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Online Supplement to the Paper: Financial Bubble Implosion and Reverse Regression[1]

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This paper supplements the main text "Financial Bubble Implosion and Reverse Regression"with additional proofs and technical material. First, this paper provides a detailed proof of the limiting distribution of the BSDF* statistic (developed in the paper) under the null hypothesis. Second, it provide proofs of supplementary lemmas that deliver the limit behavior of the recursive BDF statistic in dating the origination and collapse of a bubble and in dating turning points of crisis episodes. Third, it provides additional simulations for the detection rate and dating accuracy of the reverse regression procedure. Fourth, it provides some sensitivity analyses for the empirical application to the Nasdaq stock market.

1. THE LIMIT BEHAVIOUR OF THE BSDF* STATISTIC UNDER THE NULL

Lemmas S.A1 and S.A2 below provide some standard partial sum asymptotics that hold under the following assumption, where the input process ε_t is assumed to be *iid* for convenience but may be extended to martingale differences with appropriate changes to the limit theory. These results mirror those given in Phillips, Shi, and Yu (2015b; PSY).

Assumption (EC) *Let $u_t = \psi(L)\varepsilon_t = \Sigma_{j=0}^{\infty}\psi_j\varepsilon_{t-j}$, where $\Sigma_{j=0}^{\infty} j\left|\psi_j\right| < \infty$ and $\{\varepsilon_t\}$ is an i.i.d sequence with mean zero, variance σ^2 and finite fourth moment.*

Lemma S.A1. *Suppose u_t satisfies error condition EC. Define $M_T(g) = 1/T\sum_{s=1}^{[Tg]} u_s$ with $r \in [g_0, 1]$ and $\xi_t = \sum_{s=1}^{t} u_s$. Let $g_2, g_w \in [g_0, 1]$ and $g_1 = g_2 - g_w$. The following hold:*

(1) $\sum_{s=1}^{t} u_s = \psi(1)\sum_{s=1}^{t}\varepsilon_s + \eta_t - \eta_0$, *where* $\eta_t = \sum_{j=0}^{\infty}\alpha_j\varepsilon_{t-j}$, $\eta_0 = \sum_{j=0}^{\infty}\alpha_j\varepsilon_{-j}$ *and* $\alpha_j = -\sum_{i=1}^{\infty}\psi_{j+i}$*, which is absolutely summable.*

(2) $\frac{1}{T}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\varepsilon_t^2 \xrightarrow{p} \sigma^2 g_w$.

(3) $T^{-1/2}\sum_{t=1}^{[Tg]}\varepsilon_t \xrightarrow{L} \sigma W(g)$.

(4) $T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t \xrightarrow{L} \sigma^2\left[\int_{g_1}^{g_2} W(s)\,dW - \frac{1}{2}g_w\right]$.

(5) $T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\varepsilon_t t \xrightarrow{L} \sigma\left[g_2 W(g_2) - g_1 W(g_1) - \int_{g_1}^{g_2} W(s)\,ds\right]$.

(6) $T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}(\eta_{t-1} - \eta_0)\varepsilon_t \xrightarrow{p} 0$.

(7) $T^{-1/2}\left(\eta_{[Tg]} - \eta_0\right) \xrightarrow{p} 0$.

(8) $\sqrt{T}M_T(g) \xrightarrow{L} \psi(1)\sigma W(g)$.

(9) $T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_{t-1} \xrightarrow{L} \psi(1)\sigma\int_{g_1}^{g_2} W(s)\,ds$.

(10) $T^{-5/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_{t-1}t \xrightarrow{L} \psi(1)\sigma\int_{g_1}^{g_2} W(s)\,s\,ds$.

(11) $T^{-2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_{t-1}^2 \xrightarrow{L} \sigma^2\psi(1)^2\int_{g_1}^{g_2} W(s)^2\,ds$.

(12) $T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_t\varepsilon_{t-j} \xrightarrow{p} 0, \forall j = 0, \pm1, \pm2, \ldots$

Lemma S.A2. *Define $X_t^* = -\alpha_T\psi(1)t + \sum_{s=1}^{t}\omega_s$, where $\alpha_T = cT^{-\eta}$ with $\eta > 1/2$ and $\omega_t = -u_{T+2-t} = \psi(L)v_t$. Let u_t satisfy condition EC. Then*

$$(a)\ T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X_{t-1}^* v_t \xrightarrow{L} \psi(1)\sigma^2\left[\int_{1-g_2}^{1-g_1} W(s)\,dW + \frac{1}{2}g_w\right].$$

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$$(b)\ T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^*_{t-1} \xrightarrow{L} \psi(1)\,\sigma\int_{1-g_2}^{1-g_1} W(s)\,ds.$$

$$(c)\ T^{-2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^{*2}_{t-1} \xrightarrow{L} \sigma^2\psi(1)^2\int_{1-g_2}^{1-g_1} W(s)^2\,ds.$$

$$(d)\ T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^*_{t-1}v_{t-j} \xrightarrow{p} 0,\quad j=0,1,\cdots.$$

$$(e)\ T^{-1/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} v_t \xrightarrow{L} -\sigma\int_{1-g_2}^{1-g_1} dW(s).$$

Proof of Lemma S.A2. (a) From (1) of Lemma S.A1,

$$\begin{aligned}
T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^*_{t-1}v_t &= -T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X_{T+2-t}\varepsilon_{T+2-t}\\
&= T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\left(\alpha_T\psi(1)(T+2-t)+\sum_{s=1}^{T+2-t}u_s\right)\varepsilon_{T+2-t}\\
&= T^{-1}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\left(\alpha_T\psi(1)t+\sum_{s=1}^{t}u_s\right)\varepsilon_t\\
&= T^{-1}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\left(\alpha_T\psi(1)t+\psi(1)\sum_{s=1}^{t}\varepsilon_s+\eta_t-\eta_0\right)\varepsilon_t\\
&= \alpha_T\psi(1)T^{1/2}\left(T^{-3/2}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}t\varepsilon_t\right)+\psi(1)\left(T^{-1}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\varepsilon_t^2\right)\\
&\quad+\psi(1)\left(T^{-1}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t\right)+\left(T^{-1}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}(\eta_t-\eta_0)\varepsilon_t\right)\\
&\xrightarrow{L}\psi(1)\,\sigma^2\left[\int_{1-g_2}^{1-g_1}W(s)\,dW+\frac{1}{2}g_w\right].
\end{aligned}$$

(b) From Lemma S.A1(10),

$$\begin{aligned}
T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^*_t &= T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X_{T+1-t}\\
&= T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\left[\alpha_T\psi(1)(T+1-t)+\sum_{s=1}^{T+1-t}u_s\right]\\
&= \alpha_T\psi(1)T^{-3/2}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor}t+T^{-3/2}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor}\xi_t\\
&\xrightarrow{L}\psi(1)\,\sigma\int_{1-g_2}^{1-g_1}W(s)\,ds.
\end{aligned}$$

(c) From (11) and (12) of Lemma S.A1,

$$\begin{aligned}
T^{-2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^{*2}_t &= T^{-2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X^2_{T+1-t} = T^{-2}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor} X_t^2\\
&= T^{-2}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor}(\alpha_T t\psi(1)+\xi_t)^2\\
&= \alpha_T^2\psi(1)^2T\left(T^{-3}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor}t^2\right)+\left(T^{-2}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor}\xi_t^2\right)
\end{aligned}$$

$$+2\alpha_T\psi(1)T^{1/2}\left(T^{-5/2}\sum_{t=T+1-\lfloor Tg_2\rfloor}^{T+1-\lfloor Tg_1\rfloor}t\xi_t\right)$$
$$\xrightarrow{L}\sigma^2\psi(1)^2\int_{1-g_2}^{1-g_1}W(s)^2\,ds.$$

(d) From (5) and (13) of Lemma S.A1,

$$\begin{aligned}
T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}X^*_{t-1}v_{t-j} &= -T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}X_{T+2-t}\varepsilon_{T+2-t+j}\\
&= -T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\left[\alpha_T\psi(1)(T+2-t)+\sum_{s=1}^{T+2-t}\varepsilon_s\right]\varepsilon_{T+2-t+j}\\
&= -T^{-3/2}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\left[\alpha_T\psi(1)t+\sum_{s=1}^{t}\varepsilon_s\right]\varepsilon_{t+j}\\
&= -\alpha_T\psi(1)\left(T^{-3/2}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}t\varepsilon_{t+j}\right)-\left(T^{-3/2}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\xi_t\varepsilon_{t+j}\right)\\
&\xrightarrow{p}0.
\end{aligned}$$

(e) By definition of $v_t=-\varepsilon_{T+2-t}$, we have

$$T^{-1/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}v_t=-T^{-1/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\varepsilon_{T+2-t}=-T^{-1/2}\sum_{t=T+2-\lfloor Tg_2\rfloor}^{T+2-\lfloor Tg_1\rfloor}\varepsilon_t$$
$$\xrightarrow{L}-\sigma\int_{1-g_2}^{1-g_1}dW(s).$$

□

With these results in hand, we can now derive the asymptotic distribution of the BSDF* statistic.

Proof of Theorem 3.1. The proof follows the same lines as that of Theorem 1 of PSY (2015a, 2015b). In this case the regression model is

$$\Delta X^*_t=\mu+\rho X^*_{t-1}+\sum_{k=1}^{p-1}\phi^k\Delta X^*_{t-k}+v_t,\text{ with }t=\lfloor Tg_1\rfloor,....,\lfloor Tg_2\rfloor.$$

Under the null hypothesis that $\mu=-cT^{-\eta}$ and $\rho=0$, we have $X^*_t=\tilde{\alpha}_Tt+\sum_{s=1}^{t}\omega_s$ and $\triangle X^*_t=\tilde{\alpha}_T+\omega_t$, where $\tilde{\alpha}_T=-\psi(1)cT^{-\eta}$ and $\omega_t=\psi(L)v_t$ with $\psi(L)=\left(1-\phi^1L-\phi^2L^2-\cdots-\phi^{p-1}L^{p-1}\right)^{-1}$.

The deviation of the OLS estimate $\hat{\theta}$ from the true value θ is given by

$$\hat{\theta}-\theta=\left[\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tZ_t'\right]^{-1}\left[\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tv_t\right],\qquad(1.1)$$

where $Z_t=[\tilde{\alpha}_T+\omega_{t-1}\ \tilde{\alpha}_T+\omega_{t-2}\ \ldots\ \tilde{\alpha}_T+\omega_{t-p+1}\ 1\ X^*_{t-1}]'$, $\theta=[\phi^1\ \phi^2\ \ldots\phi^{p-1}\ \mu\ \rho]'$. Notice that the estimated model parameters should depend on the sample range g_1 and g_2, however we use, for instance, $\hat{\theta}$ and θ instead of $\hat{\theta}_{g_1,g_2}$ and θ_{g_1,g_2}, for ease of notation. The probability limit of $\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tZ_t'$ is block diagonal from (d) of Lemma S.A2. Therefore, we only need to obtain the last 2×2 components of $\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tZ_t'$ and the last 2×1 component of $\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tv_t$ to calculate the DF statistics, which are

$$\begin{bmatrix}\Sigma'1 & \Sigma X^*_{t-1}\\ \Sigma'X^*_{t-1} & \Sigma X^{*2}_{t-1}\end{bmatrix}\text{ and }\begin{bmatrix}\Sigma'v_t\\ \Sigma'X^*_{t-1}v_t\end{bmatrix},$$

respectively, where Σ' denotes summation over $t=\lfloor Tg_1\rfloor,\lfloor Tg_1\rfloor+1,\cdots,\lfloor Tg_2\rfloor$. Based on (3) in Lemma S.A1 and (a) in Lemma S.A2, the scaling matrix should be $\Upsilon_T=diag\left(\sqrt{T},T\right)$. Pre-multiplying equation (1.1) by Υ_T, results in

$$\Upsilon_T\begin{bmatrix}\widehat{\mu}-\mu\\ \widehat{\rho}-\rho\end{bmatrix}=\left\{\Upsilon_T^{-1}\left[\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tZ_t'\right]_{(-2)\times(-2)}\Upsilon_T^{-1}\right\}^{-1}\left\{\Upsilon_T^{-1}\left[\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}Z_tv_t\right]_{(-2)\times1}\right\}.\qquad(1.2)$$

Consider the matrix $\Upsilon_T^{-1}\left[\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} Z_t Z_t'\right]_{(-2)\times(-2)}\Upsilon_T^{-1}$, whose partitioned form is

$$\begin{bmatrix}\sqrt{T} & 0\\ 0 & T\end{bmatrix}^{-1}\begin{bmatrix}\Sigma'1 & \Sigma X_{t-1}^*\\ \Sigma' X_{t-1}^* & \Sigma X_{t-1}^{*2}\end{bmatrix}\begin{bmatrix}\sqrt{T} & 0\\ 0 & T\end{bmatrix}^{-1}=\begin{bmatrix}T^{-1}\Sigma'1 & T^{-3/2}\Sigma' X_{t-1}^*\\ T^{-3/2}\Sigma' X_{t-1}^* & T^{-2}\Sigma' X_{t-1}^{*2}\end{bmatrix}$$
$$\xrightarrow{L}\begin{bmatrix}g_w & \psi(1)\sigma\int_{1-g_2}^{1-g_1}W(s)\,ds\\ \psi(1)\sigma\int_{1-g_2}^{1-g_1}W(s)\,ds & \sigma^2\psi(1)^2\int_{1-g_2}^{1-g_1}W(s)^2\,ds\end{bmatrix}$$

and the matrix $\Upsilon_T^{-1}\left[\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor} X_t\varepsilon_t\right]_{(-2)\times 1}$, for which

$$\begin{bmatrix}T^{-1/2}\Sigma v_t\\ T^{-1}\Sigma X_{t-1}^* v_t\end{bmatrix}\xrightarrow{L}\begin{bmatrix}-\sigma\int_{1-g_2}^{1-g_1}dW(s)\\ \psi(1)\sigma^2\left[\int_{1-g_2}^{1-g_1}W(s)\,dW+\frac{1}{2}g_w\right]\end{bmatrix}.$$

Under the null hypothesis that $\mu=-cT^{-\eta}$ and $\rho=0$,

$$\begin{bmatrix}\sqrt{T}(\hat{\mu}-\mu)\\ T\hat{\rho}\end{bmatrix}\xrightarrow{L}\begin{bmatrix}g_w & A\\ A & B\end{bmatrix}^{-1}\begin{bmatrix}C\\ D\end{bmatrix},$$

where

$$\begin{aligned}A&=\psi(1)\sigma\int_{1-g_2}^{1-g_1}W(s)\,ds,\\ B&=\sigma^2\psi(1)^2\int_{1-g_2}^{1-g_1}W(s)^2\,ds,\\ C&=-\sigma\int_{1-g_2}^{1-g_1}dW(s),\\ D&=\psi(1)\sigma^2\left[\int_{1-g_2}^{1-g_1}W(s)\,dW+\frac{1}{2}g_w\right].\end{aligned}$$

Therefore, $\hat{\rho}$ converges at rate T to the following limit variate

$$T\hat{\rho}\xrightarrow{L}\frac{AC-g_wD}{A^2-g_wB}.$$

To calculate the t-statistic $t=\frac{\hat{\rho}}{se(\hat{\rho})}$ of $\hat{\rho}$, we first find the standard error $se(\hat{\rho})$. We have

$$var\left(\begin{bmatrix}\hat{\mu}\\ \hat{\rho}\end{bmatrix}\right)=\sigma^2\begin{bmatrix}\Sigma'1 & \Sigma' X_{t-1}^*\\ \Sigma' X_{t-1}^* & \Sigma' X_{t-1}^{*2}\end{bmatrix}^{-1},$$

so the variance of $T\hat{\beta}$ can be calculated from

$$\begin{aligned}&var\left(\begin{bmatrix}\sqrt{T}(\hat{\mu}-\mu)\\ T\hat{\rho}\end{bmatrix}\right)\\ &=\sigma^2\left\{\begin{bmatrix}\sqrt{T} & 0\\ 0 & T\end{bmatrix}^{-1}\begin{bmatrix}\Sigma'1 & \Sigma' X_{t-1}^*\\ \Sigma' X_{t-1}^* & \Sigma' X_{t-1}^{*2}\end{bmatrix}\begin{bmatrix}\sqrt{T} & 0\\ 0 & T\end{bmatrix}^{-1}\right\}^{-1}\\ &=\sigma^2\begin{bmatrix}T^{-1}\Sigma'1 & T^{-3/2}\Sigma' X_{t-1}^*\\ T^{-3/2}\Sigma' X_{t-1}^* & T^{-2}\Sigma' X_{t-1}^{*2}\end{bmatrix}^{-1}\xrightarrow{L}\sigma^2\begin{bmatrix}g_w & A\\ A & B\end{bmatrix}^{-1}.\end{aligned}$$

It follows that the sub-sample DF statistic (i.e. t-statistic t of $\hat{\beta}$) satisfies

$$\begin{aligned}DF_{g_1,g_2}&=\frac{g_wD-AC}{\sigma g_w^{1/2}(g_wB-A^2)^{1/2}}\\ &\xrightarrow{L}\frac{g_w\left[\int_{1-g_2}^{1-g_1}W(s)\,dW+\frac{1}{2}g_w\right]+\int_{1-g_2}^{1-g_1}W(s)\,ds.\int_{1-g_2}^{1-g_1}dW}{g_w^{1/2}\left\{g_w\int_{1-g_2}^{1-g_1}W(s)^2\,ds-\left[\int_{1-g_2}^{1-g_1}W(s)\,ds\right]^2\right\}^{1/2}}.\end{aligned}\tag{1.3}$$

By a continuous mapping argument just as in the proof of theorem 3.1 of PSY (2015b), the asymptotic distribution of the backward sup DF statistic $BSDF^*_{g_2}(g_0)$ is found to be

$$\sup_{g_1\in[0,g_2-g_0]}\left\{\frac{g_w\left[\int_{1-g_2}^{1-g_1}W(s)\,dW+\frac{1}{2}g_w\right]+\int_{1-g_2}^{1-g_1}W(s)\,ds.\int_{1-g_2}^{1-g_1}dW}{g_w^{1/2}\left\{g_w\int_{1-g_2}^{1-g_1}W(s)^2\,ds-\left[\int_{1-g_2}^{1-g_1}W(s)\,ds\right]^2\right\}^{1/2}}\right\}, \tag{1.4}$$

giving the stated result. However, as mentioned above, derivation of (1.4) is not an immediate application of the continuous mapping theorem, which would require tightness of the random function sequence $\{DF_{g_1,g_2}\}$ as well as the finite dimensional limit theory given above in the one dimensional case (1.3). Instead, as in the proof of theorem 1 of Zivot and Andrews (1992), a rigorous proof of (1.4) is more easily accomplished by the application of the continuous mapping theorem to a functional of the sample partial sum process $Z_T^0(g)=\sqrt{T}N_T(g)=T^{-1/2}\sum_{s=1}^{\lfloor Tg\rfloor}v_s=T^{-1/2}X^{*0}_{\lfloor Tg\rfloor}$ and the error variance estimate $\hat{\sigma}^2$. We proceed to construct this functional and derive the limit result (1.4). First, note that under the null we have $X_t^*=\sum_{s=1}^t v_s+O_p\left(c\frac{t}{T^\eta}\right)=:X_t^{*0}+O_p\left(T^{1-\eta}\right)$ so that $Z_T(g):=T^{-1/2}X^*_{\lfloor Tg\rfloor}=T^{-1/2}X^{*0}_{\lfloor Tg\rfloor}+O_p\left(T^{1/2-\eta}\right)=Z_T^0(g)+o_p(1)$ uniformly. Since under the null hypothesis $\psi(L)=1$ and $Z_T^0(g)=T^{-1/2}\sum_{s=1}^{\lfloor Tg\rfloor}v_s\Rightarrow -\sigma W(1-g)$ (from Lemma S.A2), we have

$$T^{-2}\left\{\Sigma' X_{t-1}^{*2}-\left(\Sigma' X_{t-1}^*\right)^2/\Sigma' 1\right\} = \int_{g_1}^{g_2}Z_T^{*0}(s)^2\,ds-\left(\int_{g_1}^{g_2}Z_T^{*0}(s)\,ds\right)^2/g_w+o_p(1) \tag{1.5}$$

$$\Rightarrow \sigma^2\left\{\int_{1-g_2}^{1-g_1}W(s)^2\,ds-\left(\int_{1-g_2}^{1-g_1}W(s)\,ds\right)^2/g_w\right\}. \tag{1.6}$$

Define the two functionals $h_{1g}\left(Z_T^0\right):=\int_0^g Z_T^0(s)\,ds$ and $h_{2g}\left(Z_T^0\right):=\int_0^g Z_T^0(s)^2\,ds$ of $Z_T^0(g)\in D[0,1]$, the Skorohod space equipped with the uniform topology. Both h_{1g} and h_{2g} are continuous functionals by standard arguments. It is convenient to write

$$\int_{g_1}^{g_2}Z_T^0(s)\,ds=h_{1g_2}\left(Z_T^0\right)-h_{1g_1}\left(Z_T^0\right),\ \int_{g_1}^{g_2}Z_T^0(s)^2\,ds=h_{2g_2}\left(Z_T^0\right)-h_{2g_1}\left(Z_T^0\right),$$

and then (1.5)-(1.6) can be written in functional form as

$$\begin{aligned}&T^{-2}\left\{\Sigma' X_{t-1}^{*2}-\left(\Sigma' X_{t-1}^*\right)^2/\Sigma' 1\right\}\\&=\left\{h_{2g_2}\left(Z_T^0\right)-h_{2g_1}\left(Z_T^0\right)\right\}-\left\{h_{1g_2}\left(Z_T^0\right)-h_{1g_1}\left(Z_T^0\right)\right\}^2/g_w+o_p(1)\\&\Rightarrow \sigma^2\left\{h_{2g_2}(W)-h_{2g_1}(W)\right\}-\left\{h_{1g_2}(W)-h_{1g_1}(W)\right\}^2/g_w.\end{aligned} \tag{1.7}$$

Since h_{1g} and h_{2g} are continuous, so is the functional

$$E_{1,g_1,g_2}\left(Z_t^0\right):=\left\{h_{2g_2}\left(Z_T^0\right)-h_{2g_1}\left(Z_T^0\right)\right\}-\left\{h_{1g_2}\left(Z_T^0\right)-h_{1g_1}\left(Z_T^0\right)\right\}^2/g_w.$$

Now observe that

$$\begin{aligned}E_{1,g_1,g_2}(W)&=\int_{1-g_2}^{1-g_1}W(s)^2\,ds-\left(\int_{1-g_2}^{1-g_1}W(s)\,ds\right)^2/g_w\\&=\int_{1-g_2}^{1-g_1}W(s)^2\,ds-g_w\left(g_w^{-1}\int_{1-g_2}^{1-g_1}W(s)\,ds\right)^2\\&=:\int_{1-g_2}^{1-g_1}\underline{W}_{g_1,g_2}(s)^2\,ds\end{aligned}$$

with $\underline{W}_{g_1,g_2}(s)=W(s)-g_w^{-1}\int_{1-g_2}^{1-g_1}W(s)\,ds$. Just as in Phillips and Hansen (1990, Lemma A2), we have that $\int_{1-g_2}^{1-g_1}\underline{W}_{g_1,g_2}(s)^2\,ds>0$ $a.s.$ since $g_w=g_2-g_1\geq g_0>0$. It follows that the functional

$$E^1_{g_1,g_2}(W):=1/E_{1,g_1,g_2}(W) \tag{1.8}$$

is well defined for all (g_1,g_2) such that $g_w=g_2-g_1\geq g_0>0$ and so the functional

$$E^1_{g_1,g_2}\left(Z_t^0\right):=\frac{1}{E_{1,g_1,g_2}\left(Z_t^0\right)}$$

is continuous with limit $E^1_{g_1,g_2}\left(Z_t^0\right)\Rightarrow E^1_{g_1,g_2}(\sigma W)=\sigma^2 E^1_{g_1,g_2}W$.

Next, from (1.2), the numerator component of the t ratio $(T\hat{\beta})$ is given by

$$\frac{\begin{bmatrix} -T^{-3/2}\Sigma' X^*_{t-1} & T^{-1}\Sigma' 1 \end{bmatrix} \begin{bmatrix} T^{-1/2}\Sigma' v_t \\ T^{-1}\Sigma' X^*_{t-1} v_t \end{bmatrix}}{\left(T^{-2}\Sigma' X^{*2}_{t-1}\right)\left(T^{-1}\Sigma' 1\right) - \left(T^{-3/2}\Sigma' X^*_{t-1}\right)^2}$$

$$= \frac{\begin{bmatrix} -\int_{g_1}^{g_2} Z_T^0(s)\,ds & g_w \end{bmatrix} \begin{bmatrix} Z_T^0(g_2) - Z_T^0(g_1) \\ \int_{g_1}^{g_2} Z_T^0(s)\,dZ_T^0(s) \end{bmatrix}}{\left(\int_{g_1}^{g_2} Z_T^0(s)^2 ds\right) g_w - \left(\int_{g_1}^{g_2} Z_T^0(s)\,ds\right)^2} + o_p(1)$$

$$= \frac{\begin{bmatrix} -\int_{g_1}^{g_2} Z_T^0(s)\,ds & g_w \end{bmatrix} \begin{bmatrix} Z_T^0(g_2) - Z_T^0(g_1) \\ \frac{1}{2}\left[Z_T^0(g_2)^2 - Z_T^0(g_1)^2 - T^{-1}\Sigma' v_t^2\right] \end{bmatrix}}{\left(\int_{g_1}^{g_2} Z_T^0(s)^2 ds\right) g_w - \left(\int_{g_1}^{g_2} Z_T^0(s)\,ds\right)^2} + o_p(1)$$

$$= \frac{\begin{bmatrix} -\left[h_{1g_2}\left(Z_T^0\right) - h_{1g_1}\left(Z_T^0\right)\right] & g_w \end{bmatrix} \begin{bmatrix} Z_T^0(g_2) - Z_T^0(g_1) \\ \frac{1}{2}\left[Z_T^0(g_2)^2 - Z_T^0(g_1)^2 - T^{-1}\Sigma' v_t^2\right] \end{bmatrix}}{g_w E_{1,g_1,g_2}\left(Z_t^0\right)} + o_p(1)$$

$$= \frac{\frac{g_w}{2}\left[Z_T^0(g_2)^2 - Z_T^0(g_1)^2 + \frac{\lfloor Tg_w \rfloor}{T}\hat{\sigma}^2\right] - \left[h_{1g_2}\left(Z_T^0\right) - h_{1g_1}\left(Z_T^0\right)\right]\left[Z_T^0(g_2) - Z_T^0(g_1)\right]}{g_w E_{1,g_1,g_2}\left(Z_t^0\right)}$$

$$+ o_p(1) \tag{1.9}$$

$$\Rightarrow \frac{\frac{g_w}{2}\left[W(1-g_1)^2 - W(1-g_2)^2 + g_w\right] + \left[h_{1g_2}(W) - h_{1g_1}(W)\right]\left[W(1-g_1) - W(1-g_2)\right]}{g_w E_{1,g_1,g_2}(W)}.$$

Define

$$E_{2,g_1,g_2}\left(Z_T^0, \hat{\sigma}^2\right)$$
$$= \frac{g_w}{2}\left[Z_T^0(g_2)^2 - Z_T^0(g_1)^2 + g_w\hat{\sigma}^2\right] - \left[h_{1g_2}\left(Z_T^0\right) - h_{1g_1}\left(Z_T^0\right)\right]\left[Z_T^0(g_2) - Z_T^0(g_1)\right] \tag{1.10}$$
$$\Rightarrow E_{2,g_1,g_2}\left(\sigma W, \sigma^2\right)$$
$$= \sigma^2 \frac{g_w}{2}\left[W(1-g_1)^2 - W(1-g_2)^2 + g_w\right] + \sigma^2\left[h_{1g_2}(W) - h_{1g_1}(W)\right]\left[W(1-g_1) - W(1-g_2)\right] \tag{1.11}$$

so that (1.9) is

$$\frac{E_{2,g_1,g_2}\left(Z_T^0, \hat{\sigma}^2\right)}{g_w E_{1,g_1,g_2}\left(Z_T^0\right)} + o_p(1) \Rightarrow \frac{E_{2,g_1,g_2}\left(\sigma W, \sigma^2\right)}{g_w E_{1,g_1,g_2}(\sigma W)}. \tag{1.12}$$

Using (1.12) and (1.7) we have

$$T^2 s_{\hat{\beta}}^2 = \frac{\hat{\sigma}^2}{T^{-2}\Sigma' X^{*2}_{t-1} - \left(T^{-1}\Sigma' X^*_{t-1}\right)^2 / \Sigma' 1}$$
$$= \frac{\hat{\sigma}^2}{E_{1,g_1,g_2}\left(Z_T^0\right)} + o_p(1). \tag{1.13}$$

