p(1)$ and

$$\begin{aligned}
\mathbb{E}\left[\left|\mathbf{1}_{\{RK^*(p)\le 0\}}\right|\right] &= \mathbb{P}\left[RK^*(p) \le 0\right] \le \mathbb{P}\Big[n^{1/4}\Big|RK^*(p) - \int_0^1 \sigma_t^2 dt\Big| \ge n^{1/4}\bar{\eta}_n\Big] \\
&\le \mathbb{E}\Big[n^{1/2}\Big(RK^*(p) - \int_0^1 \sigma_t^2 dt\Big)^2\Big] n^{-1/2}\bar{\eta}_n^{-2} = O_p(n^{-1/2+2\epsilon}) = o_p(n^{-1/4})
\end{aligned}$$

by Chebyshev's inequality, implying $\mathbf{1}_{\{RK^*(p)\le 0\}} \le o_p(n^{-1/4})$, this provides the final result. $\square$

5.5 Proof of Theorem A.1

First, similarly to proof for the TSRK in Section 4 above and the corresponding proofs in Ikeda (2015, Sections A.3.1 and A.3.2), the asymptotic properties of the $TSN(p)$ estimator are deduced using Lemma 1. Specifically, under the conditions for Lemma 1, use the same notation as in Section 4 to write the $\mathcal{H}_1$-conditional expectation of $TSN(p)$ as

$$\begin{aligned}
\mathbb{E}[TSN(p)|\mathcal{H}_1] &= (1-\tau^2)^{-1}\left(|\lambda^{(2)}(0)|nG^{-2}\right)^{-1}\left(\mathcal{B}_n(G) - \mathcal{B}_n(H) + o_p(1)\right) \\
&= \Omega + \frac{\lambda_q^{(2)}}{|\lambda^{(2)}(0)|} G^{-q}\sum_{h\in\mathbb{Z}} |h|^q \Omega(h) + 2n^{-1/2}\sum_{h\in\mathbb{Z}} |h|\Omega^{(ee)}(h),
\end{aligned}$$

which reduces to $\mathbb{E}[TSN(p)|\mathcal{H}_1] = \Omega + O_p(G^{-q}) + O_p(n^{-1/2}) = \Omega + O_p(G^{-q}) + o_p(1)$ since the last bias term will never affect the asymptotics. For the end-point terms, $\mathcal{E}_n(G)$ and $\mathcal{E}_n(H)$, the respective conditions on the jittering rate ξ in Theorem 1 suffices for the $TSN(p)$ estimator to eliminate end-effects as well since multiplication of the scale $\bar{s}_n = (1-\tau^2)^{-1}\left(|\lambda^{(2)}(0)|nG^{-2}\right)^{-1} = O(G^2/n)$ where $G = n^g$ with $g \in [1/(2q+1), 1/2]$ ensures that the two terms are smaller than or equal to $O_p(\mathcal{E}_n(H))$. Next, as the mixed normal distribution is additive, Lemma C.1 (d) gives

$$\mathcal{Z}_{\mathcal{N}} \equiv \bar{s}_n\left(\mathcal{Z}_n(G) - \mathcal{Z}_n(H)\right) \overset{d_s(\mathcal{H}_1)}{\to} MN\left(0, \lim_{n\to\infty}\mathcal{V}_{\mathcal{N},n}(\lambda)\right).$$

where

$$\mathcal{V}_{\mathcal{N},n}(\lambda) = O_p\left(Gn^{-1}\right) + O_p\left(HG^4n^{-3}\right) + O_p\left(G^3n^{-2}\right),$$

and for which the asymptotic orders follow by multiplying $\mathcal{V}_n(\lambda, G)$ and $\mathcal{V}_n(\lambda, H)$ with $O(\bar{s}_n^2)$. Finally, the moment and stable central limit theory results for $PRV(p)$ are established using exactly the same arguments as in the proof of Jacod, Li, Mykland, Podolskij & Vetter (2009, Theorem 3.1) and Podolskij & Vetter (2009, Theorem 4) along with the Cauchy-Schwarz inequality to bound the cross-covariance terms, $\mathcal{C}_{\mathcal{N},n}(\lambda)$. The justification for extending the results to an α-mixing-dependent exogenous MMS noise is the same as for Hautsch & Podolskij (2013, Theorem 1) and in the proof of Lemma 2. □

6 Results for the Proofs of Theorems 3 and 4

This section supplements the proofs of Theorems 3 and 4 in Section B.3 by providing the first and second-order Taylor expansions, leading to equation (B.15). Moreover, it establishes that higher-order Taylor expansion effects are, indeed, of lower stochastic order than the corresponding first and second-order effects and may, thus, be disregarded.

