Supplementary Material to
Characteristic Function Based Testing for Conditional Independence: A Nonparametric Regression Approach

Xia Wang[a,b], Yongmiao Hong[c,d]

[a] Lingnan (University) College, Sun Yat-Sen University

[b] Fujian Key Laboratory of Statistical Science (Xiamen University)

[c] Department of Economics and Department of Statistical Sciences, Cornell University

[d] Wang Yanan Institute for Studies in Economics, Xiamen University

This Supplementary Material contains three appendices. Appendix A provides the detailed proofs of Theorems 1-3, Appendix B contains the detailed simulation results which are not reported in the paper, and Appendix C shows the figures for the data series used in Section 7.

Appendix A Mathematical Proofs of Theorems 1-3

Throughout this Supplementary Material, we denote

$$\hat{M}_h = nh^{d_x/2}\hat{M} = h^{d_x/2}\sum_{t=1}^{n}\iint |\hat{\sigma}(u,v,X_t)|^2 a(X_t)dW_1(u)dW_2(v),$$

and $\varepsilon_{yz}(u,v,X_s) = e^{\mathbf{i}(u'Y_s+v'Z_s)} - \phi_{yz}(u,v,X_s)$, $\varepsilon_y(u,X_s) = \varepsilon_{yz}(u,0,X_s)$, $\varepsilon_z(v,X_s) = \varepsilon_{yz}(0,v,X_s)$, $\bar{\phi}_{yz}(u,v,x) = E[\hat{\phi}_{yz}(u,v,x)]$, $\bar{\phi}_y(u,x) = E[\hat{\phi}_y(u,x)]$, $\bar{\phi}_z(v,x) = E[\hat{\phi}_z(v,x)]$. In addition, $\xi_t = (X_t', Y_t', Z_t')'$, $C \in (0,\infty)$ is a generic bounded constant that may vary from case to case, A^* denotes the conjugate of A, and $Re(A)$ denotes the real part of A.

A.1 Proof of Theorem 1

Under $\mathbb{H}_0 : \phi_{yz}(u,v,x) = \phi_y(u,x)\phi_z(v,x)$, we can decompose $\hat{\sigma}(u,v,x)$ as follows:

$$\begin{aligned}\hat{\sigma}(u,v,x) = &\left[\hat{\phi}_{yz}(u,v,x) - \phi_{yz}(u,v,x)\right] - \phi_z(v,x)\left[\hat{\phi}_y(u,x) - \phi_y(u,x)\right] \\ &- \phi_y(u,x)\left[\hat{\phi}_z(v,x) - \phi_z(v,x)\right] - \left[\hat{\phi}_y(u,x) - \phi_y(u,x)\right]\left[\hat{\phi}_z(v,x) - \phi_z(v,x)\right]. \qquad (A.1)\end{aligned}$$

According to $(A.1)$, we decompose $\hat{M}_h$ as follows:

$$\hat{M}_h = h^{d_x/2} \sum_{t=1}^{n} \iint \Big\{ |\hat{\phi}_{yz} - \phi_{yz}|^2 + |\phi_y|^2 |\hat{\phi}_z - \phi_z|^2 + |\phi_z|^2 |\hat{\phi}_y - \phi_y|^2 + 2Re\left[\phi_y \phi_z^* (\hat{\phi}_z - \phi_z)(\hat{\phi}_y - \phi_y)^*\right]$$
$$- 2Re\left[(\hat{\phi}_{yz} - \phi_{yz})\phi_y^* (\hat{\phi}_z - \phi_z)^*\right] - 2Re\left[(\hat{\phi}_{yz} - \phi_{yz})\phi_z^* (\hat{\phi}_y - \phi_y)^*\right] + |(\hat{\phi}_y - \phi_y)(\hat{\phi}_z - \phi_z)|^2$$
$$- 2Re\left[(\hat{\phi}_{yz} - \phi_{yz})(\hat{\phi}_y - \phi_y)^* (\hat{\phi}_z - \phi_z)^*\right] + 2Re\left[(\hat{\phi}_y - \phi_y)\phi_y^*\right] |\hat{\phi}_z - \phi_z|^2$$
$$+ 2Re\left[(\hat{\phi}_z - \phi_z)\phi_z^*\right] |\hat{\phi}_y - \phi_y|^2 \Big\} a(X_t) dW_1(u) dW_2(v) = \sum_{i=1}^{10} T_i, \text{ say,} \tag{A.2}$$

where $\hat{\phi}_{yz} \equiv \hat{\phi}_{yz}(u, v, X_t)$, $\hat{\phi}_y \equiv \hat{\phi}_y(u, X_t)$, $\hat{\phi}_z \equiv \hat{\phi}_z(v, X_t)$, $\phi_{yz} \equiv \phi_{yz}(u, v, X_t)$, $\phi_y \equiv \phi_y(u, X_t)$, $\phi_z \equiv \phi_z(v, X_t)$. We shall analyze each of these terms $\{T_i\}_{i=1}^{10}$ in $(A.2)$ to identify the leading terms that determine the asymptotic distribution of our test. The leading terms of $\{T_i\}_{i=1}^{10}$ are given by Propositions 0.1 - 0.7 below.

Proposition 0.1 *Under the conditions of Theorem 1,* $T_1 = B_1 + \tilde{U}_1 + o_P(1)$, *where*

$$B_1 = h^{-d_x/2} \iiint a(x) \left[1 - |\phi_{yz}(u, v, x)|^2\right] dW_1(u) dW_2(v) dx \int D^2(p, \tau) K(\tau)^2 d\tau,$$
$$\tilde{U}_1 = \frac{2}{nh^{3d_x/2}} \sum_{1 \le s < r \le n} U_1(\xi_s, \xi_r),$$

and $U_1(\xi_s, \xi_r) = \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) Re\left[\varepsilon_{yz}(u, v, X_s) \varepsilon_{yz}(u, v, X_r)^*\right] dW_1(u) dW_2(v) dx.$

Proposition 0.2 *Under the conditions of Theorem 1,* $T_2 = B_2 + \tilde{U}_2 + o_P(1)$, *where*

$$B_2 = h^{-d_x/2} \iiint a(x) |\phi_y(u, x)|^2 \left[1 - |\phi_z(v, x)|^2\right] dW_1(u) dW_2(v) dx \int D^2(p, \tau) K(\tau)^2 d\tau,$$
$$\tilde{U}_2 = \frac{2}{nh^{3d_x/2}} \sum_{1 \le s < r \le n} U_2(\xi_s, \xi_r),$$

and $U_2(\xi_s, \xi_r) = \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) |\phi_y(u, x)|^2 Re\left[\varepsilon_z(v, X_s) \varepsilon_z(v, X_r)^*\right] \times dW_1(u) dW_2(v) dx.$

Proposition 0.3 *Under the conditions of Theorem 1,* $T_3 = B_3 + \tilde{U}_3 + o_P(1)$, *where*

$$B_3 = h^{-d_x/2} \iiint a(x) |\phi_z(v, x)|^2 \left[1 - |\phi_y(u, x)|^2\right] dW_1(u) dW_2(v) dx \int D^2(p, \tau) K(\tau)^2 d\tau,$$
$$\tilde{U}_3 = \frac{2}{nh^{3d_x/2}} \sum_{1 \le s < r \le n} U_3(\xi_s, \xi_r),$$

and $U_3(\xi_s, \xi_r) = \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) |\phi_z(v, x)|^2 Re\left[\varepsilon_y(u, X_s) \varepsilon_y(u, X_r)^*\right] \times dW_1(u) dW_2(v) dx.$

Proposition 0.4 *Under the conditions of Theorem 1,* $T_4 = \tilde{U}_4 + o_P(1)$, *where* $\tilde{U}_4 = \frac{2}{nh^{3d_x/2}} \sum_{s \ne r} U_4(\xi_s, \xi_r)$, *and* $U_4(\xi_s, \xi_r) = \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) Re\left[\phi_y(u, x) \phi_z(v, x)^* \varepsilon_z(v, X_s) \varepsilon_y(u, X_r)^*\right]$ $\times dW_1(u) dW_2(v) dx.$

Proposition 0.5 *Under the conditions of Theorem 1,* $T_5 = B_5 + \tilde{U}_5 + o_P(1)$, *where*

$$B_5 = -2h^{-d_x/2} \iiint a(x) |\phi_y(u, x)|^2 \left[1 - |\phi_z(v, x)|^2\right] dW_1(u) dW_2(v) dx \int D^2(p, \tau) K(\tau)^2 d\tau,$$
$$\tilde{U}_5 = \frac{2}{nh^{3d_x/2}} \sum_{s \ne r} U_5(\xi_s, \xi_r),$$

and $U_5(\xi_s, \xi_r) = -\iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) Re\left[\phi_y(u, x)^* \varepsilon_{yz}(u, v, X_s) \varepsilon_z(v, X_r)^*\right] \times dW_1(u) dW_2(v) dx.$

Proposition 0.6 *Under the conditions of Theorem 1, $T_6 = B_6 + \tilde{U}_6 + o_P(1)$, where*

$$B_6 = -2h^{-d_x/2} \iiint a(x)|\phi_z(v,x)|^2 \left[1 - |\phi_y(u,x)|^2\right] dW_1(u)dW_2(v)dx \int D^2(p,\tau)K(\tau)^2 d\tau,$$

$$\tilde{U}_6 = \frac{2}{nh^{3d_x/2}} \sum_{s \neq r} U_6(\xi_s, \xi_r),$$

and $U_6(\xi_s,\xi_r) = -\iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) Re\left[\phi_z(v,x)^* \varepsilon_{yz}(u,v,X_s)\varepsilon_y(u,X_r)^*\right] \times dW_1(u)dW_2(v)dx$
.

Proposition 0.7 *Under the conditions of Theorem 1, $T_7 + T_8 + T_9 + T_{10} = o_P(1)$.*

Based on Propositions 0.1 - 0.7, we can obtain the asymptotic centering factor B, and the leading term U that determines the asymptotic distribution of the test statistic:

$$B = \sum_{i=1}^{6} B_i = h^{-d_x/2} \iiint a(x)(1 - |\phi_y(u,x)|^2)(1 - |\phi_z(v,x)|^2)dxdW_1(u)dW_2(v) \int K^2(\tau)D^2(\tau)d\tau, \tag{A.3}$$

$$U = \sum_{i=1}^{6} \tilde{U}_i = \frac{2}{nh^{3d_x/2}} \sum_{1 \le s < r \le n} U(\xi_s, \xi_r),$$

where $U(\xi_s,\xi_r) = \sum_{i=1}^{3} U_i(\xi_s,\xi_r) + 2\sum_{i=4}^{6} U_i(\xi_s,\xi_r)$.

Proposition 0.8 provides the asymptotic distribution of U, which is a second order degenerate U-statistic.

Proposition 0.8 *Under the conditions of Theorem 1, $U/\sqrt{V} \xrightarrow{d} N(0,1)$, where*

$$V = 2\int \left[\iint |\Phi_y(u_1+u_2,x)|^2 dW_1(u_1)dW_1(u_2) \iint |\Phi_z(v_1+v_2,x)|^2 dW_2(v_1)dW_2(v_2)\right] a^2(x)dx$$
$$\times \int \left[\int D(p,\tau)D(p,\tau+\eta)K(\tau)K(\tau+\eta)d\tau\right]^2 d\eta, \tag{A.4}$$

with $\Phi_s(a_1+a_2,x) = \phi_s(a_1+a_2,x) - \phi_s(a_1,x)\phi_s(a_2,x)$ for $s = y$ or z.

As the test statistic is obtained by replacing the asymptotic centering factor B and scaling factor V by their estimators $\hat{B}$ and $\hat{V}$, which are given in (13) and (14) respectively, we shall show that replacing B and V by $\hat{B}$ and $\hat{V}$ does not affect the limiting distribution of the test statistic.

Proposition 0.9 *Under the conditions of Theorem 1, $\hat{B} - B = o_P(1)$ and $\hat{V} - V = o_P(1)$.*

The proof of Theorem 1 will be completed provided Propositions 0.1 - 0.9 are proven, which we turn to next.
Proof of Proposition 0.1**.** We first decompose T_1 as follows:

$$\begin{aligned}
T_1 &= h^{d_x/2} \sum_{t=1}^{n} \iint \left|\hat{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right|^2 a(X_t)dW_1(u)dW_2(v) \\
&= h^{d_x/2} \sum_{t=1}^{n} \iint \left|\hat{\phi}_{yz}(u,v,X_t) - \bar{\phi}_{yz}(u,v,X_t)\right|^2 a(X_t)dW_1(u)dW_2(v) \\
&+ 2h^{d_x/2} \sum_{t=1}^{n} \iint Re\left\{\left[\hat{\phi}_{yz}(u,v,X_t) - \bar{\phi}_{yz}(u,v,X_t)\right]\left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v) \\
&+ h^{d_x/2} \sum_{t=1}^{n} \iint \left|\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right|^2 a(X_t)dW_1(u)dW_2(v) \\
&= A_1 + 2R_1 + R_2, \text{ say.}
\end{aligned} \tag{A.5}$$

The proof of Proposition 0.1 consists of the proofs of Lemmas 0.10 - 0.12 below.

Lemma 0.10 *Under the conditions of Theorem 1,* $A_1 = B_1 + \tilde{U}_1 + o_P(1)$.

Lemma 0.11 *Let* R_1 *be defined as in* (A.5)*. Then* $R_1 = o_P(1)$.

Lemma 0.12 *Let* R_2 *be defined as in* (A.5)*. Then* $R_2 = o_P(1)$.

Proof of Lemma 0.10**.** We decompose A_1 as follows:

$$
\begin{aligned}
A_1 &= h^{d_x/2}\sum_{t=1}^{n}\iint\left|\sum_{s=1}^{n}\frac{1}{nh^{d_x}g(X_t)}D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\varepsilon_{yz}(u,v,X_s)\right|^2 a(X_t)dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&= \frac{1}{n^2h^{3d_x/2}}\sum_{t=1}^{n}\frac{a(X_t)}{g^2(X_t)}K^2(0)\iint|\varepsilon_{yz}(u,v,X_t)|^2\,dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{2}{n^2h^{3d_x/2}}\sum_{s\neq t}\frac{a(X_t)}{g^2(X_t)}K(0)D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\iint Re\left[\varepsilon_{yz}(u,v,X_s)\varepsilon_{yz}(u,v,X_t)^*\right]dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{1}{n^2h^{3d_x/2}}\sum_{s\neq t}\frac{a(X_t)}{g^2(X_t)}D^2\left(p,\frac{X_s-X_t}{h}\right)K^2\left(\frac{X_s-X_t}{h}\right)\iint|\varepsilon_{yz}(u,v,X_s)|^2\,dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{1}{n^2h^{3d_x/2}}\sum_{t\neq s\neq r}\frac{a(X_t)}{g^2(X_t)}D\left(p,\frac{X_s-X_t}{h}\right)D\left(p,\frac{X_r-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)K\left(\frac{X_r-X_t}{h}\right) \\
&\times\iint Re\left[\varepsilon_{yz}(u,v,X_s)\varepsilon_{yz}(u,v,X_r)^*\right]dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&= A_{11}+A_{12}+A_{13}+A_{14},\ \text{say.}
\end{aligned}
$$

We shall first prove $A_{11} = o_P(1)$. Define

$$
\eta_1(X_t) = \frac{a(X_t)}{g^2(X_t)}\iint|\varepsilon_{yz}(u,v,X_t)|^2\,dW_1(u)dW_2(v).
$$

Then we have

$$
A_{11} \quad = \quad \frac{K^2(0)}{nh^{3d_x/2}}E[\eta_1(X_t)] + \frac{K^2(0)}{n^2h^{3d_x/2}}\sum_{t=1}^{n}\{\eta_1(X_t)-E[\eta_1(X_t)]\} = A_{11}^{(1)} + A_{11}^{(2)},\ \text{say.}
$$

Given assumptions A.3 and A.4, we obtain the constant $A_{11}^{(1)} = O(n^{-1}h^{-3d_x/2}) = o(1)$. Since $E(A_{11}^{(2)}) = 0$, and

$$
\begin{aligned}
\text{var}(A_{11}^{(2)}) &\leq K^4(0)n^{-4}h^{-3d_x}\sum_{t=1}^{n}\text{var}[\eta_1(X_t)] + K^4(0)n^{-3}h^{-3d_x}\sum_{i=1}^{n-1}|\text{cov}\,[\eta_1(X_1),\eta_1(X_{1+i})]| \\
&= O(n^{-3}h^{-3d_x}) + O(n^{-3}h^{-3d_x})\sum_{i=1}^{\infty}j^2\beta(j)^{\delta/(1+\delta)} = O(n^{-3}h^{-3d_x}),
\end{aligned}
$$

we have $A_{11}^{(2)} = o_P(1)$ by Chebyshev's inequality. Consequently, $A_{11} = A_{11}^{(1)} + A_{11}^{(2)} = o_P(1)$.