It follows from (1.9) - (1.13) that the DF statistic (or t ratio) can be written as

$$DF_{g_1,g_2} = \frac{T\hat{\beta}}{\left(T^2 s_{\hat{\beta}}^2\right)^{1/2}} = \frac{E_{2,g_1,g_2}\left(Z_T^0, \hat{\sigma}^2\right)}{g_w E_{1,g_1,g_2}\left(Z_T^0\right)}\left(\frac{E_{1,g_1,g_2}\left(Z_T^0\right)}{\hat{\sigma}^2}\right)^{1/2} + o_p(1)$$
$$= \frac{E_{2,g_1,g_2}\left(Z_T^0, \hat{\sigma}^2\right)}{g_w\left(E_{1,g_1,g_2}\left(Z_T^0\right)\right)^{1/2}\hat{\sigma}} + o_p(1) =: E_{g_1,g_2}\left(Z_T^0, \hat{\sigma}^2\right) + o_p(1),$$

which defines the required functional $E_{g_1,g_2}(\cdot,\cdot)$ representing the t ratio in terms of $\left(Z_T^0, \hat{\sigma}^2\right)$. Since $\hat{\sigma}^2 \to_p \sigma^2$, we have

$$DF_{g_1,g_2} = E_{g_1,g_2}\left(Z_T^0, \hat{\sigma}^2\right) \Rightarrow E_{g_1,g_2}\left(\sigma W, \sigma^2\right) = \frac{E_{2,g_1,g_2}(\sigma W, \sigma)}{g_w\left(E_{1,g_1,g_2}(\sigma W)\right)^{1/2}\sigma} = \frac{E_{2,g_1,g_2}(W, 1)}{g_w E_{1,g_1,g_2}(W)^{1/2}}$$

$$= \frac{\frac{1}{2}g_w\left\{W\left(1-g_1\right)^2-W\left(1-g_2\right)^2+g_w\right\}+\left(\int_{1-g_2}^{1-g_1}W\left(s\right)ds\right)\left\{W\left(1-g_1\right)-W\left(1-g_2\right)\right\}}{g_w\left\{g_w\int_{1-g_2}^{1-g_1}W\left(s\right)^2ds-\left[\int_{1-g_2}^{1-g_1}W\left(s\right)ds\right]^2\right\}^{1/2}}$$

$$= E_{g_1,g_2}\left(W,1\right).$$

In view of the continuity of $E_{2,g_1,g_2}\left(Z_T^0,\hat{\sigma}^2\right)$ and $1/E_{1,g_1,g_2}\left(Z_T^0,\right)$, the functional $E_{g_1,g_2}\left(Z_T^0,\hat{\sigma}^2\right)$ is continuous for all (g_1,g_2) such that $g_w=g_2-g_1\geq g_0>0$. The continuous functional $E_{g_1,g_2}\left(\cdot,\cdot\right)$ maps $D[0,1]\times\mathbb{R}^+$ onto a function defined on $\Lambda_0=\{(g_1,g_2):1\geq g_2\geq g_1+g_0 \text{ and } 1-g_0\geq g_1\geq 0\}$. Define the double sup functional $E^*\left(E_{g_1,g_2}\right)=\sup_{(g_1,g_2)\in\Lambda_0}E_{g_1,g_2}$ which maps functions defined on Λ_0 onto $\mathbb{R}$. Let E_{g_1,g_2} and $\tilde{E}_{g_1,g_2}$ be two functions defined on Λ_0 such that $\sup_{(g_1,g_2)\in\Lambda_0}\left|E_{g_1,g_2}-\tilde{E}_{g_1,g_2}\right|<\varepsilon$ for some given $\varepsilon>0$. The function $E^*\left(E_{g_1,g_2}\right)$ is continuous with respect to the uniform norm on its domain because

$$\left|E^*\left(E_{g_1,g_2}\right)-E^*\left(\tilde{E}_{g_1,g_2}\right)\right|=\left|\sup_{(g_1,g_2)\in\Lambda_0}\left[E_{g_1,g_2}-\tilde{E}_{g_1,g_2}\right]\right|\leq\sup_{(g_1,g_2)\in\Lambda_0}\left|E_{g_1,g_2}-\tilde{E}_{g_1,g_2}\right|<\varepsilon.$$

We therefore deduce by continuous mapping the weak convergence

$$\begin{aligned}\sup_{(g_1,g_2)\in\Lambda_0}DF_{g_1,g_2} &= \sup_{(g_1,g_2)\in\Lambda_0}E^*\left(E_{g_1,g_2}\left(\left(Z_T^0,\hat{\sigma}^2\right)\right)\right)\\ &\Rightarrow \sup_{(g_1,g_2)\in\Lambda_0}E^*\left(E_{g_1,g_2}\left(\left(W,1\right)\right)\right)=\sup_{\substack{g_2\in[g_0,1]\\ g_1\in[0,g_2-g_0]}}E_{g_1,g_2}\left(W,1\right),\end{aligned}$$

giving (1.4) as required. □

2. THE LIMIT BEHAVIOUR OF THE BSDF* STATISTIC UNDER THE ALTERNATIVE

2.1. Notation

- The bubble period $B=[T_e,T_c]$, where $T_e=\lfloor Tf_e\rfloor$ and $T_c=\lfloor Tf_c\rfloor$.
- The crisis period $C=(T_c,T_r]$, where $T_r=\lfloor Tf_r\rfloor$.
- The normal market periods $N_0=[1,T_e)$ and $N_1=[T_r+1,T]$, where T is the last observation of the sample.
- The data generating process is specified as

$$X_t=\begin{cases} cT^{-\eta}+X_{t-1}+\varepsilon_t,\text{ constant } c,\ \eta>1/2, & t\in N_0\cup N_1\\ \delta_T X_{t-1}+\varepsilon_t, & t\in B\\ \gamma_T X_{t-1}+\varepsilon_t, & t\in C\end{cases}, \qquad (2.14)$$

  where $\varepsilon_t\sim N\left(0,\sigma^2\right)$, $X_0=o_p(1)$, $\delta_T=1+c_1T^{-\alpha}$ and $\gamma_T=1-c_2T^{-\beta}$ with $c_1,c_2>0$ and $\alpha,\beta\in[0,1)$. If $\alpha>\beta$, the rate of bubble collapse is faster than that of bubble expansion. If $\alpha<\beta$, the rate of bubble collapse is slower than that of bubble expansion.
- Let $X_t^*=X_{T+1-t}$. The dynamic of X_t^* is

$$X_t^*=\begin{cases} -cT^{-\eta}+X_{t-1}^*+v_t,\text{ constant } c,\ \eta>1/2, & t\in N_0\cup N_1\\ \gamma_T^{-1}X_{t-1}^*+\gamma_T^{-1}v_t, & t\in C\\ \delta_T^{-1}\left[X_{t-1}^*+v_t\right], & t\in B\end{cases}, \qquad (2.15)$$

  where $v_t=-\varepsilon_{T+2-t}\sim N\left(0,\sigma^2\right)$ and $X_0^*=o_p(1)$.
- Let $\tau_1=\lfloor Tg_1\rfloor$ and $\tau_2=\lfloor Tg_2\rfloor$ be the starting and ending point of the regression. We have $T_1=T+1-\tau_2$, $T_2=T+1-\tau_1$ and $\tau_w=\lfloor Tg_w\rfloor$ be the regression window size.
- Let $\tau_e=\lfloor Tg_e\rfloor$, $\tau_r=\lfloor Tg_r\rfloor$, and $\tau_c=\lfloor Tg_c\rfloor$, where $g_e=1-f_r$, $g_c=1-f_c$, $g_r=1-f_e$. This suggests that $N_1=[1,\tau_e)$, $C=[\tau_e,\tau_c)$, $B=[\tau_c,\tau_r]$, $N_0=(\tau_r,T]$.

Under the stated conditions, partial sums of ε_t satisfy the functional law

$$T^{-1/2}\sum_{t=1}^{\lfloor T\cdot\rfloor}\varepsilon_t\Rightarrow B(\cdot):=\sigma W(\cdot), \qquad (2.16)$$

where W is standard Brownian motion. We follow the approach developed in Phillips and Yu (2009) and PSY (2015a&b). The additional regime on bubble collapsing leads to a much lengthier derivation for a single bubble case. We focus on deriving the limit behaviour of the recursive BDF statistic in a model with a single bubble episode. The results continue to hold in models with more than one bubble episode. The consistency of the PSY strategy is provided in Section 2 for dating bubble expansion and Section 3 for dating bubble collapsing.

With minor modifications, the results continue to hold under correspondingly general conditions on the innovations

ε_t for which the weak convergence (2.16) applies as well as the limit theory for mildly explosive processes given in Phillips and Magdalinos (2007a, 2007b). To keep this supplement to manageable length we do not go into the details of those extensions here.

2.2. *Dating Bubble Expansion*

Lemma S.B1. *Under the data generating process,*

(1) *For* $t \in N_0$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p)$.

(2) *For* $t \in B$, $X_{t=\lfloor Tp \rfloor} = \delta_T^{t-T_e} X_{T_e} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-T_e} B(f_e)$.

(3) *For* $t \in C$,

$$X_{t=\lfloor Tp \rfloor} = \gamma_T^{t-T_c} X_{T_c} + \sum_{j=0}^{t-T_c-1} \gamma_T^j \varepsilon_{t-j} \sim_a T^{1/2} \delta_T^{T_c-T_e} \gamma_T^{t-T_c} B(f_e) + T^{\beta/2} X_{c_2}.$$

(4) *For* $t \in N_1$,

$$X_{t=\lfloor Tp \rfloor} = \begin{cases} \sum_{j=0}^{t-T_r-1} \varepsilon_{t-j} \{1 + o_p(1)\} \sim_a T^{1/2} [B(p) - B(f_r)] & \text{if } \alpha > \beta \\ \gamma_T^{T_r-T_c} X_{T_c} \sim_a T^{1/2} \gamma_T^{T_r-T_c} \delta_T^{T_c-T_e} B(f_e) & \text{if } \alpha < \beta \end{cases}.$$

Proof. (1) For $t \in N_0$,

$$X_t = X_0 + cT^{-\eta} t + \sum_{s=1}^{t} \varepsilon_s.$$

Since $T^{-1/2} \sum_{s=1}^{t} \varepsilon_s \xrightarrow{L} B(p)$ as $T \to \infty$,

$$X_t = cT^{1-\eta} \left(\frac{t}{T} \right) + T^{1/2} \left(T^{-1/2} \sum_{s=1}^{t} \varepsilon_s \right) \sim_a T^{1/2} B(p).$$

(2) For $t \in B$, the data generating process

$$X_t = \delta_T X_{t-1} + \varepsilon_t = \delta_T^{t-T_e} X_{T_e} + \sum_{j=0}^{t-T_e-1} \delta_T^j \varepsilon_{t-j}.$$

Based on Phillips and Magdalinos (2007, lemma 4.2), we know that for $\alpha < 1$,

$$T^{-\alpha/2} \sum_{j=0}^{t-T_e-1} \delta_T^{-(t-T_e)+j} \varepsilon_{t-j} \xrightarrow{L} X_{c_1} \equiv N\left(0, \sigma^2/2c_1\right)$$

as $t - T_e \to \infty$. Furthermore, we know that $T^{-1/2} X_{T_e} \xrightarrow{L} B(p)$. Therefore,

$$\begin{aligned} \delta_T^{-(t-T_e)} T^{-1/2} X_t &= T^{-1/2} X_{T_e} + T^{-1/2} \sum_{j=0}^{t-T_e-1} \delta_T^{-(t-T_e)+j} \varepsilon_{t-j} \\ &= T^{-1/2} X_{T_e} + T^{-(1-\alpha)/2} T^{-\alpha/2} \sum_{j=0}^{t-T_e-1} \delta_T^{-(t-T_e)+j} \varepsilon_{t-j} \xrightarrow{L} B(f_e). \end{aligned}$$

This implies that the first term has a higher order than the second term and hence

$$X_t = \delta_T^{t-T_e} X_{T_e} \left\{ 1 + \frac{\sum_{j=0}^{t-T_e-1} \delta_T^j \varepsilon_{t-j}}{\delta_T^{t-T_e} X_{T_e}} \right\} = \delta_T^{t-T_e} X_{T_e} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-T_e} B(f_e).$$

(3) For $t \in C$, the data generating process

$$X_t = \gamma_T X_{t-1} + \varepsilon_t = \gamma_T^{t-T_c} X_{T_c} + \sum_{j=0}^{t-T_c-1} \gamma_T^j \varepsilon_{t-j}.$$

Since

$$\begin{aligned} E\left[\left(T^{-\beta/2} \sum_{j=0}^{t-T_c-1} \gamma_T^j \varepsilon_{t-j} \right)^2 \right] &= T^{-\beta} \sum_{j=0}^{t-T_c-1} \gamma_T^{2j} E\left(\varepsilon_{t-j}^2\right) = \sigma^2 T^{-\beta} \frac{\gamma_T^{2(t-T_c)} - 1}{\gamma_T^2 - 1} \\ &= \sigma^2 \frac{e^{-2c_2(p-f_c)T^{1-\beta}} - 1}{-2c_2 + c_2^2 T^{-\beta}} \to \frac{\sigma^2}{2c_2}. \end{aligned}$$

we have

$$T^{-\beta/2}\sum_{j=0}^{t-T_c-1}\gamma_T^j\varepsilon_{t-j}\xrightarrow{L}X_{c_2}\equiv N\left(0,\sigma^2/2c_2\right).$$

Therefore,

$$X_t=\gamma_T^{t-T_c}X_{T_c}+\sum_{j=0}^{t-T_c-1}\gamma_T^j\varepsilon_{t-j}\sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c}B(f_e)+T^{\beta/2}X_{c_2}$$

due to the fact that

$$\begin{aligned}\gamma_T^{t-T_c}X_{T_c}&\sim{}_aT^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c}B(f_e),\\ \sum_{j=0}^{t-T_c-1}\gamma_T^j\varepsilon_{t-j}&\sim{}_aT^{\beta/2}X_{c_2}.\end{aligned}$$

(4) For $t\in N_1$, we have

$$\begin{aligned}X_t&=(t-T_r)\,cT^{-\eta}+X_{T_r}+\sum_{j=0}^{t-T_r-1}\varepsilon_{t-j}\\ &=\left(\frac{t-T_r}{T}\right)cT^{1-\eta}+X_{T_r}+T^{1/2}\left(T^{-1/2}\sum_{j=0}^{t-T_r-1}\varepsilon_{t-j}\right)\\ &=\gamma_T^{T_r-T_c}X_{T_c}+\sum_{j=0}^{T_r-T_c-1}\gamma_T^j\varepsilon_{T_r-j}+\sum_{j=0}^{t-T_r-1}\varepsilon_{t-j}\left\{1+o_p(1)\right\}\\ &=\begin{cases}\sum_{j=0}^{t-T_r-1}\varepsilon_{t-j}\left\{1+o_p(1)\right\}\sim_a T^{1/2}\left[B(p)-B(f_r)\right] & \text{if } \alpha>\beta\\ \gamma_T^{T_r-T_c}X_{T_c}\sim_a T^{1/2}\gamma_T^{T_r-T_c}\delta_T^{T_c-T_e}B(f_e) & \text{if } \alpha<\beta\end{cases}\end{aligned}$$

due the fact that $\eta>1/2$ and

$$\begin{aligned}\gamma_T^{T_r-T_c}X_{T_c}&\sim_a T^{1/2}\gamma_T^{T_r-T_c}\delta_T^{T_c-T_e}B(f_e)\\ \sum_{j=0}^{T_r-T_c-1}\gamma_T^j\varepsilon_{T_r-j}&\sim_a T^{\beta/2}X_{c_2}\\ \sum_{j=0}^{t-T_r-1}\varepsilon_{t-j}&\sim_a T^{1/2}\left[B(p)-B(f_r)\right].\end{aligned}$$

□

Lemma S.B2. *Under the data generating process,*

(1) *For $T_1\in N_0$ and $T_2\in B$,*

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j=\frac{T^{\alpha}\delta_T^{T_2-T_e}}{T_wc_1}X_{T_e}\left\{1+o_p(1)\right\}\sim_a T^{\alpha-1/2}\delta_T^{T_2-T_e}\frac{1}{f_wc_1}B(f_e).$$

(2) *For $T_1\in N_0$ and $T_2\in C$,*

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j=\begin{cases}\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\left\{1+o_p(1)\right\}\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e) & \text{if } \alpha>\beta\\ X_{T_c}\frac{T^{\beta}}{T_wc_2}\left\{1+o_p(1)\right\}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{B(f_e)}{f_wc_2} & \text{if } \alpha<\beta\end{cases}.$$

(3) *For $T_1\in N_0$ and $T_2\in N_1$,*

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j=\begin{cases}\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\left\{1+o_p(1)\right\}\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e) & \text{if } \alpha>\beta\\ \frac{T^{\beta}}{T_wc_2}X_{T_c}\left\{1+o_p(1)\right\}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_2}B(f_e) & \text{if } \alpha<\beta\end{cases}.$$

(4) *For $T_1\in B$ and $T_2\in C$,*

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j=\begin{cases}\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\left\{1+o_p(1)\right\}\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e) & \text{if } \alpha>\beta\\ X_{T_c}\frac{T^{\beta}}{T_wc_2}\left\{1+o_p(1)\right\}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_2}B(f_e) & \text{if } \alpha<\beta\end{cases}.$$

(5) *For* $T_1 \in B$ *and* $T_2 \in N_1$,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^{\alpha}\delta_T^{T_c-T_1}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e) & \text{if } \alpha>\beta \\ X_{T_c}\frac{T^{\beta}}{T_w c_2}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e) & \text{if } \alpha<\beta \end{cases}.$$

(6) *For* $T_1 \in C$ *and* $T_2 \in N_1$,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}\sum_{i=0}^{j-T_r-1}\varepsilon_{j-i}\left\{1+o_p(1)\right\} \sim_a T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds & \text{if } \alpha>\beta \\ X_{T_c}\frac{\gamma_T^{T_1-T_c}T^{\beta}}{T_w c_2}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{1}{c_2 f_w}B(f_e) & \text{if } \alpha<\beta \end{cases}.$$

Proof. (1) For $T_1 \in N_0$ and $T_2 \in B$, we have

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \frac{1}{T_w}\sum_{j=T_1}^{T_e-1} X_j + \frac{1}{T_w}\sum_{j=T_e}^{T_2} X_j.$$

The first term is

$$\begin{aligned} \frac{1}{T_w}\sum_{j=T_1}^{T_e-1} X_j &= T^{1/2}\frac{T_e-T_1}{T_w}\left(\frac{1}{T_e-T_1}\sum_{j=T_1}^{T_e-1}\frac{X_j}{\sqrt{T}}\right) \\ &\sim_a T^{1/2}\frac{f_e-f_1}{f_w}\int_{f_1}^{f_e} B(s)\,ds. \end{aligned} \tag{2.17}$$

The second term is

$$\begin{aligned} \frac{1}{T_w}\sum_{j=T_e}^{T_2} X_j &= \frac{X_{T_e}}{T_w}\sum_{j=T_e}^{T_2}\delta_T^{j-T_e}\left\{1+o_p(1)\right\} \text{ from Lemma S.B1} \\ &= \frac{X_{T_e}}{T_w}\frac{\delta_T^{T_2-T_e+1}-1}{\delta_T-1}\left\{1+o_p(1)\right\} \\ &= X_{T_e}\frac{T^{\alpha}\delta_T^{T_2-T_e}+c_1\delta_T^{T_2-T_e}-T^{\alpha}}{T_w c_1}\left\{1+o_p(1)\right\} \\ &= X_{T_e}\frac{T^{\alpha}\delta_T^{T_2-T_e}}{T_w c_1}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{T_2-T_e}\frac{B(f_e)}{f_w c_1}. \end{aligned} \tag{2.18}$$

Furthermore, we have

$$\frac{T^{\alpha-1/2}\delta_T^{T_2-T_e}}{T^{1/2}} = \frac{\delta_T^{T_2-T_e}}{T^{1-\alpha}} = \frac{e^{c(f_2-f_e)T^{1-\alpha}}}{T^{1-\alpha}} > 1.$$

This implies that $T_w\sum_{j=T_e}^{T_2} X_j$ has a higher order than $T_w\sum_{j=T_1}^{T_e-1} X_j$ and hence

$$\begin{aligned} \frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j &= \frac{1}{T_w}\sum_{j=T_e}^{T_2} X_j\left\{1+o_p(1)\right\} \\ &= \frac{T^{\alpha}\delta_T^{T_2-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} \text{ from equation (2.18)} \\ &\sim_a T^{\alpha-1/2}\delta_T^{T_2-T_e}\frac{1}{f_w c_1}B(f_e). \end{aligned}$$

(2) For $T_1 \in N_0$ and $T_2 \in C$, we have

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \frac{1}{T_w}\sum_{j=T_1}^{T_e-1} X_j + \frac{1}{T_w}\sum_{j=T_e}^{T_c} X_j + \frac{1}{T_w}\sum_{j=T_c+1}^{T_2} X_j.$$

The first term is

$$\frac{1}{T_w}\sum_{j=T_1}^{T_e-1} X_j \sim_a T^{1/2}\frac{f_e-f_1}{f_w}\int_{f_1}^{f_e} B(s)\,ds \text{ from equation (2.17)}$$

The second term is

$$\frac{1}{T_w}\sum_{j=T_e}^{T_c} X_j = \frac{X_{T_e}}{T_w}\sum_{j=T_e}^{T_c}\delta_T^{j-T_e}\left\{1+o_p(1)\right\} \text{ from Lemma S.B1}$$

$$= \frac{X_{T_e}}{T_w} \frac{\delta_T^{T_c - T_e + 1} - 1}{\delta_T - 1} \{1 + o_p(1)\}$$
$$= X_{T_e} \frac{T^\alpha \delta_T^{T_c - T_e} + c_1 \delta_T^{T_c - T_e} - T^\alpha}{T_w c_1} \{1 + o_p(1)\}$$
$$= X_{T_e} \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_1} B(f_e). \tag{2.19}$$

For the third term,

$$\frac{1}{T_w} \sum_{j=T_c+1}^{T_2} X_j = \frac{1}{T_w} \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{j-T_c} X_{T_c} + \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l} \right] \text{ from Lemma S.B1}$$
$$= \frac{1}{T_w} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-T_c} X_{T_c} + \frac{1}{T_w} \sum_{j=T_c+1}^{T_2} \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l}$$
$$= \frac{1}{T_w} \frac{T^\beta}{c_2} X_{T_c} + T^{1/2+\beta} \frac{(f_2 - f_c)^{1/2}}{T_w} \sqrt{\frac{c_2}{2}} \left[T^{-1/2-\beta} \sqrt{\frac{2}{c_2 (f_2 - f_c)}} \sum_{j=T_c+1}^{T_2} \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l} \right]$$
$$= \frac{T^\beta}{T_w c_2} X_{T_c} \sim_a T^{\beta - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e)$$

due to the fact that

$$\sum_{j=T_c+1}^{T_2} \gamma_T^{j-T_c} = \frac{\gamma_T \left(\gamma_T^{T_2 - T_c} - 1 \right)}{\gamma_T - 1} = \frac{T^\beta}{c_2}$$
$$T^{-1/2-\beta} \sqrt{\frac{2}{c_2 (f_2 - f_c)}} \sum_{j=T_c+1}^{T_2} \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l} \xrightarrow{L} X_{c_2} \text{ (see Lemma S.B7 for the proof)}$$

and

$$\frac{1}{T_w} \frac{T^\beta}{c_2} X_{T_c} \sim_a T^{\beta - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e)$$
$$T^{1/2+\beta} \frac{(f_2 - f_c)^{1/2}}{T_w} \sqrt{\frac{c_2}{2}} \left[T^{-1/2-\beta} \sqrt{\frac{2}{c_2 (f_2 - f_c)}} \sum_{j=T_c+1}^{T_2} \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l} \right] \sim_a T^{\beta - 1/2} \frac{(f_2 - f_c)^{1/2}}{f_w} \sqrt{\frac{c_2}{2}} X_{c_2}.$$

Therefore,

$$\frac{1}{T_w} \sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_1} B(f_e) & \text{if } \alpha > \beta \\ X_{T_c} \frac{T^\beta}{T_w c_2} \{1 + o_p(1)\} \sim_a T^{\beta - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e) & \text{if } \alpha < \beta \end{cases}$$

(3) For $T_1 \in N_0$ and $T_2 \in N_1$,

$$\frac{1}{T_w} \sum_{j=T_1}^{T_2} X_j = \frac{1}{T_w} \sum_{j=T_1}^{T_e - 1} X_j + \frac{1}{T_w} \sum_{j=T_e}^{T_c} X_j + \frac{1}{T_w} \sum_{j=T_c+1}^{T_r} X_j + \frac{1}{T_w} \sum_{j=T_r+1}^{T_2} X_j.$$

The first term is

$$\frac{1}{T_w} \sum_{j=T_1}^{T_e - 1} X_j \sim_a T^{1/2} \frac{f_e - f_1}{f_w} \int_{f_1}^{f_e} B(s)\, ds \text{ from equation (2.17).}$$

The second term is

$$\frac{1}{T_w} \sum_{j=T_e}^{T_c} X_j \sim_a T^{\alpha - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_1} B(f_e) \text{ from equation (2.19).}$$

For the third term,

$$\frac{1}{T_w} \sum_{j=T_c+1}^{T_r} X_j = \frac{T^\beta}{T_w c_2} X_{T_c} \sim_a T^{\beta - 1/2} \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e).$$

For the fourth term, if $\alpha > \beta$

$$\frac{1}{T_w} \sum_{j=T_r+1}^{T_2} X_j = \frac{1}{T_w} \sum_{j=T_r+1}^{T_2} \sum_{l=0}^{j-T_r-1} \varepsilon_{j-l} \{1 + o_p(1)\}$$

$$= T^{1/2}\frac{T_2-T_r}{T_w}\left[\frac{1}{T_2-T_r}\sum_{j=T_r+1}^{T_2}\left(T^{-1/2}\sum_{i=0}^{j-T_r-1}\varepsilon_{j-i}\right)\right]\{1+o_p(1)\}$$
$$\sim_a T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}[B(s)-B(f_r)]\,ds$$

if $\alpha<\beta$,

$$\frac{1}{T_w}\sum_{j=T_r+1}^{T_2}X_j = \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}\gamma_T^{T_r-T_c}X_{T_c}\{1+o_p(1)\}$$
$$= \frac{T_2-T_r}{T_w}\gamma_T^{T_r-T_c}X_{T_c}\{1+o_p(1)\}\sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{T_r-T_c}\frac{f_2-f_r}{f_w}B(f_e).$$

Therefore,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j = \begin{cases}\frac{T^\alpha\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\{1+o_p(1)\}\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e) & \text{if } \alpha>\beta\\ \frac{T^\beta}{T_wc_2}X_{T_c}\{1+o_p(1)\}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_2}B(f_e) & \text{if } \alpha<\beta\end{cases}.$$

(4) For $T_1\in B$ and $T_2\in C$, we have

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j = \frac{1}{T_w}\sum_{j=T_1}^{T_c}X_j + \frac{1}{T_w}\sum_{j=T_c+1}^{T_2}X_j.$$

The first term is

$$\frac{1}{T_w}\sum_{j=T_1}^{T_c}X_j = \frac{X_{T_e}}{T_w}\sum_{j=T_1}^{T_c}\delta_T^{j-T_e}\{1+o_p(1)\}\text{ from Lemma S.B1}$$
$$= \frac{X_{T_e}}{T_w}\frac{\delta_T^{T_1-T_e}\left(\delta_T^{T_c-T_1+1}-1\right)}{\delta_T-1}\{1+o_p(1)\}$$
$$= \frac{X_{T_e}}{T_w}\frac{T^\alpha\delta_T^{T_c-T_e}+c_1\delta_T^{T_c-T_e}-T^\alpha\delta_T^{T_1-T_e}}{c_1}\{1+o_p(1)\}$$
$$= \frac{T^\alpha\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\{1+o_p(1)\}$$
$$\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e). \tag{2.20}$$

For the second term,

$$\frac{1}{T_w}\sum_{j=T_c+1}^{T_2}X_j = \frac{T^\beta}{T_wc_2}X_{T_c}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_2}B(f_e).$$

Therefore,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j = \begin{cases}\frac{T^\alpha\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\{1+o_p(1)\}\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e) & \text{if } \alpha>\beta\\ \frac{T^\beta}{T_wc_2}X_{T_c}\{1+o_p(1)\}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_2}B(f_e) & \text{if } \alpha<\beta\end{cases}.$$

(5) For $T_1\in B$ and $T_2\in N_1$, we have

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j = \frac{1}{T_w}\sum_{j=T_1}^{T_c}X_j + \frac{1}{T_w}\sum_{j=T_c+1}^{T_r}X_j + \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}X_j.$$

The first term is

$$\frac{1}{T_w}\sum_{j=T_1}^{T_c}X_j = \frac{T^\alpha\delta_T^{T_c-T_e}}{T_wc_1}X_{T_e}\{1+o_p(1)\}\sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_wc_1}B(f_e)\text{ from equation (2.20)}$$

For the second term,

$$\frac{1}{T_w}\sum_{j=T_c+1}^{T_r}X_j = \frac{T^\beta}{T_wc_2}X_{T_c}\{1+o_p(1)\}\sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{B(f_e)}{f_wc_2}$$

The third term is

$$\frac{1}{T_w}\sum_{j=T_r+1}^{T_2}X_j\sim_a\begin{cases}T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}[B(s)-B(f_r)]\,ds & \text{if } \alpha>\beta\\ T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{T_r-T_c}\frac{f_2-f_r}{f_w}B(f_e) & \text{if } \alpha<\beta\end{cases}$$

Therefore,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e) & \text{if } \alpha>\beta \\ \frac{T^\beta}{T_w c_2} X_{T_c}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e) & \text{if } \alpha<\beta \end{cases}.$$

(6) For $T_1 \in C$ and $T_2 \in N_1$,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \frac{1}{T_w}\sum_{j=T_1}^{T_r} X_j + \frac{1}{T_w}\sum_{j=T_r+1}^{T_2} X_j.$$

For the first term,

$$\begin{aligned} \frac{1}{T_w}\sum_{j=T_1}^{T_r} X_j &= \frac{1}{T_w}\sum_{j=T_1}^{T_r}\left[\gamma_T^{j-T_c}X_{T_c} + \sum_{l=0}^{j-T_c-1}\gamma_T^l \varepsilon_{j-l}\right] \text{ from Lemma S.B1} \\ &= \frac{1}{T_w}X_{T_c}\sum_{j=T_1}^{T_r}\gamma_T^{j-T_c} + \frac{1}{T_w}\sum_{j=T_1}^{T_r}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l} \\ &= X_{T_c}\frac{T^\beta\gamma_T^{T_1-T_c}}{T_w c_2} + T^{1/2+\beta}\frac{1}{T_w}\sqrt{\frac{c_2(f_r-f_1)}{2}}\left[T^{-1/2-\beta}\sqrt{\frac{2}{c_2(f_r-f_1)}}\sum_{j=T_1}^{T_r}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right] \\ &\sim_a \begin{cases} T^{\beta-1/2}\frac{1}{f_w}\sqrt{\frac{c_2(f_r-f_1)}{2}}X_{c_2} & \text{if } \alpha>\beta \\ T^{\beta-1/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{1}{c_2 f_w}B(f_e) & \text{if } \alpha<\beta \end{cases} \end{aligned}$$

due to the fact that

$$X_{T_c}\frac{T^\beta\gamma_T^{T_1-T_c}}{T_w c_2} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{1}{c_2 f_w}B(f_e)$$

$$T^{1/2+\beta}\frac{1}{T_w}\sqrt{\frac{c_2(f_r-f_1)}{2}}\left[T^{-1/2-\beta}\sqrt{\frac{2}{c_2(f_r-f_1)}}\sum_{j=T_1}^{T_r}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right] \sim_a T^{\beta-1/2}\frac{1}{f_w}\sqrt{\frac{c_2(f_r-f_1)}{2}}X_{c_2}.$$

The second term is

$$\frac{1}{T_w}\sum_{j=T_r+1}^{T_2} X_j \sim_a \begin{cases} T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds & \text{if } \alpha>\beta \\ T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{T_r-T_c}\frac{f_2-f_r}{f_w}B(f_e) & \text{if } \alpha<\beta \end{cases}$$

Therefore,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}\sum_{i=0}^{j-T_r-1}\varepsilon_{j-i}\left\{1+o_p(1)\right\} \sim_a T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds & \text{if } \alpha>\beta \\ X_{T_c}\frac{\gamma_T^{T_1-T_c}T^\beta}{T_w c_2}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{1}{c_2 f_w}B(f_e) & \text{if } \alpha<\beta \end{cases}.$$

□

Lemma S.B3. *Suppose the centered quantity* $\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j$.