6.1 Collection of Taylor Expansion Terms

Recall, the function $h(x_i, x_{i-1}, \ldots, x_{i-B+1}) = x_i - (x_i x_{i-1} \ldots x_{i-B+1})^{1/B}$ has $h(z, z, \ldots, z) = 0$ along with first and second derivatives, $h_i^{(1)}(z, z, \ldots, z) = 1 - B^{-1}$, $h_{i-s}^{(1)}(z, z, \ldots, z) = -B^{-1}$ for $s > 0$, and

$$h_{i-s,i-s}^{(2)}(z, z, \ldots, z) = \frac{B-1}{B^2}\frac{1}{z}, \qquad h_{i-s,i-g}^{(2)}(z, z, \ldots, z) = \frac{-1}{B^2}\frac{1}{z}, \; s \neq g.$$

where $(s, g) \in [0, 1, \ldots, B-1]^2$. Then, for the first-order effect, denoted $\mathcal{E}_1$, from a Taylor expansion of $h(RK_i^T(p), RK_{i-1}^T(p), \ldots, RK_{i-B+1}^T(p))$ around $h(\sigma^2_{\tau_{i-B}}\Delta\tau, \sigma^2_{\tau_{i-B}}\Delta\tau, \ldots, \sigma^2_{\tau_{i-B}}\Delta\tau)$,

$$\begin{aligned}
\mathcal{E}_1 &= (1 - B^{-1})\left(RK_i^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right) - B^{-1}\sum_{j=1}^{B-1}\left(RK_{i-j}^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right) \\
&= \frac{1}{B}\sum_{j=1}^{B-1}\left(RK_i^T(p) - RK_{i-j}^T(p)\right).
\end{aligned}$$

For the second-order effect, denoted $\mathcal{E}_2$, it follows

$$\begin{aligned}
2B^2\mathcal{E}_2 &= \frac{(B-1)}{\sigma^2_{\tau_{i-B}}\Delta\tau}\left(RK_i^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right)^2 + \frac{(B-1)}{\sigma^2_{\tau_{i-B}}\Delta\tau}\sum_{j=1}^{B-1}\left(RK_{i-j}^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right)^2 \\
&\quad - \frac{2}{\sigma^2_{\tau_{i-B}}\Delta\tau}\sum_{j_1=0}^{B-2}\sum_{j=1+j_1}^{B-1}\left(RK_{i-j_1}^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right)\left(RK_{i-j}^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right) \\
&= \frac{1}{\sigma^2_{\tau_{i-B}}\Delta\tau}\sum_{j=1}^{B-1}\left(RK_i^T(p) - RK_{i-j}^T(p)\right)^2 + \frac{(B-2)}{\sigma^2_{\tau_{i-B}}\Delta\tau}\sum_{j=1}^{B-1}\left(RK_{i-j}^T(p) - \sigma^2_{\tau_{i-B}}\Delta\tau\right)^2
\end{aligned}$$

$$
- \frac{2}{\sigma^2_{\tau_{i-B}} \Delta\tau} \sum_{j_1=1}^{B-2} \sum_{j=1+j_1}^{B-1} \left(RK^T_{i-j_1}(p) - \sigma^2_{\tau_{i-B}} \Delta\tau \right) \left(RK^T_{i-j}(p) - \sigma^2_{\tau_i - B} \Delta\tau \right)
$$

$$
= \frac{1}{\sigma^2_{\tau_{i-B}} \Delta\tau} \sum_{j_1=0}^{B-2} \sum_{j=1+j_1}^{B-1} \left(RK^T_{i-j_1}(p) - RK^T_{i-j}(p) \right)^2
$$

where the last equality follows by repeated use of the same collection of terms as between the first and second equality. This provides the representation in equation (B.15). □