Now, we prove $A_{12} = o_P(1)$. Define

$$
\begin{aligned}
\eta_2(\xi_t,\xi_s) = &\frac{a(X_t)}{g^2(X_t)}K(0)D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\iint Re\left[\varepsilon_{yz}(u,v,X_s)\varepsilon_{yz}(u,v,X_t)^*\right]dW_1(u)dW_2(v) \\
&+ \frac{a(X_s)}{g^2(X_s)}K(0)D\left(p,\frac{X_t-X_s}{h}\right)K\left(\frac{X_t-X_s}{h}\right)\iint Re\left[\varepsilon_{yz}(u,v,X_t)\varepsilon_{yz}(u,v,X_s)^*\right]dW_1(u)dW_2(v).
\end{aligned}
$$

We can rewrite $A_{12} = \frac{2}{n^2h^{3d_x/2}}\sum_{1\leq t<s\leq n}\eta_2(\xi_t,\xi_s)$. In addition, for any given $\tilde{\xi}\in\mathbb{R}^{d_x+d_y+d_z}$, we have $\int\eta_2(\xi_t,\tilde{\xi})dP(\xi_t) = 0$. By Lemma A(ii) of Hjellvik *et al.* (1998), we have

$$
E|A_{12}|^2 \leq \frac{C}{n^2h^{3d_x}}\left[E|\eta_2(\xi_t,\xi_s)|^{2(1+\delta)}\right]^{1/(1+\delta)}\sum_{j=1}^{n-1}j^2\beta(j)^{\delta/(1+\delta)} = O(n^{-2}h^{-3d_x+d_x/(1+\delta)}) = o(1).
$$

Then, $A_{12} = o_P(1)$ by Chebyshev's inequality.

Next, we prove $A_{13} = B_1 + o_P(1)$. Define

$$\eta_3(\xi_t, \xi_s) = \frac{a(X_t)}{g^2(X_t)} \iint D^2\left(p, \frac{X_s - X_t}{h}\right) K^2\left(\frac{X_s - X_t}{h}\right) |\varepsilon_{yz}(u, v, X_s)|^2 dW_1(u) dW_2(v)$$
$$+ \frac{a(X_s)}{g^2(X_s)} \iint D^2\left(p, \frac{X_s - X_t}{h}\right) K^2\left(\frac{X_t - X_s}{h}\right) |\varepsilon_{yz}(u, v, X_t)|^2 dW_1(u) dW_2(v)$$

and $\eta_3(\xi_t) = \int \eta_3(\xi_t, \xi) dP(\xi)$, $\eta_3 = \int \eta_3(\xi) dP(\xi)$. Then A_{13} could be rewritten as follows:

$$\begin{aligned}
A_{13} &= \frac{1}{n^2 h^{3d_x/2}} \sum_{1 \le t < s \le n} \eta_3(\xi_t, \xi_s) \\
&= \frac{1}{n^2 h^{3d_x/2}} \sum_{1 \le t < s \le n} [\eta_3(\xi_t, \xi_s) - \eta_3(\xi_t) - \eta_3(\xi_s) + \eta_3] + \frac{n-1}{n^2 h^{3d_x/2}} \sum_{t=1}^{n} [\eta_3(\xi_t) - \eta_3] + \frac{n-1}{2nh^{3d_x/2}} \eta_3 \\
&= A_{13}^{(1)} + A_{13}^{(2)} + A_{13}^{(3)}, \text{ say.}
\end{aligned}$$

By Lemma A(ii) of Hjellvik *et al.* (1998), we have

$$E[A_{13}^{(1)}]^2 \le \frac{C}{n^2 h^{3d_x}} [E|\eta_3(\xi_t, \xi_s) - \eta_3(\xi_t) - \eta_3(\xi_s) + \eta_3|^{2(1+\delta)}]^{1/(1+\delta)} \sum_{j=1}^{n} j^2 \beta(j)^{\delta/(1+\delta)} = O(n^{-2} h^{-3d_x + d_x/(1+\delta)}).$$

Then, $A_{13}^{(1)} = o_P(1)$ follows from Chebyshev's inequality. In addition,

$$\begin{aligned}
E[A_{13}^{(2)}]^2 &\le \frac{1}{n^2 h^{3d_x}} \sum_{t=1}^{n} \text{var}[\eta_3(\xi_t)] + \frac{2}{nh^{3d_x}} \sum_{j=1}^{n-1} |\text{cov}\,[\eta_3(\xi_1), \eta_3(\xi_{1+j})]| \\
&\le \frac{C}{nh^{d_x}} + \frac{C}{nh^{d_x}} \sum_{j=1}^{n-1} j^2 \beta(j)^{\delta/(1+\delta)} = O(n^{-1} h^{-d_x}),
\end{aligned}$$

where we have used the mixing inequality, change of variable and Assumption A.1. Thus, $A_{13}^{(2)} = o_P(1)$ by Chebyshev's inequality. Moreover,

$$\begin{aligned}
A_{13}^{(3)} &= \frac{n-1}{nh^{3d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2\left(p, \frac{X_2 - X_1}{h}\right) K^2\left(\frac{x_2 - x_1}{h}\right) E[|\varepsilon(u, v, x_2)|^2 | x_2] g(x_2) dx_1 dx_2 dW_1(u) dW_2(v) \\
&= \frac{n-1}{nh^{3d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2\left(p, \frac{X_2 - X_1}{h}\right) K^2\left(\frac{x_2 - x_1}{h}\right) [1 - |\phi_{yz}(u, v, x_2)|^2] g(x_2) dx_1 dx_2 dW_1(u) dW_2(v) \\
&= \frac{n-1}{nh^{d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2(p, \tau) K^2(\tau) [1 - |\phi_{yz}(u, v, x_1 + \tau h)|^2] g(x_1 + \tau h) dx_1 d\tau dW_1(u) dW_2(v) \\
&= B_1 + o_P(1),
\end{aligned}$$

where the last equality follows by the continuity of $g(\cdot)$ and $\phi_{yz}(u, v, \cdot)$, and the second order Taylor expansion. Hence, we have proved $A_{13} = B_1 + o_P(1)$.

Finally, we consider the A_{14} term. Define

$$\eta_4(\xi_t,\xi_s,\xi_r) = \iint \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s - X_t}{h}\right) D\left(p, \frac{X_r - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) K\left(\frac{X_r - X_t}{h}\right)$$
$$\times\, Re\left[\varepsilon_{yz}(u,v,X_s)\varepsilon_{yz}(u,v,X_r)^*\right] dW_1(u)dW_2(v)$$
$$+ \iint \frac{a(X_s)}{g^2(X_s)} D\left(p, \frac{X_t - X_s}{h}\right) D\left(p, \frac{X_r - X_s}{h}\right) K\left(\frac{X_t - X_s}{h}\right) K\left(\frac{X_r - X_s}{h}\right)$$
$$\times\, Re\left[\varepsilon_{yz}(u,v,X_r)\varepsilon_{yz}(u,v,X_t)^*\right] dW_1(u)dW_2(v)$$
$$+ \iint \frac{a(X_r)}{g^2(X_r)} D\left(p, \frac{X_s - X_r}{h}\right) D\left(p, \frac{X_t - X_r}{h}\right) K\left(\frac{X_s - X_r}{h}\right) K\left(\frac{X_t - X_r}{h}\right)$$
$$\times\, Re\left[\varepsilon_{yz}(u,v,X_t)\varepsilon_{yz}(u,v,X_s)^*\right] dW_1(u)dW_2(v)$$

and

$$\eta_4(\xi_s,\xi_r) = \int \eta_4(\xi,\xi_s,\xi_r)dP(\xi)$$
$$= \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right)$$
$$\times\, Re\left[\varepsilon_{yz}(u,v,X_s)\varepsilon_{yz}(u,v,X_r)^*\right] dx dW_1(u)dW_2(v).$$

Then, A_{14} could be rewritten as

$$A_{14} = \frac{1}{3n^2h^{3d_x/2}} \sum_{s\neq r\neq t} \left[\eta_4(\xi_t,\xi_s,\xi_r) - \eta_4(\xi_t,\xi_s) - \eta_4(\xi_t,\xi_r) - \eta_4(\xi_s,\xi_r)\right]$$
$$- \frac{2}{n^2h^{3d_x/2}} \sum_{s\neq r} \eta_4(\xi_s,\xi_r) + \frac{1}{nh^{3d_x/2}} \sum_{s\neq r} \eta_4(\xi_s,\xi_r)$$
$$= A_{14}^{(1)} + A_{14}^{(2)} + \tilde{U}_1, \text{ say.}$$

The proof of Lemma 1 will be completed provided $A_{14}^{(1)} = o_P(1)$ and $A_{14}^{(2)} = o_P(1)$ hold. We note that

$$M = \int |\eta_4(\xi_t,\xi_s,\xi_r)|^2 dP = O(h^{2d_x}),$$

where P denotes any probability measure in the set $\{P(\xi_t,\xi_s,\xi_r), P(\xi_t)P(\xi_s,\xi_r), P(\xi_t,\xi_r)P(\xi_s), P(\xi_t)P(\xi_s)P(\xi_r)\}$. By Lemma A(i) of Hjellvik *et al.* (1998), we have

$$E\left[A_{14}^{(1)}\right]^2 \leq Cn^{-1}h^{-3d_x}M^{1/(1+\delta)} \sum_{j=1}^{\infty} j^2\beta(j)^{\delta/(1+\delta)} = O(n^{-1}h^{-3d_x+2d_x/(1+\delta)}).$$

Thus, $A_{14}^{(1)} = o_P(1)$ by Chebyshev's inequality. In addition, $A_{14}^{(2)} = o_P(1)$ by the fact that $\tilde{U}_1 = O_P(1)$, which is proved in Proposition 8. Therefore, we have finished the proof of Lemma 1.

Proof of Lemma 0.11. Firstly, we decompose R_1 as follows:

$$R_1 = h^{d_x/2} \iint \sum_{t=1}^{n} \frac{a(X_t)}{nh^{d_x}g(X_t)} Re\left\{\sum_{s=1}^{n} K\left(\frac{X_s - X_t}{h}\right)\left[e^{\mathbf{i}(u'Y_s+v'Z_s)} - \phi_{yz}(u,v,X_s)\right]\right.$$
$$\left.\times \left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right]^*\right\} dW_1(u)dW_2(v)\left[1 + o_P(1)\right]$$
$$= \frac{K(0)}{nh^{d_x/2}} \sum_{t=1}^{n} \iint \frac{a(X_t)}{g(X_t)} Re\left\{\varepsilon_{yz}(u,v,X_t)\left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right]^*\right\} dW_1(u)dW_2(v)\left[1 + o_P(1)\right]$$
$$+ \frac{1}{nh^{d_x/2}} \sum_{s\neq t} \iint \frac{a(X_t)}{g(X_t)} D\left(p, \frac{X_s - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) Re\left\{\varepsilon_{yz}(u,v,X_s)\left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right]^*\right\} dW_1(u)dW_2(v)\left[1 + o_P(1)\right]$$
$$= (R_{11} + R_{12})\left[1 + o_P(1)\right], \text{ say.} \tag{A.6}$$

We shall first prove $R_{11} = o_P(1)$. By Masry (1996a, b), we know the bias $\bar{\phi}_{yz}(u,v,x) - \phi_{yz}(u,v,x) = O(h^{p+1})$ uniformly in $(u,v) \in \mathbb{R}^{d_y+d_z}$ and in $x \in \mathbb{G}$. By Assumptions A.3 and A.4, we have $E(R_{11}) = 0$ and $\text{var}(R_{11}) = O(n^{-1}h^{2p+2-d_x}) = o(1)$. Hence we obtain $R_{11} = o_P(1)$ by Chebyshev's inequality.

Next, we prove $R_{12} = o_P(1)$. Define

$$\Psi(\xi_s,\xi_t) = \iint \frac{a(X_t)}{g(X_t)} D\left(p, \frac{X_s - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) Re\left\{\varepsilon_{yz}(u,v,X_s)\left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right]^*\right\} dW_1(u)dW_2(v)$$
$$+ \iint \frac{a(X_s)}{g(X_s)} D\left(p, \frac{X_t - X_s}{h}\right) K\left(\frac{X_t - X_s}{h}\right) Re\left\{\varepsilon_{yz}(u,v,X_t)\left[\bar{\phi}_{yz}(u,v,X_s) - \phi_{yz}(u,v,X_s)\right]^*\right\} dW_1(u)dW_2(v)$$

and

$$\Psi(\xi_s) = \int \Psi(\xi_s,\xi)dP(\xi)$$
$$= \iiint a(x) D\left(p, \frac{X_s - x}{h}\right) K\left(\frac{X_s - x}{h}\right) Re\left\{\varepsilon_{yz}(u,v,X_s)\left[\bar{\phi}_{yz}(u,v,x) - \phi_{yz}(u,v,x)\right]^*\right\} dW_1(u)dW_2(v)dx,$$

where we have used the fact that $E\left[\varepsilon_{yz}(u,v,X_t)|X_t\right] = 0$. Then

$$R_{12} = \frac{1}{nh^{d_x/2}} \sum_{1\le t<s\le n} [\Psi(\xi_s,\xi_t) - \Psi(\xi_s) - \Psi(\xi_t)] + \frac{2(n-1)}{nh^{d_x/2}} \sum_{t=1}^{n} \Psi(\xi_t) = R_{12}^{(1)} + R_{12}^{(2)}, \text{ say.} \quad \text{(A.7)}$$

Obviously, $E[\Psi(\xi_s,\xi_t) - \Psi(\xi_s) - \Psi(\xi_t)] = 0$, which implies $E[R_{12}^{(1)}] = 0$. By Lemma A(ii) of Hjellvik *et al.* (1998), we have

$$\text{var}\left(R_{12}^{(1)}\right) \le \frac{C}{n^2h^{d_x}} n^2 E\left[|\Psi(\xi_s,\xi_t) - \Psi(\xi_s) - \Psi(\xi_t)|^{2(1+\delta)}\right]^{\frac{1}{1+\delta}} \sum_{j=1}^{\infty} j^2 \beta(j)^{\delta/1+\delta} = O(h^{2p+2-\delta d_x/(1+\delta)}) = o(1).$$

In addition,

$$\text{var}\left(R_{12}^{(2)}\right) \le \frac{4(n-1)^2}{n^2h^{d_x}} \sum_{t=1}^{n} \text{var}[\Psi(\xi_t)] + \frac{4(n-1)^2}{n^2h^{d_x}} n \sum_{j=1}^{n-1} |\text{cov}\left[\Psi(\xi_1),\Psi(\xi_{1+j})\right]|$$
$$\le Cnh^{-d_x} O(h^{2p+2+2d_x}) + Cnh^{-d_x} \sum_{j=1}^{\infty} j^2 \beta^{\frac{\delta}{1+\delta}}(j) O(h^{2p+2+2d_x}) = O(nh^{2p+2+d_x}) = o(1).$$

Hence, we have $R_{12}^{(1)} = o_P(1), R_{12}^{(2)} = o_P(1)$ by Chebyshev's inequality. We have finished the proof of Lemma 0.11.