(1) *For* $T_1 \in N_0$ *and* $T_2 \in B$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha\delta_T^{T_2-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t\in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^\alpha\delta_T^{T_2-T_e}}{T_w c_1}\right]X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t\in B \end{cases}.$$

(2) *For* $T_1 \in N_0$ *and* $T_2 \in C$, *if* $\alpha > \beta$

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t\in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^\alpha\delta_T^{T_c-T_e}}{T_w c_1}\right]X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t\in B \\ \left[\gamma_T^{t-T_c}X_{T_c} - \frac{T^\alpha\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t\in C \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\beta}{T_w c_2}X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t\in N_0 \\ \left[\delta_T^{t-T_e}X_{T_e} - \frac{T^\beta}{T_w c_2}X_{T_c}\right]\left\{1+o_p(1)\right\} & \text{if } t\in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right]X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t\in C \end{cases}.$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$, if $\alpha > \beta$*

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \left\{1 + o_p(1)\right\} & if\, t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1}\right] X_{T_e} \left\{1 + o_p(1)\right\} & if\, t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e}\right] \left\{1 + o_p(1)\right\} & if\, t \in C \\ -\frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \left\{1 + o_p(1)\right\} & if\, t \in N_1 \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\beta}{T_w c_2} X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in N_0 \\ \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right] \left\{1 + o_p(1)\right\} & if\, t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in C \\ -\frac{T^\beta}{T_w c_2} X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in N_1 \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$, if $\alpha > \beta$*

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1}\right] X_{T_e} \left\{1 + o_p(1)\right\} & if\, t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e}\right] \left\{1 + o_p(1)\right\} & if\, t \in C \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} X_{T_e} - X_{T_c} \frac{T^\beta}{T_w c_2}\right] \left\{1 + o_p(1)\right\} & if\, t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in C \end{cases}$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$, if $\alpha > \beta$*

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1}\right] X_{T_e} \left\{1 + o_p(1)\right\} & if\, t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e}\right] \left\{1 + o_p(1)\right\} & if\, t \in C \\ -\frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \left\{1 + o_p(1)\right\} & if\, t \in N_1 \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right] \left\{1 + o_p(1)\right\} & if\, t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in C \\ -\frac{T^\beta}{T_w c_2} X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in N_1 \end{cases}$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$, if $\alpha > \beta$,*

$$\tilde{X}_t = \begin{cases} \left[\gamma_T^{t-T_c} X_{T_c} - \frac{1}{T_w} \sum_{j=T_r+1}^{T_2} \sum_{i=0}^{j-T_r-1} \varepsilon_{j-i}\right] \left\{1 + o_p(1)\right\} & if\, t \in C \\ \left[\sum_{j=0}^{t-T_r-1} \varepsilon_{t-j} - \frac{1}{T_w} \sum_{j=T_r+1}^{T_2} \sum_{i=0}^{j-T_r-1} \varepsilon_{j-i}\right] \left\{1 + o_p(1)\right\} & if\, t \in N_1 \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} \left[\gamma_T^{t-T_c} - \frac{\gamma_T^{T_1 - T_c} T^\beta}{T_w c_2}\right] X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in C \\ -\frac{\gamma_T^{T_1 - T_c} T^\beta}{T_w c_2} X_{T_c} \left\{1 + o_p(1)\right\} & if\, t \in N_1 \end{cases}.$$

Proof. (1) Suppose $T_1 \in N_0$ and $T_2 \in B$. We have

$$\tilde{X}_t = X_t - T_w^{-1} \sum_{j=T_1}^{T_2} X_j = \begin{cases} -\frac{T^\alpha \delta_T^{T_2 - T_e}}{T_w c_1} X_{T_e} \left\{1 + o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_2 - T_e}}{T_w c_1}\right] X_{T_e} \left\{1 + o_p(1)\right\} & \text{if } t \in B \end{cases}, \tag{2.21}$$

This is due to the fact that

$$X_t \sim_a T^{1/2} B(p) \text{ if } t \in N_0$$

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j \sim_a T^{\alpha-1/2}\delta_T^{T_2-T_e}\frac{1}{f_w c_1}B(f_e).$$

(2) Suppose $T_1 \in N_0$, $T_2 \in C$ and $\alpha > \beta$. We have

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} -\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1}\right] X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \end{cases}. \tag{2.22}$$

Suppose $T_1 \in N_0$, $T_2 \in C$ and $\alpha < \beta$. We have

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} -\frac{T^\beta}{T_w c_2} X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right]\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in C \end{cases}. \tag{2.23}$$

Those are due to the fact that

$$\begin{aligned} & X_t \sim_a T^{1/2}B(p) \text{ if } t \in N_0 \\ & X_t \sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c}B(f_e) + T^{\beta/2}X_{c_2} \text{ if } t \in C \\ & \frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j \sim_a \begin{cases} T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e) & \text{if } \alpha > \beta \\ T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e) & \text{if } \alpha < \beta \end{cases}. \end{aligned}$$

(3) Suppose $T_1 \in N_0$, $T_2 \in N_1$ and $\alpha > \beta$.

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} -\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1}\right] X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \\ -\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases};$$

Suppose $T_1 \in N_0$, $T_2 \in N_1$ and $\alpha < \beta$.

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} -\frac{T^\beta}{T_w c_2} X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right]\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in C \\ -\frac{T^\beta}{T_w c_2} X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases}.$$

Those are due to the fact that

$$\begin{aligned} & X_t \sim_a T^{1/2}B(p) \text{ if } t \in N_0 \\ & X_t \sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c}B(f_e) + T^{\beta/2}X_{c_2} \text{ if } t \in C \\ & X_t \sim_a \begin{cases} T^{1/2}\left[B(p) - B(f_r)\right] & \text{if } \alpha > \beta \\ T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{T_r-T_c}B(f_e) & \text{if } \alpha < \beta \end{cases} \text{ if } t \in N_1 \\ & \frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j \sim_a \begin{cases} T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e) & \text{if } \alpha > \beta \\ T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e) & \text{if } \alpha < \beta \end{cases}. \end{aligned}$$

(4) Suppose $T_1 \in B$, $T_2 \in C$ and $\alpha > \beta$. We have

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1}\right] X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \end{cases}$$

Suppose $T_1 \in B$, $T_2 \in C$ and $\alpha < \beta$. We have

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right]\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in C \end{cases}$$

Those are due to the fact that

$$X_t \sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c} B(f_e) + T^{\beta/2} X_{c_2} \text{ if } t \in C$$

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j \sim_a \begin{cases} T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e) & \text{if } \alpha > \beta \\ T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2} B(f_e) & \text{if } \alpha < \beta \end{cases}.$$

(5) Suppose $T_1 \in B$, $T_2 \in N_1$ and $\alpha > \beta$. We have

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1}\right] X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \\ -\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases}.$$

Suppose $T_1 \in B$, $T_2 \in N_1$ and $\alpha < \beta$. We have

$$\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right]\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in C \\ -\frac{T^\beta}{T_w c_2} X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases}.$$

Those are due to the fact that

$$X_t \sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c} B(f_e) + T^{\beta/2} X_{c_2} \text{ if } t \in C$$

$$X_t \sim_a \begin{cases} T^{1/2}\left[B(p) - B(f_r)\right] & \text{if } \alpha > \beta \\ T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{T_r-T_c} B(f_e) & \text{if } \alpha < \beta \end{cases} \text{ if } t \in N_1$$

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j \sim_a \begin{cases} T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e) & \text{if } \alpha > \beta \\ T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2} B(f_e) & \text{if } \alpha < \beta \end{cases}.$$

(6) Suppose $T_1 \in C$, $T_2 \in N_1$ and $\alpha > \beta$. We have

$$\begin{aligned} \tilde{X}_t &= X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j \\ &= \begin{cases} \left[\gamma_T^{t-T_c} X_{T_c} - \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}\sum_{i=0}^{j-T_r-1}\varepsilon_{t-i}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \\ \left[\sum_{j=0}^{t-T_r-1}\varepsilon_{t-j} - \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}\sum_{i=0}^{j-T_r-1}\varepsilon_{j-i}\right]\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases}. \end{aligned}$$

Suppose $T_1 \in C$, $T_2 \in N_1$ and $\alpha < \beta$. We have

$$\begin{aligned} \tilde{X}_t &= X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j \\ &= \begin{cases} \left[\gamma_T^{t-T_c} - \frac{\gamma_T^{T_1-T_c} T^\beta}{T_w c_2}\right] X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in C \\ -\frac{\gamma_T^{T_1-T_c} T^\beta}{T_w c_2} X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases} \end{aligned}$$

Those are due to the fact that

$$X_t \sim_a T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{t-T_c} B(f_e) + T^{\beta/2} X_{c_2} \text{ if } t \in C$$

$$X_t \sim_a \begin{cases} T^{1/2}\left[B(p) - B(f_r)\right] & \text{if } \alpha > \beta \\ T^{1/2}\delta_T^{T_c-T_e}\gamma_T^{T_r-T_c} B(f_e) & \text{if } \alpha < \beta \end{cases} \text{ if } t \in N_1$$

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j \sim_a \begin{cases} T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right] ds & \text{if } \alpha > \beta \\ T^{\beta-1/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{1}{c_2 f_w} B(f_e) & \text{if } \alpha < \beta \end{cases}.$$

□

Lemma S.B4. *The sum of squared $\tilde{X}_t$ terms are as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a T^{1+\alpha} \delta_T^{2(T_2-T_e)} \frac{1}{2c_1} B(f_e)^2 .$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha \leq \beta \end{cases} .$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^2 (f_2 - f_r) \left\{ \int_{f_r}^{f_2} [B(s) - B(f_r)]^2 ds - \frac{f_2 - f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s) - B(f_r)] ds \right]^2 \right\} & \text{if } \alpha > \beta \\ T^{1+\beta} \delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

Proof. (1) For $T_1 \in N_0$ and $T_2 \in B$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_e} \tilde{X}_{j-1}^2 + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}^2.$$

The first term is

$$\begin{aligned} \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 &= \sum_{j=T_1}^{T_e-1} \frac{T^{2\alpha} \delta_T^{2(T_2-T_e)}}{T_w^2 c_1^2} X_{T_e}^2 \left\{1 + o_p(1)\right\} \text{ from Lemma S.B3} \\ &= \frac{T_e - T_1}{T_w^2 c_1^2} T^{2\alpha} \delta_T^{2(T_2-T_e)} X_{T_e}^2 \left\{1 + o_p(1)\right\} \\ &\sim_a \frac{f_e - f_1}{f_w^2 c_1} T^{2\alpha} \delta_T^{2(T_2-T_e)} B(f_e)^2 . \end{aligned}$$

Given that

$$\begin{aligned} \sum_{j=T_e}^{T_2} \delta_T^{2(j-1-T_e)} &= \frac{\delta_T^{2(T_2-T_e)} - \delta_T^{-2}}{\delta_T^2 - 1} = \frac{T^\alpha \delta_T^{2(T_2-T_e)}}{2c_1} \left\{1 + o_p(1)\right\} \\ \sum_{j=T_e}^{T_2} \delta_T^{j-1-T_e} &= \frac{\delta_T^{T_2-T_e} - \delta_T^{-1}}{\delta_T - 1} = \frac{T^\alpha \delta_T^{T_2-T_e}}{c_1} \left\{1 + o_p(1)\right\}, \end{aligned}$$

the second term is

$$\begin{aligned} &\sum_{j=T_e}^{T_2} \tilde{X}_{j-1}^2 \\ &= \sum_{j=T_e}^{T_2} \left[\delta_T^{j-1-T_e} - \frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1} \right]^2 X_{T_e}^2 \left\{1 + o_p(1)\right\} \\ &= \sum_{j=\tau e}^{T_2} \left[\delta_T^{2(j-1-T_e)} - 2\delta_T^{j-1-T_e} \frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1} + \frac{T^{2\alpha} \delta_T^{2(T_2-T_e)}}{T_w c_1^2} \right] X_{T_e}^2 \left\{1 + o_p(1)\right\} \end{aligned}$$

$$= \left[\frac{T^{\alpha}\delta_T^{2(T_2-T_e)}}{2c_1} - 2\frac{T^{2\alpha-1}\delta_T^{2(T_2-T_e)}}{f_w c_1^2} + \frac{f_2 - f_e + \frac{1}{T}}{f_w c_1^2} T^{2\alpha-1}\delta_T^{2(T_2-T_e)} \right] X_{T_e}^2 \left\{1 + o_p(1)\right\}$$
$$= \frac{T^{\alpha}\delta_T^{2(T_2-T_e)}}{2c_1} X_{T_e}^2 \left\{1 + o_p(1)\right\} \text{ (since } \alpha > 2\alpha - 1)$$
$$\sim_a \frac{T^{1+\alpha}\delta_T^{2(T_2-T_e)}}{2c_1} B(f_e)^2 .$$

Since $1+\alpha > 2\alpha$, the quantity $\sum_{j=T_e}^{T_2} \tilde{X}_{j-1}^2$ dominates $\sum_{j=T_1}^{T_e} \tilde{X}_{j-1}^2$. Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}^2 \left\{1 + o_p(1)\right\} = \frac{T^{\alpha}\delta_T^{2(T_2-T_e)}}{2c_1} X_{T_e}^2 \left\{1 + o_p(1)\right\}$$
$$\sim_a \frac{T^{1+\alpha}\delta_T^{2(T_2-T_e)}}{2c_1} B(f_e)^2 .$$

(2) For $T_1 \in N_0$, $T_2 \in C$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_e} \tilde{X}_{j-1}^2 + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2 + \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 .$$

Suppose $\alpha > \beta$. The first term is

$$\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_e-1} \frac{T^{2\alpha}\delta_T^{2(T_c-T_e)}}{T_w^2 c_1^2} X_{T_e}^2 \left\{1 + o_p(1)\right\} \text{ from Lemma S.B3}$$
$$= \frac{T_e - T_1}{T_w^2 c_1^2} T^{2\alpha}\delta_T^{2(T_c-T_e)} X_{T_e}^2 \left\{1 + o_p(1)\right\}$$
$$\sim_a T^{2\alpha}\delta_T^{2(T_c-T_e)} \frac{f_e - f_1}{f_w^2 c_1} B(f_e)^2 . \tag{2.24}$$

The second term is

$$\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2$$
$$= \sum_{j=T_e}^{T_c} \left[\delta_T^{j-1-T_e} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} \right]^2 X_{T_e}^2 \left\{1 + o_p(1)\right\}$$
$$= \sum_{j=T_e}^{T_c} \left[\delta_T^{2(j-1-T_e)} - 2\delta_T^{j-1-T_e} \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} + \frac{T^{2\alpha}\delta_T^{2(T_c-T_e)}}{T_w c_1^2} \right] X_{T_e}^2 \left\{1 + o_p(1)\right\}$$
$$= \left[\frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1} - 2\frac{T^{2\alpha-1}\delta_T^{2(T_c-T_e)}}{f_w c_1^2} + \frac{f_c - f_e + \frac{1}{T}}{f_w c_1^2} T^{2\alpha-1}\delta_T^{2(T_c-T_e)} \right] X_{T_e}^2 \left\{1 + o_p(1)\right\}$$
$$= \frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1 + o_p(1)\right\} \text{ (since } \alpha > 2\alpha - 1)$$
$$\sim_a \frac{T^{1+\alpha}\delta_T^{2(T_c-T_e)}}{2c_1} B(f_e)^2 \tag{2.25}$$

due to the fact that

$$\sum_{j=T_e}^{T_c} \delta_T^{2(j-1-T_e)} = \frac{\delta_T^{2(T_c-T_e)} - \delta_T^{-2}}{\delta_T^2 - 1} = \frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1} \left\{1 + o_p(1)\right\}$$
$$\sum_{j=T_e}^{T_c} \delta_T^{j-1-T_e} = \frac{\delta_T^{T_c-T_e} - \delta_T^{-1}}{\delta_T - 1} = \frac{T^{\alpha}\delta_T^{T_c-T_e}}{c_1} \left\{1 + o_p(1)\right\} .$$

The third term is

$$\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2$$

$$
\begin{aligned}
&= \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{j-1-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \right]^2 \{1+o_p(1)\} \\
&= \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{2(j-1-T_c)} X_{T_c}^2 - 2\gamma_T^{j-1-T_c} \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} X_{T_c} + \frac{T^{2\alpha} \delta_T^{2(T_c-T_e)}}{T_w c_1^2} X_{T_e}^2 \right] \{1+o_p(1)\} \\
&= \left[X_{T_c}^2 \sum_{j=T_c+1}^{T_2} \gamma_T^{2(j-1-T_c)} - 2\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} X_{T_c} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c} + T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{T_2-T_c}{T_w c_1^2} X_{T_e}^2 \right] \{1+o_p(1)\} \\
&= \left[X_{T_c}^2 \frac{T^\beta}{2c_2} - 2\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} X_{T_c} \frac{T^\beta}{c_2} + T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{T_2-T_c}{T_w c_1^2} X_{T_e}^2 \right] \{1+o_p(1)\} \\
&\sim_a \begin{cases} \frac{T_2-T_c}{T_w c_1^2} T^{2\alpha} \delta_T^{2(T_c-T_e)} X_{T_e}^2 \{1+o_p(1)\} \sim_a T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \beta < 2\alpha - 1 \\ X_{T_c}^2 \frac{T^\beta}{2c_2} \{1+o_p(1)\} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } 2\alpha - 1 < \beta < \alpha \end{cases}
\end{aligned}
$$

This is due to the fact that

$$
\sum_{j=T_c+1}^{T_2} \gamma_T^{2(j-1-T_c)} = \frac{\gamma_T^{2(T_2-T_c)} - 1}{\gamma_T^2 - 1} = \frac{T^\beta}{2c_2}
$$

$$
\sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c} = \frac{\gamma_T^{T_2-T_c} - 1}{\gamma_T - 1} = \frac{T^\beta}{c_2}
$$

and

$$
X_{T_c}^2 \frac{T^\beta}{2c_2} \sim {}_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2
$$

$$
2\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} X_{T_c} \frac{T^\beta}{c_2} \sim {}_a T^{\alpha+\beta} \delta_T^{2(T_c-T_e)} \frac{2}{f_w c_1 c_2} B(f_e)^2
$$

$$
T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{T_2-T_c}{T_w c_1^2} X_{T_e}^2 \sim {}_a T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2
$$

Therefore,

$$
\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \{1+o_p(1)\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(T_c-T_e)}}{2c_1} B(f_e)^2 .
$$

Suppose $\alpha < \beta$. The first term is

$$
\begin{aligned}
\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 &= \sum_{j=T_1}^{T_e-1} \frac{T^{2\beta}}{T_w^2 c_2^2} X_{T_c}^2 \{1+o_p(1)\} \text{ from Lemma S.B3} \\
&= \frac{T_e - T_1}{T_w^2 c_2^2} T^{2\beta} X_{T_c}^2 \{1+o_p(1)\} \\
&\sim_a \frac{f_e - f_1}{f_w^2 c_2^2} T^{2\beta} \delta_T^{2(T_c-T_e)} B(f_e)^2 .
\end{aligned} \tag{2.26}
$$

The second term is

$$
\begin{aligned}
&\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2 \\
&= \sum_{j=T_e}^{T_c} \left[\delta_T^{j-1-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c} \right]^2 \{1+o_p(1)\} \\
&= \sum_{j=T_e}^{T_c} \left[\delta_T^{2(j-1-T_e)} X_{T_e}^2 - 2\delta_T^{j-1-T_e} \frac{T^\beta}{T_w c_2} X_{T_e} X_{T_c} + \frac{T^{2\beta}}{T_w c_2^2} X_{T_c}^2 \right] \{1+o_p(1)\} \\
&= \left[\frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 - 2\frac{T^{\alpha+\beta-1} \delta_T^{T_c-T_e}}{f_w c_1 c_2} X_{T_e} X_{T_c} + T^{2\beta-1} \frac{f_c - f_e}{f_w c_2^2} X_{T_c}^2 \right] \{1+o_p(1)\} \\
&= \begin{cases} \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \{1+o_p(1)\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(T_c-T_e)}}{2c_1} B(f_e)^2 & \text{if } 1+\alpha > 2\beta \\ T^{2\beta-1} \frac{f_c-f_e}{f_w c_2^2} X_{T_c}^2 \sim_a T^{2\beta} \delta_T^{2(T_c-T_e)} \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } 1+\alpha < 2\beta \end{cases}
\end{aligned} \tag{2.27}
$$

due to the fact that

$$\sum_{j=T_e}^{T_c} \delta_T^{2(j-1-T_e)} = \frac{\delta_T^{2(T_c-T_e)} - \delta_T^{-2}}{\delta_T^2 - 1} = \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} \left\{1 + o_p(1)\right\}$$

$$\sum_{j=T_e}^{T_c} \delta_T^{j-1-T_e} = \frac{\delta_T^{T_c-T_e} - \delta_T^{-1}}{\delta_T - 1} = \frac{T^\alpha \delta_T^{T_c-T_e}}{c_1} \left\{1 + o_p(1)\right\},$$

and

$$\frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \sim_a T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2$$

$$2\frac{T^{\alpha+\beta-1} \delta_T^{T_c-T_e}}{f_w c_1 c_2} X_{T_e} X_{T_c} \sim_a T^{\alpha+\beta} \delta_T^{2(T_c-T_e)} \frac{2}{f_w c_1 c_2} B(f_e)^2$$

$$T^{2\beta-1} \frac{f_c - f_e}{f_w c_2^2} X_{T_c}^2 \sim_a T^{2\beta} \delta_T^{2(T_c-T_e)} \frac{f_c - f_e}{f_w c_2^2} B(f_e)^2.$$

The third term is

$$\begin{aligned}
\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 &= \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{j-1-T_c} - \frac{T^\beta}{T_w c_2}\right]^2 X_{T_c}^2 \left\{1 + o_p(1)\right\} \\
&= \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{2(j-1-T_c)} - 2\frac{T^\beta}{T_w c_2} \gamma_T^{j-1-T_c} + \frac{T^{2\beta}}{T_w c_2^2}\right] X_{T_c}^2 \left\{1 + o_p(1)\right\} \\
&= \left[\sum_{j=T_c+1}^{T_2} \gamma_T^{2(j-1-T_c)} - 2\frac{T^\beta}{T_w c_2} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c} + T^{2\beta} \frac{T_2 - T_c}{T_w c_2^2}\right] X_{T_c}^2 \left\{1 + o_p(1)\right\} \\
&= \left[\frac{T^\beta}{2c_2} - 2\frac{T^{2\beta-1}}{f_w c_2^2} + T^{2\beta-1} \frac{f_2 - f_c}{f_w c_2^2}\right] X_{T_c}^2 \left\{1 + o_p(1)\right\} \\
&= \frac{T^\beta}{2c_2} X_{T_c}^2 \left\{1 + o_p(1)\right\} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2.
\end{aligned}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(T_c-T_e)}}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ \frac{T^\beta}{2c_2} X_{T_c}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{1+\beta}}{2c_2} \delta_T^{2(T_c-T_e)} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(3) For $T_1 \in N_0$, $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_e} \tilde{X}_{j-1}^2 + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2 + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2.$$

Suppose $\alpha > \beta$. The first term is

$$\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 \sim_a T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{f_e - f_1}{f_w^2 c_1} B(f_e)^2 \text{ from equation (2.24).}$$

The second term is

$$\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1 + o_p(1)\right\} \sim_a T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 \text{ from equation (2.25).}$$

The third term is

$$\begin{aligned}
\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 &= \sum_{j=T_c+1}^{T_r} \left[\gamma_T^{j-1-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\right]^2 \left\{1 + o_p(1)\right\} \\
&\sim_a \begin{cases} \frac{T_r - T_c}{T_w c_1^2} T^{2\alpha} \delta_T^{2(T_c-T_e)} X_{T_e}^2 \left\{1 + o_p(1)\right\} \sim_a T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{f_r - f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \beta < 2\alpha - 1 \\ X_{T_c}^2 \frac{T^\beta}{2c_2} \left\{1 + o_p(1)\right\} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } 2\alpha - 1 < \beta < \alpha \end{cases}.
\end{aligned}$$

The fourth term is

$$\begin{aligned}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2 &= \sum_{j=T_r+1}^{T_2}\frac{T^{2\alpha}\delta_T^{2(T_c-T_e)}}{T_w^2c_1^2}X_{T_e}^2\left\{1+o_p(1)\right\} \text{ from Lemma S.B3}\\ &= \frac{T_2-T_r}{T_w^2c_1^2}T^{2\alpha}\delta_T^{2(T_c-T_e)}X_{T_e}^2\left\{1+o_p(1)\right\}\\ &\sim_a T^{2\alpha}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w^2c_1^2}B(f_e)^2.\end{aligned}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2 = \sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2\left\{1+o_p(1)\right\} = \frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\left\{1+o_p(1)\right\} \sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2.$$

Suppose $\alpha<\beta$. The first term is

$$\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 = \frac{T_e-T_1}{T_w^2c_2^2}T^{2\beta}X_{T_c}^2\left\{1+o_p(1)\right\} \sim_a T^{2\beta}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1}{f_w^2c_2^2}B(f_e)^2.$$

The second term is

$$\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2 = \begin{cases}\frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\left\{1+o_p(1)\right\} \sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } 1+\alpha>2\beta\\ T^{2\beta-1}\frac{f_c-f_e}{f_wc_2^2}X_{T_c}^2\left\{1+o_p(1)\right\} \sim_a T^{2\beta}\delta_T^{2(T_c-T_e)}\frac{f_c-f_e}{f_wc_2^2}B(f_e)^2 & \text{if } 1+\alpha<2\beta\end{cases}.$$

The third term is

$$\begin{aligned}\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2 &= \sum_{j=T_c+1}^{T_r}\left[\gamma_T^{j-1-T_c}-\frac{T^{\beta}}{T_wc_2}\right]^2X_{T_c}^2\left\{1+o_p(1)\right\}\\ &= \frac{T^{\beta}}{2c_2}X_{T_c}^2\left\{1+o_p(1)\right\} \sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2.\end{aligned}$$

The fourth term is

$$\begin{aligned}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2 &= \sum_{j=T_r+1}^{T_2}\frac{T^{2\beta}}{T_w^2c_2^2}X_{T_c}^2\left\{1+o_p(1)\right\} \text{ from Lemma S.B3}\\ &= \frac{T_2-T_r}{T_w^2c_2^2}T^{2\beta}X_{T_c}^2\left\{1+o_p(1)\right\} \sim_a T^{2\beta}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w^2c_2^2}B(f_e)^2.\end{aligned}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2 = \begin{cases}\frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\left\{1+o_p(1)\right\} \sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta\\ \frac{T^{\beta}}{2c_2}X_{T_c}^2\left\{1+o_p(1)\right\} \sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha\le\beta\end{cases}.$$

(4) For $T_1\in B$ and $T_2\in C$,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2 + \sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2.$$

Suppose $\alpha>\beta$. The first term is

$$\begin{aligned}&\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2\\ &= \sum_{j=T_1}^{T_c}\left[\delta_T^{j-1-T_e}-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_wc_1}\right]^2X_{T_e}^2\left\{1+o_p(1)\right\}\\ &= \sum_{j=T_1}^{T_c}\left[\delta_T^{2(j-1-T_e)}-2\delta_T^{j-1-T_e}\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_wc_1}+\frac{T^{2\alpha}\delta_T^{2(T_c-T_e)}}{T_wc_1^2}\right]X_{T_e}^2\left\{1+o_p(1)\right\}\\ &= \left[\frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1}-2\frac{T^{2\alpha-1}\delta_T^{2(T_c-T_e)}}{f_wc_1^2}+\frac{f_c-f_1}{f_wc_1^2}T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right]X_{T_e}^2\left\{1+o_p(1)\right\}\\ &= \frac{T^{\alpha}\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\left\{1+o_p(1)\right\} \text{ (since } \alpha>2\alpha-1)\end{aligned}$$

$$\sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2$$

due to the fact that

$$\sum_{j=T_1}^{T_c}\delta_T^{2(j-1-T_e)} = \frac{\delta_T^{2(T_1-1-T_e)}\left(\delta_T^{2(T_c-T_1+1)}-1\right)}{\delta_T^2-1} = \frac{T^\alpha\delta_T^{2(T_c-T_e)}}{2c_1}\{1+o_p(1)\}$$

$$\sum_{j=T_1}^{T_c}\delta_T^{j-1-T_e} = \frac{\delta_T^{T_1-1-T_e}\left(\delta_T^{T_c-T_1+1}-1\right)}{\delta_T-1} = \frac{T^\alpha\delta_T^{T_c-T_e}}{c_1}\{1+o_p(1)\},$$

The second term is

$$\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2 = \sum_{j=T_c+1}^{T_2}\left[\gamma_T^{j-1-T_c}X_{T_c} - \frac{T^\alpha\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\right]^2\{1+o_p(1)\}$$

$$\sim_a \begin{cases} \frac{T_2-T_c}{T_w c_1^2}T^{2\alpha}\delta_T^{2(T_c-T_e)}X_{T_e}^2\{1+o_p(1)\} \sim_a T^{2\alpha}\delta_T^{2(T_c-T_e)}\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \beta<2\alpha-1 \\ X_{T_c}^2\frac{T^\beta}{2c_2}\{1+o_p(1)\} \sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } 2\alpha-1<\beta<\alpha \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2 = \frac{T^\alpha\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\{1+o_p(1)\} \sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2.$$

Suppose $\alpha<\beta$. The first term is

$$\begin{aligned}
&\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2 \\
&= \sum_{j=T_1}^{T_c}\left[\delta_T^{j-1-T_e}X_{T_e} - \frac{T^\beta}{T_w c_2}X_{T_c}\right]^2\{1+o_p(1)\} \\
&= \sum_{j=T_1}^{T_c}\left[\delta_T^{2(j-1-T_e)}X_{T_e}^2 - 2\delta_T^{j-1-T_e}\frac{T^\beta}{T_w c_2}X_{T_e}X_{T_c} + \frac{T^{2\beta}}{T_w c_2^2}X_{T_c}^2\right]\{1+o_p(1)\} \\
&= \left[\frac{T^\alpha\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2 - 2\frac{T^{\alpha+\beta-1}\delta_T^{T_c-T_e}}{f_w c_1 c_2}X_{T_e}X_{T_c} + T^{2\beta-1}\frac{f_c-f_1}{f_w c_2^2}X_{T_c}^2\right]\{1+o_p(1)\} \\
&= \begin{cases} \frac{T^\alpha\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\{1+o_p(1)\} \sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } 1+\alpha>2\beta \\ T^{2\beta-1}\frac{f_c-f_1}{f_w c_2^2}X_{T_c}^2\{1+o_p(1)\} \sim_a T^{2\beta}\delta_T^{2(T_c-T_e)}\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } 1+\alpha\le 2\beta \end{cases}
\end{aligned}$$

due to the fact that

$$\frac{T^\alpha\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2 \sim_a \frac{T^{1+\alpha}\delta_T^{2(T_c-T_e)}}{2c_1}B(f_e)^2$$

$$2\frac{T^{\alpha+\beta-1}\delta_T^{T_c-T_e}}{f_w c_1 c_2}X_{T_e}X_{T_c} \sim_a 2\frac{T^{\alpha+\beta}\delta_T^{2(T_c-T_e)}}{f_w c_1 c_2}B(f_e)^2$$

$$T^{2\beta-1}\frac{f_c-f_1}{f_w c_2^2}X_{T_c}^2 \sim_a T^{2\beta}\delta_T^{2(T_c-T_e)}\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2.$$

The second term is

$$\begin{aligned}
\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2 &= \sum_{j=T_c+1}^{T_2}\left[\gamma_T^{j-1-T_c} - \frac{T^\beta}{T_w c_2}\right]^2 X_{T_c}^2\{1+o_p(1)\} \\
&= \frac{T^\beta}{2c_2}X_{T_c}^2\{1+o_p(1)\} \sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2.
\end{aligned}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha\delta_T^{2(T_c-T_e)}}{2c_1}X_{T_e}^2\{1+o_p(1)\} \sim_a T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ \frac{T^\beta}{2c_2}X_{T_c}^2\{1+o_p(1)\} \sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(5) For $T_1 \in B$ and $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2.$$

Suppose $\alpha > \beta$. We know that

$$\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2$$

and

$$\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 =\sim_a \begin{cases} \frac{T_r-T_c}{T_w c_1^2} T^{2\alpha} \delta_T^{2(T_c-T_e)} X_{T_e}^2 \left\{1+o_p(1)\right\} \sim_a T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \beta < 2\alpha - 1 \\ X_{T_c}^2 \frac{T^\beta}{2c_2} \left\{1+o_p(1)\right\} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } 2\alpha - 1 < \beta < \alpha \end{cases}.$$

The third term is

$$\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2 = T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{T_2-T_r}{T_w^2 c_2^2} X_{T_e}^2 \left\{1+o_p(1)\right\} \sim_a T^{2\alpha} \delta_T^{2(T_c-T_e)} \frac{f_2-f_r}{f_w^2 c_2^2} B(f_e)^2.$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 \ .$$

Suppose $\alpha < \beta$. The first term is

$$\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } 1+\alpha > 2\beta \\ T^{2\beta-1} \frac{f_c-f_1}{f_w c_2^2} X_{T_c}^2 \left\{1+o_p(1)\right\} \sim_a T^{2\beta} \delta_T^{2(T_c-T_e)} \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } 1+\alpha \le 2\beta \end{cases}.$$

The second and the third terms are

$$\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 = \frac{T^\beta}{2c_2} X_{T_c}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2,$$

$$\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2 = \frac{T_2-T_r}{T_w^2 c_2^2} T^{2\beta} X_{T_c}^2 \left\{1+o_p(1)\right\} \sim_a T^{2\beta} \delta_T^{2(T_c-T_e)} \frac{f_2-f_r}{f_w^2 c_2^2} B(f_e)^2.$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2(T_c-T_e)}}{2c_1} X_{T_e}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\alpha} \delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ \frac{T^\beta}{2c_2} X_{T_c}^2 \left\{1+o_p(1)\right\} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(6) For $T_1 \in C$, $T_2 \in N_1$.