6.2 Higher-order Taylor Expansion Effect

The arguments for bounding higher-order Taylor expansion terms are similar to those provided in the proof of Mykland et al. (2012, Theorem 10). For the third-order derivatives,

$$
h^{(3)}_{i-s,i-s,i-s}(z, z \ldots, z) = -\frac{B-1}{B^2} \left(1 + \frac{B-1}{B} \right) \frac{1}{z^2}, \quad h^{(3)}_{i-s,i-s,i-g}(z, z \ldots, z) = \frac{B-1}{B^2} \frac{1}{B} \frac{1}{z^2}, \; s \neq q,
$$

$$
\text{and} \quad h^{(3)}_{i-s,i-g,i-v}(z, z \ldots, z) = -\frac{1}{B^3} \frac{1}{z^2}, \; s \neq g \neq v,
$$

for some $v \in [0, 1 \ldots, B-1]$. Hence, the uniform stochastic order of the third-order Taylor expansion effect, $\mathcal{E}_3$, is proportional to that of the proxy variable $\tilde{\mathcal{E}}_3$ where

$$
\tilde{\mathcal{E}}_3 = \frac{1}{\Delta\tau^2} \sum_{i=B}^{n_L} \sigma^{-4}_{\tau_{i-b}} \left| RK^*_{i-j}(p) - \sigma^2_{\tau_{i-B}} \Delta\tau \right|^3 \leq \frac{1}{\Delta\tau^2} \frac{1}{\tilde{k}^2} \sum_{i=B}^{n_L} \left| RK^*_{i-j}(p) - \sigma^2_{\tau_{i-B}} \Delta\tau \right|^3, \tag{F.5}
$$

using, again, $\tilde{k} = \inf_{i=1,\ldots,n_L} \sigma^2_{\tau_{i-B}}$, $\tilde{k} > 0$. Then, by defining the term

$$
\tilde{\mathcal{L}}_{i-j} = RK^*_{i-j}(p) - \sigma^2_{\tau_{i-B}} \Delta\tau = [\sigma^2_{\tau_{i-j}} - \sigma^2_{\tau_{i-B}}] \Delta\tau + L_{\tau_{i-j}},
$$

where $L_{\tau_{i-j}}$ is defined as in the proof of Lemma C.10, it follows by applying the Burkholder-Davis-Gundy (BDG) inequality, see, e.g., Protter (2004, p. 195), that

$$
\mathbb{E}\left[|\tilde{\mathcal{L}}_{i-j}|^3 \right] \leq K \mathbb{E}\left[[\tilde{\mathcal{L}}_{i-j}, \tilde{\mathcal{L}}_{i-j}]^{3/2} \right]. \tag{F.6}
$$

Due to the similarity between $\tilde{\mathcal{L}}_{i-j}$ and the corresponding terms in (C.2), it follows by nearly identical derivations that the quadratic variation at time τ_{i-j} is bounded as $[\tilde{\mathcal{L}}_{i-j}, \tilde{\mathcal{L}}_{i-j}] \leq O_p(n^{-1/2-\beta}) + O_p(n^{-3\beta})$.[7] Then, using this result in conjunction with the inequalities (F.5) and (F.6), the uniform stochastic order of the proxy variable $\tilde{\mathcal{E}}_3 \leq O_p(n^{-3/2(1/2-\beta)}) + O_p(n^{-3/2\beta})$ is deduced. Hence, $\tilde{\mathcal{E}}_3 = o_p(n^{-1/2+\beta}) + o_p(n^{-\beta})$ when $\beta \in (0, 1/2)$, that is, the third-order Taylor expansion effect is of strictly lower stochastic order than the corresponding second-order effect in Lemma C.10.