Proof of Lemma 0.12. Define

$$\Upsilon(X_t) = \iint \left|\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right|^2 a(X_t) dW_1(u)dW_2(v).$$

Then we can write

$$R_2 = h^{d_x/2} \sum_{t=1}^{n} \{\Upsilon(X_t) - E[\Upsilon(X_t)]\} + nh^{d_x/2} E[\Upsilon(X_t)] = R_{21} + R_{22}, \text{ say.}$$

Firstly, we prove $R_{21} = o_P(1)$. Since

$$\text{var}(R_{21}) \le h^{d_x} \sum_{t=1}^{n} \text{var}\left[\Upsilon(X_t)\right] + 2nh^{d_x} \sum_{j=1}^{n-1} |\text{cov}\left[\Upsilon(X_1),\Upsilon(X_{1+j})\right]|$$
$$\le Cnh^{d_x} O(h^{4p+4}) + Cnh^{d_x} \sum_{j=1}^{\infty} j^2 \beta(j)^{\delta/1+\delta} O(h^{4p+4}) = O(nh^{4p+4+d_x}) = o(1),$$

we have $R_{21} = o_P(1)$ by Chebyshev's inequality.

Finally, the term R_{22} is a nonnegative constant of order $R_{22} = O(nh^{2p+2+d_x/2}) = o_P(1)$.

Proof of Proposition 0.2**.** Similar to the proof of Proposition 0.1, we first decompose T_2 as follows:

$$
\begin{aligned}
T_2 &= h^{d_x/2}\sum_{t=1}^{n}\iint |\phi_y(u,X_t)|^2|\hat{\phi}_z(v,X_t)-\phi_z(v,X_t)|^2 a(X_t)dW_1(u)dW_2(v)\\
&= h^{d_x/2}\sum_{t=1}^{n}\iint |\phi_y(u,X_t)|^2|\hat{\phi}_z(v,X_t)-\bar{\phi}_z(v,X_t)|^2 a(X_t)dW_1(u)dW_2(v)\\
&+ 2h^{d_x/2}\sum_{t=1}^{n}\iint |\phi_y(u,X_t)|^2 Re\left\{\left[\hat{\phi}_z(v,X_t)-\bar{\phi}_z(v,X_t)\right]\left[\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v)\\
&+ h^{d_x/2}\sum_{t=1}^{n}\iint |\phi_y(u,X_t)|^2|\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)|^2 a(X_t)dW_1(u)dW_2(v)\\
&= A_2 + 2R_3 + R_4, \text{ say.}
\end{aligned}
\tag{A.8}
$$

Then the proof of Proposition 0.2 consists of the proofs of Lemmas 0.13- 0.15 below.

Lemma 0.13 *Under the conditions of Theorem 1,* $A_2 = B_2 + \tilde{U}_2 + o_P(1)$*.*

Lemma 0.14 *Let* R_3 *be defined as in* (A.8)*. Then* $R_3 = o_P(1)$ *under the conditions of Theorem 1.*

Lemma 0.15 *Let* R_4 *be defined as in* (A.8)*. Then* $R_4 = o_P(1)$ *under the conditions of Theorem 1.*

Proof of Lemma 0.13**.** We decompose A_2 as follows:

$$
\begin{aligned}
A_2 &= h^{d_x/2}\sum_{t=1}^{n}\iint |\phi_y(u,X_t)|^2\left|\sum_{s=1}^{n}\frac{1}{nh^{d_x}g(X_t)}D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\varepsilon_z(v,X_s)\right|^2 a(X_t)dW_1(u)dW_2(v)\cdot[1+o_P(1)]\\
&= \frac{1}{n^2h^{3d_x/2}}\sum_{t=1}^{n}\frac{a(X_t)}{g^2(X_t)}K^2(0)\iint |\phi_y(u,X_t)|^2|\varepsilon_z(v,X_t)|^2 dW_1(u)dW_2(v)\cdot[1+o_P(1)]\\
&+ \frac{2}{n^2h^{3d_x/2}}\sum_{s\neq t}\frac{a(X_t)}{g^2(X_t)}K(0)D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\iint |\phi_y(u,X_t)|^2 Re\left[\varepsilon_z(v,X_s)\varepsilon_z(v,X_t)^*\right]dW_1(u)dW_2(v)\cdot[1+o_P(1)]\\
&+ \frac{1}{n^2h^{3d_x/2}}\sum_{s\neq t}\frac{a(X_t)}{g^2(X_t)}D^2\left(p,\frac{X_s-X_t}{h}\right)K^2\left(\frac{X_s-X_t}{h}\right)\iint |\phi_y(u,X_t)|^2|\varepsilon_z(v,X_s)|^2 dW_1(u)dW_2(v)\cdot[1+o_P(1)]\\
&+ \frac{1}{n^2h^{3d_x/2}}\sum_{t\neq s\neq r}\frac{a(X_t)}{g^2(X_t)}D\left(p,\frac{X_s-X_t}{h}\right)D\left(p,\frac{X_r-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)K\left(\frac{X_r-X_t}{h}\right)\\
&\times\iint |\phi_y(u,X_t)|^2 Re\left[\varepsilon_z(v,X_s)\varepsilon_z(v,X_r)^*\right]dW_1(u)dW_2(v)\cdot[1+o_P(1)]\\
&= A_{21}+A_{22}+A_{23}+A_{24}, \text{ say.}
\end{aligned}
$$

Since $|\phi_y(u,X_t)|^2$ is bounded, we can easily show $A_{21} = o_P(1)$ and $A_{22} = o_P(1)$ following analogous reasoning to the proofs for $A_{11} = o_P(1)$ and $A_{12} = o_P(1)$. Now we consider the A_{23} term. Define

$$
\begin{aligned}
\eta_5(\xi_t,\xi_s) &= \frac{a(X_t)}{g^2(X_t)}D^2\left(p,\frac{X_s-X_t}{h}\right)K^2\left(\frac{X_s-X_t}{h}\right)\iint |\phi_y(u,X_t)|^2|\varepsilon_z(v,X_s)|^2 dW_1(u)dW_2(v)\\
&+ \frac{a(X_s)}{g^2(X_s)}D^2\left(p,\frac{X_t-X_s}{h}\right)K^2\left(\frac{X_t-X_s}{h}\right)\iint |\phi_y(u,X_s)|^2|\varepsilon_z(v,X_t)|^2 dW_1(u)dW_2(v)
\end{aligned}
$$

and $\eta_5(\xi_t) = \int \eta_5(\xi_t, \xi) dP(\xi)$, $\eta_5 = \int \eta_5(\xi) dP(\xi)$. Then A_{23} could be rewritten as

$$\begin{aligned}
A_{23} &= \frac{1}{n^2 h^{3d_x/2}} \sum_{1\le t<s\le n} \eta_5(\xi_t, \xi_s) \\
&= \frac{1}{n^2 h^{3d_x/2}} \sum_{1\le t<s\le n} [\eta_5(\xi_t, \xi_s) - \eta_5(\xi_t) - \eta_5(\xi_s) + \eta_5] + \frac{n-1}{n^2 h^{3d_x/2}} \sum_{t=1}^{n} [\eta_5(\xi_t) - \eta_5] + \frac{n-1}{2nh^{3d_x/2}} \eta_5 \\
&= A_{23}^{(1)} + A_{23}^{(2)} + A_{23}^{(3)}, \text{ say.}
\end{aligned}$$

Following analogous analysis of $A_{13}^{(1)}$, $A_{13}^{(2)}$ and $A_{13}^{(3)}$, we can show $E[A_{23}^{(1)}]^2 = O(n^{-2} h^{-3d_x + d_x/(1+\delta)})$, $E[A_{23}^{(2)}]^2 = O(n^{-1} h^{-d_x})$, and

$$\begin{aligned}
A_{23}^{(3)} &= \frac{n-1}{nh^{3d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2\left(p, \frac{x_2 - x_1}{h}\right) K^2\left(\frac{x_2 - x_1}{h}\right) |\phi_y(u, x_1)|^2 E[|\varepsilon_z(v, x_2)|^2 | x_2] g(x_2) dx_1 dx_2 dW_1(u) dW_2(v) \\
&= \frac{n-1}{nh^{3d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2\left(p, \frac{x_2 - x_1}{h}\right) K^2\left(\frac{x_2 - x_1}{h}\right) |\phi_y(u, x_1)|^2 [1 - |\phi_z(v, x_2)|^2] g(x_2) dx_1 dx_2 dW_1(u) dW_2(v) \\
&= \frac{n-1}{nh^{d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2(p, \tau) K^2(\tau) |\phi_y(u, x_1)|^2 [1 - |\phi_z(v, x_1 + \tau h)|^2] g(x_1 + \tau h) dx_1 d\tau dW_1(u) dW_2(v) \\
&= B_2 + o_p(1),
\end{aligned}$$

where the last equality follows by the continuity of $g(\cdot)$ and $\phi_z(v, \cdot)$, and the second order Taylor expansion. Hence we have $A_{23} = B_2 + o_P(1)$.

Finally, we prove $A_{24} = \tilde{U}_2 + o_P(1)$. Define

$$\begin{aligned}
\eta_6(\xi_t, \xi_s, \xi_r) &= \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s - X_t}{h}\right) D\left(p, \frac{X_r - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) K\left(\frac{X_r - X_t}{h}\right) \\
&\quad \times \iint |\phi_y(u, X_t)|^2 \, Re\left[\varepsilon_z(v, X_s) \varepsilon_z(v, X_r)^*\right] dW_1(u) dW_2(v) \\
&\quad + \frac{a(X_s)}{g^2(X_s)} D\left(p, \frac{X_t - X_s}{h}\right) D\left(p, \frac{X_r - X_s}{h}\right) K\left(\frac{X_t - X_s}{h}\right) K\left(\frac{X_r - X_s}{h}\right) \\
&\quad \times \iint |\phi_y(u, X_s)|^2 \, Re\left[\varepsilon_z(v, X_r) \varepsilon_z(v, X_t)^*\right] dW_1(u) dW_2(v) \\
&\quad + \frac{a(X_r)}{g^2(X_r)} D\left(p, \frac{X_s - X_r}{h}\right) D\left(p, \frac{X_t - X_r}{h}\right) K\left(\frac{X_s - X_r}{h}\right) K\left(\frac{X_t - X_r}{h}\right) \\
&\quad \times \iint |\phi_y(u, X_r)|^2 \, Re\left[\varepsilon_z(v, X_t) \varepsilon_z(v, X_s)^*\right] dW_1(u) dW_2(v)
\end{aligned}$$

and

$$\begin{aligned}
\eta_6(\xi_s, \xi_r) &= \int \eta_6(\xi, \xi_s, \xi_r) dP(\xi) \\
&= \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_r - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_r - x}{h}\right) \\
&\quad \times |\phi_y(u, x)|^2 \, Re\left[\varepsilon_z(v, X_s) \varepsilon_z(v, X_r)^*\right] dW_1(u) dW_2(v) dx.
\end{aligned}$$

Then, A_{24} could be rewritten as

$$\begin{aligned}
A_{24} &= \frac{1}{3n^2 h^{3d_x/2}} \sum_{s\ne r\ne t} [\eta_6(\xi_t, \xi_s, \xi_r) - \eta_6(\xi_t, \xi_s) - \eta_6(\xi_t, \xi_r) - \eta_6(\xi_s, \xi_r)] \\
&\quad - \frac{2}{n^2 h^{3d_x/2}} \sum_{s\ne r} \eta_6(\xi_s, \xi_r) + \frac{1}{nh^{3d_x/2}} \sum_{s\ne r} \eta_6(\xi_s, \xi_r) \\
&= A_{24}^{(1)} + A_{24}^{(2)} + \tilde{U}_2.
\end{aligned}$$

Following analogous analysis of A_{14}, we can show $A_{24}^{(1)} = o_P(1)$ and $A_{24}^{(2)} = o_P(1)$. Thus, we have finished the proof of Lemma 0.13.

Proof of Lemma 0.14**.** Firstly, we decompose R_3 as follows:

$$R_3 = h^{d_x/2}\sum_{t=1}^{n}\iint |\phi_y(u,X_t)|^2 \, Re\left\{\sum_{s=1}^{n}\frac{a(X_t)}{nh^{d_x}g(X_t)}D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\left[e^{\mathbf{i}v'Z_s}-\phi_z(v,X_s)\right]\right.$$
$$\times\left.\left[\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\}dW_1(u)dW_2(v)[1+o_P(1)]$$
$$=\frac{K(0)}{nh^{d_x/2}}\sum_{t=1}^{n}\frac{a(X_t)}{g(X_t)}\iint |\phi_y(u,X_t)|^2\, Re\left\{\varepsilon_z(v,X_t)\left[\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\}dW_1(u)dW_2(v)[1+o_P(1)]$$
$$+\frac{1}{nh^{d_x/2}}\sum_{s\neq t}\frac{a(X_t)}{g(X_t)}D\left(p,\frac{X_s-X_t}{h}\right)K\left(\frac{X_s-X_t}{h}\right)\iint |\phi_y(u,X_t)|^2 Re\left\{\varepsilon_z(v,X_s)\left[\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\}$$
$$\times dW_1(u)dW_2(v)[1+o_P(1)]$$
$$=[R_{31}+R_{32}][1+o_P(1)],\ \text{say}.$$

Following analogous analysis of R_{11} and R_{12}, we can show $E(R_{31}) = 0$, $\text{var}(R_{31}) = O(n^{-1}h^{2p+2-d_x}) = o(1)$, and $E(R_{32}^2) = O(nh^{d_x+2p+2} + h^{2p+2-\delta d_x/(1+\delta)}) = o(1)$. Thus, $R_{31} = o_P(1)$ and $R_{32} = o_P(1)$ by Chebyshev's inequality.

Proof of Lemma 0.15**.** The proof is the same as that of Lemma 0.12 except that we replace $\Upsilon_1(X_t)$ with

$$\Upsilon_2(X_t) = \iint |\phi_y(u,X_t)|^2\left|\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right|^2 a(X_t)dW_1(u)dW_2(v).$$

Proof of Proposition 0.3**.** The proof of Proposition 0.3 is the same as the proof of Proposition 0.2 except that we replace $\phi_y(u,X_t)$ and $\hat{\phi}_z(v,X_t)-\phi_z(v,X_t)$ with $\phi_z(v,X_t)$ and $\hat{\phi}_y(u,X_t)-\phi_y(u,X_t)$ respectively.

Proof of Proposition 0.4**.** We first decompose T_4 as follows:

$$T_4 = 2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\left[\hat{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]\left[\hat{\phi}_y(u,X_t)-\phi_y(u,X_t)\right]^*\right\}a(X_t)dW_1(u)dW_2(v)$$
$$= 2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\left[\hat{\phi}_z(v,X_t)-\bar{\phi}_z(v,X_t)\right]\left[\hat{\phi}_y(u,X_t)-\bar{\phi}_y(u,X_t)\right]^*\right\}a(X_t)dW_1(u)dW_2(v)$$
$$+ 2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\left[\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]\left[\hat{\phi}_y(u,X_t)-\bar{\phi}_y(u,X_t)\right]^*\right\}a(X_t)dW_1(u)dW_2(v)$$
$$+ 2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\left[\hat{\phi}_z(v,X_t)-\bar{\phi}_z(v,X_t)\right]\left[\bar{\phi}_y(u,X_t)-\phi_y(u,X_t)\right]^*\right\}a(X_t)dW_1(u)dW_2(v)$$
$$+ 2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\left[\bar{\phi}_z(v,X_t)-\phi_z(v,X_t)\right]\left[\bar{\phi}_y(u,X_t)-\phi_y(u,X_t)\right]^*\right\}a(X_t)dW_1(u)dW_2(v)$$
$$= A_4 + R_5 + R_6 + R_7,\ \text{say}. \tag{A.9}$$

Then the proof of Proposition 0.4 consists of the proofs of Lemmas 0.16- 0.19 below.

Lemma 0.16 *Under the conditions of Theorem 1,* $A_4 = \tilde{U}_4 + o_P(1)$.