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 = \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}^2 + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2.$$

Suppose $\alpha > \beta$. The first term is

$$\begin{aligned} &\sum_{j=T_1}^{T_r} \tilde{X}_{j-1}^2 \\ &= \sum_{j=T_1}^{T_r} \left[\gamma_T^{j-1-T_c} X_{T_c} - \frac{1}{T_w} \sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i}\right]^2 \left\{1+o_p(1)\right\} \\ &= \sum_{j=T_1}^{T_r} \left[\gamma_T^{2(j-1-T_c)} X_{T_c}^2 - 2\gamma_T^{j-1-T_c} X_{T_c} \frac{1}{T_w} \sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i} + \frac{1}{T_w^2} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i}\right)^2\right] \left\{1+o_p(1)\right\} \\ &= \left[X_{T_c}^2 \sum_{j=T_1}^{T_r} \gamma_T^{2(j-1-T_c)} - 2\frac{X_{T_c}}{T_w} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i}\right) \left(\sum_{j=T_1}^{T_r} \gamma_T^{j-1-T_c}\right) \right. \\ &\left. + \frac{T_r-T_1}{T_w^2} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i}\right)^2\right] \left\{1+o_p(1)\right\} \end{aligned}$$

$$= \left[X_{T_c}^2 \frac{T^\beta \gamma_T^{2(T_1-1-T_c)}}{2c_2} - 2\frac{T^\beta \gamma_T^{T_1-1-T_c}}{c_2 T_w} X_{T_c} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i} \right) + \frac{T_r - T_1}{T_w^2} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i} \right)^2 \right] \left\{ 1 + o_p(1) \right\}$$

$$= \frac{T_r - T_1}{T_w^2} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i} \right)^2 \left\{ 1 + o_p(1) \right\}$$

$$\sim_a T^2 \frac{(f_r - f_1)(f_2 - f_r)^2}{f_w^2} \left[\int_{f_r}^{f_2} [B(s) - B(f_r)] ds \right]^2$$

due to the fact that

$$\sum_{j=T_1}^{T_r} \gamma_T^{2(j-1-T_c)} = \frac{\gamma_T^{2(T_1-1-T_c)} \left(\gamma_T^{2(T_r-T_1+1)} - 1 \right)}{\gamma_T^2 - 1} = \frac{T^\beta \gamma_T^{2(T_1-T_c)}}{2c_2}$$

$$\sum_{j=T_1}^{T_r} \gamma_T^{j-1-T_c} = \frac{\gamma_T^{T_r-T_c} - \gamma_T^{T_1-1-T_c}}{\gamma_T - 1} = \frac{T^\beta \gamma_T^{T_1-T_c}}{c_2}$$

and

$$X_{T_c}^2 \frac{T^\beta \gamma_T^{2(T_1-1-T_c)}}{2c_2} \sim_a T^{1+\beta} \delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-1-T_c)} \frac{1}{2c_2} B(f_e)^2$$

$$2\frac{T^\beta \gamma_T^{T_1-1-T_c}}{c_2} X_{T_c} \left(\frac{1}{T_w} \sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i} \right) \sim_a T^{1+\beta} \delta_T^{T_c-T_e} \gamma_T^{T_1-1-T_c} \frac{2(f_2 - f_r)}{c_2 f_w} \int_{f_r}^{f_2} [B(s) - B(f_r)] ds B(f_e)$$

$$\frac{T_r - T_1}{T_w} \left(\sum_{l=T_r+1}^{T_2} \sum_{i=0}^{l-T_r-1} \varepsilon_{l-i} \right)^2 \sim_a T^2 \frac{(f_r - f_1)(f_2 - f_r)^2}{f_w} \left[\int_{f_r}^{f_2} [B(s) - B(f_r)] ds \right]^2$$

The second term is

$$\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2$$

$$= \sum_{j=T_r+1}^{T_2} \left[\sum_{i=0}^{j-1-T_r-1} \varepsilon_{j-i-1} - \frac{1}{T_w} \sum_{k=T_r+1}^{T_2} \sum_{i=0}^{k-T_r-1} \varepsilon_{k-i} \right]^2 \left\{ 1 + o_p(1) \right\} \text{ from Lemma S.B3}$$

$$= T(T_2 - T_r) \left[\frac{1}{T_2 - T_r} \sum_{j=T_r+1}^{T_2} \left(T^{-1/2} \sum_{i=0}^{j-T_r-2} \varepsilon_{j-i-1} \right)^2 \right]$$

$$- 2T \frac{(T_2 - T_r)^2}{T_w} \left[\frac{1}{T_2 - T_r} \sum_{j=T_r+1}^{T_2} \left(T^{-1/2} \sum_{i=0}^{j-T_r-1} \varepsilon_{t-i} \right) \right]^2$$

$$+ T \frac{(T_2 - T_r)^3}{T_w^2} \left[\frac{1}{T_2 - T_r} \sum_{k=T_r+1}^{T_2} \left(T^{-1/2} \sum_{i=0}^{k-T_r-1} \varepsilon_{k-i} \right) \right]^2$$

$$\sim_a T^2 (f_2 - f_r) \left\{ \int_{f_r}^{f_2} [B(s) - B(f_r)]^2 ds + \frac{(f_2 - f_r)(f_2 - f_r - 2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} [B(s) - B(f_r)] ds \right]^2 \right\}.$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a T^2 (f_2 - f_r) \left\{ \int_{f_r}^{f_2} [B(s) - B(f_r)]^2 ds - \frac{f_2 - f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s) - B(f_r)] ds \right]^2 \right\}.$$

Suppose $\alpha < \beta$. The first term is

$$\sum_{j=T_1}^{T_r} \tilde{X}_{j-1}^2$$

$$= \sum_{j=T_1}^{T_r} \left[\gamma_T^{j-1-T_c} - \frac{\gamma_T^{T_1-T_c} T^\beta}{T_w c_2} \right]^2 X_{T_c}^2 \left\{ 1 + o_p(1) \right\}$$

$$
\begin{aligned}
&= \left[\sum_{j=T_1}^{T_r} \gamma_T^{2(j-1-T_c)} - 2\frac{\gamma_T^{T_1-T_c}T^{\beta}}{T_w c_2}\sum_{j=T_1}^{T_r}\gamma_T^{j-1-T_c} + \sum_{j=T_1}^{T_r}\frac{\gamma_T^{2(T_1-T_c)}T^{2\beta}}{T_w c_2^2}\right] X_{T_c}^2\left\{1+o_p(1)\right\} \\
&= \left[T^{\beta}\gamma_T^{2(T_1-T_c)}\frac{1}{2c_2} - 2\gamma_T^{2(T_1-T_c)}T^{2\beta-1}\frac{1}{f_w c_2^2} + \gamma_T^{2(T_1-T_c)}T^{2\beta-1}\frac{f_r-f_1}{f_w c_2^2}\right] X_{T_c}^2\left\{1+o_p(1)\right\} \\
&= T^{\beta}\gamma_T^{2(T_1-T_c)}\frac{1}{2c_2}X_{T_c}^2\left\{1+o_p(1)\right\} \\
&\sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2c_2}B(f_e)^2.
\end{aligned}
$$

The second term is

$$
\begin{aligned}
\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2 &= \sum_{j=T_r+1}^{T_2}\left[-\frac{\gamma_T^{T_1-T_c}T^{\beta}}{T_w c_2}\right]^2 X_{T_c}^2\left\{1+o_p(1)\right\} \\
&= \sum_{j=T_r+1}^{T_2}\frac{\gamma_T^{2(T_1-T_c)}T^{2\beta}}{T_w c_2^2}X_{T_c}^2\left\{1+o_p(1)\right\} \\
&= T^{2\beta-1}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2^2}X_{T_c}^2\left\{1+o_p(1)\right\} \\
&\sim_a T^{2\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2^2}B(f_e)^2 .
\end{aligned}
$$

Therefore,

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2 \sim_a T^{1+\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2c_2}B(f_e)^2.
$$

□

Lemma S.B5. *The sums of cross-products of $\tilde{X}_t$ and ε_t are as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e).
$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.
$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.
$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.
$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.
$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$
\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2 - \frac{1}{2}(f_2-f_r)\sigma^2 \right. & \\ \left. -2\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.
$$

Proof. (1) For $T_1 \in N_0$ and $T_2 \in B$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=T_1}^{T_e-1} -\frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1} X_{T_e}\varepsilon_j \{1+o_p(1)\} \\ &= -\frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1} X_{T_e} \sum_{j=T_1}^{T_e-1} \varepsilon_j \{1+o_p(1)\} \\ &= -\frac{T^\alpha \delta_T^{T_2-T_e}}{f_w c_1} \left(T^{-1/2} X_{T_e}\right) \left(T^{-1/2} \sum_{j=T_1}^{T_e-1} \varepsilon_j\right) \{1+o_p(1)\} \\ &\sim_a -\frac{T^\alpha \delta_T^{T_2-T_e}}{f_w c_1} B(f_e)\left[B(f_e)-B(f_1)\right].\end{aligned}$$

The second term is

$$\begin{aligned}&\sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j \\ &= \sum_{j=T_e}^{T_2} \left[\delta_T^{j-1-T_e} - \frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1}\right] X_{T_e}\varepsilon_j \{1+o_p(1)\} \\ &= \left[\sum_{j=T_e}^{T_2} \delta_T^{j-1-T_e}\varepsilon_j - \frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1} \sum_{j=T_e}^{T_2} \varepsilon_j\right] X_{T_e} \{1+o_p(1)\} \\ &= \left[T^{\alpha/2}\delta_T^{T_2-T_e}\left(\frac{1}{T^{\alpha/2}} \sum_{j=T_e}^{T_2} \delta_T^{-(T_2-j+1)}\varepsilon_j\right) - \frac{\delta_T^{T_2-T_e}}{T^{1/2-\alpha} f_w c_1}\left(\frac{1}{\sqrt{T}} \sum_{j=T_e}^{T_2} \varepsilon_j\right)\right] X_{T_e} \{1+o_p(1)\} \\ &= T^{\alpha/2}\delta_T^{T_2-T_e}\left(T^{-\alpha/2} \sum_{j=T_e}^{T_2} \delta_T^{-(T_2-j+1)}\varepsilon_j\right) X_{T_e} \{1+o_p(1)\} \text{ (since } \alpha/2 > \alpha - 1/2) \\ &\sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e} X_{c_1} B(f_e).\end{aligned}$$

Since $(\alpha+1)/2 > \alpha$, the component $\sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\varepsilon_j$. Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e} X_{c_1} B(f_e).$$

(2) For $T_1 \in N_0$, $T_2 \in C$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\alpha > \beta$. The first term is

$$\begin{aligned}\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=T_1}^{T_e-1} -\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\varepsilon_j \{1+o_p(1)\} \\ &= -\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \sum_{j=T_1}^{T_e-1} \varepsilon_j \{1+o_p(1)\} \\ &= -\frac{T^\alpha \delta_T^{T_c-T_e}}{f_w c_1} \left(T^{-1/2} X_{T_e}\right) \left(T^{-1/2} \sum_{j=T_1}^{T_e-1} \varepsilon_j\right) \{1+o_p(1)\} \\ &\sim_a -\frac{T^\alpha \delta_T^{T_c-T_e}}{f_w c_1} B(f_e)\left[B(f_e)-B(f_1)\right].\end{aligned}$$

The second term is

$$\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\varepsilon_j$$

$$
\begin{aligned}
&= \sum_{j=T_e}^{T_c} \left[\delta_T^{j-1-T_e} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} \right] X_{T_e}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[\sum_{j=T_e}^{T_c} \delta_T^{j-1-T_e}\varepsilon_j - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} \sum_{j=T_e}^{T_c} \varepsilon_j \right] X_{T_e} \left\{1+o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{T_c-T_e} \left(\frac{1}{T^{\alpha/2}} \sum_{j=T_e}^{T_c} \delta_T^{-(T_c-j+1)}\varepsilon_j \right) - \frac{T^{\alpha-1/2}\delta_T^{T_c-T_e}}{f_w c_1} \left(\frac{1}{\sqrt{T}} \sum_{j=T_e}^{T_c} \varepsilon_j \right) \right] X_{T_e} \left\{1+o_p(1)\right\} \\
&= T^{(1+\alpha)/2}\delta_T^{T_c-T_e} \left(\frac{1}{T^{\alpha/2}} \sum_{j=T_e}^{T_c} \delta_T^{-(T_c-j+1)}\varepsilon_j \right) \left(T^{-1/2} X_{T_e} \right) \left\{1+o_p(1)\right\} \\
&\sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e).
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{j-1-T_c} X_{T_c} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \right] \varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[X_{T_c} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c}\varepsilon_j - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \sum_{j=T_c+1}^{T_2} \varepsilon_j \right] \left\{1+o_p(1)\right\} \\
&= \left[T^{\beta/2} X_{T_c} \left(T^{-\beta/2} \sum_{l=0}^{T_2-T_c-1} \gamma_T^{l}\varepsilon_{l+T_c+1} \right) - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{f_w c_1} \left(T^{-1/2} X_{T_e} \right) \left(T^{-1/2} \sum_{j=T_c+1}^{T_2} \varepsilon_j \right) \right] \left\{1+o_p(1)\right\} \\
&\sim_a \begin{cases} -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e)\left[B(f_2)-B(f_c)\right] & \text{if } 2\alpha > 1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } 2\alpha < 1+\beta \end{cases}
\end{aligned}
$$

due to the fact that

$$
\begin{aligned}
X_{T_c} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c}\varepsilon_j &\sim {}_a T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} \\
\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \sum_{j=T_c+1}^{T_2} \varepsilon_j &\sim {}_a T^{\alpha}\delta_T^{T_c-T_e} \frac{1}{f_w c_1} B(f_e)\left[B(f_2)-B(f_c)\right]
\end{aligned}
$$

Therefore,

$$
\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\varepsilon_j \left\{1+o_p(1)\right\} \sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e).
$$

Suppose $\alpha < \beta$. The first term is

$$
\begin{aligned}
\sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=T_1}^{T_e-1} -\frac{T^{\beta}}{T_w c_2} X_{T_c}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= -\frac{T^{\beta}}{T_w c_2} X_{T_c} \sum_{j=T_1}^{T_e-1} \varepsilon_j \left\{1+o_p(1)\right\} \\
&= -\frac{T^{\beta}\delta_T^{T_c-T_e}}{f_w c_1} \left(T^{-1/2}\delta_T^{-(T_c-T_e)} X_{T_c} \right) \left(T^{-1/2} \sum_{j=T_1}^{T_e-1} \varepsilon_j \right) \left\{1+o_p(1)\right\} \\
&\sim_a -T^{\beta}\delta_T^{T_c-T_e} \frac{1}{f_w c_2} B(f_e)\left[B(f_e)-B(f_1)\right].
\end{aligned}
$$

The second term is

$$
\begin{aligned}
&\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=T_e}^{T_c} \left[\delta_T^{j-1-T_e} X_{T_e} - \frac{T^{\beta}}{T_w c_2} X_{T_c} \right] \varepsilon_j \left\{1+o_p(1)\right\}
\end{aligned}
$$

$$= \left[X_{T_e} \sum_{j=T_e}^{T_c} \delta_T^{j-1-T_e} \varepsilon_j - \frac{T^\beta}{T_w c_2} X_{T_c} \sum_{j=T_e}^{T_c} \varepsilon_j \right] \{1 + o_p(1)\}$$

$$= \left[T^{(1+\alpha)/2} \delta_T^{T_c - T_e} \left(T^{-1/2} X_{T_e} \right) \left(\frac{1}{T^{\alpha/2}} \sum_{j=T_e}^{T_c} \delta_T^{-(T_c - j + 1)} \varepsilon_j \right) \right.$$

$$\left. - \frac{T^\beta \delta_T^{T_c - T_e}}{f_w c_2} \left(T^{-1/2} \delta_T^{-(T_c - T_e)} X_{T_c} \right) \left(\frac{1}{\sqrt{T}} \sum_{j=T_e}^{T_c} \varepsilon_j \right) \right] \{1 + o_p(1)\}$$

$$\sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c - T_e} X_{c_1} B(f_e). & \text{if } 1+\alpha > 2\beta \\ -T^\beta \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e) [B(f_c) - B(f_e)] & \text{if } 1+\alpha < 2\beta \end{cases}.$$

The third term is

$$\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1} \varepsilon_j$$

$$= \sum_{j=T_c+1}^{T_2} \left[\gamma_T^{j-1-T_c} - \frac{T^\beta}{T_w c_2} \right] X_{T_c} \varepsilon_j \{1 + o_p(1)\}$$

$$= \left[X_{T_c} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c} \varepsilon_j - \frac{T^\beta}{T_w c_2} X_{T_c} \sum_{j=T_c+1}^{T_2} \varepsilon_j \right] \{1 + o_p(1)\}$$

$$= \left[T^{\beta/2} X_{T_c} \left(T^{-\beta/2} \sum_{j=l}^{T_2 - T_c - 1} \gamma_T^{l} \varepsilon_{l+1+T_c} \right) - \frac{T^\beta \delta_T^{T_c - T_e}}{f_w c_2} \left(T^{-1/2} \delta_T^{-(T_c - T_e)} X_{T_c} \right) \left(T^{-1/2} \sum_{j=T_c+1}^{T_2} \varepsilon_j \right) \right] \{1 + o_p(1)\}$$

$$= X_{T_c} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c} \varepsilon_j \{1 + o_p(1)\}$$

$$\sim_a T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2}$$

due to the fact that

$$X_{T_c} \sum_{j=T_c+1}^{T_2} \gamma_T^{j-1-T_c} \varepsilon_j \sim_a T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2}$$

$$\frac{T^\beta}{T_w c_2} X_{T_c} \sum_{j=T_c+1}^{T_2} \varepsilon_j \sim_a T^\beta \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e) [B(f_2) - B(f_c)]$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1} \varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(3) For $T_1 \in N_0, T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=T_1}^{T_e - 1} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1} \varepsilon_j.$$

Suppose $\alpha > \beta$. The first term is

$$\sum_{j=T_1}^{T_e - 1} \tilde{X}_{j-1} \varepsilon_j \sim_a -T^\alpha \delta_T^{T_c - T_e} \frac{1}{f_w c_1} B(f_e) [B(f_e) - B(f_1)].$$

The second term is

$$\sum_{j=T_e}^{T_c} \tilde{X}_{j-1} \varepsilon_j \sim_a T^{(1+\alpha)/2} \delta_T^{T_c - T_e} X_{c_1} B(f_e).$$

The third term is

$$\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1} \varepsilon_j$$

$$= \sum_{j=T_c+1}^{T_r} \left[\gamma_T^{j-1-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \right] \varepsilon_j \{1 + o_p(1)\}$$

$$= \left[X_{T_c} \sum_{j=T_c+1}^{T_r} \gamma_T^{j-1-T_c} \varepsilon_j - \frac{T^{\alpha} \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \sum_{j=T_c+1}^{T_r} \varepsilon_j \right] \left\{ 1 + o_p(1) \right\}$$

$$= \left[T^{\beta/2} X_{T_c} \left(T^{-\beta/2} \sum_{j=T_c+1}^{T_r} \gamma_T^{l} \varepsilon_{l+T_c+1} \right) - \frac{T^{\alpha} \delta_T^{T_c - T_e}}{f_w c_1} \left(T^{-1/2} X_{T_e} \right) \left(T^{-1/2} \sum_{j=T_c+1}^{T_r} \varepsilon_j \right) \right] \left\{ 1 + o_p(1) \right\}$$

$$\sim_a \begin{cases} -T^{\alpha} \delta_T^{T_c - T_e} \frac{1}{f_w c_1} B(f_e) \left[B(f_r) - B(f_c) \right] & \text{if } 2\alpha > 1 + \beta \\ T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } 2\alpha < 1 + \beta \end{cases}$$

due to the fact that

$$X_{T_c} \sum_{j=T_c+1}^{T_r} \gamma_T^{j-1-T_c} \varepsilon_j \sim_a T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2}$$

$$\frac{T^{\alpha} \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \sum_{j=T_c+1}^{T_r} \varepsilon_j \sim_a T^{\alpha} \delta_T^{T_c - T_e} \frac{1}{f_w c_1} B(f_e) \left[B(f_r) - B(f_c) \right]$$

The fourth term is

$$\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=T_r+1}^{T_2} -\frac{T^{\alpha} \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \varepsilon_j \left\{ 1 + o_p(1) \right\}$$

$$= -\frac{T^{\alpha} \delta_T^{T_c - T_e}}{f_w c_1} \left(T^{-1/2} X_{T_e} \right) \left(T^{-1/2} \sum_{j=T_r+1}^{T_2} \varepsilon_j \right) \left\{ 1 + o_p(1) \right\}$$

$$\sim_a -\frac{T^{\alpha} \delta_T^{T_c - T_e}}{f_w c_1} B(f_e) \left[B(f_2) - B(f_r) \right].$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=T_e}^{T_c} \tilde{X}_{j-1} \varepsilon_j \left\{ 1 + o_p(1) \right\} \sim_a T^{(1+\alpha)/2} \delta_T^{T_c - T_e} X_{c_1} B(f_e).$$

Suppose $\alpha < \beta$. The first term is

$$\sum_{j=T_1}^{T_e - 1} \tilde{X}_{j-1} \varepsilon_j \sim_a -\frac{T^{\beta} \delta_T^{T_c - T_e}}{f_w c_2} B(f_e) \left[B(f_e) - B(f_1) \right].$$

The second term is

$$\sum_{j=T_e}^{T_c} \tilde{X}_{j-1} \varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c - T_e} X_{c_1} B(f_e) & \text{if } 1 + \alpha > 2\beta \\ -T^{\beta} \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e) \left[B(f_c) - B(f_e) \right] & \text{if } 1 + \alpha < 2\beta \end{cases}.$$

The third term is

$$\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=T_c+1}^{T_r} \left[\gamma_T^{j-1-T_c} - \frac{T^{\beta}}{T_w c_2} \right] X_{T_c} \varepsilon_j \left\{ 1 + o_p(1) \right\}$$

$$\sim_a T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2}.$$

The fourth term is

$$\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1} \varepsilon_j = \sum_{j=T_r+1}^{T_2} -\frac{T^{\beta}}{T_w c_2} X_{T_c} \varepsilon_j \left\{ 1 + o_p(1) \right\}$$

$$= -\frac{T^{\beta} \delta_T^{T_c - T_e}}{f_w c_2} \left(T^{-1/2} \delta_T^{-(T_c - T_e)} X_{T_c} \right) \left(T^{-1/2} \sum_{j=T_r+1}^{T_2} \varepsilon_j \right) \left\{ 1 + o_p(1) \right\}$$

$$\sim_a -T^{\beta} \delta_T^{T_c - T_e} \frac{1}{f_w c_2} B(f_e) \left[B(f_2) - B(f_r) \right].$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1} \varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(4) For $T_1 \in B$ and $T_2 \in C$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\alpha > \beta$. The first term is

$$\begin{aligned}
\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=T_1}^{T_c} \left[\delta_T^{j-1-T_e} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\right] X_{T_e}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= X_{T_e}\left[\sum_{j=T_1}^{T_c} \delta_T^{j-1-T_e}\varepsilon_j - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\sum_{j=T_1}^{T_c}\varepsilon_j\right]\left\{1+o_p(1)\right\} \\
&= X_{T_e}\left[T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_1}^{T_c}\delta_T^{j-1-T_c}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}\sum_{j=T_1}^{T_c}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_1}^{T_c}\delta_T^{j-1-T_c}\varepsilon_j\right) X_{T_e}\left\{1+o_p(1)\right\} \\
&\sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e).
\end{aligned}$$

The second term is

$$\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e)\left[B(f_2)-B(f_c)\right] & \text{if } 2\alpha > 1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } 2\alpha < 1+\beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e).$$

Suppose $\alpha < \beta$. The first term is

$$\begin{aligned}
\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=T_1}^{T_c}\left[\delta_T^{j-1-T_e} X_{T_e} - \frac{T^{\beta}}{T_w c_2} X_{T_c}\right]\varepsilon_j\left\{1+o_p(1)\right\} \\
&\sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e) & \text{if } 1+\alpha > 2\beta \\ -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2} B(f_e)\left[B(f_c)-B(f_1)\right] & \text{if } 1+\alpha < 2\beta \end{cases}.
\end{aligned}$$

The second term is

$$\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_c+1}^{T_2}\left[\gamma_T^{j-1-T_c} - \frac{T^{\beta}}{T_w c_2}\right] X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\} \sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(5) For $T_1 \in B$ and $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j.$$

Suppose $\alpha > \beta$. The first term is

$$\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\varepsilon_j = T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_1}^{T_c}\delta_T^{j-1-T_c}\varepsilon_j\right) X_{T_e}\left\{1+o_p(1)\right\} \sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e).$$

The second term is

$$\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e)\left[B(f_r)-B(f_c)\right] & \text{if } 2\alpha > 1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } 2\alpha < 1+\beta \end{cases}$$

The third term is

$$\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_r+1}^{T_2} -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\varepsilon_j\left\{1+o_p(1)\right\}$$

$$\sim_a -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_2)-B(f_r)\right].$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

Suppose $\alpha<\beta$. The first term is

$$\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e) & \text{if } 1+\alpha>2\beta \\ -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_c)-B(f_1)\right] & \text{if } 1+\alpha<2\beta \end{cases}.$$

The second term is

$$\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_c+1}^{T_r}\left[\gamma_T^{j-1-T_c}-\frac{T^{\beta}}{T_w c_2}\right]X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}$$
$$\sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.$$

The third term is

$$\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_r+1}^{T_2}-\frac{T^{\beta}}{T_w c_2}X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}$$
$$\sim_a -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_2)-B(f_r)\right].$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

(6) For $T_1\in C$, $T_2\in N_1$ and $\alpha>\beta$,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j = \sum_{j=T_1}^{T_r}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}\varepsilon_j = \left[X_{T_c}\sum_{j=T_1}^{T_r}\gamma_T^{j-1-T_c}\varepsilon_j - \frac{1}{T_w}\sum_{l=T_r+1}^{T_2}\left(\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right)\left(\sum_{j=T_1}^{T_r}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\}$$
$$\sim_a -T\frac{f_2-f_r}{f_w}\left[B(f_r)-B(f_1)\right]\int_{f_r}^{f_2}\left[B(r_s)-B(f_r)\right]ds.$$

due to the fact that

$$X_{T_c}\sum_{j=T_1}^{T_r}\gamma_T^{j-1-T_c}\varepsilon_j = X_{T_c}T^{\beta/2}\gamma_T^{T_1-T_c-1}\left(T^{-\beta/2}\sum_{j=T_1}^{T_r}\gamma_T^{j-T_1}\varepsilon_j\right) \sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c-1}B(f_e)X_{c_2}$$
$$\frac{1}{T_w}\sum_{l=T_r+1}^{T_2}\left(\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right)\left(\sum_{j=T_1}^{T_r}\varepsilon_j\right) \sim_a T\frac{f_2-f_r}{f_w}\left[B(f_r)-B(f_1)\right]\int_{f_r}^{f_2}\left[B(r_s)-B(f_r)\right]ds.$$