For the fourth-order Taylor expansion term, it readily follows that all fourth-order (cross) deriva-

[7]Notice, there is no summation nor scaling with $\Delta\tau^{-1} = O(n^\beta)$. Hence, the terms are an order $O_p(n^{-2\beta})$ lower.

tives, evaluated at z, are of the form $c_{s,g,v,u}z^{-3}$ for some constant $c_{s,g,v,u}$. Hence, the uniform stochastic order of the resulting fourth-order effect is, similarly to the third-order effect above, proportional to that of a proxy variable $\tilde{\mathcal{E}}_4$, where

$$\tilde{\mathcal{E}}_4 = \frac{1}{\Delta\tau^3}\sum_{i=B}^{n_L}\sigma_{\tau_{i-b}}^{-6}\left(RK_{i-j}^*(p) - \sigma_{\tau_{i-B}}^2\Delta\tau\right)^4 \le \frac{1}{\Delta\tau^3}\frac{1}{\tilde{k}^3}\sum_{i=B}^{n_L}\left(RK_{i-j}^*(p) - \sigma_{\tau_{i-B}}^2\Delta\tau\right)^4. \tag{F.7}$$

Then, by the same arguments as above, the stochastic order $\tilde{\mathcal{E}}_4 \le O_p(n^{-2(1/2-\beta)}) + O_p(n^{-2\beta})$ is deduced, implying that the fourth-order effect will not impact the asymptotic results. By the usual stopping and truncation arguments for Taylor expansions, this proves that the higher-order Taylor expansion effect in equation (B.15) is uniformly of strictly lower stochastic order than the first and second-order effects, that is, $|\sum_{i=B}^{n_L}\mathcal{R}_{i,h}| = o_p(n^{-1/2+\beta}) + o_p(n^{-\beta})$, when $\beta \in (0, 1/2)$. □

7 Notes on Locally Stationary Processes

This section details how the endogenous MMS noise component, e, relates to the locally stationary processes analyzed in Dahlhaus & Polonik (2009) and Dahlhaus (2009). Specifically, it explains how to map assumptions, definitions, and three asymptotic results from said papers to the present setting.

7.1 Mapping Assumptions and Definitions

First, for direct comparability with Dahlhaus & Polonik (2009) and Dahlhaus (2009), this section adopts their notation and subsequently explains how it differs from the notation used in the main text. Hence, let $X_{t,n}$ $(t = 1, \ldots, n)$ denote a locally stationary process, which has representation

$$X_{t,n} = \sum_{j=-\infty}^{\infty} a_{t,n}(j)\epsilon_{t-j},$$

where $a_{t,n}(j)$ models its time-varying parameters and ϵ_t its innovations, both are detailed below. Moreover, for some $\kappa > 0$, let

$$\ell(j) = \mathbf{1}_{\{|j|\le 1\}} + \left(|j|\log^{1+\kappa}|j|\right)\mathbf{1}_{\{|j|>1\}},$$

and suppose there exists a sequence of functions $a(\cdot, j) : (0, 1] \to \mathbb{R}$. Then, $X_{t,n}$ is assumed to satisfy the following five regularity conditions:

L1: $\sup_{t,n}|a_{t,n}(j)| \le K/\ell(j)$;

L2: $\sup_u |a(u, j)| \le K/\ell(j)$;

L3: $\sup_j \sum_{t=1}^n \left|a_{t,n}(j) - a(t/n, j)\right| \le K$;

L4: $\sup\left\{\sum_{k=1}^m |a(x_k, j) - a(x_{k-1}, j)| : 0 \le x_0 < \cdots < x_m \le 1,\ m \in \mathbb{N}\right\} \le K/\ell(j)$;

L5: ϵ_t is i.i.d. with $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{E}[\epsilon_t^2] = 1$. In addition, all moments of ϵ_t are assumed to exist.

This representation and its associated regularity conditions have direct mappings to the endogenous noise component and Assumption 3. Specifically, the parameter $\theta(t'_i, g)$ corresponds to $a_{t,n}(g)$, the function $\theta_t(g)$ to $a(t, g)$, and since $(\Delta t'_{i-g})^{-1/2}\Delta\tilde{W}_{t'_{i-g}} \overset{d}{=} N(0, 1)$ and i.i.d., this maps the innovations of the endogenous noise to ϵ_{t-g}, noting that the former satisfy the regularity conditions in **L.5**. Instead of defining a sequence $\ell(j)$ and constructing various bounds using $K/\ell(j)$, Assumption 3 defines

$$\alpha_e(g) = O(1)\mathbf{1}_{\{|g|\le 1\}} + O\left(|g|^{-(1+r_e+\epsilon)}\right)\mathbf{1}_{\{|g|>1\}}, \quad r_e \in \mathbb{N}^+,$$