Lemma 0.17 *Let* R_5 *be defined as in* (A.9)*. Then* $R_5 = o_P(1)$ *under the conditions of Theorem 1.*

Lemma 0.18 *Let* R_6 *be defined as in* (A.9)*. Then* $R_6 = o_P(1)$ *under the conditions of Theorem 1.*

Lemma 0.19 *Let R_7 be defined as in (A.9). Then $R_7 = o_P(1)$ under the conditions of Theorem 1.*

Proof of Lemma 0.16**.** We decompose A_4 as follows:

$$
\begin{aligned}
A_4 &= \frac{2}{n^2 h^{3d_x/2}} \sum_{t=1}^{n} \frac{a(X_t)}{g^2(X_t)} K^2(0) \iint Re\left[\phi_y(u,X_t)\phi_z^*(v,X_t)\varepsilon_y(u,X_t)^*\varepsilon_z(v,X_t)\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&\quad + \frac{2}{n^2 h^{3d_x/2}} \sum_{s\neq t} \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) K(0) K\left(\frac{X_s-X_t}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_t)\phi_z^*(v,X_t)\varepsilon_y(u,X_t)^*\varepsilon_z(v,X_s)\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&\quad + \frac{2}{n^2 h^{3d_x/2}} \sum_{s\neq t} \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) K(0) K\left(\frac{X_s-X_t}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_t)\phi_z^*(v,X_t)\varepsilon_y(u,X_s)^*\varepsilon_z(v,X_t)\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&\quad + \frac{2}{n^2 h^{3d_x/2}} \sum_{s\neq t} \frac{a(X_t)}{g^2(X_t)} D^2\left(p, \frac{X_s-X_t}{h}\right) K^2\left(\frac{X_s-X_t}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_t)\phi_z^*(v,X_t)\varepsilon_y(u,X_s)^*\varepsilon_z(v,X_s)\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&\quad + \frac{2}{n^2 h^{3d_x/2}} \sum_{s\neq r\neq t} \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) D\left(p, \frac{X_r-X_t}{h}\right) K\left(\frac{X_s-X_t}{h}\right) K\left(\frac{X_r-X_t}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_t)\phi_z^*(v,X_t)\varepsilon_y(u,X_s)^*\varepsilon_z(v,X_r)\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&= A_{41}+A_{42}+A_{43}+A_{44}+A_{45}, \text{ say.}
\end{aligned}
$$

We shall first prove $A_{41} = o_P(1)$. It is obvious that $E(A_{41}) = 0$ and $\text{var}(A_{41}) = O(n^{-3}h^{-3d_x})$. Thus, we have $A_{41} = o_P(1)$. Under $\mathbb{H}_0$, we obtain $E(A_{42}) = 0$, $E(A_{43}) = 0$, and $E(A_{44}) = 0$. By Lemma A(ii) of Hjellvik *et al.* (1998), we have $\text{var}(A_{42}) = O(n^{-2}h^{-3d_x+d_x/(1+\delta)})$, $\text{var}(A_{43}) = O(n^{-2}h^{-3d_x+d_x/(1+\delta)})$, $\text{var}(A_{44}) = O(n^{-2}h^{-3d_x+d_x/(1+\delta)})$. It follows that $A_{42} = o_P(1)$, $A_{43} = o_P(1)$ and $A_{44} = o_P(1)$ by Chebyshev's inequality.

Now, let us consider the A_{45} term. We can show $A_{45} = \tilde{U}_4 + o_P(1)$ following analogous analysis of A_{14} except that we replace $\eta_4(\xi_t,\xi_s,\xi_r)$ and $\eta_4(\xi_s,\xi_r)$ with

$$
\begin{aligned}
\eta_7(\xi_t,\xi_s,\xi_r) &= \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) D\left(p, \frac{X_r-X_t}{h}\right) K\left(\frac{X_s-X_t}{h}\right) K\left(\frac{X_r-X_t}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_t)\phi_z^*(v,X_t)\varepsilon_y(u,X_s)^*\varepsilon_z(v,X_r)\right] dW_1(u)dW_2(v) \\
&\quad + \frac{a(X_s)}{g^2(X_s)} D\left(p, \frac{X_t-X_s}{h}\right) D\left(p, \frac{X_r-X_s}{h}\right) K\left(\frac{X_t-X_s}{h}\right) K\left(\frac{X_r-X_s}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_s)\phi_z^*(v,X_s)\varepsilon_y(u,X_r)^*\varepsilon_z(v,X_t)\right] dW_1(u)dW_2(v) \\
&\quad + \frac{a(X_r)}{g^2(X_r)} D\left(p, \frac{X_s-X_r}{h}\right) D\left(p, \frac{X_t-X_r}{h}\right) K\left(\frac{X_s-X_r}{h}\right) K\left(\frac{X_t-X_r}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_r)\phi_z^*(v,X_r)\varepsilon_y(u,X_t)^*\varepsilon_z(v,X_s)\right] dW_1(u)dW_2(v)
\end{aligned}
$$

and

$$\eta_7(\xi_s,\xi_r) = \int \eta_7(\xi,\xi_s,\xi_r)dP(\xi)$$
$$= \iint \frac{a(x)}{g(x)} D\left(p, \frac{X_s-x}{h}\right) D\left(p, \frac{X_r-x}{h}\right) K\left(\frac{X_s-x}{h}\right) K\left(\frac{X_r-x}{h}\right)$$
$$\times Re\left[\phi_y(u,x)\phi_z^*(v,x)\varepsilon_y(u,X_s)^*\varepsilon_z(v,X_r)\right] dx dW_1(u) dW_2(v).$$

Proof of Lemma 0.17**.** Firstly, R_5 could be decomposed as follows:

$$R_5 = \frac{2K(0)}{nh^{d_x/2}} \sum_{t=1}^{n} \iint \frac{a(X_t)}{g(X_t)} Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\varepsilon_y(u,X_t)^*\left[\bar\phi_z(v,X_t)-\phi_z(v,X_t)\right]\right\} dW_1(u)dW_2(v)\cdot[1+o_P(1)]$$
$$+ \frac{2}{nh^{d_x/2}} \sum_{s\neq t} \iint \frac{a(X_t)}{g(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) K\left(\frac{X_s-X_t}{h}\right)$$
$$\times Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\varepsilon_y(u,X_s)^*\left[\bar\phi_z(v,X_t)-\phi_z(v,X_t)\right]\right\} dW_1(u)dW_2(v)\cdot[1+o_P(1)]$$
$$= (R_{51}+R_{52})\cdot[1+o_P(1)], \text{say.}$$

Since $\bar\phi_z(v,x)-\phi_z(v,x) = O(h^{p+1})$, we have $E(R_{51}) = 0$ and $\text{var}(R_{51}) = O(n^{-1}h^{2p+2-d_x})$. It follows that $R_{51} = o_P(1)$ by Chebyshev's inequality.

Next, we show $R_{52} = o_P(1)$. It is the same as the proof of $R_{12} = o_P(1)$ except that we replace $\Psi(\xi_s,\xi_t)$ and $\Psi(\xi_s)$ with

$$\Psi_5(\xi_s,\xi_t) = \iint \frac{a(X_t)}{g(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) K\left(\frac{X_s-X_t}{h}\right) Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\varepsilon_y(u,X_s)^*\left[\bar\phi_z(v,X_t)-\phi_z(v,X_t)\right]\right\} dW_1(u)dW_2(v)$$
$$+ \iint \frac{a(X_s)}{g(X_s)} D\left(p, \frac{X_t-Xs}{h}\right) K\left(\frac{X_t-X_s}{h}\right) Re\left\{\phi_y(u,X_s)\phi_z(v,X_s)^*\varepsilon_y(u,X_t)^*\left[\bar\phi_z(v,X_s)-\phi_z(v,X_s)\right]\right\} dW_1(u)dW_2(v)$$

and

$$\Psi_5(\xi_s) = \int \Psi_5(\xi_s,\xi)dP(\xi)$$
$$= \iiint a(x) D\left(p, \frac{X_s-x}{h}\right) K\left(\frac{X_s-x}{h}\right) Re\left\{\phi_y(u,x)\phi_z(v,x)^*\varepsilon_y(u,X_s)^*\left[\bar\phi_z(v,x)-\phi_z(v,x)\right]\right\} dx dW_1(u)dW_2(v).$$

Proof of Lemma 0.18**.** The proof of Lemma 0.18 is the same as that of Lemma 0.12 except that we replace $\Upsilon_1(X_t)$ with

$$\Upsilon_5(X_t) = \iint Re\left\{\phi_y(u,X_t)\phi_z(v,X_t)^*\left[\bar\phi_z(v,X_t)-\phi_z(v,X_t)\right]\left[\bar\phi_y(u,X_t)-\phi_y(u,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v).$$

Proof of Proposition 0.5**.** We first decompose T_5 as follows:

$$T_5 = -2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y^*(u,X_t)\left[\hat\phi_{yz}(u,v,X_t)-\phi_{yz}(u,v,X_t)\right]\left[\hat\phi_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v)$$
$$= -2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y^*(u,X_t)\left[\hat\phi_{yz}(u,v,X_t)-\bar\phi_{yz}(u,v,X_t)\right]\left[\hat\phi_z(v,X_t)-\bar\phi_z(v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v)$$
$$-2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y^*(u,X_t)\left[\bar\phi_{yz}(u,v,X_t)-\phi_{yz}(u,v,X_t)\right]\left[\hat\phi_z(v,X_t)-\bar\phi_z(v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v)$$
$$-2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y^*(u,X_t)\left[\hat\phi_{yz}(u,v,X_t)-\bar\phi_{yz}(u,v,X_t)\right]\left[\bar\phi_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v)$$
$$-2h^{d_x/2}\sum_{t=1}^{n}\iint Re\left\{\phi_y^*(u,X_t)\left[\bar\phi_{yz}(u,v,X_t)-\phi_{yz}(u,v,X_t)\right]\left[\bar\phi_z(v,X_t)-\phi_z(v,X_t)\right]^*\right\} a(X_t)dW_1(u)dW_2(v)$$
$$= A_5+R_8+R_9+R_{10}, \text{ say.} \tag{A.10}$$

Then the proof of Proposition 0.5 consists of the proofs of Lemmas 0.20 -0.23 below.

Lemma 0.20 *Under the conditions of Theorem 1,* $A_5 = B_5 + \tilde{U}_5 + o_P(1)$.

Lemma 0.21 *Let* R_8 *be defined as in (A.10). Then* $R_8 = o_P(1)$ *under the conditions of Theorem 1.*

Lemma 0.22 *Let* R_9 *be defined as in (A.10). Then* $R_9 = o_P(1)$ *under the conditions of Theorem 1.*

Lemma 0.23 *Let* R_{10} *be defined as in (A.10). Then* $R_{10} = o_P(1)$ *under the conditions of Theorem 1.*

Proof of Lemma 0.20**.** We decompose A_5 as follows:

$$\begin{aligned}
A_5 &= \frac{-2}{n^2 h^{3d_x/2}} \sum_{t=1}^{n} \frac{a(X_t)}{g^2(X_t)} K^2(0) \iint Re\left[\phi_y(u,X_t)^* \varepsilon_{yz}(u,v,X_t)\varepsilon_z(v,X_t)^*\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{-2}{n^2 h^{3d_x/2}} \sum_{s\neq t} \frac{a(X_t)}{g^2(X_t)} K(0) D\left(p, \frac{X_s - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) \\
&\times \iint Re\left[\phi_y(u,X_t)^* \varepsilon_{yz}(u,v,X_s)\varepsilon_z(v,X_t)^*\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{-2}{n^2 h^{3d_x/2}} \sum_{s\neq t} \frac{a(X_t)}{g^2(X_t)} K(0) D\left(p, \frac{X_s - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) \\
&\times \iint Re\left[\phi_y(u,X_t)^* \varepsilon_{yz}(u,v,X_t)\varepsilon_z(v,X_s)^*\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{-2}{n^2 h^{3d_x/2}} \sum_{s\neq t} \frac{a(X_t)}{g^2(X_t)} D^2\left(p, \frac{X_s - X_t}{h}\right) K^2\left(\frac{X_s - X_t}{h}\right) \\
&\times \iint Re\left[\phi_y(u,X_t)^* \varepsilon_{yz}(u,v,X_s)\varepsilon_z(v,X_s)^*\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&+ \frac{-2}{n^2 h^{3d_x/2}} \sum_{s\neq r\neq t} \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s - X_t}{h}\right) D\left(p, \frac{X_r - X_t}{h}\right) K\left(\frac{X_s - X_t}{h}\right) K\left(\frac{X_r - X_t}{h}\right) \\
&\times \iint Re\left[\phi_y(u,X_t)^* \varepsilon_{yz}(u,v,X_s)\varepsilon_z(v,X_r)^*\right] dW_1(u)dW_2(v)\cdot[1+o_P(1)] \\
&= A_{51} + A_{52} + A_{53} + A_{54} + A_{55}, \text{ say.}
\end{aligned}$$

We can show $A_{51} = o_P(1)$, $A_{52} = o_P(1)$, and $A_{53} = o_P(1)$ by following analogous reasoning to the proofs for $A_{11} = o_P(1)$ and $A_{12} = o_P(1)$. Now we consider the A_{54} term. Define

$$\begin{aligned}
\eta_8(\xi_t,\xi_s) &= \frac{a(X_t)}{g^2(X_t)} D^2\left(p, \frac{X_s - X_t}{h}\right) K^2\left(\frac{X_s - X_t}{h}\right) \iint Re\left[\phi_y(u,X_t)^* \varepsilon_{yz}(u,v,X_s)\varepsilon_z(v,X_s)^*\right] dW_1(u)dW_2(v) \\
&+ \frac{a(X_s)}{g^2(X_s)} D^2\left(p, \frac{X_t - X_s}{h}\right) K^2\left(\frac{X_t - X_s}{h}\right) \iint Re\left[\phi_y(u,X_s)^* \varepsilon_{yz}(u,v,X_t)\varepsilon_z(v,X_t)^*\right] dW_1(u)dW_2(v)
\end{aligned}$$

and $\eta_8(\xi_t) = \int \eta_8(\xi_t,\xi)dP(\xi)$, $\eta_8 = \int \eta_8(\xi)dP(\xi)$. Then A_{54} could be rewritten as

$$\begin{aligned}
A_{54} &= \frac{-2}{n^2 h^{3d_x/2}} \sum_{1\le t<s\le n} \eta_8(\xi_t,\xi_s) \\
&= \frac{-2}{n^2 h^{3d_x/2}} \sum_{1\le t<s\le n} \left[\eta_8(\xi_t,\xi_s) - \eta_8(\xi_t) - \eta_8(\xi_s) + \eta_8\right] - \frac{2(n-1)}{n^2 h^{3d_x/2}} \sum_{t=1}^{n} \left[\eta_8(\xi_t) - \eta_8\right] - \frac{(n-1)}{n h^{3d_x/2}} \eta_8 \\
&= A_{54}^{(1)} + A_{54}^{(2)} + A_{54}^{(3)}, \text{ say.}
\end{aligned}$$

Following analogous analysis of $A_{13}^{(1)}$ and $A_{13}^{(2)}$, we can show $E[A_{54}^{(1)}]^2 = O(n^{-2}h^{-3d_x+d_x/(1+\delta)})$ and $E[A_{54}^{(2)}]^2 = O(n^{-1}h^{-d_x})$. Thus, $A_{54}^{(1)} = o_P(1)$ and $A_{54}^{(2)} = o_P(1)$ by Chebyshev's inequality. For the $A_{54}^{(3)}$ term, we have

$$\begin{aligned}
A_{54}^{(3)} &= \frac{-2(n-1)}{nh^{3d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2\left(p, \frac{x_2-x_1}{h}\right) K^2\left(\frac{x_2-x_1}{h}\right) \\
&\quad \times Re\left\{\phi_y(u,x_1)^* E\left[\varepsilon_{yz}(u,v,x_2)\varepsilon_z(v,x_2)^*|x_2\right]\right\} g(x_2)dx_1dx_2dW_1(u)dW_2(v) \\
&= \frac{-2(n-1)}{nh^{3d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2\left(p, \frac{x_2-x_1}{h}\right) K^2\left(\frac{x_2-x_1}{h}\right) \\
&\quad \times Re\left[\phi_y(u,x_2)\phi_y(u,x_1)^*\right]\left[1-|\phi_z(v,x_2)|^2\right] g(x_2)dx_1dx_2dW_1(u)dW_2(v) \\
&= \frac{-2(n-1)}{nh^{d_x/2}} \iiiint \frac{a(x_1)}{g(x_1)} D^2(p,\tau)K^2(\tau)\, Re\left[\phi_y(u,x_1+\tau h)\phi_y(u,x_1)^*\right] \\
&\quad \times \left[1-|\phi_z(v,x_1+\tau h)|^2\right] g(x_1+\tau h)dx_1d\tau dW_1(u)dW_2(v) \\
&= B_5 + o_P(1),
\end{aligned}$$

where the last equality follows by the continuity of $g(\cdot)$, $\phi_y(u,\cdot)$ and $\phi_z(u,\cdot)$, and the second order Taylor expansion.