The second term is

$$\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j$$
$$= \sum_{j=T_r+1}^{T_2}\left[\sum_{l=0}^{j-1-T_r-1}\varepsilon_{j-1-l} - \frac{1}{T_w}\sum_{l=T_r+1}^{T_2}\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right]\varepsilon_j\left\{1+o_p(1)\right\}$$
$$= T\left(T^{-1}\sum_{j=T_r+1}^{T_2}\sum_{l=0}^{j-1-T_r-1}\varepsilon_{j-1-l}\varepsilon_j\right)$$
$$-T\frac{T_2-T_r}{T_w}\left[\frac{1}{T_2-T_r}\sum_{l=T_r+1}^{T_2}\left(T^{-1/2}\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right)\right]\left(T^{-1/2}\sum_{j=T_r+1}^{T_2}\varepsilon_j\right)\left\{1+o_p(1)\right\}$$
$$\sim_a T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2-\frac{1}{2}(f_2-f_r)\sigma^2-\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\}.$$

due to the fact that

$$\sum_{j=T_r+1}^{T_2}\sum_{l=0}^{j-1-T_r-1}\varepsilon_{j-1-l}\varepsilon_j=\sum_{j=T_r+1}^{T_2}Z_{j-1}\varepsilon_j=\frac{1}{2}T\left[T^{-1}Z_{T_2}^2-\sum_{j=T_r+1}^{T_2}\varepsilon_j^2\right]$$
$$\sim_a T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2-\frac{1}{2}(f_2-f_r)\sigma^2\right\}$$

where $Z_{j-1}=\sum_{l=0}^{j-1-T_r-1}\varepsilon_{j-1-l}$. Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j\sim_a T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2-\frac{1}{2}(f_2-f_r)\sigma^2-2\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\}.$$

For $T_1\in C$, $T_2\in N_1$ and $\alpha<\beta$,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}\varepsilon_j+\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
&\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}\varepsilon_j\\
&=\sum_{j=T_1}^{T_r}\left[\gamma_T^{j-1-T_c}-\frac{\gamma_T^{T_1-T_c}T^{\beta}}{T_wc_2}\right]X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}\\
&=X_{T_c}\left[T^{\beta/2}\gamma_T^{T_1-T_c-1}\left(T^{-\beta/2}\gamma_T^{-(T_1-T_c-1)}\sum_{j=T_1}^{T_r}\gamma_T^{j-1-T_c}\varepsilon_j\right)-\frac{\gamma_T^{T_1-T_c}T^{\beta-1/2}}{f_wc_2}\left(T^{-1/2}\sum_{j=T_1}^{T_r}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\}\\
&=X_{T_c}T^{\beta/2}\gamma_T^{T_1-T_c-1}\left(T^{-\beta/2}\gamma_T^{-(T_1-T_c-1)}\sum_{j=T_1}^{T_r}\gamma_T^{j-1-T_c}\varepsilon_j\right)\left\{1+o_p(1)\right\}\\
&\sim_a T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j&=X_{T_c}\sum_{j=T_r+1}^{T_2}\left[-\frac{\gamma_T^{T_1-T_c}T^{\beta}}{T_wc_2}\right]\varepsilon_j\left\{1+o_p(1)\right\}\\
&=-X_{T_c}\frac{\gamma_T^{T_1-T_c}T^{\beta-1/2}}{f_wc_2}\left(T^{-1/2}\sum_{j=T_r+1}^{T_2}\varepsilon_j\right)\left\{1+o_p(1)\right\}\\
&\sim_a -T^{\beta}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{B(f_2)-B(f_r)}{f_wc_2}B(f_e).
\end{aligned}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j\sim_a T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.$$

□

Lemma S.B6. *The sums of cross-products of $\tilde{X}_{j-1}$ and $\tilde{X}_j-\delta_T\tilde{X}_{j-1}$ are as follows.*

(1) *For $T_1\in N_0$ and $T_2\in B$,*

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right)\sim_a -T^{\alpha}\delta_T^{2(T_2-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2.$$

(2) *For $T_1\in N_0$ and $T_2\in C$,*

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right)\sim_a\begin{cases}-T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha\\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta\end{cases}.$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2} \left[\int_{f_r}^{f_2} \left[B(s) - B(f_r)\right] ds\right]^2 & \text{if } \alpha > \beta \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

Proof. (1) When $T_1 \in N_0$ and $T_2 \in B$,

$$\begin{aligned} \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) && (2.28) \\ &= \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j - c_1 T^{-\alpha} \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 && (2.29) \end{aligned}$$

The first term is

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e} X_{c_1} B(f_e) \text{ (from Lemma S.B5).}$$

The second term is

$$-c_1 T^{-\alpha} \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 \sim_a -\frac{f_e - f_1}{f_w} T^{\alpha} \delta_T^{2(T_2-T_e)} B(f_e)^2.$$

The second term dominates the first terms and hence

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim_a -T^{\alpha}\delta_T^{2(T_2-T_e)} \frac{f_e - f_1}{f_w} B(f_e)^2.$$

(2) When $T_1 \in N_0$ and $T_2 \in C$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) \\ &+ \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T X_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (1-\delta_T) \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}^2 + (\gamma_T - \delta_T) \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 \end{aligned}$$

The first term is

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

For the second term,

$$(1-\delta_T)\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 = -c_1T^{-\alpha}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1} \sim_a \begin{cases} -T^{\alpha}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2 & \text{if } \alpha>\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1}{f_wc_2^2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

For the third term,

$$(\gamma_T-\delta_T)\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(3) When $T_1\in N_0$ and $T_2\in N_1$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\delta_T\tilde{X}_{j-1}\right)+\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1}+\varepsilon_j-\delta_T\tilde{X}_{j-1}\right) \\ &+\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1}+\varepsilon_j-\delta_T\tilde{X}_{j-1}\right)+\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\delta_T\tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j+(1-\delta_T)\left[\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2+\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2\right]+(\gamma_T-\delta_T)\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2 \end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

The second term

$$\begin{aligned} &(1-\delta_T)\left[\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2+\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2\right] \\ &\sim_a \begin{cases} -T^{\alpha}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1+f_2-f_r}{f_wc_1}B(f_e)^2 & \text{if } \alpha>\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1+f_2-f_r}{f_wc_2^2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases} \end{aligned}$$

The third term

$$(\gamma_T-\delta_T)\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(4) When $T_1 \in B$ and $T_2 \in C$,

$$\begin{aligned}
&\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\gamma_T - \delta_T) \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 .
\end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases} .$$

The second term

$$(\gamma_T - \delta_T) \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(5) When $T_1 \in B$ and $T_2 \in N_1$,

$$\begin{aligned}
&\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) \\
&+ \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \delta_T \tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\gamma_T - \delta_T) \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 + (1-\delta_T) \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2 .
\end{aligned}$$

The first term is

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases} .$$

For the second term,

$$(\gamma_T - \delta_T) \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

For the third term,

$$(1-\delta_T) \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2 \sim^a \begin{cases} -T^{\alpha} \delta_T^{2(T_c-T_e)} c_1 \frac{f_2-f_r}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha > \beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} c_1 \frac{f_2-f_r}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(6) When $T_1 \in C$ and $T_2 \in N_1$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right)+\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1}+\varepsilon_j-\delta_T\tilde{X}_{j-1}\right)+\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\delta_T\tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j+(\gamma_T-\delta_T)\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}^2+(1-\delta_T)\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2 \end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2-\frac{1}{2}(f_2-f_r)\sigma^2 \right. \\ \left. \quad -\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

The second term

$$(\gamma_T-\delta_T)\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2-\beta}c_2\frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2 & \text{if } \alpha>\beta \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

Therefore, we have

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2-\beta}c_2\frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2 & \text{if } \alpha>\beta \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

□

Lemma S.B7. *The sums of cross-products of $\tilde{X}_{j-1}$ and $\tilde{X}_j-\gamma_T\tilde{X}_{j-1}$ are as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \sim^a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(5) *For* $T_1 \in B$ *and* $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) *For* $T_1 \in C$ *and* $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right)$$

$$\sim_a \begin{cases} T^{2-\beta}c_2(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\} & \text{if } \alpha > \beta \\ T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

Proof. (1) When $T_1 \in N_0$ and $T_2 \in B$,

$$\begin{aligned}\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (1-\gamma_T)\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 + (\delta_T - \gamma_T)\sum_{j=T_e}^{T_2}\tilde{X}_{j-1}^2\end{aligned}$$

The first term is

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e) \text{ (from Lemma S.B5).}$$

The second term is

$$(1-\gamma_T)\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 \sim_a T^{2\alpha-\beta}\delta_T^{2(T_2-T_e)}c_2\frac{f_e-f_1}{f_w c_1}B(f_e)^2$$

The third term

$$(\delta_T - \gamma_T)\sum_{j=T_e}^{T_2}\tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

The second term dominates the first terms and hence

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(2) When $T_1 \in N_0$ and $T_2 \in C$,

$$\begin{aligned}&\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) \\ &+ \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \gamma_T X_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (1-\gamma_T)\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 + (\delta_T - \gamma_T)\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2\end{aligned}$$

The first term i

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(1-\gamma_T)\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 = c_2 T^{-\beta}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1} \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_e-f_1}{f_w c_1} B(f_e)^2 & \text{if } \alpha>\beta \\ T^{\beta}\delta_T^{2(T_c-T_e)} \frac{f_e-f_1}{f_w c_2} B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

The third term

$$(\delta_T-\gamma_T)\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(3) When $T_1 \in N_0$ and $T_2 \in N_1$,

$$\begin{aligned}
&\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\gamma_T\tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_c}\tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1}+\varepsilon_j-\gamma_T\tilde{X}_{j-1}\right) \\
&+ \sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1}+\varepsilon_j-\gamma_T\tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\gamma_T\tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j + (1-\gamma_T)\left[\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 + \sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2\right] + (\delta_T-\gamma_T)\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2
\end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

The second term

$$\begin{aligned}
&(1-\gamma_T)\left[\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2 + \sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2\right] \\
&\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_e-f_1+f_2-f_r}{f_w c_1} B(f_e)^2 & \text{if } \alpha>\beta \\ T^{\beta}\delta_T^{2(T_c-T_e)} \frac{f_e-f_1+f_2-f_r}{f_w c_2} B(f_e)^2 & \text{if } \alpha<\beta \end{cases}
\end{aligned}$$

The third term

$$(\delta_T-\gamma_T)\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(4) When $T_1 \in B$ and $T_2 \in C$,

$$\begin{aligned}
&\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\gamma_T\tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_c}\tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1}+\varepsilon_j-\gamma_T\tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1}+\varepsilon_j-\gamma_T\tilde{X}_{j-1}\right)
\end{aligned}$$

$$= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\delta_T - \gamma_T) \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2.$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(\delta_T - \gamma_T) \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) When $T_1 \in B$ and $T_2 \in N_1$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\delta_T \tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) \\ &+ \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\delta_T - \gamma_T) \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 + (1-\gamma_T) \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2. \end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(\delta_T - \gamma_T) \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

The third term

$$(1-\gamma_T) \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2 \sim^a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{f_2-f_r}{f_w c_2} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{\beta}\delta_T^{2(T_c-T_e)} \frac{f_2-f_r}{f_w c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) When $T_1 \in C$ and $T_2 \in N_1$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \end{aligned}$$

$$= \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \gamma_T \tilde{X}_{j-1}\right)$$
$$= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (1-\gamma_T)\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2 - \frac{1}{2}(f_2-f_r)\sigma^2 \right. & \\ \left. -\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(1-\gamma_T)\sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}^2$$
$$\sim_a \begin{cases} T^{2-\beta}c_2(f_2-f_r)\left\{\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2\right\} & \text{if } \alpha > \beta \\ T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore, we have

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right)$$
$$\sim_a \begin{cases} T^{2-\beta}c_2(f_2-f_r)\left\{\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2\right\} & \text{if } \alpha > \beta \\ T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

□

Lemma S.B8. *The sums of cross-products of $\tilde{X}_{j-1}$ and $\tilde{X}_j - \tilde{X}_{j-1}$ are as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2.$$

(2) *For $T_1 \in N_0$ and $T_2 \in C, \geq$*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

Proof. (1) When $T_1 \in N_0$ and $T_2 \in B$,

$$\begin{aligned} \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) \quad (2.30) \\ &= \sum_{j=T_e}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\delta_T - 1)\sum_{j=T_e}^{T_2} \tilde{X}_{j-1}^2 \quad (2.31) \end{aligned}$$

The first term is

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e} X_{c_1} B(f_e) \text{ (from Lemma S.B5).}$$

The second term is

$$(\delta_T - 1)\sum_{j=T_e}^{T_2} \tilde{X}_{j-1}^2 \sim_a T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2.$$

The second term dominates the first terms and hence

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2.$$

(2) When $T_1 \in N_0$ and $T_2 \in C$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_e-1} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_c} \tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1} + \varepsilon_j - X_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\delta_T - 1)\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2 + (\gamma_T - 1)\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 \end{aligned}$$

The first term is

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

For the second term,

$$(\delta_T - 1)\sum_{j=T_e}^{T_c} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{f_c-f_e}{f_w c_1^{-1} c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

For the third term,

$$(\gamma_T - 1)\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(3) When $T_1 \in N_0$ and $T_2 \in N_1$,

$$\begin{aligned}
&\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\tilde{X}_{j-1}\right) + \sum_{j=T_e}^{T_c}\tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1}+\varepsilon_j-\tilde{X}_{j-1}\right) \\
&+ \sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1}+\varepsilon_j-\tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_{j-1}+\varepsilon_j-\tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j + (\delta_T-1)\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2 + (\gamma_T-1)\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2
\end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

The second term

$$\begin{aligned}
&(\delta_T-1)\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2 \\
&\sim_a \begin{cases} T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}
\end{aligned}$$

The third term

$$(\gamma_T - 1)\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(4) When $T_1 \in B$ and $T_2 \in C$,

$$\begin{aligned}
&\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}\left(\tilde{X}_j-\tilde{X}_{j-1}\right) \\
&= \sum_{j=T_1}^{T_c}\tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1}+\varepsilon_j-\tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1}+\varepsilon_j-\tilde{X}_{j-1}\right)
\end{aligned}$$

$$= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\delta_T - 1)\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 + (\gamma_T - 1)\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(\delta_T - 1)\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

The third term

$$(\gamma_T - 1)\sum_{j=T_c+1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) When $T_1 \in B$ and $T_2 \in N_1$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_c} \tilde{X}_{j-1}\left(\delta_T\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) + \sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}\left(\gamma_T\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) \\ &+ \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\delta_T - 1)\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 + (\gamma_T - 1)\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2. \end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(\delta_T - 1)\sum_{j=T_1}^{T_c} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

The third term

$$(\gamma_T - 1)\sum_{j=T_c+1}^{T_r} \tilde{X}_{j-1}^2 \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{f_r-f_c}{f_w c_1}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) When $T_1 \in C$ and $T_2 \in N_1$,

$$\begin{aligned} &\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_r} \tilde{X}_{j-1}\left(\gamma_T \tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) + \sum_{j=T_r+1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_{j-1} + \varepsilon_j - \tilde{X}_{j-1}\right) \\ &= \sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j + (\gamma_T - 1)\sum_{j=T_1}^{T_r} \tilde{X}_{j-1}^2 \end{aligned}$$

The first term

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T\left\{\frac{1}{2}\left[B(f_2) - B(f_r)\right]^2 - \frac{1}{2}(f_2 - f_r)\sigma^2 \right. \\ \left. -\frac{f_2-f_r}{f_w}\left[B(f_2) - B(f_r)\right]\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right\} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$(\gamma_T - 1)\sum_{j=T_1}^{T_r} \tilde{X}_{j-1}^2 \sim_a \begin{cases} -T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right]^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore, we have

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right]^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

□

2.2.1. Test asymptotics The fitted regression model for the recursive unit root tests is

$$X_t = \hat{\mu}_{f_1,f_2} + \hat{\rho}_{f_1,f_2} X_{t-1} + \hat{\varepsilon}_t,$$

where the intercept $\hat{\mu}_{f_1,f_2}$ and slope coefficient $\hat{\rho}_{f_1,f_2}$ are obtained using data over the subperiod $[f_1, f_2]$.

Remark . *Based on Lemma S.B4 and Lemma S.B5, we can obtain the limit distribution of $\hat{\delta}_T - \delta_T$ using*

$$\hat{\rho}_{f_1,f_2} - \delta_T = \frac{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right)}{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2}.$$

When $T_1 \in N_0$ and $T_2 \in B$

$$\hat{\rho}_{f_1,f_2} - \delta_T \sim_a -\frac{1}{T} 2c_1 \frac{f_e - f_1}{f_w}.$$

When $T_1 \in C$ and $T_2 \in N_1$,

$$\hat{\rho}_{f_1,f_2} - \delta_T \sim_a \begin{cases} -T^{-\beta}\frac{c_2 \frac{(f_r-f_1)(f_2-f_r)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2}{\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\}} & \text{if } \alpha > \beta \\ -T^{-\alpha} c_1 & \text{if } \alpha < \beta \end{cases}.$$

and for all other cases

$$\hat{\rho}_{f_1,f_2} - \delta_T \sim_a \begin{cases} -T^{\alpha-\beta-1} 2c_2 \frac{f_2-f_c}{f_w c_1} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T^{-\alpha} c_1 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{-\alpha} c_1 & \text{if } \alpha < \beta \end{cases}$$

Remark . *Based on Lemma S.B4 and Lemma S.B6, we can obtain the limit distribution of $\hat{\delta}_T - \gamma_T$ using*

$$\hat{\rho}_{f_1,f_2} - \gamma_T = \frac{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1} \left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1} \right)}{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2}.$$

When $T_1 \in N_0$ and $T_2 \in B$

$$\hat{\rho}_{f_1,f_2} - \gamma_T \sim_a \begin{cases} T^{-\beta} c_2 & \text{if } \alpha > \beta \\ T^{-\alpha} c_1 & \text{if } \alpha < \beta \end{cases}.$$

When $T_1 \in C$ and $T_2 \in N_1$,

$$\hat{\rho}_{f_1,f_2} - \gamma_T \sim_a \begin{cases} T^{-\beta} c_2 \frac{\left\{ \int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2 \right\}}{\int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2} & \text{if } \alpha > \beta \\ 2 \frac{1}{T} \frac{f_2-f_r}{f_w} & \text{if } \alpha < \beta \end{cases}.$$

and for all other cases

$$\hat{\rho}_{f_1,f_2} - \gamma_T \sim_a \begin{cases} T^{-\beta} c_2 & \text{if } \alpha > \beta \\ T^{-\beta} c_2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{\beta-\alpha-1} 2c_1 \frac{f_c-f_e}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Remark . *Based on Lemma S.B4 and Lemma S.B7, we can obtain the limit distribution of $\hat{\delta}_T - 1$ using*

$$\hat{\rho}_{f_1,f_2} - 1 = \frac{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1} \left(\tilde{X}_j - \tilde{X}_{j-1} \right)}{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2}.$$

When $T_1 \in N_0$ and $T_2 \in B$

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \frac{c_1}{T^\alpha}.$$

When $T_1 \in C$ and $T_2 \in N_1$

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \begin{cases} -T^{-\beta} c_2 \frac{\frac{(f_r-f_1)(f_2-f_r)}{f_w^2} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2}{\left\{ \int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2 \right\}} & \text{if } \alpha > \beta \\ -T^{-\beta} c_2 & \text{if } \alpha < \beta \end{cases}.$$

And for all other cases

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \begin{cases} -T^{\alpha-\beta-1} 2c_2 \frac{f_2-f_c}{f_w c_1} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T^{-\alpha}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T^{-\beta}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{\beta-\alpha-1} 2c_1 \frac{f_e-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Based on the above three remarks, one can see that the quantity $\hat{\rho}_{f_1,f_2} - \delta_T$ diverges to negative infinity and the quantity $\hat{\rho}_{f_1,f_2} - \gamma_T$ diverges to positive infinity. In other words, the estimated value of $\hat{\delta}_T$ is bounded by δ_T and γ_T. Furthermore, the quantity $\hat{\rho}_{f_1,f_2} - 1$ diverges to positive infinity when $T_1 \in N_0$ and $T_2 \in B$ and negative infinity when $T_1 \in C$ and $T_2 \in N_1$.

To obtain the asymptotic distributions of the Dickey-Fuller t-statistic, we first obtain the equation standard error of the regression over $[T_1, T_2]$, which is

$$\hat{\sigma}_{f_1 f_2} = \left\{ T_w^{-1} \sum_{j=T_1}^{T_2} \left(\tilde{X}_j - \hat{\rho}_{f_1,f_2} \tilde{X}_{j-1} \right)^2 \right\}^{1/2}.$$

To obtain the asymptotic distributions of the Dickey-Fuller t-statistic, we need to estimate the standard error of $\hat{\delta}_T$.
(1) When $T_1 \in N_0$ and $T_2 \in B$,

$$\hat{\sigma}_{f_1 f_2}^2$$

$$
\begin{aligned}
&= T_w^{-1}\sum_{j=T_1}^{T_2}\left(\tilde{X}_j-\hat{\rho}_{f_1,f_2}\tilde{X}_{j-1}\right)^2\\
&= T_w^{-1}\left[\sum_{j=T_1}^{T_e-1}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-1\right)\tilde{X}_{j-1}\right]^2+\sum_{j=T_e}^{T_2}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2\right]\\
&= T_w^{-1}\sum_{j=T_1}^{T_2}\varepsilon_j^2+\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2+\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w^{-1}\sum_{j=T_e}^{T_2}\tilde{X}_{j-1}^2\\
&-2\left(\hat{\rho}_{f_1,f_2}-1\right)T_w^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)T_w^{-1}\sum_{j=T_e}^{T_2}\tilde{X}_{j-1}\varepsilon_j\\
&= \left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2\left\{1+o_p(1)\right\}=O_p\left(T^{-1}\delta_T^{2(T_2-T_e)}\right)
\end{aligned}
$$

due to the fact that

$$
\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2=O_p\left(T^{-2\alpha}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(T_2-T_e)}\right)=O_p\left(T^{-1}\delta_T^{2(T_2-T_e)}\right),
$$

$$
\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w^{-1}\sum_{j=T_e}^{T_2}\tilde{X}_{j-1}^2=O_p\left(T^{-2}\right)O_p\left(T^{\alpha}\delta_T^{2(T_2-T_e)}\right)=O_p\left(T^{\alpha-2}\delta_T^{2(T_2-T_e)}\right),
$$

and the sum squared terms $\sum\tilde{X}_{j-1}^2$ always dominate the sum cross product terms $\sum\tilde{X}_{j-1}\varepsilon_j$ (see lemma 5 and Lemma 4).

(2) When $T_1\in N_0$ and $T_2\in C$,

$$
\begin{aligned}
&\hat{\sigma}_{f_1f_2}^2\\
&= T_w^{-1}\sum_{j=T_1}^{T_2}\left(\tilde{X}_j-\hat{\rho}_{f_1,f_2}\tilde{X}_{j-1}\right)^2\\
&= T_w^{-1}\left[\sum_{j=T_1}^{T_e-1}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-1\right)\tilde{X}_{j-1}\right]^2+\sum_{j=T_e}^{T_c}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2+\sum_{j=T_c+1}^{T_2}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)\tilde{X}_{j-1}\right]^2\right]\\
&= T_w^{-1}\sum_{j=T_1}^{T_2}\varepsilon_j^2+\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2+\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w^{-1}\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2+\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2\\
&-2\left(\hat{\rho}_{f_1,f_2}-1\right)T_w^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)T_w^{-1}\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)T_w^{-1}\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}^2\\
&=\begin{cases}
O_p\left(T^{2(\alpha-\beta-1)}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{4\alpha-2\beta-3}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\
o_p\left(T^{-2\alpha}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right)=o_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha\\
o_p\left(T^{-2\beta}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right)=o_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\
O_p\left(T^{2(\beta-\alpha-1)}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{4\beta-2\alpha-3}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta
\end{cases},
\end{aligned}
$$

$$
\begin{aligned}
&\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w^{-1}\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}^2\\
&=\begin{cases}
O_p\left(T^{2(\alpha-\beta-1)}\right)O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{3\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta\\
O_p\left(T^{-2\alpha}\right)O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\
O_p\left(T^{-2\alpha}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta
\end{cases},
\end{aligned}
$$

$$
\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2
$$

$$= \begin{cases} O_p\left(T^{-2\beta}\right) O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

we have

$$\hat{\sigma}^2_{f_1f_2} \sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(3) When $T_1 \in N_0$ and $T_2 \in N_1$,

$$\begin{aligned} &\hat{\sigma}^2_{f_1f_2} \\ &= T_w^{-1}\sum_{j=T_1}^{T_2}\left(\tilde{X}_j - \hat{\rho}_{f_1,f_2}\tilde{X}_{j-1}\right)^2 \\ &= T_w^{-1}\left[\sum_{j=T_1}^{T_e-1}\left[\varepsilon_j - \left(\hat{\rho}_{f_1,f_2}-1\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=T_e}^{T_c}\left[\varepsilon_j - \left(\hat{\rho}_{f_1,f_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2\right. \\ &\left. + \sum_{j=T_c+1}^{T_r}\left[\varepsilon_j - \left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=T_r+1}^{T_2}\left[\varepsilon_j - \left(\hat{\rho}_{f_1,f_2}-1\right)\tilde{X}_{j-1}\right]^2\right] \\ &= T_w^{-1}\sum_{j=T_1}^{T_2}\varepsilon_j^2 + \left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\left[\sum_{j=T_1}^{T_e-1}\tilde{X}^2_{j-1} + \sum_{j=T_r+1}^{T_2}\tilde{X}^2_{j-1}\right] + \left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w^{-1}\sum_{j=T_e}^{T_c}\tilde{X}^2_{j-1} \\ &+ \left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_r}\tilde{X}^2_{j-1} - 2\left(\hat{\rho}_{f_1,f_2}-1\right)T_w^{-1}\left[\sum_{j=T_1}^{T_e-1}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j\right] \\ &- 2\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)T_w^{-1}\sum_{j=T_e}^{T_c}\tilde{X}_{j-1}\varepsilon_j - 2\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)T_w^{-1}\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}\varepsilon_j. \end{aligned}$$

Since

$$\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w{}^{-1}\left[\sum_{j=T_1}^{T_e-1}\tilde{X}^2_{j-1} + \sum_{j=T_r+1}^{T_2}\tilde{X}^2_{j-1}\right]$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\alpha-2\beta-3}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T^{-2\alpha}\right) O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right) = o_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T^{-2\beta}\right) O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right) = o_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\beta-2\alpha-3}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases},$$

$$\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w^{-1}\sum_{j=T_e}^{T_c}\tilde{X}^2_{j-1}$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases},$$

$$\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_r}\tilde{X}^2_{j-1}$$

$$= \begin{cases} O_p\left(T^{-2\beta}\right) O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

we have

$$\hat{\sigma}^2_{f_1f_2} \sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(4)When $T_1 \in B$ and $T_2 \in C$,

$$\begin{aligned} &\hat{\sigma}^2_{f_1f_2} \\ &= T_w^{-1}\sum_{j=T_1}^{T_2}\left(\tilde{X}_j - \hat{\rho}_{f_1,f_2}\tilde{X}_{j-1}\right)^2 \\ &= T_w^{-1}\left[\sum_{j=T_1}^{T_c}\left[\varepsilon_j - \left(\hat{\rho}_{f_1,f_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=T_c+1}^{T_2}\left[\varepsilon_j - \left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)\tilde{X}_{j-1}\right]^2\right] \\ &= T_w^{-1}\sum_{j=T_1}^{T_2}\varepsilon_j^2 + \left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2 + \left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2 \\ &\quad -2\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)T_w^{-1}\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}\varepsilon_j - 2\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)T_w^{-1}\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j. \end{aligned}$$

Since

$$\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases},$$

$$\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_2}\tilde{X}_{j-1}^2$$

$$= \begin{cases} O_p\left(T^{-2\beta}\right) O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\hat{\sigma}^2_{f_1f_2} \sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(5) When $T_1 \in B$ and $T_2 \in N_1$,

$$\begin{aligned} &\hat{\sigma}^2_{f_1f_2} \\ &= T_w^{-1}\sum_{j=T_1}^{T_2}\left(\tilde{X}_j - \hat{\rho}_{f_1,f_2}\tilde{X}_{j-1}\right)^2 \end{aligned}$$

$$= T_w^{-1}\left\{\sum_{j=T_1}^{T_c}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2+\sum_{j=T_c+1}^{T_r}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)\tilde{X}_{j-1}\right]^2\right.$$

$$\left.+\sum_{j=T_r+1}^{T_2}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-1\right)\tilde{X}_{j-1}\right]\right\}$$

$$= T_w^{-1}\sum_{j=T_1}^{T_2}\varepsilon_j^2+\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2+\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2$$

$$+\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w^{-1}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2-2\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)T_w^{-1}\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}\varepsilon_j$$

$$-2\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)T_w^{-1}\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{f_1,f_2}-1\right)T_w^{-1}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j$$

Since

$$\left(\hat{\rho}_{f_1,f_2}-\delta_T\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_c}\tilde{X}_{j-1}^2$$

$$=\begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right)O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{3\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta\\ O_p\left(T^{-2\alpha}\right)O_p\left(T^{\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ O_p\left(T^{-2\alpha}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w^{-1}\sum_{j=T_c+1}^{T_r}\tilde{X}_{j-1}^2$$

$$=\begin{cases} O_p\left(T^{-2\beta}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\ O_p\left(T^{-2\beta}\right)O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha\\ O_p\left(T^{-2\beta}\right)O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ O_p\left(T^{2(\beta-\alpha-1)}\right)O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{3\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w^{-1}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2$$

$$=\begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{4\alpha-2\beta-3}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\ o_p\left(T^{-2\alpha}\right)O_p\left(T^{2\alpha-1}\delta_T^{2(T_c-T_e)}\right)=o_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha\\ o_p\left(T^{-2\beta}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right)=o_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ O_p\left(T^{2(\beta-\alpha-1)}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{4\beta-2\alpha-3}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\hat{\sigma}_{f_1f_2}^2\sim_a\begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha\\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(6) When $T_1\in C$ and $T_2\in N_1$,

$$\hat{\sigma}_{f_1f_2}^2$$

$$= T_w^{-1}\sum_{j=T_1}^{T_2}\left(\tilde{X}_j-\hat{\rho}_{f_1,f_2}\tilde{X}_{j-1}\right)^2$$

$$= T_w^{-1}\left\{\sum_{j=T_1}^{T_r}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)\tilde{X}_{j-1}\right]^2+\sum_{j=T_r+1}^{T_2}\left[\varepsilon_j-\left(\hat{\rho}_{f_1,f_2}-1\right)\tilde{X}_{j-1}\right]^2\right\}$$

$$= T_w^{-1}\sum_{j=T_1}^{T_2}\varepsilon_j^2+\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 T_w{}^{-1}\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}^2+\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w^{-1}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2$$

$$-2\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)T_w^{-1}\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}\varepsilon_j-2\left(\hat{\rho}_{f_1,f_2}-1\right)T_w^{-1}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}\varepsilon_j$$

Since

$$\left(\hat{\rho}_{f_1,f_2}-\gamma_T\right)^2 {T_w}^{-1}\sum_{j=T_1}^{T_r}\tilde{X}_{j-1}^2$$
$$=\begin{cases} O_p\left(T^{-2\beta}\right)O_p\left(T\right)=O_p\left(T^{1-2\beta}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-2\alpha}\right)O_p\left(T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right)=O_p\left(T^{\beta-2\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha<\beta \end{cases},$$

$$\left(\hat{\rho}_{f_1,f_2}-1\right)^2 T_w^{-1}\sum_{j=T_r+1}^{T_2}\tilde{X}_{j-1}^2$$
$$=\begin{cases} O_p\left(T^{-2\beta}\right)O_p\left(T\right)=O_p\left(T^{1-2\beta}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-2\beta}\right)O_p\left(T^{2\beta-1}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right)=O_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha<\beta \end{cases},$$

Therefore,

$$\hat{\sigma}_{f_1f_2}^2\sim_a\begin{cases} O_p\left(T^{1-2\beta}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{\beta-2\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha<\beta \end{cases}.$$

The asymptotic distribution of the Dickey-Fuller t statistic can be calculated as follows

$$DF_{f_1,f_2}^t=\left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2}{\hat{\sigma}_{f_1f_2}^2}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right).$$

Notice that the sign of the DF statistics depend on that of $\hat{\rho}_{f_1,f_2}-1$. (1)When $T_1\in N_0$ and $T_2\in B$,

$$DF_{f_1,f_2}^t=\left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2}{\hat{\sigma}_{f_1f_2}^2}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right)=O_p\left(T^{1-\alpha/2}\right)\to+\infty.$$

When $T_1\in C$ and $T_2\in N_1$

$$DF_{f_1,f_2}^t=\left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2}{\hat{\sigma}_{f_1f_2}^2}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right)$$
$$=\begin{cases} O_p\left(T^{1/2}\right)\to-\infty & \text{if } \alpha>\beta \\ O_p\left(T^{1/2+\alpha-\beta}\right)\to-\infty & \text{if } \alpha<\beta \end{cases}.$$

When $T_1\in N_0$ and $T_2\in C$

$$DF_{f_1,f_2}^t=\left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2}{\hat{\sigma}_{f_1f_2}^2}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right)$$
$$=\begin{cases} O_p\left(T^{\alpha/2}\right)\to-\infty & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right)\to-\infty & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right)\to-\infty & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{\beta/2}\right)\to+\infty & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

For all other cases

$$DF_{f_1,f_2}^t=\left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}_{j-1}^2}{\hat{\sigma}_{f_1f_2}^2}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right)$$
$$=\begin{cases} O_p\left(T^{\alpha/2}\right)\to-\infty & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T^{(1-\alpha+\beta)/2}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T^{(1-\beta+\alpha)/2}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{\beta/2}\right)\to+\infty & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

2.2.2. The Consistency of f_e and f_c Given that $f_2 = f$ and $f_1 \in [0, f - f_0]$, the asymptotic distributions of the backward sup DF statistic under the alternative hypothesis are:

$$BSDF_f(f_0) \sim \begin{cases} F_f(W, f_0) & \text{if } f \in N_0 \\ O_p\left(T^{1-\alpha/2}\right) \to +\infty & \text{if } f \in B \\ O_p\left(T^{\omega(\alpha,\beta)}\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} & \text{if } f \in C \end{cases}$$

This proves Theorem 3.2.