to serve a similar purpose. The polynomial decay in $\alpha_e(g)$ is slightly stronger than the corresponding decay of $1/\ell(j)$. Not only is the former needed to establish the maximal inequalities in Lemma C.4, the specification also facilitates direct comparability with the α-mixing rate for the exogenous MMS noise component, $\alpha_u(g)$, allowing the two noise components to be treated similarly. Moreover, the conditions imposed in Assumptions 3(2)-3(5) are equivalent to conditions L1-L4, respectively. Finally, Assumption 3 imposes a few additional conditions on the standard Brownian motion, $\Delta\tilde{W}_{t'_i}$ such that the latter facilitates correlations with increments of the efficient price process.

Remark 8. *The use of locally stationary processes to describe dependence in the MMS noise maps naturally into the asymptotic framework for high-frequency financial data. Dependence is defined in tick time, here denoted by $t = 1, \ldots, n$, yet inference is carried out using functions $a(\cdot, j)$ at rescaled time points $t/n \in [0, 1]$, that is, in its infill asymptotic limit.*

Next, to ease the applications of Dahlhaus & Polonik (2009, Proposition 5.4) and Dahlhaus (2009, Theorems 2.4 and 3.2) in the present analysis, write, in their notation, the time-varying spectral density of $X_{t,n}$ at the rescaled time point u as

$$f(u, \lambda) = \frac{1}{2\pi}|A(u, \lambda)|^2, \quad A(u, \lambda) = \sum_{j=-\infty}^{\infty} a(u, j)\exp(-\mathrm{i}\lambda j),$$

and let the time-varying covariance at lag h be defined as

$$c(u, h) = \int_{-\pi}^{\pi} f(u, \lambda)\exp(\mathrm{i}\lambda h)d\lambda = \sum_{j=-\infty}^{\infty} a(u, h + j)a(u, j),$$

noting that $c(u, h)$ is equivalent to writing $\Omega_u^{(ee)}(h)$ in Definition 1. Moreover, define the generalized spectral measure as

$$F(\phi) = \int_0^1 \int_{-\pi}^{\pi} \phi(u, \lambda)f(u, \lambda)\, d\lambda\, du,$$

where $\phi(\cdot)$ is a weight function to be discussed below, and write its empirical counterpart as

$$F_n(\phi) = \frac{1}{n}\sum_{t=1}^{n}\int_{-\pi}^{\pi}\phi(t/n,\lambda)J_n(t/n,\lambda)\,d\lambda$$

with $J_n(t/n,\lambda)$ being defined as

$$J_n(t/n,\lambda) = \frac{1}{2\pi}\sum_{k:1\le[t+1/2\pm k/2]\le n} X_{[t+1/2+k/2],n}X_{[t+1/2-k/2],n}\exp(-\mathrm{i}\lambda k),$$

which may be seen as a raw estimate of $f(t/n,\lambda)$. In the present analysis, for the purpose of developing asymptotic results for flat-top realized kernel estimators, these definitions simplify, since the weight function is time-invariant. To see this, write $\phi(u,\lambda) = \tilde{\phi}(\lambda)$, then $F_n(\phi)$ may be rewritten as

$$F_n(\phi) = \int_{-\pi}^{\pi}\tilde{\phi}(\lambda)\frac{1}{n}\sum_{t=1}^{n}J_n(t/n,\lambda)\,d\lambda,$$

for which

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}J_n(t/n,\lambda) &= \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{k:1\le[t+1/2\pm k/2]\le n} X_{[t+1/2+k/2],n}X_{[t+1/2-k/2],n}\exp(-\mathrm{i}\lambda k)\\ &= \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}X_{t,n}X_{s,n}\exp(-\mathrm{i}\lambda(t-s)) = \frac{1}{2\pi n}\Big|\sum_{t=1}^{n}x_{t,n}\exp(-\mathrm{i}\lambda t)\Big|^2,\end{aligned}$$

which is the classical periodogram, $\mathcal{I}_n(\lambda, X)$, adopting the notation from the proof of Lemma B.2. Hence, in this case, $F_n(\phi)$ is a standard spectral density estimator with spectral window $\tilde{\phi}(\lambda)$. Moreover, this estimator converges to