Finally, we prove $A_{55} = \tilde{U}_5 + o_P(1)$. Define

$$\begin{aligned}
\eta_9(\xi_t,\xi_s,\xi_r) &= \frac{a(X_t)}{g^2(X_t)} D\left(p, \frac{X_s-X_t}{h}\right) D\left(p, \frac{X_r-X_t}{h}\right) K\left(\frac{X_s-X_t}{h}\right) K\left(\frac{X_r-X_t}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_t)^*\varepsilon_{yz}(u,v,X_s)\varepsilon_z(v,X_r)^*\right] dW_1(u)dW_2(v) \\
&\quad + \frac{a(X_s)}{g^2(X_s)} D\left(p, \frac{X_t-X_s}{h}\right) D\left(p, \frac{X_r-X_s}{h}\right) K\left(\frac{X_t-X_s}{h}\right) K\left(\frac{X_r-X_s}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_s)^*\varepsilon_{yz}(u,v,X_r)\varepsilon_z(v,X_t)^*\right] dW_1(u)dW_2(v) \\
&\quad + \frac{a(X_r)}{g^2(X_r)} D\left(p, \frac{X_s-X_r}{h}\right) D\left(p, \frac{X_t-X_r}{h}\right) K\left(\frac{X_s-X_r}{h}\right) K\left(\frac{X_t-X_r}{h}\right) \\
&\quad \times \iint Re\left[\phi_y(u,X_r)^*\varepsilon_{yz}(u,v,X_t)\varepsilon_z(v,X_s)^*\right] dW_1(u)dW_2(v)
\end{aligned}$$

and

$$\begin{aligned}
\eta_9(\xi_s,\xi_r) &= \int \eta_8(\xi,\xi_s,\xi_r)dP(\xi) \\
&= \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_s-x}{h}\right) D\left(p, \frac{X_r-x}{h}\right) K\left(\frac{X_s-x}{h}\right) K\left(\frac{X_r-x}{h}\right) \\
&\quad \times Re\left[\phi_y(u,x)^*\varepsilon_{yz}(u,v,X_s)\varepsilon_z(v,X_r)^*\right] dxdW_1(u)dW_2(v).
\end{aligned}$$

Then, A_{55} could be rewritten as:

$$\begin{aligned}
A_{55} &= \frac{-2}{3n^2h^{3d_x/2}} \sum_{s\neq r\neq t} \left[\eta_9(\xi_t,\xi_s,\xi_r) - \eta_9(\xi_t,\xi_s) - \eta_9(\xi_t,\xi_r) - \eta_9(\xi_s,\xi_r)\right] \\
&\quad + \frac{4}{n^2h^{3d_x/2}} \sum_{s\neq r} \eta_9(\xi_s,\xi_r) - \frac{2}{nh^{3d_x/2}} \sum_{s\neq r} \eta_9(\xi_s,\xi_r) \\
&= A_{55}^{(1)} + A_{55}^{(2)} + \tilde{U}_5, \text{ say.}
\end{aligned}$$

Following analogous analysis of A_{14}, we can show $A_{55}^{(1)} = o_P(1)$ and $A_{55}^{(2)} = o_P(1)$. Thus, we have proved Lemma 0.20.

Proof of Lemma 0.21**.** Firstly, we decompose R_8 as follows:

$$\begin{aligned}
R_8 &= \frac{-2K(0)}{nh^{d_x/2}} \sum_{t=1}^{n} \frac{a(X_t)}{g(X_t)} \iint Re\left\{ \left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t) \right] \phi_y(u,X_t)^* \varepsilon_z(v,X_t)^* \right\} dW_1(u) dW_2(v) [1 + o_P(1)] \\
&+ \frac{-2}{nh^{d_x/2}} \sum_{s \neq t} \frac{a(X_t)}{g(X_t)} K\left(\frac{X_s - X_t}{h} \right) \iint Re\left\{ \left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t) \right] \phi_y(u,X_t)^* \varepsilon_z(v,X_s)^* \right\} \\
&\times dW_1(u) dW_2(v) [1 + o_P(1)] \\
&= [R_{81} + R_{82}][1 + o_P(1)], \text{ say.}
\end{aligned}$$

Following analogous analysis of R_{11} and R_{12}, we have $E(R_{81}^2) = O(n^{-1}h^{2p+2-d_x})$ and $E(R_{82}^2) = O(nh^{d_x+2p+2} + h^{2p+2-\delta d_x/(1+\delta)})$. Thus, $R_{81} = o_P(1)$ and $R_{82} = o_P(1)$ by Chebyshev's inequality.

Proof of Lemma 0.22**.** The proof is the same as that of Lemma 0.21 except that we replace $\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)$ and $\varepsilon_z(v,X_t)$ with $\bar{\phi}_z(v,X_t) - \phi_z(v,X_t)$ and $\varepsilon_{yz}(u,v,X_t)$ respectively.

Proof of Lemma 0.23**.** The proof is the same as that of Lemma 3 except that we replace $\Upsilon_1(X_t)$ with

$$\Upsilon_8(X_t) = \iint Re\left\{ \phi_y^*(u,X_t) \left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t) \right] \left[\bar{\phi}_z(v,X_t) - \phi_z(v,X_t) \right]^* \right\} a(X_t) dW_1(u) dW_2(v).$$

Proof of Proposition 0.6**.** The proof of Proposition 0.6 is the same as that of Proposition 0.5 except that we replace $\hat{\phi}_y(u,X_t)^*$ and $\hat{\phi}_z(v,X_t) - \phi_z(v,X_t)$ with $\hat{\phi}_z(v,X_t)^*$ and $\hat{\phi}_y(u,X_t) - \phi_y(u,X_t)$ respectively.

Proof of Proposition 0.7**.** The proof of Proposition 0.7 is rather tedious but straightforward. The decomposition and analysis are similar to those in the proof of Proposition 0.1. Take T_7 as an example. By decomposing $\hat{\phi}_y(u,X_t) - \phi_y(u,X_t)$ and $\hat{\phi}_z(v,X_t) - \phi_z(v,X_t)$ into $[\hat{\phi}_y(u,X_t) - \bar{\phi}_y(u,X_t)] + [\bar{\phi}_y(u,X_t) - \phi_y(u,X_t)]$ and $[\hat{\phi}_z(v,X_t) - \bar{\phi}_z(v,X_t)] + [\bar{\phi}_z(v,X_t) - \phi_z(v,X_t)]$, we could decompose T_7 into ten terms, where the first term $T_{71} = h^{d_x/2} \sum_{t=1}^{n} \iint |\hat{\phi}_y(u,X_t) - \bar{\phi}_y(u,X_t)|^2 |\hat{\phi}_z(v,X_t) - \bar{\phi}_z(v,X_t)|^2 a(X_t) dW_1(u) dW_2(v)$ could be further decomposed into sixteen terms. We can prove that each term is a higher order term following analogous reasoning to the proof of Proposition 1. We omit the details of the proof here.

Proof of Proposition 0.8**.** Because $E[U(\xi_s, \xi)] = E[U(\xi', \xi_r)] = 0$ for any given ξ and ξ', $U = \sum_{1 \le s < r \le n} U(\xi_s, \xi_r)$ is a second order degenerate U-statistic. Following Tenreiro's (1997) central limit theorem for degenerate U-statistics in a time series context, we have $\sigma_n^{-1} \sum_{1 \le s < r \le n} U(\xi_s, \xi_r) \xrightarrow{d} N(0,1)$ if the following conditions hold: For some constants $\delta_0 > 0$, $\gamma_0 < \frac{1}{2}$ and $\gamma_1 > 0$, (i) $u_n(4 + \delta_0) = O(n^{\gamma_0})$; (ii) $v_n(2) = o(1)$; (iii) $w_n(2 + \delta_0/2) = o(n^{1/2})$, and (iv) $z_n(2) n^{\gamma_1} = O(1)$, where $\sigma_n^2 = \sum_{1 \le s < r \le n} \text{var}[U(\xi_s, \xi_r)]$,

$$\begin{aligned}
u_n(q) &= \max\left\{ \max_{1 \le i \le n} \|U(\xi_i, \xi_1)\|_q, \|U(\xi_1, \bar{\xi}_1)\|_q \right\}, v_n(q) = \max\left\{ \max_{1 \le i \le n} \|G_{n1}(\xi_i, \xi_1)\|_q, \|G_{n1}(\xi_1, \bar{\xi}_1)\|_q \right\}, \\
w_n(q) &= \|G_{n1}(\xi_1, \xi_1)\|_q, z_n(q) = \max_{1 \le i \le n} \max_{1 \le j \le n} \left\{ \|G_{nj}(\xi_i, \xi_1)\|_q, \|G_{nj}(\xi_1, \xi_i)\|_q, \|G_{nj}(\xi_1, \bar{\xi}_1)\|_q \right\},
\end{aligned}$$

$G_{ni}(\eta, \tau) = E\left[U(\xi_i, \eta) U(\xi_1, \tau) \right]$, $\bar{\xi}_1$ is an independent copy of ξ_1, and $\|\cdot\|_q = \{E|\cdot|^q\}^{1/q}$ for $q \ge 1$.

First, we calculate the asymptotic variance of $U(\xi_s, \xi_r)$, namely $\sigma_0^2 = \text{var}\left[U(\xi_s, \xi_r) \right] = \iint U(\xi_s, \xi_r)^2 dP(\xi_s) dP(\xi_r)$. Since $U(\xi_s, \xi_r)$ contains six terms, we need to calculate the individual variances of these six terms as well as their fifteen pairwise covariances. In calculation, we have used the following facts: (1) under $\mathbb{H}_0$, Y_t is independent of Z_t conditional on X_t; (2) the weighting functions $W_1(u)$ and $W_2(v)$ weigh sets symmetric about the origin equally, implying $\iint \Phi_y(u_1+u_2,x)dW_1(u_1)dW_1(u_2) = \iint \Phi_y(u_1-u_2,x)dW_1(u_1)dW_1(u_2)$, $\iint \Phi_z(v_1+v_2,x)dW_2(v_1)dW_2(v_2) = \iint \Phi_z(v_1-v_2,x)dW_2(v_1)dW_2(v_2)$; (3) similarly, $\iiiint \Phi_{yz}(u_1+u_2,v_1+v_2,x)dW_1(u_1)dW_1(u_2)dW_2(v_1)dW_2(v_2) = \iiiint [\phi_{yz}(u_1+u_2,v_1+v_2,x) - \phi_{yz}(u_1,v_1,x)\phi_{yz}(u_2,v_2,x)]\, dW_1(u_1)dW_1(u_2)\, dW_2(v_1)dW_2(v_2) = \iiiint [\phi_z(v_1+v_2,x)\Phi_y(u_1+u_2,x) + \phi_y(u_1+u_2,x)\Phi_z(v_1+v_2,x)]\, dW_1(u_1)dW_1(u_2)\, dW_2(v_1)dW_2(v_2)$. By tedious but straightforward algebra, we obtain

$$\sigma_0^2 = h^{3d_x} \int \left[\iint |\Phi_y(u_1+u_2,x)|^2 dW_1(u_1)dW_1(u_2) \iint |\Phi_z(v_1+v_2,x)|^2 dW_2(v_1)dW_2(v_2) \right] a^2(x)dx \\ \times \int \left[\int D(p,\tau)D(p,\tau+\eta)K(\tau)K(\tau+\eta)d\tau \right]^2 d\eta.$$

It follows that $\sigma_n^2 = \sum_{1\le s<r\le n} \mathrm{var}[U(\xi_s,\xi_r)] = \frac{n^2}{2}\sigma_0^2[1+o(1)]$. Hence, we have $V = \mathrm{var}\,(U) = \frac{4}{n^2 h^{3d_x}}\sigma_n^2 = \frac{2}{h^{3d_x}}\sigma_0^2[1+o(1)]$.

Now, we verify Conditions (i)-(iv). Since $U(\xi_s,\xi_r)$ is a sum of six terms, the product $U(\xi_i,\eta)U(\xi_j,\tau)$ contains 36 terms. All these terms have the same order of magnitude, and here we verify the first term $U_1(\xi_i,\eta)U_1(\xi_j,\tau)$ only. For $i \neq 1$,

$$\begin{aligned} & E|U(\xi_i,\xi_1)|^q \\ & \sim E\left| \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_i - x}{h}\right) D\left(p, \frac{X_1 - x}{h}\right) K\left(\frac{X_i - x}{h}\right) K\left(\frac{X_1 - x}{h}\right) \right. \\ & \quad \times Re\left[\varepsilon_{yz}(u,v,X_i)\varepsilon_{yz}(u,v,X_1)^*\right] dW_1(u)dW_2(v)dx|^q \\ & = h^{qd_x} E\left| \iiint \frac{a(X_i-\tau h)}{g(X_i-\tau h)} D(p,\tau)\, D\left(p, \frac{X_1-X_i}{h}\right) K(\tau)\, K\left(\tau + \frac{X_1-X_i}{h}\right) \right. \\ & \quad \times Re\left[\varepsilon_{yz}(u,v,X_i)\varepsilon_{yz}(u,v,X_1)^*\right] dW_1(u)dW_2(v)d\tau|^q \\ & = O\left(h^{(q+1)d_x}\right), \end{aligned}$$

so we have $\|U(\xi_i,\xi_1)\|_q = O\left(h^{d_x + d_x/q}\right)$. By a similar argument, we can obtain the same order of magnitude for $\|U(\xi_1,\bar{\xi}_1)\|_q$, where $\bar{\xi}_1$ is an independent copy of ξ_1. Hence, Condition (i) holds for any $\delta_0 > 0$ and $\gamma_0 < \frac{1}{2}$.

Next, we verify Condition (ii). Since for $i \neq 1$,

$$\begin{aligned} & E\,|G_{n1}(\xi_i,\xi_1)|^q = E\,|U(\xi_1,\xi_i)U(\xi_1,\xi_1)|^q \\ & \sim E\left| \iiint \frac{a(x)}{g(x)} D\left(p, \frac{X_1 - x}{h}\right) D\left(p, \frac{X_i - x}{h}\right) K\left(\frac{X_1 - x}{h}\right) K\left(\frac{X_i - x}{h}\right) Re\left[\varepsilon_{yz}(u_1,v_1,X_1)\varepsilon_{yz}(u_1,v_1,X_i)^*\right] \right. \\ & \quad \left. \times\, dW_1(u_1)dW_2(v_1)dx \iiint \frac{a(x')}{g(x')} D^2\left(p, \frac{X_1 - x'}{h}\right) K^2\left(\frac{X_1 - x'}{h}\right) |\varepsilon_{yz}(u_2,v_2,X_1)|^2\, dW_1(u_2)dW_2(v_2)dx' \right|^q \\ & = h^{2qd_x} E\left| \iiint \frac{a(X_1-\tau h)}{g(X_1-\tau h)} D(p,\tau)\, D\left(p, \tau + \frac{X_i - X_1}{h}\right) K(\tau)\, K\left(\tau + \frac{X_i - X_1}{h}\right) Re\left[\varepsilon_{yz}(u_1,v_1,X_1)\varepsilon_{yz}(u_1,v_1,X_i)^*\right] \right. \\ & \quad \left. \times\, dW_1(u_1)dW_2(v_1)d\tau \iiint \frac{a(X_1-\eta h)}{g(X_1-\eta h)} D^2(p,\eta) K^2(\eta)\, |\varepsilon_{yz}(u_2,v_2,X_1)|^2\, dW_1(u_2)dW_2(v_2)d\eta \right|^q \\ & = O\left(h^{(2q+1)d_x}\right), \end{aligned}$$

we have $\|G_{n1}(\xi_i,\xi_1)\|_q = O\left(h^{(2+1/q)d_x}\right)$. By a similar argument, we can obtain the same order of magnitude for $\|G_{n1}(\xi_1,\bar{\xi}_1)\|_q$. Consequently, Condition (ii) is satisfied.