The origination of the bubble expansion and bubble collapse are identified as

$$f_e = \inf_{f \in [f_0, 1]} \left\{ f : BSDF_f(f_0) > scv^{\beta_T} \right\},$$
$$\hat{f}_c = \inf_{f \in [f_e + L_T, 1]} \left\{ f : BSDF_f(f_0) < scv^{\beta_T} \right\}.$$

We know that when $\beta_T \to 0$, $scv^{\beta_T} \to \infty$.

It is obvious that if $f \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{ BSDF_f(f_0) > scv^{\beta_T} \right\} = \Pr\left\{ F_{f_0}(W) = \infty \right\} = 0.$$

If $f \in B$, $\lim_{T\to\infty} \Pr\left\{ BSDF_f(f_0) > scv^{\beta_T} \right\} = 1$ provided that $\frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $f \in C$,

$$\lim_{T\to\infty} \Pr\left\{ BSDF_f(f_0) < scv^{\beta_T} \right\} = 1$$

provided that

$$\begin{cases} \frac{T^{\alpha/2}}{scv^{\beta_T}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^{\beta_T}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^{\beta_T}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^{\beta_T}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{ \hat{f}_e > f_e + \eta \right\} \to 0 \text{ and } \Pr\left\{ \hat{f}_c < f_c - \gamma \right\} \to 0,$$

since $\Pr\left\{ BSDF_{f_e + a_\eta}(f_0) > scv^{\beta_T} \right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{ BSDF_{f_c - a_\gamma}(f_0) > scv^{\beta_T} \right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary, $\Pr\left\{ \hat{f}_e < f_e \right\} \to 0$ and $\Pr\left\{ \hat{f}_c > f_c \right\} \to 0$, we deduce that $\Pr\left\{ |\hat{f}_e - f_e| > \eta \right\} \to 0$ and $\Pr\left\{ |\hat{f}_c - f_c| > \gamma \right\} \to 0$ as $T \to \infty$, provided that

$$\begin{cases} \frac{T^{\alpha/2}}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Therefore, $\hat{f}_e$ and $\hat{f}_c$ are consistent estimators of f_e and f_c. This proves Theorem 3.3.

2.2.3. Auxiliary Lemmas

Lemma S.B9. *Under the stated conditions, we have*

$$T^{-1/2-\beta} \sqrt{\frac{2}{c_2(f_c - f_1)}} \sum_{j=T_1}^{T_c} \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l} \xrightarrow{L} X_{c_2} \equiv N\left(0, \frac{\sigma^2}{2c_2}\right).$$
$$T^{-1/2-\beta} \sqrt{\frac{2}{c_2(f_2 - f_c)}} \sum_{j=T_c+1}^{T_2} \sum_{l=0}^{j-T_c-1} \gamma_T^l \varepsilon_{j-l} \xrightarrow{L} X_{c_2} \equiv N\left(0, \frac{\sigma^2}{2c_2}\right).$$

Proof.

$$
\begin{aligned}
&\sum_{j=T_1}^{T_c}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\\
&=\sum_{j=T_1}^{T_c}\sum_{k=j}^{T_c+1}\gamma_T^{j-k}\varepsilon_k=\sum_{j=T_1}^{T_c}\left(\gamma_T^{j-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{j-T_c-2}\varepsilon_{T_c+2}+...+\gamma_T\varepsilon_{j-1}+\gamma_T^0\varepsilon_j\right)\\
&=\left(\gamma_T^{T_c-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_c-T_c-2}\varepsilon_{T_c+2}+...+\gamma_T\varepsilon_{T_c-1}+\gamma_T^0\varepsilon_{T_c}\right)\\
&\quad+\left(\gamma_T^{T_c-1-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_c-1-T_c-2}\varepsilon_{T_c+2}+...+\gamma_T\varepsilon_{T_c-2}+\gamma_T^0\varepsilon_{T_c-1}\right)+...\\
&\quad+\left(\gamma_T^{T_1+1-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_1+1-T_c-2}\varepsilon_{T_c+2}+...+\gamma_T\varepsilon_{T_1}+\gamma_T^0\varepsilon_{T_1+1}\right)\\
&\quad+\left(\gamma_T^{T_1-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_1-T_c-2}\varepsilon_{T_c+2}+...+\gamma_T\varepsilon_{T_1-1}+\gamma_T^0\varepsilon_{T_1}\right)\\
&=\varepsilon_{T_c+1}\left(\gamma_T^{T_c-T_c-1}+\gamma_T^{T_c-1-T_c-1}+...+\gamma_T^{T_1-T_c-1}\right)\\
&\quad+\varepsilon_{T_c+2}\left(\gamma_T^{T_c-T_c-2}+\gamma_T^{T_c-1-T_c-2}+...+\gamma_T^{T_1-T_c-2}\right)+...\\
&\quad+\varepsilon_{T_1}\left(\gamma_T^{T_c-T_1}+\gamma_T^{T_c-1-T_1}+...+\gamma_T^{T_1-T_1}\right)\\
&\quad+\varepsilon_{T_1+1}\left(\gamma_T^{T_c-T_1-1}+\gamma_T^{T_c-1-T_1-1}+...+\gamma_T^{T_1+1-T_1-1}\right)+...+\varepsilon_{T_c}\gamma_T^{T_c-T_c}\\
&=\varepsilon_{T_c+1}\frac{\gamma_T^{T_1-T_c-1}\left(\gamma_T^{T_c-T_1+1}-1\right)}{\gamma_T-1}+\varepsilon_{T_c+2}\frac{\gamma_T^{T_1-T_c-2}\left(\gamma_T^{T_c-T_1+1}-1\right)}{\gamma_T-1}+...\\
&\quad+\varepsilon_{T_1}\frac{\gamma_T^{T_c-T_1+1}-1}{\gamma_T-1}+\varepsilon_{T_1+1}\frac{\gamma_T^{T_c-T_1}-1}{\gamma_T-1}+...+\varepsilon_{T_c}\frac{\gamma_T-1}{\gamma_T-1}\\
&=\varepsilon_{T_c+1}\frac{\gamma_T^{T_1-T_c-1}}{c_2T^{-\beta}}+\varepsilon_{T_c+2}\frac{\gamma_T^{T_1-T_c-2}}{c_2T^{-\beta}}+...+\varepsilon_{T_1}\frac{1}{c_2T^{-\beta}}+\varepsilon_{T_1+1}\frac{1}{c_2T^{-\beta}}+...+\varepsilon_{T_c}\frac{1}{c_2T^{-\beta}}
\end{aligned}
$$

Therefore, we know that

$$
\begin{aligned}
&E\left[\left(\sum_{j=T_1}^{T_c}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right)^2\right]\\
&=E\left[\left(\varepsilon_{T_c+1}\frac{\gamma_T^{T_1-T_c-1}}{c_2T^{-\beta}}+\varepsilon_{T_c+2}\frac{\gamma_T^{T_1-T_c-2}}{c_2T^{-\beta}}+...+\varepsilon_{T_1}\frac{1}{c_2T^{-\beta}}+\varepsilon_{T_1+1}\frac{1}{c_2T^{-\beta}}+...+\varepsilon_{T_c}\frac{1}{c_2T^{-\beta}}\right)^2\right]\\
&=\left\{\frac{T^{2\beta}}{c_2^2}\left[\gamma_T^{2(T_1-T_c-1)}+\gamma_T^{2(T_1-T_c-2)}+...+1\right]+\frac{T^{2\beta}}{c_2^2}(T_c-T_1)\right\}\sigma^2\\
&=\left[\frac{T^{2\beta}}{c_2^2}\frac{\gamma_T^{2(T_1-T_c)}-1}{\gamma_T^2-1}+\frac{T^{2\beta}}{c_2^2}(T_c-T_1)\right]\sigma^2\\
&=\left[\frac{T^{3\beta}}{2c_2^3}+\frac{T^{1+2\beta}}{c_2^2}(f_c-f_1)\right]\sigma^2\left\{1+o_p(1)\right\}\\
&=T^{2\beta}(T_c-T_1)\frac{\sigma^2}{c_2^2}\left\{1+o_p(1)\right\}
\end{aligned}
$$

Therefore, we have

$$
T^{-1/2-\beta}\sqrt{\frac{2}{c_2(f_c-f_1)}}\sum_{j=T_1}^{T_c}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\xrightarrow{L}X_{c_2}\equiv N\left(0,\frac{\sigma^2}{2c_2}\right).
$$

The proof of the second equation is similar to that of the first equation. □

Lemma S.B10. *Under the stated conditions, we have*

$$
T^{-3\beta/2}\gamma_T^{-(T_1-T_c-1)}2c_2\sum_{j=T_1}^{T_c}\gamma_T^{j-1-T_c}\left(\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right)\xrightarrow{L}X_{c_2}\equiv N\left(0,\frac{\sigma^2}{2c_2}\right),
$$

$$T^{-3\beta/2}2c_2\sum_{j=T_c+1}^{T_2}\gamma_T^{j-1-T_c}\left(\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right)\xrightarrow{L}X_{c_2}\equiv N\left(0,\frac{\sigma^2}{2c_2}\right).$$

Proof.

$$\begin{aligned}
&\sum_{j=T_1}^{T_c}\gamma_T^{j-1-T_c}\left(\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right)\\
&=\sum_{j=T_1}^{T_c}\gamma_T^{j-1-T_c}\left(\sum_{k=T_c+1}^{j}\gamma_T^{j-k}\varepsilon_k\right)=\sum_{j=T_1}^{T_c}\gamma_T^{j-1-T_c}\left(\gamma_T^{j-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{j-T_c-2}\varepsilon_{T_c+2}+\ldots+\gamma_T\varepsilon_{j-1}+\gamma_T^0\varepsilon_j\right)\\
&=\gamma_T^{T_c-1-T_c}\left(\gamma_T^{T_c-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_c-T_c-2}\varepsilon_{T_c+2}+\ldots+\gamma_T\varepsilon_{T_c-1}+\gamma_T^0\varepsilon_{T_c}\right)\\
&\quad+\gamma_T^{T_c-1-1-T_c}\left(\gamma_T^{T_c-1-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_c-1-T_c-2}\varepsilon_{T_c+2}+\ldots+\gamma_T\varepsilon_{T_c-2}+\gamma_T^0\varepsilon_{T_c-1}\right)\\
&\quad+\ldots\\
&\quad+\gamma_T^{T_1+1-1-T_c}\left(\gamma_T^{T_1+1-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_1+1-T_c-2}\varepsilon_{T_c+2}+\ldots+\gamma_T\varepsilon_{T_1}+\gamma_T^0\varepsilon_{T_1+1}\right)\\
&\quad+\gamma_T^{T_1-1-T_c}\left(\gamma_T^{T_1-T_c-1}\varepsilon_{T_c+1}+\gamma_T^{T_1-T_c-2}\varepsilon_{T_c+2}+\ldots+\gamma_T\varepsilon_{T_1-1}+\gamma_T^0\varepsilon_{T_1}\right)\\
&=\varepsilon_{T_c+1}\left(\gamma_T^{T_c-1-T_c}\gamma_T^{T_c-T_c-1}+\gamma_T^{T_c-1-1-T_c}\gamma_T^{T_c-1-T_c-1}+\ldots+\gamma_T^{T_1-1-T_c}\gamma_T^{T_1-T_c-1}\right)\\
&\quad+\varepsilon_{T_c+2}\left(\gamma_T^{T_c-1-T_c}\gamma_T^{T_c-T_c-2}+\gamma_T^{T_c-1-1-T_c}\gamma_T^{T_c-1-T_c-2}+\ldots+\gamma_T^{T_1-1-T_c}\gamma_T^{T_1-T_c-2}\right)\\
&\quad+\ldots\\
&\quad+\varepsilon_{T_1}\left(\gamma_T^{T_c-1-T_c}\gamma_T^{T_c-T_1}+\gamma_T^{T_c-1-1-T_c}\gamma_T^{T_c-1-T_1}+\ldots+\gamma_T^{T_1-1-T_c}\gamma_T^{T_1-T_1}\right)\\
&\quad+\varepsilon_{T_1+1}\left(\gamma_T^{T_c-1-T_c}\gamma_T^{T_c-T_1-1}+\gamma_T^{T_c-1-1-T_c}\gamma_T^{T_c-1-T_1-1}+\ldots+\gamma_T^{T_1+1-1-T_c}\gamma_T^{T_1+1-T_1-1}\right)\\
&\quad+\varepsilon_{T_1+2}\left(\gamma_T^{T_c-1-T_c}\gamma_T^{T_c-T_1-2}+\gamma_T^{T_c-1-1-T_c}\gamma_T^{T_c-1-T_1-2}+\ldots+\gamma_T^{T_1+2-1-T_c}\gamma_T^{T_1+2-T_1-2}\right)\\
&\quad+\ldots+\varepsilon_{T_c}\gamma_T^{T_c-1-T_c}\gamma_T^{T_c-T_c}\\
&=\varepsilon_{T_c+1}\frac{\gamma_T^{2\tau_1-2\tau_f-2}\left(\gamma_T^{2(T_c-T_1+1)}-1\right)}{\gamma_T^2-1}+\varepsilon_{T_c+2}\frac{\gamma_T^{2\tau_1-2\tau_f-3}\left(\gamma_T^{2(T_c-T_1+1)}-1\right)}{\gamma_T^2-1}+\ldots\\
&\quad+\varepsilon_{T_1}\frac{\gamma_T^{2\tau_1-T_c-T_1-1}\left(\gamma_T^{2(T_c-T_1+1)}-1\right)}{\gamma_T^2-1}+\varepsilon_{T_1+1}\frac{\gamma_T^{T_1-T_c}\left(\gamma_T^{2(T_c-T_1)}-1\right)}{\gamma_T^2-1}+\\
&\quad+\varepsilon_{T_1+2}\frac{\gamma_T^{T_1-T_c+1}\left(\gamma_T^{2(T_c-T_1-1)}-1\right)}{\gamma_T^2-1}\ldots+\varepsilon_{T_c}\frac{\gamma_T^{T_c-1-T_c}\left(\gamma_T^2-1\right)}{\gamma_T^2-1}\\
&=\left[\varepsilon_{T_c+1}\frac{\gamma_T^{2\tau_1-2\tau_f-2}}{2c_2T^{-\beta}}+\varepsilon_{T_c+2}\frac{\gamma_T^{2\tau_1-2\tau_f-3}}{2c_2T^{-\beta}}+\ldots+\varepsilon_{T_1}\frac{\gamma_T^{T_1-T_c-1}}{2c_2T^{-\beta}}+\varepsilon_{T_1+1}\frac{\gamma_T^{T_1-T_c}}{2c_2T^{-\beta}}\ldots+\varepsilon_{T_c}\frac{\gamma_T^{T_c-1-T_c}}{2c_2T^{-\beta}}\right]\left\{1+o_p(1)\right\}
\end{aligned}$$

Therefore, we know that

$$\begin{aligned}
&E\left[\left(\sum_{j=T_1}^{T_c}\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right)^2\right]\\
&=E\left[\left(\varepsilon_{T_c+1}\frac{\gamma_T^{2\tau_1-2\tau_f-2}}{2c_2T^{-\beta}}+\varepsilon_{T_c+2}\frac{\gamma_T^{2\tau_1-2\tau_f-3}}{2c_2T^{-\beta}}+\ldots+\varepsilon_{T_1}\frac{\gamma_T^{T_1-T_c-1}}{2c_2T^{-\beta}}+\varepsilon_{T_1+1}\frac{\gamma_T^{T_1-T_c}}{2c_2T^{-\beta}}\ldots+\varepsilon_{T_c}\frac{\gamma_T^{T_c-1-T_c}}{2c_2T^{-\beta}}\right)^2\right]\\
&=\frac{T^{2\beta}}{4c_2^2}\left[\left(\gamma_T^{2\left(2\tau_1-2\tau_f-2\right)}+\gamma_T^{2\left(2\tau_1-2\tau_f-3\right)}+\ldots+\gamma_T^{2(T_1-T_c-1)}\right)+\left(\gamma_T^{2(T_1-T_c)}+\ldots+\gamma_T^{2(T_c-1-T_c)}\right)\right]\sigma^2\\
&=\frac{T^{2\beta}}{4c_2^2}\left\{\frac{\gamma_T^{2(T_1-T_c-1)}\left[\gamma_T^{2(T_1-T_c)}-1\right]}{\gamma_T^2-1}+\frac{\gamma_T^{2(T_1-T_c)}\left[\gamma_T^{2(T_c-T_1)}-1\right]}{\gamma_T^2-1}\right\}\sigma^2\\
&=\frac{T^{2\beta}}{4c_2^2}\left\{\frac{\gamma_T^{2(T_1-T_c-1)}}{2c_2T^{-\beta}}+\frac{\gamma_T^{2(T_1-T_c)}}{2c_2T^{-\beta}}\right\}\sigma^2\left\{1+o_p(1)\right\}
\end{aligned}$$

$$= T^{3\beta}\gamma_T^{2(T_1-T_c-1)}\frac{\sigma^2}{8c_2^3}\left\{1+o_p(1)\right\}$$

Therefore,

$$T^{-3\beta/2}\gamma_T^{-(T_1-T_c-1)}2c_2\sum_{j=T_1}^{T_c}\gamma_T^{j-1-T_c}\left(\sum_{l=0}^{j-T_c-1}\gamma_T^l\varepsilon_{j-l}\right)\xrightarrow{L}X_{c_2}\equiv N\left(0,\frac{\sigma^2}{2c_2}\right).$$

The proof of the second equation is similar to that of the first equation. □

2.3. Dating Bubble Implosion

Define the demean quantity as $\tilde{X}_t^*\equiv X_t^*-\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j^*$. Since $\tau_w=T_w$ and

$$\sum_{j=\tau_1}^{\tau_2}X_j^*=\sum_{j=\tau_1}^{\tau_2}X_{T+1-j}=\sum_{i=T+1-\tau_2}^{T+1-\tau_1}X_i=\sum_{i=T_1}^{T_2}X_i,$$

we have

$$\tilde{X}_t^*=X_t^*-\frac{1}{T_w}\sum_{j=T_1}^{T_2}X_j^*=X_{T+1-t}-\frac{1}{T_w}\sum_{i=T_1}^{T_2}X_i=\tilde{X}_{T+1-t}.$$

Based on this linkage, we derive the next three lemmas.

Lemma S.C1. *The sum of squared $\tilde{X}_t^*$ terms are as follows.*

(1) *For $\tau_1\in B$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\sum_{j=T_1}^{T_2}\tilde{X}_{j+1}^2\sim_a T^{1+\alpha}\delta_T^{2(T_2-T_e)}\frac{1}{2c_1}B(f_e)^2 .$$

(2) *For $\tau_1\in C$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\sum_{j=T_1}^{T_2}\tilde{X}_{j+1}^2\sim_a\begin{cases}T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta\\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta\end{cases}.$$

(3) *For $\tau_1\in N_1$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\sum_{j=T_1}^{T_2}\tilde{X}_{j+1}^2\sim_a\begin{cases}T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta\\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha\le\beta\end{cases}.$$

(4) *For $\tau_1\in C$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\sum_{j=T_1}^{T_2}\tilde{X}_{j+1}^2\sim_a\begin{cases}T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta\\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta\end{cases}.$$

(5) *For $\tau_1\in N_1$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\sum_{j=T_1}^{T_2}\tilde{X}_{j+1}^2\sim_a\begin{cases}T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta\\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta\end{cases}.$$

(6) *For $\tau_1\in N_1$ and $\tau_2\in C$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\sum_{j=T_1}^{T_2}\tilde{X}_{j+1}^2$$

$$\sim_a\begin{cases}T^2(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2ds-\frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\} & \text{if } \alpha>\beta\\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta\end{cases}.$$

Lemma S.C2. *The sums of cross-products of $\tilde{X}_t^*$ and v_t are as follows.*

(1) *For $\tau_1\in B$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*v_j\sim_a -T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e).$$

(2) *For* $\tau_1 \in C$ *and* $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(3) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(4) *For* $\tau_1 \in C$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(5) *For* $\tau_1 \in C$ *and* $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(6) *For* $\tau_1 \in C$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T\left\{\frac{1}{2}\left[B(f_2) - B(f_r)\right]^2 + \frac{1}{2}(f_2 - f_r)\sigma^2 \right. \\ \left. -\frac{f_2 - f_r}{f_w}\left[B(f_2) - 2B(f_r) + B(f_1)\right]\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right\} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\gamma_T^{T_1 - T_c}\delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

Proof. By construction, we have

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j = -\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{T+2-j} \varepsilon_{T+2-j} = -\sum_{j=T_1+1}^{T_2+1} \tilde{X}_j \varepsilon_j.$$

(1) For $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j = -\sum_{j=T_1+1}^{T_e} \tilde{X}_j \varepsilon_j - \sum_{j=T_e+1}^{T_2+1} \tilde{X}_j \varepsilon_j$$

The first term

$$\begin{aligned} \sum_{j=T_1+1}^{T_e} \tilde{X}_j \varepsilon_j &= \sum_{j=T_1+1}^{T_e} \tilde{X}_j \varepsilon_j \\ &= \sum_{j=T_1+1}^{T_e} -\frac{T^{\alpha}\delta_T^{T_2 - T_e}}{T_w c_1} X_{T_e} \varepsilon_j \left\{1 + o_p(1)\right\} \\ &= -\frac{T^{\alpha}\delta_T^{T_2 - T_e}}{T_w c_1} X_{T_e} \sum_{j=T_1+1}^{T_e} \varepsilon_j \left\{1 + o_p(1)\right\} \\ &= -\frac{T^{\alpha}\delta_T^{T_2 - T_e}}{f_w c_1}\left(T^{-1/2} X_{T_e}\right)\left(T^{-1/2}\sum_{j=T_1+1}^{T_e} \varepsilon_j\right)\left\{1 + o_p(1)\right\} \\ &\sim_a -\frac{T^{\alpha}\delta_T^{T_2 - T_e}}{f_w c_1} B(f_e)\left[B(f_e) - B(f_1)\right]. \end{aligned}$$

The second term

$$\begin{aligned} &\sum_{j=T_e+1}^{T_2+1} \tilde{X}_j \varepsilon_j \\ &= \sum_{j=T_e+1}^{T_2+1}\left[\delta_T^{j-T_e} - \frac{T^{\alpha}\delta_T^{T_2 - T_e}}{T_w c_1}\right] X_{T_e} \varepsilon_j \left\{1 + o_p(1)\right\} \\ &= \left[\sum_{j=T_e+1}^{T_2+1} \delta_T^{j-T_e} \varepsilon_j - \frac{T^{\alpha}\delta_T^{T_2 - T_e}}{T_w c_1}\sum_{j=T_e+1}^{T_2+1} \varepsilon_j\right] X_{T_e}\left\{1 + o_p(1)\right\} \end{aligned}$$

$$= \left[T^{\alpha/2}\delta_T^{T_2-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_e+1}^{T_2+1}\delta_T^{-T_2+j}\varepsilon_j\right) - \frac{\delta_T^{T_2-T_e}}{T^{1/2-\alpha}f_w c_1}\left(\frac{1}{\sqrt{T}}\sum_{j=T_e+1}^{T_2+1}\varepsilon_j\right)\right] X_{T_e}\left\{1+o_p(1)\right\}$$

$$= T^{\alpha/2}\delta_T^{T_2-T_e}\left(T^{-\alpha}\sum_{j=T_e+1}^{T_2+1}\delta_T^{-(T_2-j)}\varepsilon_j\right)X_{T_e}\left\{1+o_p(1)\right\} \text{ (since } \alpha/2>\alpha-1/2)$$

$$\sim_a T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e).$$

This is due to the fact that

$$E\left[\left(\sum_{j=T_e+1}^{T_2+1}\delta_T^{j-T_e}\varepsilon_j\right)^2\right] = \sum_{j=T_e+1}^{T_2+1}\delta_T^{2(j-T_e)}E\left(\varepsilon_j^2\right) = \sigma^2\frac{\delta_T^2\left[\delta_T^{2(T_2-T_e+1)}-1\right]}{\delta_T^2-1}$$

$$= \sigma^2\frac{\left(1+2c_1T^{-\alpha}+c_2^2T^{-2\alpha}\right)\left[\delta_T^{2(T_2-T_e+1)}-1\right]}{2c_1T^{-\alpha}+c_2^2T^{-2\alpha}}$$

$$= T^{\alpha}\delta_T^{2(T_2-T_e)}\frac{\sigma^2}{2c_1}\left\{1+o_p(1)\right\}$$

$$\frac{1}{T^{\alpha/2}}\sum_{j=T_e+1}^{T_2+1}\delta_T^{-T_2+j}\varepsilon_j \xrightarrow{L} X_{c_1} \equiv N\left(0,\sigma^2/2c_1\right)$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a -T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e)$$

(2)For $T_1 \in C$, $T_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j = -\sum_{j=T_1+1}^{T_e}\tilde{X}_j\varepsilon_j - \sum_{j=T_e+1}^{T_c}\tilde{X}_j\varepsilon_j - \sum_{j=T_c+1}^{T_2+1}\tilde{X}_j\varepsilon_j$$

Suppose $\alpha > \beta$. The first term

$$\sum_{j=T_1+1}^{T_e}\tilde{X}_j\varepsilon_j = \sum_{j=T_1+1}^{T_e} -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\varepsilon_j\left\{1+o_p(1)\right\}$$

$$= -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\sum_{j=T_1+1}^{T_e}\varepsilon_j\left\{1+o_p(1)\right\}$$

$$= -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}X_{T_e}\right)\left(T^{-1/2}\sum_{j=T_1+1}^{T_e}\varepsilon_j\right)\left\{1+o_p(1)\right\}$$

$$\sim_a -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{f_w c_1}B(f_e)\left[B(f_e)-B(f_1)\right].$$

The second term

$$\sum_{j=T_e+1}^{T_c}\tilde{X}_j\varepsilon_j$$

$$= \sum_{j=T_e+1}^{T_c}\left[\delta_T^{j-T_e} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\right]X_{T_e}\varepsilon_j\left\{1+o_p(1)\right\}$$

$$= \left[\sum_{j=T_e}^{T_c}\delta_T^{j-T_e}\varepsilon_j - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\sum_{j=T_e+1}^{T_c}\varepsilon_j\right]X_{T_e}\left\{1+o_p(1)\right\}$$

$$= \left[T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_e+1}^{T_c}\delta_T^{-(T_c-j+1)}\varepsilon_j\right) - \frac{T^{\alpha-1/2}\delta_T^{T_c-T_e}}{f_w c_1}\left(\frac{1}{\sqrt{T}}\sum_{j=T_e+1}^{T_c}\varepsilon_j\right)\right]X_{T_e}\left\{1+o_p(1)\right\}$$

$$= T^{(1+\alpha)/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_e+1}^{T_c}\delta_T^{-(T_c-j+1)}\varepsilon_j\right)\left(T^{-1/2}X_{T_e}\right)\left\{1+o_p(1)\right\}$$

$$\sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

The third term is

$$\begin{aligned}
&\sum_{j=T_c+1}^{T_2+1} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=T_c+1}^{T_2+1} \left[\gamma_T^{j-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\right] \varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[X_{T_c} \sum_{j=T_c+1}^{T_2+1} \gamma_T^{j-T_c}\varepsilon_j - \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \sum_{j=T_c+1}^{T_2+1}\varepsilon_j\right] \left\{1+o_p(1)\right\} \\
&= \left[T^{\beta/2} X_{T_c}\left(T^{-\beta/2}\sum_{j=T_c+1}^{T_2+1}\gamma_T^{j-T_c}\varepsilon_j\right) - \frac{T^\alpha \delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}X_{T_e}\right)\left(T^{-1/2}\sum_{j=T_c+1}^{T_2+1}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\} \\
&\sim_a \begin{cases} -T^\alpha \delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e)\left[B(f_2)-B(f_c)\right] & \text{if } 2\alpha > 1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } 2\alpha < 1+\beta \end{cases}
\end{aligned}$$

due to the fact that

$$E\left[\left(\sum_{j=T_c+1}^{T_2+1}\gamma_T^{j-T_c}\varepsilon_j\right)^2\right] = \sum_{j=T_c+1}^{T_2+1}\gamma_T^{2(j-T_c)} E\left(\varepsilon_j^2\right) \sim_a T^\beta \frac{\sigma^2}{2c_2}$$

$$T^{\beta/2} X_{T_c}\left(T^{-\beta/2}\sum_{j=T_c+1}^{T_2+1}\gamma_T^{j-T_c}\varepsilon_j\right) \sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2}$$

$$\frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \sum_{j=T_c+1}^{T_2}\varepsilon_j \sim_a T^\alpha \delta_T^{T_c-T_e}\frac{1}{f_w c_1} B(f_e)\left[B(f_2)-B(f_c)\right]$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* v_j \sim_a -T^{(1+\alpha)/2}\delta_T^{T_c-T_e} X_{c_1} B(f_e).$$