$$F(\phi) = \int_{-\pi}^{\pi}\tilde{\phi}(\lambda)\left(\int_0^1 f(u,\lambda)\,du\right)d\lambda,$$

that is, the integrated, or average, spectral density. In the present setting, this will correspond to the average spectral density for the endogenous MMS noise component, thereby ($1/2\pi$ times) its average long run variance, $\Omega^{(ee)}$, at frequency $\lambda = 0$. Specifically, applying the notation from the proof of Lemma B.2, the direct impact of serial dependence in the endogenous noise component, e, on the asymptotic distribution of the flat-top realized kernel estimators is

$$A(e) = s_n\int_{-\pi}^{\pi}\mathcal{K}_n(\delta)\mathcal{I}_n(\delta,e)\,d\delta, \quad s_n = 2\pi\lambda^{(2)}(0)nH^{-2},$$

where $\mathcal{K}_n(\delta)$ is the spectral window and s_n is a deterministic scale, thus clearly illustrating its similarity with the spectrum estimator, $F_n(\phi)$, for a locally stationary process.[8] The endogenous noise also

[8] See also Priestley (1981, Section 6.2.3) for corresponding details on spectrum estimators for stationary processes.

impacts the asymptotic distribution for flat-top realized kernels in Theorems 1 and 2 through other channels, e.g., through correlations with increments of the efficient price process. However, these limits are derived without relying on spectrum results from Dahlhaus (2009).

Remark 9. *Whereas the current form of $A(e)$ facilitates long-run variance estimation for the endogenous noise, this may be generalized to other frequencies by modifying the spectral window, $\mathcal{K}_n(\delta)$.*

7.2 Application of Dahlhaus & Polonik (2009, Proposition 5.4)

The results in Dahlhaus & Polonik (2009, Proposition 5.4) are used to establish convergence of moments for the endogenous MMS noise component in Lemma C.6, among others. Although stated with a data taper on Dahlhaus & Polonik (2009, p. 19), they make a prior remark stating that the proposition for ordinary covariances is recovered as the special case of *no* data tapering. Hence, the latter is dropped here for notational convenience. Under conditions L1-L5, they show that

(i) $\sup_t |\mathrm{Cov}[X_{t,n}, X_{t+k,n}]| \leq K/\ell(k)$,

(ii) $\sup_u |c(u,k)| \leq K/\ell(k)$,

(iii) $\sum_{t=1}^{n} |\,\mathrm{Cov}[X_{t+k_1,n}, X_{t-k_2,n}] - c(t/n, k_1+k_2)| \leq K\,(1+\min\{|k_1|, n\}/\ell(k_1+k_2))$

(iv) $\sup\big\{\sum_{j=1}^{m} |c(x_j,k) - c(x_{j-1},k)| : 0 \leq x_0 < \cdots < x_m \leq 1,\ m \in \mathbb{N}\big\} \leq K/\ell(k)$.

Since the endogenous noise assumption in the present setting satisfies conditions L1-L5, the corresponding results may readily be translated into the notation used in the main text:

(i) $\sup_{t_i} \big|\mathrm{Cov}[e_{t_i}, e_{t_{i+k}}]\big| \leq \alpha_e(k)$,

(ii) $\sup_u |\Omega_u^{(ee)}(k)| \leq \alpha_e(k)$,

(iii) $\sum_{i=1}^{n} |\,\mathrm{Cov}[e_{t_{i-h}}, e_{t_{i+s}}] - \Omega_{t_i}^{(ee)}(h+s)| \leq K(1+\min\{|h|, n\}\alpha_e(h+s))$

(iv) $\sup\big\{\sum_{j=1}^{m} |\Omega_{x_j}^{(ee)}(k) - \Omega_{x_{j-1}}^{(ee)}(k)| : 0 \leq x_0 < \cdots < x_m \leq 1,\ m \in \mathbb{N}\big\} \leq \alpha_e(k)$.