Now, we verify Condition (iii). Since

$$
\begin{aligned}
&E\left|G_{n1}(\xi_1,\xi_1)\right|^q = E\left|U(\xi_1,\xi_1)U(\xi_1,\xi_1)\right|^q \\
&\sim E\left|\iiint \frac{a(x)}{g(x)} D^2\left(p,\frac{X_1-x}{h}\right) K^2\left(\frac{X_1-x}{h}\right)\left|\varepsilon_{yz}(u_1,v_1,X_1)\right|^2 dW_1(u_1)dW_2(v_1)dx\right. \\
&\quad\times \left.\iiint \frac{a(x')}{g(x')} D^2\left(p,\frac{X_1-x'}{h}\right) K^2\left(\frac{X_1-x'}{h}\right)\left|\varepsilon_{yz}(u_2,v_2,X_1)\right|^2 dW_1(u_2)dW_2(v_2)dx'\right|^q \\
&= h^{2qd_x} E\left|\iiint \frac{a(X_1-\tau h)}{g(X_1-\tau h)} D^2(p,\tau) K^2(\tau)\left|\varepsilon_{yz}(u_1,v_1,X_1)\right|^2 dW_1(u_1)dW_2(v_1)d\tau\right. \\
&\quad\times \left.\iiint \frac{a(X_1-\eta h)}{g(X_1-\eta h)} D^2(p,\eta) K^2(\eta)\left|\varepsilon_{yz}(u_2,v_2,X_1)\right|^2 dW_1(u_2)dW_2(v_2)d\eta\right|^q \\
&= O\left(h^{2qd_x}\right),
\end{aligned}
$$

we have $w_n(q) = O\left(h^{2d_x}\right) = o(1)$. Hence, Condition (iii) holds.

To verify Condition (iv), for $i \neq j \neq 1$, we first calculate

$$
\begin{aligned}
&E|G_{nj}(\xi_i,\xi_1)|^q = E\left|E_j\left[U(\xi_j,\xi_i)U(\xi_1,\xi_1)\right]\right|^q \\
&\sim E\left|\int\left[\iiint \frac{a(x)}{g(x)} D\left(p,\frac{X_j-x}{h}\right) D\left(p,\frac{X_i-x}{h}\right) K\left(\frac{X_j-x}{h}\right) K\left(\frac{X_i-x}{h}\right) Re\left[\varepsilon_{yz}(u_1,v_1,X_j)\varepsilon_{yz}(u_1,v_1,X_i)^*\right]\right.\right. \\
&\quad\times\left.\left. dW_1(u_1)dW_2(v_1)dx \iiint \frac{a(x')}{g(x')} D^2\left(p,\frac{X_1-x'}{h}\right) K^2\left(\frac{X_1-x'}{h}\right)\left|\varepsilon_{yz}(u_2,v_2,X_1)\right|^2 dW_1(u_2)dW_2(v_2)dx'\right] dP(\xi_j)\right|^q \\
&= h^{2qd_x} E\left|\int\left[\iiint \frac{a(X_j-\tau h)}{g(X_j-\tau h)} D(p,\tau) D\left(p,\tau+\frac{X_i-X_j}{h}\right) K(\tau) K\left(\tau+\frac{X_i-X_j}{h}\right) Re\left[\varepsilon_{yz}(u_1,v_1,X_j)\varepsilon_{yz}(u_1,v_1,X_i)^*\right]\right.\right. \\
&\quad\times\left.\left. dW_1(u_1)dW_2(v_1)d\tau \iiint \frac{a(X_1-\eta h)}{g(X_1-\eta h)} D^2(p,\eta) K^2(\eta)\left|\varepsilon_{yz}(u_2,v_2,X_1)\right|^2 dW_1(u_2)dW_2(v_2)d\eta\right] dP(\xi_j)\right|^q \\
&= O\left(h^{3qd_x}\right).
\end{aligned}
$$

By a similar argument, we can obtain $E|G_{nj}(\xi_1,\xi_i)|^q = O\left(h^{3qd_x+d_x}\right)$ and $E|G_{nj}(\xi_1,\bar{\xi}_1)|^q = O\left(h^{3qd_x+d_x}\right)$. It follows that $z_n(p) = O\left(h^{3d_x}\right)$, and Condition (iv) holds by setting $\gamma_1 = 3\lambda d_x > 0$. The desired asymptotic normality follows immediately.

Proof of Proposition 0.9. We should prove: (i) $\hat{B} - B = o_P(1)$, and (ii) $\hat{V} - V = o_P(1)$. Since the proofs of (i) and (ii) are similar, we focus on the proof of (i) here. We first decompose

$$
\begin{aligned}
&\hat{B} - B \\
&= h^{-d_x/2}\iiint a(x)\left\{\left[1-|\hat{\phi}_y(u,x)|^2\right]\left[1-|\hat{\phi}_z(v,x)|^2\right] - \left[1-|\phi_y(u,x)|^2\right]\left[1-|\phi_z(v,x)|^2\right]\right\} \\
&\quad\times dW_1(u)dW_2(v)dx\int D^2(p,\tau)K^2(\tau)d\tau \\
&= h^{-d_x/2}\iiint a(x)\left[|\phi_y(u,x)|^2-1\right]\left[|\hat{\phi}_z(v,x)|^2-|\phi_z(v,x)|^2\right] dW_1(u)dW_2(v)dx\int D^2(p,\tau)K^2(\tau)d\tau \\
&\quad+ h^{-d_x/2}\iiint a(x)\left[|\phi_z(v,x)|^2-1\right]\left[|\hat{\phi}_y(u,x)|^2-|\phi_y(u,x)|^2\right] dW_1(u)dW_2(v)dx\int D^2(p,\tau)K^2(\tau)d\tau \\
&\quad+ h^{-d_x/2}\iiint a(x)\left[|\hat{\phi}_y(u,x)|^2-|\phi_y(u,x)|^2\right]\left[|\hat{\phi}_z(v,x)|^2-|\phi_z(v,x)|^2\right] dW_1(u)dW_2(v)dx\int D^2(p,\tau)K^2(\tau)d\tau \\
&= H_1 + H_2 + H_3, \text{ say.}
\end{aligned}
$$

To show $\hat{B} - B = o_P(1)$, we should prove $H_i = o_P(1)$ for $i = 1, 2, 3$. Since the proofs of $H_i,\ i = 1, 2, 3$, are rather similar, we focus on the proof of $H_1 = o_P(1)$. We decompose H_1 as follows:

$$\begin{aligned}
H_1 &= h^{-d_x/2} \iiint \left[|\phi_y(u,x)|^2 - 1\right] |\hat{\phi}_z(v,x) - \phi_z(v,x)|^2 a(x) dW_1(u) dW_2(v) dx \int D^2(p,\tau) K^2(\tau) d\tau \\
&\quad + 2h^{-d_x/2} \iiint \left[|\phi_y(u,x)|^2 - 1\right] Re\left\{\left[\hat{\phi}_z(v,x) - \phi_z(v,x)\right] \phi_z(v,x)^*\right\} a(x) dW_1(u) dW_2(v) dx \int D^2(p,\tau) K^2(\tau) d\tau \\
&= H_{11} + H_{12},\ \text{say.}
\end{aligned}$$

We further decompose H_{11} as follows:

$$\begin{aligned}
H_{11} &\le 2h^{-d_x/2} \iiint \left[|\phi_y(u,x)|^2 - 1\right] \left|\hat{\phi}_z(v,x) - \bar{\phi}_z(v,x)\right|^2 a(x) dW_1(u) dW_2(v) dx \int D^2(p,\tau) K^2(\tau) d\tau \\
&\quad + 2h^{-d_x/2} \iiint \left[|\phi_y(u,x)|^2 - 1\right] \left|\bar{\phi}_z(v,x) - \phi_z(v,x)\right|^2 a(x) dW_1(u) dW_2(v) dx \int D^2(p,\tau) K^2(\tau) d\tau \\
&= H_{11}^{(1)} + H_{11}^{(2)},\ \text{say,}
\end{aligned}$$

and

$$\begin{aligned}
H_{11}^{(1)} &= \frac{2}{n^2 h^{5d_x/2}} \sum_{s=1}^{n} \iiint \frac{a(x)}{g^2(x)} D^2\left(p, \frac{X_s - x}{h}\right) K^2\left(\frac{X_s - x}{h}\right) \left[|\phi_y(u,x)|^2 - 1\right] |\varepsilon_z(v, X_s)|^2 dW_1(u) dW_2(v) dx \int D^2(p,\tau) K^2(\tau) d\tau \\
&\quad + \frac{4}{n^2 h^{5d_x/2}} \sum_{l<s} \iiint \frac{a(x)}{g^2(x)} D\left(p, \frac{X_s - x}{h}\right) D\left(p, \frac{X_l - x}{h}\right) K\left(\frac{X_s - x}{h}\right) K\left(\frac{X_l - x}{h}\right) \left[|\phi_y(u,x)|^2 - 1\right] \\
&\quad \times Re\left[\varepsilon_z(v, X_s) \varepsilon_z(v, X_l)^*\right] dW_1(u) dW_2(v) dx \int D^2(p,\tau) K^2(\tau) d\tau \\
&= H_{11}^{(1,1)} + H_{11}^{(1,2)},\ \text{say.}
\end{aligned}$$

It is straightforward to show $E|H_{11}^{(1,1)}| = O(n^{-1} h^{-3d_x/2})$. Put $H_{11}^{(1,2)} = \frac{4}{n^2 h^{5d_x/2}} U_H$. Following analogous reasoning to the proof of Proposition 0.8, we can show that U_H is a second order degenerate U-statistic satisfying $E|U_H|^2 = O(n^2 h^{3d_x})$. Thus, $E|H_{11}^{(1,2)}|^2 = O(n^{-2} h^{-2d_x})$. Hence we have $H_{11}^{(1,1)} = o_P(1)$ and $H_{11}^{(1,2)} = o_P(1)$ by Markov's inequality and Chebyshev's inequality respectively. Following analogous reasoning to the proof of Lemma 3, we obtain the squared bias term $H_{11}^{(2)} = O(h^{-d_x/2 + 2p + 2}) = o(1)$.

Now, we consider the H_{12} term. We decompose H_{12} as follows:

$$\begin{aligned}
H_{12} &= 2h^{-d_x/2} \iiint \left[|\phi_y(u,x)|^2 - 1\right] Re\left\{\left[\hat{\phi}_z(v,x) - \bar{\phi}_z(v,x)\right] \phi_z(v,x)^*\right\} a(x) dW_1(u) dW_2(v) dx \\
&\quad + 2h^{-d_x/2} \iiint \left[|\phi_y(u,x)|^2 - 1\right] Re\left\{\left[\bar{\phi}_z(v,x) - \phi_z(v,x)\right] \phi_z(v,x)^*\right\} a(x) dW_1(u) dW_2(v) dx \\
&= H_{12}^{(1)} + H_{12}^{(2)},\ \text{say.}
\end{aligned}$$

Since $E(H_{12}^{(1)})^2 = O(n^{-1} h^{-d_x})$, we have $H_{12} = o_P(1)$ by Chebyshev's inequality. The bias term $H_{12}^{(2)} = O(h^{p+1-d_x/2}) = o(1)$. Thus, we have proved $H_1 = o_P(1)$.

A.2 Proof of Theorem 2

Under $\mathbb{H}_1(a_n)$, where $\sigma(u,v,x) = \phi_{yz}(u,v,x) - \phi_y(u,x)\phi_z(v,x) = a_n\delta(u,v,x)$, we can decompose

$$\begin{aligned}
\hat{M}_h &= h^{d_x/2} \sum_{t=1}^{n} \iint |\hat{\sigma}(u,v,X_t)|^2 a(X_t) dW_1(u) dW_2(v) \\
&= h^{d_x/2} \sum_{t=1}^{n} \iint |\hat{\sigma}(u,v,X_t) - \sigma(u,v,X_t)|^2 a(X_t) dW_1(u) dW_2(v) \\
&\quad + 2\sum_{t=1}^{n} \iint Re\left\{[\hat{\sigma}(u,v,X_t) - \sigma(u,v,X_t)]\,\sigma(u,v,X_t)^*\right\} a(X_t) dW_1(u) dW_2(v) \\
&\quad + h^{d_x/2} \sum_{t=1}^{n} \iint |\sigma(u,v,X_t)|^2 a(X_t) dW_1(u) dW_2(v) \\
&= h^{d_x/2} \sum_{t=1}^{n} \iint \left|(\hat{\phi}_{yz} - \phi_{yz}) - \phi_y(\hat{\phi}_z - \phi_z) - \phi_z(\hat{\phi}_y - \phi_y) - (\hat{\phi}_y - \phi_y)(\hat{\phi}_z - \phi_z)\right|^2 a(X_t) dW_1(u) dW_2(v) \\
&\quad + 2a_n h^{d_x/2} \sum_{t=1}^{n} \iint Re\left\{\left[(\hat{\phi}_{yz} - \phi_{yz}) - \phi_y(\hat{\phi}_z - \phi_z) - \phi_z(\hat{\phi}_y - \phi_y) - (\hat{\phi}_y - \phi_y)(\hat{\phi}_z - \phi_z)\right] \delta(u,v,X_t)^*\right\} a(X_t) dW_1(u) dW_2(v) \\
&\quad + a_n^2 h^{d_x/2} \sum_{t=1}^{n} \iint |\delta(u,v,X_t)|^2 a(X_t) dW_1(u) dW_2(v) \\
&= \sum_{i=1}^{10} T_i + M_1 + M_2, \text{ say,}
\end{aligned}$$

where $\{T_i\}_{i=1}^{10}$ are defined as in $(A.2)$. Following the proof of Theorem 1, we can show that $(\sum_{i=1}^{10} T_i - B)/\sqrt{V} \xrightarrow{d} N(0,1)$ as $n \to \infty$ under $\mathbb{H}_1(a_n)$, where B and V are given by $(A.3)$ and $(A.4)$ respectively.

Now, we show $M_1 = o_P(1)$. We decompose M_1 into four terms, denoted as $M_1^{(i)}, i = 1, \cdots, 4$, and show $M_1^{(i)} = o_P(1)$ for $i = 1, \cdots, 4$. Since the proofs of the $M_1^{(i)}$ are similar, we focus on the proof of $M_1^{(1)} = o_P(1)$. We decompose

$$\begin{aligned}
M_1^{(1)} &= 2a_n h^{d_x/2} \sum_{t=1}^{n} \iint Re\left\{\left[\hat{\phi}_{yz}(u,v,X_t) - \bar{\phi}_{yz}(u,v,X_t)\right] \delta(u,v,X_t)^*\right\} a(X_t) dW_1(u) dW_2(v) \\
&\quad + 2a_n h^{d_x/2} \sum_{t=1}^{n} \iint Re\left\{\left[\bar{\phi}_{yz}(u,v,X_t) - \phi_{yz}(u,v,X_t)\right] \delta(u,v,X_t)^*\right\} a(X_t) dW_1(u) dW_2(v) \\
&= M_1^{(1,1)} + M_1^{(1,2)}, \text{ say.}
\end{aligned}$$

It is straightforward to show $E|M_1^{(1,1)}|^2 = O(h^{d_x/2})$ and $E|M_1^{(1,2)}| = O(n^{1/2}h^{p+1+d_x/4})$. Therefore, $M_1^{(1,1)} = o_P(1)$ and $M_1^{(1,2)} = o_P(1)$ by Chebyshev's inequality and Markov's inequality respectively. It follows that $\hat{M}^{(1)} = o_P(1)$. Similarly, we can also show $M_1^{(i)} = o_P(1)$ for $i = 2, 3, 4$. Therefore, $M_1 = o_P(1)$.

We now turn to M_2. By the weak law of large numbers, we have, as $n \to \infty$,

$$\begin{aligned}
M_2 &= n^{-1} \sum_{t=1}^{n} \iint |\delta(u,v,X_t)|^2 a(X_t) dW_1(u) dW_2(v) \\
&\xrightarrow{p} \gamma = \iiint |\delta(u,v,x)|^2 a(x) g(x) dx dW_1(u) dW_2(v).
\end{aligned}$$

In addition, under $\mathbb{H}_1(a_n)$, the asymptotic variance of $\hat{M}_h$, $\text{avar}(\hat{M}_h) = \text{avar}(\sum_{i=1}^{10} T_i) = V$ given $M_1 = o_P(1)$ and $M_2 - \gamma = o_P(1)$. Consequently, we obtain the desired result of Theorem 2.