If $\alpha < \beta$, the first term

$$\begin{aligned}
\sum_{j=T_1+1}^{T_e} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=T_1+1}^{T_e} -\frac{T^\beta}{T_w c_2} X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\} \\
&= -\frac{T^\beta}{T_w c_2} X_{T_c}\sum_{j=T_1+1}^{T_e}\varepsilon_j\left\{1+o_p(1)\right\} \\
&= -\frac{T^\beta \delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}\delta_T^{-(T_c-T_e)} X_{T_c}\right)\left(T^{-1/2}\sum_{j=T_1+1}^{T_e}\varepsilon_j\right)\left\{1+o_p(1)\right\} \\
&\sim_a -T^\beta \delta_T^{T_c-T_e}\frac{1}{f_w c_2} B(f_e)\left[B(f_e)-B(f_1)\right].
\end{aligned}$$

The second term

$$\begin{aligned}
&\sum_{j=T_e+1}^{T_c} \tilde{X}_j\varepsilon_j \\
&= \sum_{j=T_e+1}^{T_c}\left[\delta_T^{j-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right]\varepsilon_j\left\{1+o_p(1)\right\} \\
&= \left[X_{T_e}\sum_{j=T_e+1}^{T_c}\delta_T^{j-T_e}\varepsilon_j - \frac{T^\beta}{T_w c_2} X_{T_c}\sum_{j=T_e+1}^{T_c}\varepsilon_j\right]\left\{1+o_p(1)\right\} \\
&= \left[T^{(1+\alpha)/2}\delta_T^{T_c-T_e}\left(T^{-1/2}X_{T_e}\right)\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_e+1}^{T_c}\delta_T^{-(T_c-j)}\varepsilon_j\right)\right. \\
&\quad \left. -\frac{T^\beta \delta_T^{T_c-T_e}}{f_w c_2}\left(T^{-1/2}\delta_T^{-(T_c-T_e)} X_{T_c}\right)\left(\frac{1}{\sqrt{T}}\sum_{j=T_e+1}^{T_c}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\}
\end{aligned}$$

$$\sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e). & \text{if } 1+\alpha>2\beta \\ -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)[B(f_c)-B(f_e)] & \text{if } 1+\alpha<2\beta \end{cases}.$$

The third term

$$\begin{aligned}
&\sum_{j=T_c+1}^{T_2+1}\tilde{X}_j\varepsilon_j \\
&=\sum_{j=T_c+1}^{T_2+1}\left[\gamma_T^{j-T_c}-\frac{T^{\beta}}{T_w c_2}\right]X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\} \\
&=\left[X_{T_c}\sum_{j=T_c+1}^{T_2+1}\gamma_T^{j-T_c}\varepsilon_j-\frac{T^{\beta}}{T_w c_2}X_{T_c}\sum_{j=T_c+1}^{T_2+1}\varepsilon_j\right]\left\{1+o_p(1)\right\} \\
&=\left[T^{\beta/2}X_{T_c}\left(T^{-\beta/2}\sum_{j=T_c+1}^{T_2+1}\gamma_T^{j-T_c}\varepsilon_j\right)-\frac{T^{\beta}\delta_T^{T_c-T_e}}{f_w c_2}\left(T^{-1/2}\delta_T^{-(T_c-T_e)}X_{T_c}\right)\left(T^{-1/2}\sum_{j=T_c+1}^{T_2+1}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\} \\
&=X_{T_c}\sum_{j=T_c+1}^{T_2}\gamma_T^{j-1-T_c}\varepsilon_j\left\{1+o_p(1)\right\} \\
&\sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.
\end{aligned}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

(3) For $\tau_1\in N_1$, $\tau_2\in N_0$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j=-\sum_{j=T_1+1}^{T_e}\tilde{X}_j\varepsilon_j-\sum_{j=T_e+1}^{T_c}\tilde{X}_j\varepsilon_j-\sum_{j=T_c+1}^{T_r+1}\tilde{X}_j\varepsilon_j-\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j$$

Suppose $\alpha>\beta$. The first term

$$\sum_{j=T_1+1}^{T_e}\tilde{X}_j\varepsilon_j\sim_a -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_e)-B(f_1)\right].$$

The second term

$$\sum_{j=T_e+1}^{T_c}\tilde{X}_j\varepsilon_j\sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

The third term

$$\begin{aligned}
&\sum_{j=T_c+1}^{T_r+1}\tilde{X}_j\varepsilon_j \\
&=\sum_{j=T_c+1}^{T_r+1}\left[\gamma_T^{j-T_c}X_{T_c}-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\right]\varepsilon_j\left\{1+o_p(1)\right\} \\
&=\left[X_{T_c}\sum_{j=T_c+1}^{T_r+1}\gamma_T^{j-T_c}\varepsilon_j-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\sum_{j=T_c+1}^{T_r+1}\varepsilon_j\right]\left\{1+o_p(1)\right\} \\
&=\left[T^{\beta/2}X_{T_c}\left(T^{-\beta/2}\sum_{j=T_c+1}^{T_r+1}\gamma_T^{j-T_c}\varepsilon_j\right)-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}X_{T_e}\right)\left(T^{-1/2}\sum_{j=T_c+1}^{T_r+1}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\} \\
&\sim_a \begin{cases} -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_r)-B(f_c)\right] & \text{if } 2\alpha>1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } 2\alpha<1+\beta \end{cases}
\end{aligned}$$

due to the fact that

$$E\left[\left(\sum_{j=T_c+1}^{T_r+1}\gamma_T^{j-T_c}\varepsilon_j\right)^2\right]=\sum_{j=T_c+1}^{T_r+1}\gamma_T^{2(j-T_c)}E\left(\varepsilon_j^2\right)\sim_a T^{\beta}\frac{\sigma^2}{2c_2}$$

$$X_{T_c}\sum_{j=T_c+1}^{T_r+1}\gamma_T^{j-1-T_c}\varepsilon_j\sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2}$$

$$\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\sum_{j=T_c+1}^{T_r+1}\varepsilon_j \sim {}_a T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_r)-B(f_c)\right]$$

The fourth term

$$\begin{aligned}\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j &= \sum_{j=T_r+2}^{T_2+1}-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\varepsilon_j\left\{1+o_p(1)\right\}\\ &= -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}X_{T_e}\right)\left(T^{-1/2}\sum_{j=T_r+2}^{T_2+1}\varepsilon_j\right)\left\{1+o_p(1)\right\}\\ &\sim_a -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{f_w c_1}B(f_e)\left[B(f_2)-B(f_r)\right].\end{aligned}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

Suppose $\alpha<\beta$. The first term

$$\sum_{j=T_1+1}^{T_e}\tilde{X}_j\varepsilon_j \sim_a -\frac{T^{\beta}\delta_T^{T_c-T_e}}{f_w c_2}B(f_e)\left[B(f_e)-B(f_1)\right].$$

The second term

$$\sum_{j=T_e+1}^{T_c}\tilde{X}_j\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e) & \text{if } 1+\alpha>2\beta\\ -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_c)-B(f_e)\right] & \text{if } 1+\alpha<2\beta\end{cases}.$$

The third term

$$\begin{aligned}\sum_{j=T_c+1}^{T_r+1}\tilde{X}_j\varepsilon_j &= \sum_{j=T_c+1}^{T_r+1}\left[\gamma_T^{j-T_c}-\frac{T^{\beta}}{T_w c_2}\right]X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}\\ &\sim {}_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.\end{aligned}$$

The fourth term

$$\begin{aligned}\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j &= \sum_{j=T_r+2}^{T_2+1}-\frac{T^{\beta}}{T_w c_2}X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}\\ &= -\frac{T^{\beta}\delta_T^{T_c-T_e}}{f_w c_2}\left(T^{-1/2}\delta_T^{-(T_c-T_e)}X_{T_c}\right)\left(T^{-1/2}\sum_{j=T_r+2}^{T_2+1}\varepsilon_j\right)\left\{1+o_p(1)\right\}\\ &\sim_a -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_2)-B(f_r)\right].\end{aligned}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta\\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta\end{cases}.$$

(4)For $\tau_1\in C$ and $\tau_2\in B$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j = -\sum_{j=T_1+1}^{T_c}\tilde{X}_j\varepsilon_j-\sum_{j=T_c+1}^{T_2+1}\tilde{X}_j\varepsilon_j$$

Suppose $\alpha>\beta$. The first term

$$\begin{aligned}\sum_{j=T_1+1}^{T_c}\tilde{X}_j\varepsilon_j &= \sum_{j=T_1+1}^{T_c}\left[\delta_T^{j-T_e}-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\right]X_{T_e}\varepsilon_j\left\{1+o_p(1)\right\}\\ &= X_{T_e}\left[\sum_{j=T_1+1}^{T_c}\delta_T^{j-T_e}\varepsilon_j-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\sum_{j=T_1+1}^{T_c}\varepsilon_j\right]\left\{1+o_p(1)\right\}\\ &= X_{T_e}\left[T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_1+1}^{T_c}\delta_T^{j-T_c}\varepsilon_j\right)-\frac{T^{\alpha-1/2}\delta_T^{T_c-T_e}}{f_w c_1}\left(T^{-1/2}\sum_{j=T_1+1}^{T_c}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\}\end{aligned}$$

$$= T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_1}^{T_c}\delta_T^{j-T_c}\varepsilon_j\right)X_{T_e}\left\{1+o_p(1)\right\}$$
$$\sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

The second term

$$\sum_{j=T_c+1}^{T_2+1}\tilde{X}_j\varepsilon_j \sim_a \begin{cases} -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_2)-B(f_c)\right] & \text{if } 2\alpha>1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } 2\alpha<1+\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

Suppose $\alpha<\beta$. The first term

$$\sum_{j=T_1+1}^{T_c}\tilde{X}_j\varepsilon_j = \sum_{j=T_1+1}^{T_c}\left[\delta_T^{j-T_e}X_{T_e}-\frac{T^{\beta}}{T_w c_2}X_{T_c}\right]\varepsilon_j\left\{1+o_p(1)\right\}$$
$$\sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e) & \text{if } 1+\alpha>2\beta \\ -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_c)-B(f_1)\right] & \text{if } 1+\alpha<2\beta \end{cases}.$$

The second term

$$\sum_{j=T_c+1}^{T_2+1}\tilde{X}_j\varepsilon_j = \sum_{j=T_c+1}^{T_2+1}\left[\gamma_T^{j-T_c}-\frac{T^{\beta}}{T_w c_2}\right]X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\} \sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

(5) For $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j = -\sum_{j=T_1+1}^{T_c}\tilde{X}_j\varepsilon_j-\sum_{j=T_c+1}^{T_r+1}\tilde{X}_j\varepsilon_j-\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j$$

Suppose $\alpha>\beta$. The first term

$$\sum_{j=T_1+1}^{T_c}\tilde{X}_j\varepsilon_j = T^{\alpha/2}\delta_T^{T_c-T_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=T_1+1}^{T_c}\delta_T^{j-T_c}\varepsilon_j\right)X_{T_e}\left\{1+o_p(1)\right\} \sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

The second term

$$\sum_{j=T_c+1}^{T_r+1}\tilde{X}_j\varepsilon_j \sim_a \begin{cases} -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_r)-B(f_c)\right] & \text{if } 2\alpha>1+\beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } 2\alpha<1+\beta \end{cases}.$$

The third term

$$\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j = \sum_{j=T_r+2}^{T_2+1}-\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\varepsilon_j\left\{1+o_p(1)\right\}$$
$$\sim_a -T^{\alpha}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B(f_e)\left[B(f_2)-B(f_r)\right].$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e).$$

Suppose $\alpha<\beta$. The first term

$$\sum_{j=T_1+1}^{T_c}\tilde{X}_j\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e}X_{c_1}B(f_e) & \text{if } 1+\alpha>2\beta \\ -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_c)-B(f_1)\right] & \text{if } 1+\alpha<2\beta \end{cases}.$$

The second term

$$\sum_{j=T_c+1}^{T_r+1}\tilde{X}_j\varepsilon_j = \sum_{j=T_c+1}^{T_r+1}\left[\gamma_T^{j-T_c}-\frac{T^{\beta}}{T_w c_2}\right]X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}$$

$$\sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.$$

The third term

$$\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j = \sum_{j=T_r+2}^{T_2+1} -\frac{T^{\beta}}{T_w c_2}X_{T_c}\varepsilon_j\left\{1+o_p(1)\right\}$$
$$\sim_a -T^{\beta}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B(f_e)\left[B(f_2)-B(f_r)\right].$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}.$$

(6) For $\tau_1 \in N_1$, $\tau_2 \in C$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j = -\sum_{j=T_1+1}^{T_r+1}\tilde{X}_j\varepsilon_j - \sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j$$

Suppose $\alpha>\beta$. The first term

$$\sum_{j=T_1+1}^{T_r+1}\tilde{X}_j\varepsilon_j = \left[X_{T_c}\sum_{j=T_1+1}^{T_r+1}\gamma_T^{j-T_c}\varepsilon_j - \frac{1}{T_w}\sum_{l=T_r+1}^{T_2}\left(\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right)\left(\sum_{l=T_r+1}^{T_2}\varepsilon_j\right)\right]\left\{1+o_p(1)\right\}$$
$$\sim_a -T\frac{f_2-f_r}{f_w}\left[B(f_r)-B(f_1)\right]\int_{f_r}^{f_2}\left[B(r_s)-B(f_r)\right]ds.$$

due to the fact that

$$E\left[\left(\sum_{j=T_1+1}^{T_r+1}\gamma_T^{j-T_c}\varepsilon_j\right)^2\right] = \sum_{j=T_1+1}^{T_r+1}\gamma_T^{2(j-T_c)}E\left(\varepsilon_j^2\right) = T^{\beta}\gamma_T^{2(T_1-T_c)}\frac{\sigma^2}{2c_2}$$

$$X_{T_c}\sum_{j=T_1}^{T_r}\gamma_T^{j-1-T_c}\varepsilon_j = X_{T_c}T^{\beta/2}\gamma_T^{T_1-T_c}\left(T^{-\beta/2}\sum_{j=T_1}^{T_r}\gamma_T^{j-T_1}\varepsilon_j\right) \sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}B(f_e)X_{c_2}$$

$$\frac{1}{T_w}\sum_{l=T_r+1}^{T_2}\left(\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right)\left(\sum_{j=T_1+1}^{T_r+1}\varepsilon_j\right) \sim_a T\frac{f_2-f_r}{f_w}\left[B(f_r)-B(f_1)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds.$$

The second term

$$\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j$$
$$= \sum_{j=T_r+2}^{T_2+1}\left[\sum_{l=0}^{j-T_r-1}\varepsilon_{j-l} - \frac{1}{T_w}\sum_{l=T_r+1}^{T_2}\sum_{i=0}^{l-T_r-1}\varepsilon_{l-i}\right]\varepsilon_j\left\{1+o_p(1)\right\}$$
$$= T\left(T^{-1}\sum_{j=T_r+2}^{T_2+1}\sum_{l=0}^{j-T_r-1}\varepsilon_{j-l}\varepsilon_j\right)$$
$$-T\frac{T_2-T_r}{T_w}\left[\frac{1}{T_2-T_r}\sum_{j=T_r+2}^{T_2+1}\left(T^{-1/2}\sum_{i=0}^{j-T_r-1}\varepsilon_{j-i}\right)\right]\left(T^{-1/2}\sum_{j=T_r+1}^{T_2}\varepsilon_j\right)\left\{1+o_p(1)\right\}$$
$$\sim_a T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2+\frac{1}{2}(f_2-f_r)\sigma^2-\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\}$$

due to the fact that

$$\sum_{j=T_r+2}^{T_2+1}\sum_{l=0}^{j-T_r-1}\varepsilon_{j-l}\varepsilon_j = \sum_{j=T_r+2}^{T_2+1}\varepsilon_j^2 + \sum_{j=T_r+2}^{T_2+1}\sum_{l=1}^{j-T_r-1}\varepsilon_{j-l}\varepsilon_j$$
$$= (T_2-T_r)\left(\frac{1}{T_2-T_r}\sum_{j=T_r+2}^{T_2+1}\varepsilon_j^2\right) + \sum_{j=T_r+2}^{T_2+1}\sum_{l=T_r+1}^{j-1}\varepsilon_l\varepsilon_j$$
$$\sim_a T\frac{1}{2}\left\{\left[B(f_2)-B(f_r)\right]^2+(f_2-f_r)\sigma^2\right\}$$

$$\sum_{j=T_r+2}^{T_2+1}\sum_{l=T_r+1}^{j-1}\varepsilon_l\varepsilon_j = \sum_{j=T_r+2}^{T_2+1}\left[Z_{j-1}-Z_{T_r}\right]\varepsilon_j \text{ with } Z_j=\sum_{l=0}^{j}\varepsilon_l$$

$$= \sum_{j=T_r+2}^{T_2+1} Z_{j-1}\varepsilon_j - Z_{T_r} \sum_{j=T_r+2}^{T_2+1} \varepsilon_j$$
$$= \frac{1}{2}\left[\sum_{j=T_r+2}^{T_2+1}\left(Z_j^2 - Z_{j-1}^2 - \varepsilon_j^2\right)\right] - Z_{T_r}\sum_{j=T_r+2}^{T_2+1}\varepsilon_j$$
$$= \frac{1}{2}\left[Z_{T_2+1}^2 - Z_{T_r+1}^2 - \sum_{j=T_r+2}^{T_2+1}\varepsilon_j^2\right] - Z_{T_r}\sum_{j=T_r+2}^{T_2+1}\varepsilon_j$$
$$\sim_a T\frac{1}{2}\left\{\left[B(f_2) - B(f_r)\right]^2 - (f_2 - f_r)\sigma^2\right\}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* v_j \sim_a -T\left\{\frac{1}{2}\left[B(f_2) - B(f_r)\right]^2 + \frac{1}{2}(f_2 - f_r)\sigma^2 - \frac{f_2 - f_r}{f_w}\left[B(f_2) - 2B(f_r) + B(f_1)\right]\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right\}.$$

Suppose $\alpha < \beta$. The first term

$$\sum_{j=T_1+1}^{T_r+1} \tilde{X}_j\varepsilon_j$$
$$= \sum_{j=T_1+1}^{T_r+1}\left[\gamma_T^{j-T_c} - \frac{\gamma_T^{T_1-T_c}T^\beta}{T_w c_2}\right] X_{T_c}\varepsilon_j\left\{1 + o_p(1)\right\}$$
$$= X_{T_c}\left[T^{\beta/2}\gamma_T^{T_1-T_c}\left(T^{-\beta/2}\gamma_T^{-(T_1-T_c)}\sum_{j=T_1+1}^{T_r+1}\gamma_T^{j-T_c}\varepsilon_j\right) - \frac{\gamma_T^{T_1-T_c}T^{\beta-1/2}}{f_w c_2}\left(T^{-1/2}\sum_{j=T_1+1}^{T_r+1}\varepsilon_j\right)\right]\left\{1 + o_p(1)\right\}$$
$$= X_{T_c}T^{\beta/2}\gamma_T^{T_1-T_c}\left(T^{-\beta/2}\gamma_T^{-(T_1-T_c)}\sum_{j=T_1}^{T_r}\gamma_T^{j-T_c}\varepsilon_j\right)\left\{1 + o_p(1)\right\}$$
$$\sim_a T^{(1+\beta)/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}B(f_e)X_{c_2}.$$

The second term

$$\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j\varepsilon_j = X_{T_c}\sum_{j=T_r+2}^{T_2+1}\left[-\frac{\gamma_T^{T_1-T_c}T^\beta}{T_w c_2}\right]\varepsilon_j\left\{1 + o_p(1)\right\}$$
$$= -X_{T_c}\frac{\gamma_T^{T_1-T_c}T^{\beta-1/2}}{f_w c_2}\left(T^{-1/2}\sum_{j=T_r+2}^{T_2+1}\varepsilon_j\right)\left\{1 + o_p(1)\right\}$$
$$\sim_a -T^\beta\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{B(f_2) - B(f_r)}{f_w c_2}B(f_e).$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^* v_j \sim_a -T^{(1+\beta)/2}\gamma_T^{T_1-T_c}\delta_T^{T_c-T_e}B(f_e)X_{c_2}.$$

□

Lemma S.C3. *The sums of cross-products of $\tilde{X}_{j-1}^*$ and $\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*$ are as follows.* (1) *For $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(2) *For $\tau_1 \in C$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(3) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(4) *For* $\tau_1 \in C$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in C$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right)$$

$$\sim_a \begin{cases} -T^{2-\beta}c_2(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\} & \text{if } \alpha > \beta \\ -T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

Proof. (1) When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_r} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_r} \tilde{X}_{j-1}^* \left(\delta_T^{-1}\tilde{X}_{j-1}^* + \delta_T^{-1}v_j - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_{j-1}^* + v_j - \gamma_T^{-1}\tilde{X}_{j-1}^*\right) \\
&= \delta_T^{-1}\sum_{j=\tau_1}^{\tau_r} \tilde{X}_{j-1}^* v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* v_j + \left(\delta_T^{-1} - \gamma_T^{-1}\right)\sum_{j=\tau_1}^{\tau_r} \tilde{X}_{j-1}^{*2} + \left(1-\gamma_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*2}
\end{aligned}$$

The first term

$$\begin{aligned}
&\left[\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r} \tilde{X}_{j-1}^* v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* v_j\right] \\
&= \left[-\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r} \tilde{X}_{T+2-j}\varepsilon_{T+2-j} - \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{T+2-j}\varepsilon_{T+2-j}\right] \\
&= \left[-\delta_T^{-1}\sum_{j=T_e+1}^{T_2+1} \tilde{X}_j\varepsilon_j - \sum_{j=T_1+1}^{T_e} \tilde{X}_j\varepsilon_j\right] \sim_a -T^{(1+\alpha)/2}\delta_T^{T_2-T_e}B(f_e)X_{c_1}
\end{aligned}$$

For the second term,

$$\begin{aligned}
&\left(\delta_T^{-1} - \gamma_T^{-1}\right)\sum_{j=\tau_1}^{\tau_r} \tilde{X}_{j-1}^{*2} \\
&= \left(\delta_T^{-1} - \gamma_T^{-1}\right)\sum_{j=T_e+1}^{T_2+1} \tilde{X}_j^2 = \left[-c_2T^{-\beta} - c_1T^{-\alpha}\right]\frac{T^{1+\alpha}\delta_T^{2(T_2-T_e)}}{2c_1}B(f_e)^2 \\
&\sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}
\end{aligned}$$

The third term

$$\left(1-\gamma_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\left(1-\gamma_T^{-1}\right)\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2\sim_a -T^{2\alpha-\beta}\delta_T^{2(T_2-T_e)}c_2\frac{f_e-f_1}{f_w c_1}B\left(f_e\right)^2$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a\begin{cases}-T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)}\frac{c_2}{2c_1}B\left(f_e\right)^2 & \text{if } \alpha>\beta\\ -T\delta_T^{2(T_2-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha<\beta\end{cases}.$$

(2) when $\tau_1\in C$ and $\tau_2\in N_0$,

$$\begin{aligned}&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\\&=\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)+\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\\&=\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^*\left(\gamma_T^{-1}\tilde{X}_{j-1}^*+\gamma_T^{-1}v_j-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)+\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^*\left(\delta_T^{-1}\tilde{X}_{j-1}^*+\delta_T^{-1}v_j-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\\&+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_{j-1}^*+v_j-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\\&=\left[\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^*v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^*v_j+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*v_j\right]+\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*2}+\left(1-\gamma_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\end{aligned}$$

The first term

$$\begin{aligned}&\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^*v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^*v_j+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*v_j\\&\sim_a\begin{cases}-T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B\left(f_e\right)X_{c_1} & \text{if } \alpha>\beta\\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B\left(f_e\right)X_{c_2} & \text{if } \alpha<\beta\end{cases}\end{aligned}$$

The second term,

$$\begin{aligned}&\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*2}=\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=T_e+1}^{T_c}\tilde{X}_j^2\\&\sim_a\begin{cases}-T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B\left(f_e\right)^2 & \text{if } \alpha>\beta\\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta\end{cases}\end{aligned}$$

The third term

$$\left(1-\gamma_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\left(1-\gamma_T^{-1}\right)\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2=\begin{cases}-T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_e-f_1}{f_w c_1}B\left(f_e\right)^2 & \text{if } \alpha>\beta\\ -T^{\beta}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1}{f_w c_2}B\left(f_e\right)^2 & \text{if } \alpha<\beta\end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a\begin{cases}-T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B\left(f_e\right)^2 & \text{if } \alpha>\beta\\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta\end{cases}.$$

(3)When $\tau_1\in N_1$ and $\tau_2\in N_0$,

$$\begin{aligned}&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)\\&=\left[\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^*v_j+\gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c}\tilde{X}_{j-1}^*v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^*v_j+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*v_j\right]+\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*2}\\&+\left(1-\gamma_T^{-1}\right)\left[\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^{*2}+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\right]\end{aligned}$$

The first term

$$\left[\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^{*}v_j+\gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c}\tilde{X}_{j-1}^{*}v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*}v_j+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right]$$

$$\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}$$

The second term

$$\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*2}=\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=T_e+1}^{T_c}\tilde{X}_j^2$$

$$\sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

The third term

$$\left(1-\gamma_T^{-1}\right)\left[\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^{*2}+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\right]=\left(1-\gamma_T^{-1}\right)\left[\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2+\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2\right]$$

$$\sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_e-f_1+f_2-f_r}{f_w c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ -T^{\beta}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1+f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\gamma_T^{-1}\tilde{X}_{j-1}^{*}\right)\sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(4) When $\tau_1\in C$ and $\tau_2\in B$,

$$\begin{aligned}&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\gamma_T^{-1}\tilde{X}_{j-1}^{*}\right)\\ &=\left[\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*}v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right]+\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\end{aligned}$$

The first term

$$\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*}v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}$$

The second term

$$\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\left(\delta_T^{-1}-\gamma_T^{-1}\right)\sum_{j=T_1+1}^{T_c}\tilde{X}_j^2$$

$$\sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\gamma_T^{-1}\tilde{X}_{j-1}^{*}\right)\sim_a \begin{cases} -T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(5) When $\tau_1\in N_1$ and $\tau_2\in B$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\gamma_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \left[\sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}^*_{j-1} v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_c} \tilde{X}^*_{j-1} v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}^*_{j-1} v_j\right] + \left(\delta_T^{-1} - \gamma_T^{-1}\right) \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}^{*2}_{j-1}$$
$$+ \left(1 - \gamma_T^{-1}\right) \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^{*2}_{j-1}$$

The first term

$$\sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}^*_{j-1} v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_c} \tilde{X}^*_{j-1} v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}^*_{j-1} v_j$$
$$\sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

The second term

$$\left(\delta_T^{-1} - \gamma_T^{-1}\right) \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}^{*2}_{j-1} = \left(\delta_T^{-1} - \gamma_T^{-1}\right) \sum_{j=T_1+1}^{T_c} \tilde{X}^2_j$$
$$\sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} c_1 \frac{f_c - f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

The third term

$$\left(1 - \gamma_T^{-1}\right) \left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^{*2}_{j-1}\right] = \left(1 - \gamma_T^{-1}\right) \sum_{j=T_r+2}^{T_2+1} \tilde{X}^2_j$$
$$\sim_a \begin{cases} -T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} \frac{f_2 - f_r}{f_w c_2} B(f_e)^2 & \text{if } \alpha > \beta \\ -T^{\beta} \delta_T^{2(T_c-T_e)} \frac{f_2 - f_r}{f_w c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore, we have

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}^*_{j-1} \left(\tilde{X}^*_j - \gamma_T^{-1} \tilde{X}^*_{j-1}\right) \sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} c_1 \frac{f_c - f_1}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6)When $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}^*_{j-1} \left(\tilde{X}^*_j - \gamma_T^{-1} \tilde{X}^*_{j-1}\right)$$
$$= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^*_{j-1} \left(\tilde{X}^*_j - \gamma_T^{-1} \tilde{X}^*_{j-1}\right) + \sum_{j=\tau_e}^{\tau_2} \tilde{X}^*_{j-1} \left(\tilde{X}^*_j - \gamma_T^{-1} \tilde{X}^*_{j-1}\right)$$
$$= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^*_{j-1} \left(\tilde{X}^*_{j-1} + v_j - \gamma_T^{-1} \tilde{X}^*_{j-1}\right) + \sum_{j=\tau_e}^{\tau_2} \tilde{X}^*_{j-1} \left(\gamma_T^{-1} \tilde{X}^*_{j-1} + \gamma_T^{-1} v_j - \gamma_T^{-1} \tilde{X}^*_{j-1}\right)$$
$$= \left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^*_{j-1} v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}^*_{j-1} v_j\right] + \left(1 - \gamma_T^{-1}\right) \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^{*2}_{j-1}$$

The first term

$$\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^*_{j-1} v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}^*_{j-1} v_j$$
$$\sim_a \begin{cases} \begin{array}{l} -T \left\{ \frac{1}{2} \left[B(f_2) - B(f_r)\right]^2 + \frac{1}{2}(f_2 - f_r)\sigma^2 \right. \\ \left. -\frac{f_2 - f_r}{f_w} \left[B(f_2) - 2B(f_r) + B(f_1)\right] \int_{f_r}^{f_2} \left[B(s) - B(f_r)\right] ds \right\} \end{array} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \gamma_T^{T_1 - T_c} \delta_T^{T_c - T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