Specifically, *(ii)* is used for Lemma C.4, and *(iii)* is used for Lemmas C.6 (a) and (c),

7.3 Application of Dahlhaus (2009, Theorems 2.4 and 3.2)

In order to state Dahlhaus (2009, Theorem 2.4), a few additional regularity conditions are imposed on the weight function $\phi(u,\lambda)$. However, since $\phi(u,\lambda) = \tilde{\phi}(\lambda)$ in the present setting, half of these conditions are redundant as there is no variation of the weight function in the time direction. The remaining two conditions are $\sup_\lambda |\tilde{\phi}(\lambda)| < \infty$ and

$$\sup_\lambda \left\{ \sum_{j=1}^{m} |\tilde{\phi}(\lambda_j) - \tilde{\phi}(\lambda_{j-1})| : -\pi \leq \lambda_0 < \ldots, < \lambda_m \leq \pi,\ m \in \mathbb{N} \right\} < \infty.$$

For the asymptotic analysis in the proof of Lemma B.2, these conditions are imposed on $\mathcal{K}_n(\delta)$, and Lemmas C.2 (b) and (c) show that they are, indeed, satisfied. Finally, before stating and translating Dahlhaus (2009, Theorem 2.4), write the empirical spectral process as

$$E_n(\phi) = \sqrt{n}(F_n(\phi) - F(\phi))$$

and restrict attention to Gaussian innovations ϵ_t, since this applies to the endogenous noise component, and it simplifies the expression for the asymptotic variance in the following central limit theorem by fixing the fourth-order cumulant of ϵ_t to zero. Under the described conditions,

$$\sqrt{n}(F_n(\phi) - F(\phi)) \xrightarrow{d} N\left(0, \text{Avar}(F_n(\phi))\right)$$

by Dahlhaus (2009, Theorem 2.4), where the asymptotic variance, $\text{Avar}(F_n(\phi))$, may be written as

$$\text{Avar}(F_n(\phi)) = 2\pi \int_{-\pi}^{\pi} \tilde{\phi}(\lambda)\left(\tilde{\phi}(\lambda) + \tilde{\phi}(-\lambda)\right)\left(\int_0^1 f^2(u,\lambda)du\right) d\lambda.$$

Hence, the use of this result in the present setting for $A(e)$, together with standard arguments in the spectrum estimation literature, cf. Priestley (1981, pp. 454-455), shows that

$$(H^3 n^{-1})^{1/2}\left(A(e) - \mathbb{E}[A(e)]\right) \xrightarrow{d} N\left(0, \text{Avar}(A(e))\right),$$

whose asymptotic variance may be written, using the notation from the proof of Lemma B.2, at frequency $\delta = 0$ as

$$\begin{aligned}
\text{Avar}(A(e)) &= \frac{H^3}{n}\frac{4\pi s_n^2}{n}\frac{1}{(2\pi)^2}\int_0^1 [\Omega_t^{(ee)}]^2 dt \int_{-\pi}^{\pi} [\mathcal{K}_n(\delta)]^2 d\delta (1 + o(1)). \\
&= 2\int_0^1 [\Omega_t^{(ee)}]^2 dt \times \left(\frac{2\pi\left(\lambda^{(2)}(0)\right)^2}{H}\int_{-\pi}^{\pi} [\mathcal{K}_n(\delta)]^2 d\delta (1 + o(1))\right) \\
&= 2\int_0^1 [\Omega_t^{(ee)}]^2 dt \times \left(\frac{1}{H}\sum_{h\in\mathbb{Z}_{n-1}} a\left(\frac{|h|}{H}\right)^2 (1 + o(1))\right) \to 4\lambda^{(22)}\int_0^1 [\Omega_t^{(ee)}]^2 dt,
\end{aligned}$$

using Parseval's theorem for the third equality, and Lemma C.2 (d) for the final convergence. Hence, by appropriately transforming $A(e)$, the discussion above shows that the present problem of calculating the asymptotic distribution theory for the contribution of the endogenous noise on the flat-top realized kernels in the proof of Lemma B.2 readily maps into the problem of spectrum estimation (at frequency zero) for locally stationary processes, and that Dahlhaus (2009, Theorem 2.4) may be invoked.

Remark 10. *The referenced Theorem 3.2 in Dahlhaus (2009) generalizes the described Theorem 2.4 to allow for data tapering and index functions $\phi_n(u,\lambda)$ that depend on n.*

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