A.3 Proof of Theorem 3

Like in the proof of Theorem 2, we can obtain the following decomposition:

$$\hat{M}_h = \sum_{i=1}^{n} T_i + M_1 + M_2,$$

where $\{T_i\}_{i=1}^{10}$, M_1 and M_2 are defined in the same way as in the proof of Theorem 2, and following the proof of Theorem 1, we can show $(\sum_{i=1}^{10} T_i - B)/\sqrt{V} \xrightarrow{d} N(0,1)$ under $\mathbb{H}_2(a_n, b_n)$. We now focus on the analysis of M_1 and M_2.

For M_1, following analogous analysis of the M_1 term in the proof of Theorem 2, we can obtain $E(M_1) = O(n^{1/2}h^{p+1+d_x/4}b_n^{1/2}) = o(1)$ as $b_n \to 0$ and $E|M_1|^2 = O(h^{d_x/2}) = o(1)$ as $h \to 0$. Hence, $M_1 = o_P(1)$ by Chebyshev's inequality.

For M_2, we have

$$
\begin{aligned}
M_2 &= a_n^2 h^{d_x/2} \sum_{t=1}^n \iint \left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t) dW_1(u) dW_2(v) \\
&= n a_n^2 h^{d_x/2} \iint E\left[\left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t)\right] dW_1(u) dW_2(v) \\
&\quad + a_n^2 h^{d_x/2} \sum_{t=1}^n \iint \left\{\left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t) - E\left[\left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t)\right]\right\} dW_1(u) dW_2(v) \\
&= M_{21} + M_{22}, \text{ say,}
\end{aligned}
$$

where

$$
\begin{aligned}
M_{21} &= n a_n^2 h^{d_x/2} \iint E\left[\left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t)\right] dW_1(u) dW_2(v) \\
&= n a_n^2 h^{d_x/2} \iiint \left|\delta\left(u, v, \frac{x - c}{b_n}\right)\right|^2 a(x) g(x) dx dW_1(u) dW_2(v) \\
&= n a_n^2 b_n h^{d_x/2} \iiint |\delta(u, v, w)|^2 a(c + w b_n) g(c + w b_n) dw dW_1(u) dW_2(v) \\
&= a(c) g(c) \iiint |\delta(u, v, w)|^2 dw dW_1(u) dW_2(v) = \kappa.
\end{aligned}
$$

Now, we consider the M_{22} term. Define

$$
\Psi_{\mathbb{H}_2}(X_t) = \iint \left\{\left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t) - E\left[\left|\delta\left(u, v, \frac{X_t - c}{b_n}\right)\right|^2 a(X_t)\right]\right\} dW_1(u) dW_2(v).
$$

Then we can write $M_{22} = a_n^2 h^{d_x/2} \sum_{t=1}^n \Psi_{\mathbb{H}_2}(X_t)$. Since $E(M_{22}) = 0$, and

$$
\begin{aligned}
\operatorname{var}(M_{22}) &\le a_n^4 h^{d_x} \sum_{t=1}^n \operatorname{var}[\Psi_{\mathbb{H}_2}(X_t)] + a_n^4 n h^{d_x} \sum_{j=1}^{n-1} |\operatorname{cov}[\Psi_{\mathbb{H}_2}(X_1), \Psi_{\mathbb{H}_2}(X_{1+j})]| \\
&\le C a_n^4 n h^{d_x} O(b_n) + C a_n^4 n h^{d_x} \sum_{j=1}^{\infty} j^2 \beta(j)^{\delta/(1+\delta)} O(b_n) \\
&= O(a_n^4 b_n n h^{d_x}) = O(a_n^2 h^{d_x/2}) = o(1), \text{ as } a_n \to 0 \text{ and } h \to 0,
\end{aligned}
$$

where we have used the fact that $a_n^2 b_n = n^{-1} h^{-d_x/2}$, we have $M_{22} = o_P(1)$ by Chebyshev's inequality. Therefore, $M_2 \xrightarrow{p} \kappa$. In addition, under $\mathbb{H}_2(a_n, b_n)$, the asymptotic variance of $\hat{M}_h$, $\operatorname{avar}(\hat{M}_h) = \operatorname{avar}(\sum_{i=1}^{10} T_i) = V$ given $M_1 = o_P(1)$ and $M_2 - \kappa = o_P(1)$. Consequently, we obtain the desired result of Theorem 3.

Appendix B Simulation Results not Reported in the Paper

Appendix B.1 We examine the finite sample performance of the tests $\widehat{SM}$, $\widehat{SM}^{(1)}$ and $\widehat{SM}^{(1,1)}$ in comparison with the tests of Granger (1969), Su and White (2007) and Nishiyama *et al.* (2011) using asymptotic critical values. Tables A.1 and A.2 below report the size and power of these tests respectively.

Table A.1 Size of Tests Based on the Asymptotic Critical Values Under DGP.S1-S3

		$SW07$		$\widehat{SM}$		$NHKJ11$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$		LIN	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
DGP.S1	$n = 100$	.317	.414	.095	.162	.038	.092	.109	.158	.099	.151	.050	.106
	$n = 200$	.278	.380	.062	.130	.178	.287	.068	.118	.070	.104	.052	.102
	$n = 500$	.217	.312	.071	.116	.339	.436	.067	.109	.050	.092	.045	.092
	$n = 1000$	.163	.252	.055	.110	.374	.474	.071	.112	.063	.100	.046	.093
DGP.S2	$n = 100$	.328	.426	.110	.179	.000	.004	.093	.145	.115	.146	.047	.103
	$n = 200$	.342	.457	.084	.128	.003	.007	.058	.114	.064	.107	.057	.108
	$n = 500$	.266	.391	.071	.119	.002	.004	.054	.091	.063	.095	.037	.100
	$n = 1000$	.243	.354	.055	.096	.002	.005	.069	.103	.060	.091	.050	.086
DGP.S3	$n = 100$	.328	.431	.096	.149	.001	.005	.075	.121	.064	.104	.042	.094
	$n = 200$	.274	.383	.076	.136	.001	.002	.069	.114	.067	.104	.046	.102
	$n = 500$	.227	.331	.070	.111	.002	.005	.068	.106	.061	.101	.053	.097
	$n = 1000$	.192	.297	.061	.115	.003	.005	.076	.109	.051	.085	.051	.101

Notes: (i) $SW07$ denotes Su and White's (2007) test, $NHKJ11$ denotes Nishiyama *et al.*'s (2011) test, and LIN denotes Granger's (1969) F test; (ii) AS and BS denote the results using asymptotic and bootstrap critical values respectively; (iii) the results using asymptotic critical values are based on 1000 iterations.

Table A.2 Power of Tests Based on the Asymptotic Critical Values Under DGP.P1-P8

		$SW07$		$\widehat{SM}$		$NHKJ11$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$		LIN	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
DGP.P1	$n = 100$	.860	.908	.964	.982	.451	.563	.986	.990	.990	.995	1.00	1.00
	$n = 200$	.974	.984	.998	.998	.791	.848	1.00	1.00	1.00	1.00	1.00	1.00
	$n = 500$	.999	1.00	1.00	1.00	.920	.951	1.00	1.00	1.00	1.00	1.00	1.00
	$n = 1000$	1.00	1.00	1.00	1.00	.923	.963	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P2	$n = 100$	.775	.835	.874	.913	.017	.029	.918	.951	.927	.955	.206	.290
	$n = 200$	.925	.953	.991	.997	.074	.102	.998	.998	.996	.997	.215	.286
	$n = 500$	.997	1.00	1.00	1.00	.116	.145	1.00	1.00	1.00	1.00	.249	.323
	$n = 1000$	1.00	1.00	1.00	1.00	.150	.181	1.00	1.00	1.00	1.00	.265	.338
DGP.P3	$n = 100$	.455	.552	.311	.417	.070	.130	.452	.555	.407	.507	.260	.357
	$n = 200$	.587	.707	.516	.619	.254	.368	.745	.811	.733	.802	.500	.617
	$n = 500$	.861	.910	.952	.976	.533	.643	.996	1.00	.997	.998	.843	.902
	$n = 1000$	.987	.994	1.00	1.00	.582	.693	1.00	1.00	1.00	1.00	.980	.989
DGP.P4	$n = 100$	.489	.594	.421	.528	.020	.046	.503	.608	.095	.131	.175	.267
	$n = 200$	.552	.662	.654	.748	.099	.147	.775	.840	.068	.120	.149	.219
	$n = 500$	.716	.821	.973	.980	.166	.241	.990	.997	.066	.101	.165	.253
	$n = 1000$	.900	.946	1.00	1.00	.177	.274	1.00	1.00	.073	.111	.181	.252
DGP.P5	$n = 100$	.911	.953	.771	.837	.005	.011	.328	.418	.077	.128	.191	.264
	$n = 200$	.990	.999	.973	.988	.020	.034	.485	.575	.087	.137	.149	.227
	$n = 500$	1.00	1.00	1.00	1.00	.110	.169	.770	.845	.069	.101	.189	.264
	$n = 1000$	1.00	1.00	1.00	1.00	.285	.380	.974	.986	.066	.117	.173	.255
DGP.P6	$n = 100$	.979	.992	.957	.974	.086	.150	.205	.285	.075	.113	.250	.344
	$n = 200$	.999	1.00	.999	1.00	.336	.438	.194	.254	.078	.118	.223	.310
	$n = 500$	1.00	1.00	1.00	1.00	.731	.815	.190	.244	.057	.093	.248	.326
	$n = 1000$	1.00	1.00	1.00	1.00	.906	.950	.168	.213	.048	.080	.267	.340
DGP.P7	$n = 100$	.830	.890	.692	.790	.001	.003	.152	.221	.078	.115	.163	.243
	$n = 200$	.953	.973	.938	.968	.002	.008	.162	.229	.080	.131	.147	.227
	$n = 500$	.999	.999	1.00	1.00	.004	.008	.121	.185	.065	.100	.195	.280
	$n = 1000$	1.00	1.00	1.00	1.00	.001	.006	.126	.170	.061	.087	.178	.255
DGP.P8	$n = 100$	.703	.810	.514	.625	.003	.003	.144	.220	.073	.118	.175	.268
	$n = 200$	.848	.917	.816	.884	.002	.004	.138	.185	.059	.095	.215	.167
	$n = 500$	.985	.989	.999	1.00	.000	.003	.113	.181	.058	.096	.167	.249
	$n = 1000$	1.00	1.00	1.00	1.00	.003	.009	.119	.174	.045	.089	.157	.241

Notes: (i) $SW07$ denotes Su and White's (2007) test, $NHKJ11$ denotes Nishiyama *et al.*'s (2011) test, and LIN denotes Granger's (1969) F test; (ii) AS and BS denote the results using asymptotic and bootstrap critical values respectively; (iii) the results using asymptotic critical values are based on 1000 iterations.

Appendix B.2 To examine the effect of increasing the dimension of conditioning variables, we test whether $Y_t \perp Z_{t-1} | (Y_{t-1}, Y_{t-2})$. This corresponds to the case of $d_x = 2$. Table A.3 below reports the size and power of the $\widehat{SM}$ test under DGP.S1-S3 and DGP.P1-P8 at 10% and 5% significance levels.

Table A.3 Size and Power of $\widehat{SM}$ Under DGP.S1-S3 and DGP.P1-P8 When $d_x = 2$

	DGP.S1, *AS*		DGP.S1, *BS*		DGP.S2, *AS*		DGP.S2, *BS*		DGP.S3, *AS*		DGP.S3, *BS*	
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
$n = 100$	.285	.392	.050	.092	.302	.402	.068	.116	.190	.298	.054	.110
$n = 200$	.200	.282	.048	.092	.186	.260	.048	.106	.118	.203	.052	.096
$n = 500$	.137	.217	–	–	.115	.194	–	–	.092	.148	–	–
$n = 1000$	.103	.171	–	–	.089	.152	–	–	.087	.141	–	–
	DGP.P1, *AS*		DGP.P1, *BS*		DGP.P2, *AS*		DGP.P2, *BS*		DGP.P3, *AS*		DGP.P3, *BS*	
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
$n = 100$	.752	.824	.472	.594	.936	.971	.622	.746	.367	.505	.128	.220
$n = 200$	.948	.961	.774	.816	.989	.993	.940	.952	.497	.631	.258	.380
$n = 500$	.998	1.00	–	–	1.00	1.00	–	–	.814	.884	–	–
$n = 1000$	1.00	1.00	–	–	1.00	1.00	–	–	.981	.994	–	–
	DGP.P4, *AS*		DGP.P4, *BS*		DGP.P5, *AS*		DGP.P5, *BS*		DGP.P6, *AS*		DGP.P6, *BS*	
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
$n = 100$	.612	.742	.242	.358	.906	.944	.478	.634	.730	.802	.452	.550
$n = 200$	.776	.866	.516	.642	.986	.994	.840	.914	.903	.925	.798	.834
$n = 500$	.963	.987	–	–	1.00	1.00	–	–	1.00	1.00	–	–
$n = 1000$	1.00	1.00	–	–	1.00	1.00	–	–	1.00	1.00	–	–
	DGP.P7, *AS*		DGP.P7, *BS*		DGP.P8, *AS*		DGP.P8, *BS*					
	5%	10%	5%	10%	5%	10%	5%	10%				
$n = 100$	.836	.890	.380	.550	.739	.834	.480	.606				
$n = 200$	.979	.987	.742	.832	.878	.936	.778	.862				
$n = 500$	1.00	1.00	–	–	1.00	1.00	–	–				
$n = 1000$	1.00	1.00	–	–	1.00	1.00	–	–				

Notes: (i) the null hypothesis for this table is $Y_t \perp Z_{t-1} | (Y_{t-1}, Y_{t-2})$; (ii) the bandwidth $h = n^{-3/11}$. We have also tried other bandwidths, and the bootstrap results are robust to the selection of bandwidth; (iii) *AS* and *BS* denote the results using asymptotic and bootstrap critical values respectively; (iv) the results using asymptotic critical values are based on 1000 iterations, while the bootstrap results are based on 500 iterations.

Appendix B.3 To compare all tests on an equal ground, we compute the size-adjusted power of Su and White's (2007) test and the tests $\widehat{SM}, \widehat{SM}^{(1)}, \widehat{SM}^{(1,1)}$ under DGP.P1-P8 at 10% and 5% levels, using the empirical critical values obtained under the null hypothesis. For DGP.P1-P3 and P6, the null model is DGP.S1, while the null model for DGP.P7 is DGP.S2, and the null models for DGP.P4, P5 and P8 are respectively

$$\begin{aligned}
\text{DGP.P4N} : Y_t &= 0.4Y_{t-1} + \varepsilon_{1,t}; \\
\text{DGP.P5N} : Y_t &= 0.3 + 0.2\log(h_t) + \sqrt{h_t}\varepsilon_{1,t}, \ h_t = 0.01 + 0.5Y_{t-1}^2; \\
\text{DGP.P8N} : Y_t &= \sqrt{h_{1,t}}\varepsilon_{1,t}, h_{1,t} = 0.01 + 0.1h_{1,t-1} + 0.4Y_{t-1}^2, \\
Z_t &= \sqrt{h_{2,t}}\varepsilon_{2,t}, h_{2,t} = 0.01 + 0.9h_{2,t-1} + 0.05Z_{t-1}^2.
\end{aligned}$$

Table A.4 Size Adjusted Power of Tests Under DGP.P1-P8

		$SW07$		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$	
		5%	10%	5%	10%	5%	10%	5%	10%
DGP.P1	$n=100$	.578	.688	.966	.979	.980	.990	.983	.997
	$n=200$	.885	.939	.998	1.00	1.00	1.00	1.00	1.00
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P2	$n=100$	.443	.533	.860	.922	.919	.953	.893	.952
	$n=200$	.697	.793	.994	.996	.997	.999	.996	.999
	$n=500$	.991	.995	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P3	$n=100$	.119	.185	.182	.295	.289	.410	.241	.374
	$n=200$	.194	.333	.395	.533	.641	.733	.620	.712
	$n=500$	.612	.742	.883	.923	.989	.993	.985	.993
	$n=1000$	.970	.983	.999	.999	1.00	1.00	1.00	1.00
DGP.P4	$n=100$	.141	.226	.362	.476	.457	.624	.088	.134
	$n=200$	.207	.303	.636	.754	.782	.879	.049	.079
	$n=500$	.433	.552	.985	.989	.997	1.00	.060	.113
	$n=1000$	.789	.877	1.00	1.00	1.00	1.00	.053	.100
DGP.P5	$n=100$	.653	.759	.566	.716	.160	.303	.035	.082
	$n=200$	.916	.969	.923	.969	.319	.478	.034	.082
	$n=500$	.999	.999	1.00	1.00	.722	.820	.034	.092
	$n=1000$	1.00	1.00	1.00	1.00	.973	.990	.031	.102
DGP.P6	$n=100$	.789	.872	.920	.970	.141	.220	.031	.076
	$n=200$	.988	.997	1.00	1.00	.159	.222	.046	.071
	$n=500$	1.00	1.00	1.00	1.00	.163	.217	.045	.085
	$n=1000$	1.00	1.00	1.00	1.00	.128	.214	.039	.085
DGP.P7	$n=100$	.449	.567	.485	.644	.116	.172	.046	.082
	$n=200$	.674	.780	.876	.951	.110	.181	.052	.098
	$n=500$	.967	.990	1.00	1.00	.125	.192	.039	.092
	$n=1000$	1.00	1.00	1.00	1.00	.080	.141	.044	.077
DGP.P8	$n=100$	.266	.398	.383	.511	.076	.148	.041	.083
	$n=200$	.475	.585	.674	.843	.087	.147	.036	.078
	$n=500$	.878	.944	.999	1.00	.088	.136	.034	.092
	$n=1000$	.993	.998	1.00	1.00	.091	.158	.043	.098

Notes: (i) $SW07$ denotes Su and White's (2007) test; (ii) the empirical critical values of each test are obtained under the null hypothesis, based on 1000 iterations; (iii) all size-corrected power results are based on 1000 iterations.

Appendix B.4 We examine the effect of using a weighting function with unbounded support for $a(x)$. Table A.5 and A.6 below report the size and power of the tests $\widehat{SM}, \widehat{SM}^{(1)}, \widehat{SM}^{(1,1)}$ under DGP.S1-S3 and DGP.P1-P8 at 10% and 5% significance levels. The Gaussian weighting function $a(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $-\infty < x < \infty$ is used.

Table A.5 Size of Tests Under DGP.S1-S3 Using the Gaussian Weighting Function for $a(x)$

		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$	
		5%	10%	5%	10%	5%	10%
DGP.S1	$n = 100$	.078	.127	.073	.115	.090	.127
	$n = 200$	.089	.136	.088	.122	.084	.118
	$n = 500$	.084	.135	.084	.124	.068	.109
	$n = 1000$	.070	.107	.069	.104	.053	.091
DGP.S2	$n = 100$	.085	.141	.092	.139	.109	.162
	$n = 200$	.093	.142	.083	.141	.084	.127
	$n = 500$	.083	.134	.087	.136	.083	.123
	$n = 1000$	.084	.122	.079	.126	.070	.104
DGP.S3	$n = 100$	.073	.115	.067	.102	.077	.119
	$n = 200$	.084	.133	.074	.115	.066	.111
	$n = 500$	.083	.120	.069	.108	.056	.087
	$n = 1000$	.086	.137	.072	.110	.052	.089

Notes: the results are based on 1000 iterations using asymptotic critical values.

Table A.6 Power of Tests Under DGP.P1-P8 Using the Gaussian Weighting Function for $a(x)$

		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$	
		5%	10%	5%	10%	5%	10%
DGP.P1	$n = 100$	.912	.936	.960	.972	.998	.999
	$n = 200$	.991	.996	.997	.998	1.00	1.00
	$n = 500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n = 1000$	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P2	$n = 100$	.767	.823	.830	.874	.954	.971
	$n = 200$	.953	.962	.957	.962	.999	1.00
	$n = 500$	.994	.995	.991	.993	1.00	1.00
	$n = 1000$	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P3	$n = 100$	.253	.336	.421	.502	.419	.507
	$n = 200$	.514	.598	.722	.785	.783	.800
	$n = 500$	.929	.956	.989	.994	.990	.994
	$n = 1000$	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P4	$n = 100$	.432	.511	.544	.641	.111	.162
	$n = 200$	.734	.806	.849	.895	.108	.145
	$n = 500$	.992	.995	1.00	1.00	.085	.124
	$n = 1000$	1.00	1.00	1.00	1.00	.065	.104
DGP.P5	$n = 100$	.709	.760	.339	.421	.106	.173
	$n = 200$	.921	.934	.551	.631	.090	.139
	$n = 500$	.996	.997	.920	.950	.074	.114
	$n = 1000$	1.00	1.00	.997	.999	.080	.120
DGP.P6	$n = 100$	.887	.910	.233	.298	.094	.138
	$n = 200$	.995	.998	.216	.273	.074	.113
	$n = 500$	1.00	1.00	.185	.237	.067	.109
	$n = 1000$	1.00	1.00	.190	.232	.060	.094
DGP.P7	$n = 100$	.728	.795	.180	.240	.089	.143
	$n = 200$	.953	.969	.178	.243	.074	.136
	$n = 500$	.999	.999	.191	.246	.062	.116
	$n = 1000$	1.00	1.00	.167	.235	.071	.116
DGP.P8	$n = 100$	.570	.659	.143	.212	.063	.114
	$n = 200$	.869	.914	.170	.241	.085	.135
	$n = 500$	.999	.999	.146	.203	.056	.097
	$n = 1000$	1.00	1.00	.140	.204	.049	.083

Notes: the results are based on 1000 iterations using asymptotic critical values.

Appendix B.5 We examine the power of the tests $\widehat{SM}, \widehat{SM}^{(1)}, \widehat{SM}^{(1,1)}$ under nonsmooth local alternatives with different shrinkage parameters b. We use different bandwidths.

Table A.7 Empirical Rejection Rates under DGP.P3 for Different Bandwidths and Parameters

		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$	
		5%	10%	5%	10%	5%	10%
$b=0.1, c=1$	$n=100$	.311	.417	.452	.555	.407	.507
	$n=200$	.516	.619	.745	.811	.733	.802
	$n=500$	.952	.976	.996	1.00	.997	.998
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.2, c=1$	$n=100$	.669	.763	.845	.883	.824	.885
	$n=200$	.951	.963	.997	.997	.996	.998
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.5, c=1$	$n=100$	.998	.999	1.00	1.00	.999	1.00
	$n=200$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.1, c=0.5$	$n=100$	.401	.530	.518	.628	.378	.517
	$n=200$	.615	.718	.818	.867	.749	.828
	$n=500$	.979	.987	.998	.999	.998	.999
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.2, c=0.5$	$n=100$	.820	.895	.916	.940	.847	.911
	$n=200$	.979	.988	1.00	1.00	.996	.999
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.5, c=0.5$	$n=100$	.999	1.00	1.00	1.00	.999	.999
	$n=200$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.1, c=1.5$	$n=100$	.270	.349	.365	.427	.364	.426
	$n=200$	.417	.508	.640	.701	.661	.720
	$n=500$	.812	.870	.969	.977	.984	.990
	$n=1000$	.997	.999	1.00	1.00	1.00	1.00
$b=0.2, c=1.5$	$n=100$	.605	.684	.774	.836	.781	.838
	$n=200$	.904	.938	.975	.983	.983	.990
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00
$b=0.5, c=1.5$	$n=100$	.994	.996	.998	.999	1.00	1.00
	$n=200$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00

Notes: (i) DGP.P3: $Y_t = 0.5Y_{t-1} + 4Z_{t-1}\varphi\left(Y_{t-1}/b\right) + \varepsilon_t$. with $b = 0.1, 0.2, 0.5$ respectively; (ii) the results are based on 1000 iterations using asymptotic critical values; (iii) the bandwidth is $h = cn^{-4/17}$ with $c = 0.5, 1, 1.5$ respectively.

Appendix B.6 We examine the impact of bandwidths on the performance of the tests $\widehat{SM}, \widehat{SM}^{(1)}$ and $\widehat{SM}^{(1,1)}$ using $h = cn^{-4/17}$ with $c = 0.5, 1, 1.5$. The simulation results with $c = 1$ have been reported in our paper. Tables A.8 and A.9 below report the size and power of the tests with $c = 0.5, 1.5$ respectively.

Table A.8 Size of Tests Under DGP.S1-S3 with $c = 0.5, 1.5$

		$c = 0.5$						$c = 1.5$					
		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
DGP.S1	$n = 100$	.145	.227	.106	.177	.050	.115	.112	.168	.109	.155	.110	.170
	$n = 200$	.098	.170	.080	.134	.050	.091	.099	.144	.077	.129	.095	.134
	$n = 500$	.072	.116	.072	.130	.036	.067	.076	.120	.073	.110	.079	.122
	$n = 1000$	.081	.125	.074	.114	.050	.085	.073	.126	.078	.125	.086	.135
DGP.S2	$n = 100$	.154	.208	.094	.165	.054	.108	.116	.173	.089	.142	.091	.138
	$n = 200$	.109	.191	.091	.135	.049	.082	.090	.141	.077	.122	.088	.129
	$n = 500$	.081	.142	.064	.113	.039	.072	.073	.118	.063	.111	.094	.128
	$n = 1000$	.068	.119	.060	.104	.036	.066	.082	.118	.077	.117	.072	.102
DGP.S3	$n = 100$	.131	.205	.082	.136	.036	.083	.103	.154	.091	.132	.106	.136
	$n = 200$	.106	.174	.091	.150	.048	.083	.076	.131	.067	.113	.078	.116
	$n = 500$	.090	.143	.081	.132	.039	.077	.077	.110	.088	.123	.075	.119
	$n = 1000$	.069	.117	.072	.122	.027	.064	.073	.122	.072	.105	.071	.107

Notes: (i) the bandwidth $h = cn^{-4/17}$ with $c = 0.5, 1.5$ in this table; (ii) the results are based on 1000 iterations using asymptotic critical values.

Table A.9 Power of Tests Under DGP.P1-P8 with $c = 0.5, 1.5$

		$c=0.5$						$c=1.5$					
		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$		$\widehat{SM}$		$\widehat{SM}^{(1)}$		$\widehat{SM}^{(1,1)}$	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
DGP.P1	$n=100$	.960	.984	.978	.992	.971	.990	.986	.991	.993	.995	.999	1.00
	$n=200$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P2	$n=100$	.842	.902	.872	.918	.828	.902	.932	.958	.974	.985	.981	.991
	$n=200$	.976	.987	.990	.993	.987	.995	.998	.999	1.00	1.00	1.00	1.00
	$n=500$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DGP.P3	$n=100$	.401	.530	.518	.628	.378	.517	.270	.349	.365	.427	.364	.426
	$n=200$	.615	.718	.818	.867	.749	.828	.417	.508	.640	.701	.661	.720
	$n=500$	.979	.987	.998	.999	.998	.999	.812	.870	.969	.977	.984	.990
	$n=1000$	1.00	1.00	1.00	1.00	1.00	1.00	.997	.999	1.00	1.00	1.00	1.00
DGP.P4	$n=100$	.442	.573	.507	.604	.078	.126	.491	.579	.614	.695	.138	.183
	$n=200$	.589	.701	.720	.797	.058	.092	.797	.852	.908	.933	.127	.168
	$n=500$	.946	.967	.989	.996	.039	.073	.991	.994	1.00	1.00	.114	.153
	$n=1000$	1.00	1.00	1.00	1.00	.038	.083	1.00	1.00	1.00	1.00	.101	.149
DGP.P5	$n=100$	.783	.878	.379	.515	.056	.117	.785	.853	.345	.433	.123	.172
	$n=200$	.950	.978	.451	.586	.038	.074	.979	.989	.512	.580	.101	.149
	$n=500$	1.00	1.00	.741	.828	.044	.076	1.00	1.00	.837	.884	.077	.103
	$n=1000$	1.00	1.00	.958	.974	.030	.067	1.00	1.00	.986	.993	.082	.117
DGP.P6	$n=100$	.945	.979	.255	.360	.047	.106	.988	.996	.203	.263	.109	.167
	$n=200$	1.00	1.00	.242	.332	.041	.069	1.00	1.00	.185	.244	.086	.131
	$n=500$	1.00	1.00	.187	.247	.022	.053	1.00	1.00	.189	.237	.095	.126
	$n=1000$	1.00	1.00	.170	.230	.020	.046	1.00	1.00	.174	.222	.063	.107
DGP.P7	$n=100$	.683	.822	.198	.324	.055	.105	.741	.827	.148	.207	.110	.153
	$n=200$	.879	.952	.181	.251	.041	.081	.959	.977	.118	.169	.080	.125
	$n=500$	.997	.999	.154	.224	.028	.074	1.00	1.00	.117	.152	.072	.097
	$n=1000$	1.00	1.00	.129	.191	.030	.049	1.00	1.00	.117	.159	.081	.123
DGP.P8	$n=100$	.555	.696	.201	.297	.050	.116	.558	.652	.124	.182	.102	.142
	$n=200$	.764	.860	.143	.206	.030	.062	.858	.922	.127	.182	.068	.103
	$n=500$	.994	.997	.138	.195	.023	.062	1.00	1.00	.138	.190	.092	.136
	$n=1000$	1.00	1.00	.127	.193	.021	.055	1.00	1.00	.103	.158	.080	.105

Notes: (i) the bandwidth $h = cn^{-4/17}$ with $c = 0.5, 1.5$ in this table; (ii) the results are based on 1000 iterations using asymptotic critical values.

Appendix C Figures for the Data Series Used in Section 7

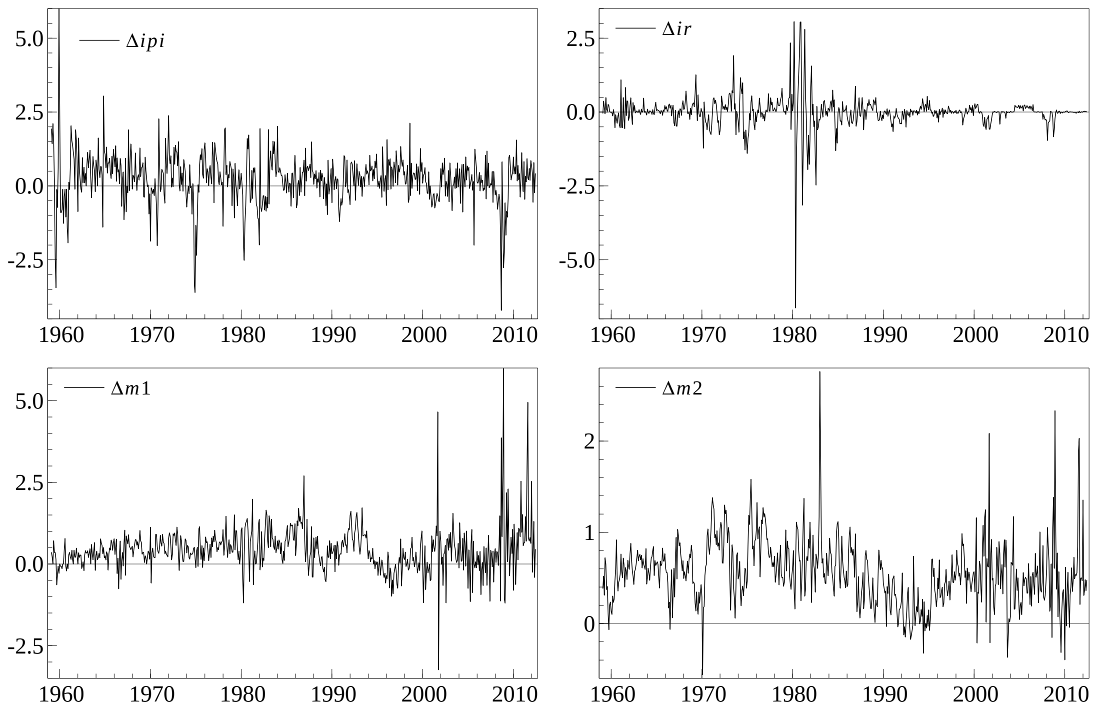

Figure 1: Data Series During the Period 1959:M01- 2012:M06