The second term

$$\left(1 - \gamma_T^{-1}\right) \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}^{*2}_{j-1} = \left(1 - \gamma_T^{-1}\right) \sum_{j=T_r+1}^{T_2} \tilde{X}^2_j$$

$$\sim_a \begin{cases} -T^{2-\beta}c_2(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2ds+\frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\} & \text{if } \alpha>\beta \\ -T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_wc_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\gamma_T^{-1}\tilde{X}_{j-1}^*\right)$$
$$\sim_a \begin{cases} -T^{2-\beta}c_2(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2ds+\frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\} & \text{if } \alpha>\beta \\ -T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_wc_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

□

Lemma S.C4. *The sums of cross-products of $\tilde{X}_{j-1}^*$ and $\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*$ are as follows.* (1) *For $\tau_1\in B$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a T^{\alpha}\delta_T^{2(T_2-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2.$$

(2) *For $\tau_1\in C$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(3) *For $\tau_1\in N_1$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(4) *For $\tau_1\in C$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(5) *For $\tau_1\in N_1$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_wc_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(6) *For $\tau_1\in N_1$ and $\tau_2\in C$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)$$
$$\sim_a \begin{cases} T^{2-\alpha}c_1(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2ds+\frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\} & \text{if } \alpha>\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}c_1\frac{f_2-f_r}{f_wc_2^2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

Proof. (1) When $\tau_1\in B$ and $\tau_2\in N_0$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)$$
$$=\sum_{j=\tau_1}^{\tau_r}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\delta_T^{-1}\tilde{X}_{j-1}^*\right)$$
$$=\sum_{j=\tau_1}^{\tau_r}\tilde{X}_{j-1}^*\left(\delta_T^{-1}\tilde{X}_{j-1}^*+\delta_T^{-1}v_j-\delta_T^{-1}\tilde{X}_{j-1}^*\right)+\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_{j-1}^*+v_j-\delta_T^{-1}\tilde{X}_{j-1}^*\right)$$

$$= \left[\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r}\tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right] + \left(1-\delta_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}$$

The first term

$$\begin{aligned}
&\left[\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r}\tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right]\\
&= \left[-\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r}\tilde{X}_{T+2-j}\varepsilon_{T+2-j} - \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{T+2-j}\varepsilon_{T+2-j}\right]\\
&= \left[-\delta_T^{-1}\sum_{j=T_e+1}^{T_2+1}\tilde{X}_j\varepsilon_j - \sum_{j=T_1+1}^{T_e}\tilde{X}_j\varepsilon_j\right] \sim_a -T^{(1+\alpha)/2}\delta_T^{T_2-T_e}B(f_e)X_{c_1}
\end{aligned}$$

The third term

$$\left(1-\delta_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2} = \left(1-\delta_T^{-1}\right)\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2 \sim_a T^{\alpha}\delta_T^{2(T_2-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2.$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \sim_a T^{\alpha}\delta_T^{2(T_2-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2.$$

(2) when $\tau_1 \in C$ and $\tau_2 \in N_0$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)\\
&= \sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)\\
&= \sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*}\left(\gamma_T^{-1}\tilde{X}_{j-1}^{*} + \gamma_T^{-1}v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*}\left(\delta_T^{-1}\tilde{X}_{j-1}^{*} + \delta_T^{-1}v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)\\
&+ \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}\left(\tilde{X}_{j-1}^{*} + v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)\\
&= \left[\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right] + \left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*2} + \left(1-\delta_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}
\end{aligned}$$

The first term

$$\begin{aligned}
&\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r}\tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\\
&\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta\\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}
\end{aligned}$$

The second term,

$$\begin{aligned}
&\left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_c}\tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=T_c+1}^{T_2+1}\tilde{X}_j^2\\
&\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha\\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha\\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}
\end{aligned}$$

The third term

$$\left(1-\delta_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2} = \left(1-\delta_T^{-1}\right)\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2 = \begin{cases} T^{\alpha}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2 & \text{if } \alpha > \beta\\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(3)When $\tau_1 \in N_1$ and $\tau_2 \in N_0$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \\
&= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \\
&+ \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \\
&= \sum_{j=\tau_1}^{\tau_e-1} \left(\tilde{X}_{j-1}^{*}+v_j-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}\left(\gamma_T^{-1}\tilde{X}_{j-1}^{*}+\gamma_T^{-1}v_j-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \\
&+ \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*}\left(\delta_T^{-1}\tilde{X}_{j-1}^{*}+\delta_T^{-1}v_j-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_{j-1}^{*}+v_j-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \\
&= \left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*}v_j + \gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*}v_j\right] + \left(1-\delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*2} \\
&+ \left(\gamma_T^{-1}-\delta_T^{-1}\right)\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*2} + \left(1-\delta_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*2}
\end{aligned}$$

The first term

$$\begin{aligned}
&\left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*}v_j + \gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*}v_j\right] \\
&\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}
\end{aligned}$$

The second term

$$\left(1-\delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*2} = \left(1-\delta_T^{-1}\right)\sum_{j=T_r+2}^{T_2+1} \tilde{X}_j^{2} \sim_a \begin{cases} T^{\alpha}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w c_1}B(f_e)^2 & \text{if } \alpha>\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_2-f_r}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

The third term

$$\left(\gamma_T^{-1}-\delta_T^{-1}\right)\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1}-\delta_T^{-1}\right)\sum_{j=T_c+1}^{T_r+1} \tilde{X}_j^{2} \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

The fourth term

$$\left(1-\delta_T^{-1}\right)\sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \left(1-\delta_T^{-1}\right)\sum_{j=T_1+1}^{T_e} \tilde{X}_j^{2} \sim_a \begin{cases} T^{\alpha}\delta_T^{2(T_c-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2 & \text{if } \alpha>\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

(4) When $\tau_1 \in C$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*}\left(\gamma_T^{-1}\tilde{X}_{j-1}^{*} + \gamma_T^{-1}v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\delta_T^{-1}\tilde{X}_{j-1}^{*} + \delta_T^{-1}v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \left[\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}v_j\right] + \left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*2}$$

The first term

$$\gamma_T^{-1}\sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

The second term

$$\left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=T_c+1}^{T_2+1} \tilde{X}_j^2$$

$$\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(5) When $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*} - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}_{j-1}^{*}\left(\tilde{X}_{j-1}^{*} + v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}\left(\gamma_T^{-1}\tilde{X}_{j-1}^{*} + \gamma_T^{-1}v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$+ \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\delta_T^{-1}\tilde{X}_{j-1}^{*} + \delta_T^{-1}v_j - \delta_T^{-1}\tilde{X}_{j-1}^{*}\right)$$

$$= \left[\sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}_{j-1}^{*}v_j + \gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}v_j\right] + \left(1 - \delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}_{j-1}^{*2} + \left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*2}$$

The first term

$$\sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}_{j-1}^{*}v_j + \gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*}v_j + \delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*}v_j$$

$$\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

The second term

$$\left(1 - \delta_T^{-1}\right)\sum_{j=\tau_1}^{\tau_{e-1}} \tilde{X}_{j-1}^{*2} = \left(1 - \delta_T^{-1}\right)\sum_{j=T_r+2}^{T_2+1} \tilde{X}_j^2 \sim_a \begin{cases} T^{\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_2-f_r}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha > \beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_2-f_r}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

The third term

$$\left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1} - \delta_T^{-1}\right)\sum_{j=T_c+1}^{T_r+1} \tilde{X}_j^2$$

$$\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Therefore, we have

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1}\tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(6)When $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$\begin{aligned} &\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1}\tilde{X}_{j-1}^*\right) \\ &= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1}\tilde{X}_{j-1}^*\right) + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1}\tilde{X}_{j-1}^*\right) \\ &= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* \left(\tilde{X}_{j-1}^* + v_j - \delta_T^{-1}\tilde{X}_{j-1}^*\right) + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* \left(\gamma_T^{-1}\tilde{X}_{j-1}^* + \gamma_T^{-1} v_j - \delta_T^{-1}\tilde{X}_{j-1}^*\right) \\ &= \left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* v_j\right] + \left(1-\delta_T^{-1}\right) \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*2} + \left(\gamma_T^{-1} - \delta_T^{-1}\right) \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^{*2} \end{aligned}$$

The first term

$$\begin{aligned} &\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* v_j \\ &\sim_a \begin{cases} -T\left\{\frac{1}{2}\left[B(f_2) - B(f_r)\right]^2 + \frac{1}{2}(f_2 - f_r)\sigma^2 \right. \\ \left. \quad -\frac{f_2-f_r}{f_w}\left[B(f_2) - 2B(f_r) + B(f_1)\right] \int_{f_r}^{f_2} \left[B(s) - B(f_r)\right] ds \right\} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \gamma_T^{T_1-T_c} \delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}. \end{aligned}$$

The second term

$$\begin{aligned} &\left(1-\delta_T^{-1}\right) \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^{*2} = \left(1-\delta_T^{-1}\right) \sum_{j=T_r+1}^{T_2} \tilde{X}_j^2 \\ &\sim_a \begin{cases} T^{2-\alpha} c_1 (f_2 - f_r) \left\{\int_{f_r}^{f_2} \left[B(s) - B(f_r)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} \left[B(s) - B(f_r)\right] ds\right]^2 \right\} & \text{if } \alpha > \beta \\ T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} c_1 \frac{f_2-f_r}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1}\tilde{X}_{j-1}^*\right) \\ &\sim_a \begin{cases} T^{2-\alpha} c_1 (f_2 - f_r) \left\{\int_{f_r}^{f_2} \left[B(s) - B(f_r)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} \left[B(s) - B(f_r)\right] ds\right]^2 \right\} & \text{if } \alpha > \beta \\ T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} c_1 \frac{f_2-f_r}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}. \end{aligned}$$

□

Lemma S.C5. *The sums of cross-products of $\tilde{X}_{j-1}^*$ and $\tilde{X}_j^* - \tilde{X}_{j-1}^*$ are as follows.*
(1) *For $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \sim_a T\delta_T^{2(T_2-T_e)} \frac{1}{2} B(f_e)^2.$$

(2) *For* $\tau_1 \in C$ *and* $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(3) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in N_0$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(4) *For* $\tau_1 \in C$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(5) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(6) *For* $\tau_1 \in N_1$ *and* $\tau_2 \in C$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \sim_a \begin{cases} T^{2-\beta}c_2\frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]\,ds\right]^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

Proof. (1) When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\begin{aligned} &\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \\ &= \sum_{j=\tau_1}^{\tau_r-1} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_r}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_j^{*}-\tilde{X}_{j-1}^{*}\right) \\ &= \sum_{j=\tau_1}^{\tau_r-1} \tilde{X}_{j-1}^{*}\left(\delta_T^{-1}\tilde{X}_{j-1}^{*}+\delta_T^{-1}v_j-\tilde{X}_{j-1}^{*}\right) + \sum_{j=\tau_r}^{\tau_2} \tilde{X}_{j-1}^{*}\left(\tilde{X}_{j-1}^{*}+v_j-\tilde{X}_{j-1}^{*}\right) \\ &= \left[\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r-1}\tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right] + \left(\delta_T^{-1}-1\right)\sum_{j=\tau_1}^{\tau_r-1}\tilde{X}_{j-1}^{*2} \end{aligned}$$

The first term

$$\left(\delta_T^{-1}-1\right)\sum_{j=\tau_1}^{\tau_r-1}\tilde{X}_{j-1}^{*2} = -c_1T^{-\alpha}\sum_{j=T_e+2}^{T_2+1}\tilde{X}_j^2 \sim_a T\delta_T^{2(T_2-T_e)}\frac{1}{2}B(f_e)^2 .$$

The second term

$$\left[\delta_T^{-1}\sum_{j=\tau_1}^{\tau_r-1}\tilde{X}_{j-1}^{*}v_j + \sum_{j=\tau_r}^{\tau_2}\tilde{X}_{j-1}^{*}v_j\right] \sim_a -T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e).$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \sim_a T\delta_T^{2(T_2-T_e)} \frac{1}{2} B(f_e)^2 .$$

(2) when $\tau_1 \in C$ and $\tau_2 \in N_0$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* \left(\gamma_T^{-1}\tilde{X}_{j-1}^* + \gamma_T^{-1} v_j - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* \left(\delta_T^{-1}\tilde{X}_{j-1}^* + \delta_T^{-1} v_j - \tilde{X}_{j-1}^*\right) \\
&+ \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* \left(\tilde{X}_{j-1}^* + v_j - \tilde{X}_{j-1}^*\right) \\
&= \left[\gamma_T^{-1} \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* v_j + \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* v_j\right] + \left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*2} + \left(\delta_T^{-1} - 1\right) \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*2}
\end{aligned}$$

The first term

$$\gamma_T^{-1} \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

The second term,

$$\begin{aligned}
&\left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1} - 1\right) \sum_{j=T_c+1}^{T_2+1} \tilde{X}_j^2 \\
&\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases}
\end{aligned}$$

The third term

$$\left(\delta_T^{-1} - 1\right) \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*2} = \left(\delta_T^{-1} - 1\right) \sum_{j=T_e+1}^{T_c} \tilde{X}_j^2 \sim_a \begin{cases} -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

(3)When $\tau_1 \in N_1$ and $\tau_2 \in N_0$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* \left(\tilde{X}_{j-1}^* + v_j - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^* \left(\gamma_T^{-1}\tilde{X}_{j-1}^* + \gamma_T^{-1} v_j - \tilde{X}_{j-1}^*\right) \\
&+ \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* \left(\delta_T^{-1}\tilde{X}_{j-1}^* + \delta_T^{-1} v_j - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_{j-1}^* + v_j - \tilde{X}_{j-1}^*\right)
\end{aligned}$$

$$= \left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^* v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* v_j \right]$$

$$+ \left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*2} + \left(\delta_T^{-1} - 1\right) \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*2}$$

The first term

$$\left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^* v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^* v_j + \sum_{j=\tau_r+1}^{\tau_2} \tilde{X}_{j-1}^* v_j \right]$$
$$\sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

The second term

$$\left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_e}^{\tau_c} \tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1} - 1\right) \sum_{j=T_c+1}^{T_r+1} \tilde{X}_j^2$$
$$\sim_a \begin{cases} T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \end{cases}$$

The third term

$$\left(\delta_T^{-1} - 1\right) \sum_{j=\tau_c+1}^{\tau_r} \tilde{X}_{j-1}^{*2} = \left(\delta_T^{-1} - 1\right) \sum_{j=T_e+1}^{T_c} \tilde{X}_j^2$$
$$\sim_a \begin{cases} -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c-T_e)} \frac{1}{2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T \delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T \delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B(f_e)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(4) When $\tau_1 \in C$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right)$$
$$= \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right)$$
$$= \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* \left(\gamma_T^{-1} \tilde{X}_{j-1}^* + \gamma_T^{-1} v_j - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^* \left(\delta_T^{-1} \tilde{X}_{j-1}^* + \delta_T^{-1} v_j - \tilde{X}_{j-1}^*\right)$$
$$= \left[\gamma_T^{-1} \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^* v_j \right] + \left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*2} + \left(\delta_T^{-1} - 1\right) \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^{*2}$$

The first term

$$\gamma_T^{-1} \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^* v_j + \delta_T^{-1} \sum_{j=\tau_c+1}^{\tau_2} \tilde{X}_{j-1}^* v_j \sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

The second term

$$\left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_1}^{\tau_c} \tilde{X}_{j-1}^{*2} = \left(\gamma_T^{-1} - 1\right) \sum_{j=T_c+1}^{T_2+1} \tilde{X}_j^2$$

$$\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}.$$

The third term

$$\left(\delta_T^{-1}-1\right)\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\left(\delta_T^{-1}-1\right)\sum_{j=T_1+1}^{T_c}\tilde{X}_j^2$$

$$\sim_a \begin{cases} -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(5) When $\tau_1\in N_1$ and $\tau_2\in B$,

$$\begin{aligned}
&\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\\
&=\sum_{j=\tau_1}^{\tau_{e-1}}\tilde{X}_{j-1}^*\left(\tilde{X}_{j-1}^*+v_j-\tilde{X}_{j-1}^*\right)+\sum_{j=\tau_1}^{\tau_{e-1}}\tilde{X}_{j-1}^*\left(\gamma_T^{-1}\tilde{X}_{j-1}^*+\gamma_T^{-1}v_j-\tilde{X}_{j-1}^*\right)\\
&+\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^*\left(\delta_T^{-1}\tilde{X}_{j-1}^*+\delta_T^{-1}v_j-\tilde{X}_{j-1}^*\right)\\
&=\left[\sum_{j=\tau_1}^{\tau_{e-1}}\tilde{X}_{j-1}^*v_j+\gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c}\tilde{X}_{j-1}^*v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^*v_j\right]+\left(\gamma_T^{-1}-1\right)\sum_{j=\tau_1}^{\tau_{e-1}}\tilde{X}_{j-1}^{*2}+\left(\delta_T^{-1}-1\right)\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*2}
\end{aligned}$$

The first term

$$\left[\sum_{j=\tau_1}^{\tau_{e-1}}\tilde{X}_{j-1}^*v_j+\gamma_T^{-1}\sum_{j=\tau_e}^{\tau_c}\tilde{X}_{j-1}^*v_j+\delta_T^{-1}\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^*v_j\right]\sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha>\beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha<\beta \end{cases}$$

The second term

$$\left(\gamma_T^{-1}-1\right)\sum_{j=\tau_1}^{\tau_{e-1}}\tilde{X}_{j-1}^{*2}=\left(\gamma_T^{-1}-1\right)\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha>\beta \\ T^{\beta}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha<\beta \end{cases}$$

The third term

$$\left(\delta_T^{-1}-1\right)\sum_{j=\tau_c+1}^{\tau_2}\tilde{X}_{j-1}^{*2}=\left(\delta_T^{-1}-1\right)\sum_{j=T_1+1}^{T_c}\tilde{X}_j^2=\begin{cases} -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha>\beta \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha\le 2\beta \end{cases}$$

Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_w c_2}B(f_e)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B(f_e)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(6)When $\tau_1\in N_1$ and $\tau_2\in C$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)$$

$$
\begin{aligned}
&= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \\
&= \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* \left(\tilde{X}_{j-1}^* + v_j - \tilde{X}_{j-1}^*\right) + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* \left(\gamma_T^{-1}\tilde{X}_{j-1}^* + \gamma_T^{-1} v_j - \tilde{X}_{j-1}^*\right) \\
&= \left[\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* v_j\right] + \left(\gamma_T^{-1} - 1\right) \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^{*2}
\end{aligned}
$$

The first term

$$
\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^* v_j + \gamma_T^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^* v_j \sim_a \begin{cases} \begin{array}{l} -T\left\{\frac{1}{2}\left[B(f_2) - B(f_r)\right]^2 + \frac{1}{2}(f_2 - f_r)\sigma^2 \right. \\ \left. -\frac{f_2-f_r}{f_w}\left[B(f_2) - 2B(f_r) + B(f_1)\right]\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right\} \end{array} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\gamma_T^{T_1-T_c}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}
$$

The second term

$$
\left(\gamma_T^{-1} - 1\right)\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^{*2} = c_2 T^{-\beta} \sum_{j=T_1+1}^{T_r+1} \tilde{X}_j^2 \sim_a \begin{cases} T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right]^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}
$$

Therefore,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s) - B(f_r)\right]ds\right]^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.
$$

□

2.3.1. Test asymptotics The fitted regression model for the recursive unit root tests is

$$
X_t^* = \hat{\mu}_{g_1,g_2} + \hat{\rho}_{g_1,g_2} X_{t-1}^* + \hat{v}_t,
$$

where the intercept $\hat{\mu}_{g_1,g_2}$ and slope coefficient $\hat{\rho}_{g_1,g_2}$ are obtained using data over the subperiod $[g_1, g_2]$.

Remark . *Based on Lemma S.C1 and Lemma S.C3, we can obtain the limit distribution of* $\hat{\gamma}_T^{-1} - \gamma_T^{-1}$ *using*

$$
\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1}\tilde{X}_{j-1}^*\right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2}}.
$$

When $\tau_1 \in B$ *and* $\tau_2 \in N_0$,

$$
\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} \sim_a \begin{cases} -c_2 T^{-\beta} & \textit{if } \alpha > \beta \\ -c_1 T^{-\alpha} & \textit{if } \alpha < \beta \end{cases}.
$$

when $\tau_1 \in N_1$ *and* $\tau_2 \in C$,

$$
\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} \sim_a \begin{cases} -T^{-\beta} c_2 \frac{\left\{\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2\right\}}{\left\{\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right]^2\right\}} & \textit{if } \alpha > \beta \\ -2T^{-1}\frac{f_2-f_r}{f_w} & \textit{if } \alpha < \beta \end{cases}
$$

for all other cases, we have

$$
\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} \sim_a \begin{cases} -T^{-\beta} c_2 & \textit{if } \alpha > \beta \\ -T^{-\beta} c_2 & \textit{if } \alpha < \beta \textit{ and } 1+\alpha > 2\beta \\ -T^{\beta-\alpha-1} 2c_1 \frac{f_c - f_e}{f_w c_2} & \textit{if } \alpha < \beta \textit{ and } 1+\alpha < 2\beta \end{cases}.
$$

Remark . *Based on Lemma S.C1 and Lemma S.C4, we can obtain the limit distribution of* $\hat{\gamma}_T^{-1} - \delta_T^{-1}$ *using*

$$
\hat{\rho}_{g_1,g_2} - \delta_T^{-1} = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(X_j^* - \delta_T^{-1}X_{j-1}^*\right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2}}.
$$

When $\tau_1 \in B$ *and* $\tau_2 \in N_0$,

$$
\hat{\rho}_{g_1,g_2} - \delta_T^{-1} \sim_a \frac{1}{T} 2c_1 \frac{f_e - f_1}{f_w};
$$

When $\tau_1 \in C$ and $\tau_2 \in N_0$,

$$\hat{\rho}_{g_1,g_2} - \delta_T^{-1} \sim_a \begin{cases} T^{-\alpha} \frac{c_1\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\}}{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2} & \text{if } \alpha > \beta \\ 2T^{\beta-\alpha-1} c_1 \frac{f_2-f_r}{f_w c_2} & \text{if } \alpha < \beta \end{cases};$$

For all other cases

$$\hat{\rho}_{g_1,g_2} - \delta_T^{-1} \sim_a \begin{cases} T^{\alpha-\beta-1}K & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ c_1 T^{-\alpha} & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ c_1 T^{-\alpha} & \text{if } \alpha < \beta \end{cases};$$

where K is a constant which equals $2c_1c_2\frac{f_r-f_c}{f_w c_1^2}$ when $\tau_1 \in N_1$ and $\tau_2 \in N_0$ and when $\tau_1 \in N_1$ and $\tau_2 \in B$ and equals $2c_1c_2\frac{f_2-f_c}{f_w c_1^2}$ when $\tau_1 \in C$ and $\tau_2 \in B$ and when $\tau_1 \in C$ and $\tau_2 \in N_0$.

Remark . *Based on Lemma S.C1 and Lemma S.C5, we can obtain the limit distribution of $\hat{\rho}_{g_1,g_2} - 1$ using*

$$\hat{\rho}_{g_1,g_2} - 1 = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^*\left(X_j^* - X_{j-1}^*\right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2}}.$$

When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\hat{\rho}_{g_1,g_2} - 1 \sim_a \frac{c_1}{T^\alpha}.$$

when $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$\hat{\rho}_{g_1,g_2} - 1 \tilde{} \begin{cases} T^{-\beta} c_2 \frac{\frac{(f_r-f_1)(f_2-f_r)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2}{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2} & \text{if } \alpha > \beta \\ T^{-\beta} c_2 & \text{if } \alpha < \beta \end{cases}$$

when $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$\hat{\rho}_{g_1,g_2} - 1 \sim_a \begin{cases} T^{\alpha-\beta-1} 2c_1 \frac{f_2-f_r}{f_w c_2} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -c_1 T^{-\alpha} & f\, \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{-\beta} c_2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{\beta-\alpha-1} 2c_1 \frac{f_c-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

for all other cases, we have

$$\hat{\rho}_{g_1,g_2} - 1 \sim_a \begin{cases} T^{\alpha-\beta-1} 2c_1 \frac{f_2-f_r}{f_w c_2} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T^{-\alpha}\right) & f\, \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T^{-\beta}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{\beta-\alpha-1} 2c_1 \frac{f_c-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Based on the above three remarks, one can see that the quantity $\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}$ diverges to negative infinity and the quantity $\hat{\rho}_{g_1,g_2} - \delta_T^{-1}$ diverges to positive infinity. In other words, the estimated value of $\hat{\rho}_{g_1,g_2}$ is bounded by γ_T^{-1} and δ_T^{-1}. Furthermore, the quantity $\hat{\rho}_{g_1,g_2} - 1$ diverges to positive infinity when $\tau_1 \in B$ and $\tau_2 \in N_0$ and when $\tau_1 \in N_1$ and $\tau_2 \in C$. For all other cases, the quantity $\hat{\rho}_{g_1,g_2} - 1$ diverges to positive infinity when bubble collapsing speed is much faster than expansion rate (i.e. $1+\beta < 2\alpha$) and to negative infinity otherwise.

To obtain the asymptotic distributions of the Dickey-Fuller t-statistic, we first obtain the equation standard error of the regression over $[T_1, T_2]$, which is

$$\hat{\sigma}_{g_1 g_2} = \left\{\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j^* - \hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2\right\}^{1/2}.$$

and its limit theory is as follows.

(1) When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\begin{aligned} &\hat{\sigma}_{g_1,g_2}^2 \\ &= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j^* - \hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2 \end{aligned}$$

$$= \tau_w^{-1}\left[\sum_{j=\tau_1}^{\tau_r}\left[\delta_T^{-1}v_j-\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2+\sum_{j=\tau_r+1}^{\tau_2}\left[v_j-\left(\hat{\rho}_{g_1,g_2}-1\right)\tilde{X}_{j-1}^*\right]^2\right]$$

$$= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=\tau_1}^{\tau_r}\tilde{X}_{j-1}^{*2}+\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\right]\left\{1+o_p(1)\right\}$$

$$= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_e+1}^{T_2+1}\tilde{X}_j^2+\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_1}^{T_e}\tilde{X}_j^2\right]\left\{1+o_p(1)\right\}$$

Notice that the terms associated with $\tilde{X}_{j-1}^{*2}$ dominate terms associated with $\tilde{X}_{j-1}^*v_j$. Since

$$\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_1}^{T_e}\tilde{X}_j^2=O_p\left(T^{-2\alpha}\right)O_p\left(T^{2\alpha}\delta_T^{2(T_2-T_e)}\right)=O_p\left(\delta_T^{2(T_2-T_e)}\right)$$

$$\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_e+1}^{T_2+1}\tilde{X}_j^2=O_p\left(T^{-2}\right)O_p\left(T^{1+\alpha}\delta_T^{2(T_2-T_e)}\right)=O_p\left(T^{\alpha-1}\delta_T^{2(T_2-T_e)}\right),$$

We have

$$Var\left(\hat{\rho}_{g_1,g_2}\right)=\left(\hat{\rho}_{g_1,g_2}-1\right)^2\tau_w^{-1}\sum_{j=T_1}^{T_e}\tilde{X}_j^2\sim_a O_p\left(T^{-1}\delta_T^{2(T_2-T_e)}\right).$$

(2) When $\tau_1\in C$ and $\tau_2\in N_0$,

$$\hat{\sigma}_{g_1,g_2}^2$$

$$=\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j^*-\hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2$$

$$=\tau_w^{-1}\left[\sum_{j=\tau_1}^{\tau_c-1}\left[\gamma_T^{-1}v_j-\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2+\sum_{j=\tau_c}^{\tau_r}\left[\delta_T^{-1}v_j-\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2\right.$$

$$\left.+\sum_{j=\tau_r+1}^{\tau_2}\left[v_j-\left(\hat{\rho}_{g_1,g_2}-1\right)\tilde{X}_{j-1}^*\right]^2\right]$$

$$=\tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=\tau_1}^{\tau_c-1}\tilde{X}_{j-1}^{*2}+\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=\tau_c}^{\tau_r}\tilde{X}_{j-1}^{*2}+\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\right]\left\{1+o_p(1)\right\}$$

$$=\tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_c+2}^{T_2+1}\tilde{X}_j^2+\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_e+1}^{T_c+1}\tilde{X}_j^2+\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2\right]\left\{1+o_p(1)\right\}$$

Since

$$\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_c+2}^{T_2+1}\tilde{X}_j^2$$

$$=\begin{cases} O_p\left(T^{-2\beta}\right)O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{2\alpha-2\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\ O_p\left(T^{-2\beta}\right)O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha\\ O_p\left(T^{-2\beta}\right)O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ O_p\left(T^{2(\beta-\alpha-1)}\right)O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{3\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_e+1}^{T_c+1}\tilde{X}_j^2$$

$$=\begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right)O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{3\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha\\ O_p\left(T^{-2\alpha}\right)O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha\\ O_p\left(T^{-2\alpha}\right)O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta\\ O_p\left(T^{-2\alpha}\right)O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right)=O_p\left(T^{2\beta-2\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_1+1}^{T_e}\tilde{X}_j^2$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T^{-2\alpha}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = o_p\left(\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T^{-2\beta}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = o_p\left(\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

we have

$$\hat{\sigma}^2_{g_1,g_2} = \begin{cases} \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_2+1}\tilde{X}_j^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_2+1}\tilde{X}_j^2 & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2 \sum_{j=T_e+1}^{T_c+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2 \sum_{j=T_e+1}^{T_c+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(3) When $\tau_1 \in N_1$ and $\tau_2 \in N_0$,

$$\begin{aligned} &\hat{\sigma}^2_{g_1,g_2} \\ &= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j^* - \hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2 \\ &= \tau_w^{-1}\left[\sum_{j=\tau_1}^{\tau_e-1}\left[v_j - \left(\hat{\rho}_{g_1,g_2}-1\right)\tilde{X}_{j-1}^*\right]^2 + \sum_{j=\tau_e}^{\tau_c-1}\left[\gamma_T^{-1}v_j - \left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2\right. \\ &\left. + \sum_{j=\tau_c}^{\tau_r}\left[\delta_T^{-1}v_j - \left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2 + \sum_{j=\tau_r+1}^{\tau_2}\left[v_j - \left(\hat{\rho}_{g_1,g_2}-1\right)\tilde{X}_{j-1}^*\right]^2\right] \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-1\right)^2\left[\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^{*2} + \sum_{j=\tau_r+1}^{\tau_2}\tilde{X}_{j-1}^{*2}\right] + \left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=\tau_e}^{\tau_c-1}\tilde{X}_{j-1}^{*2}\right. \\ &\left. + \sum_{j=\tau_c}^{\tau_r}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\tilde{X}_{j-1}^{*2}\right]\left\{1+o_p(1)\right\} \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-1\right)^2\left[\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2 + \sum_{j=T_1+1}^{T_e}\tilde{X}_j^2\right] + \left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_c+2}^{T_r+1}\tilde{X}_j^2\right. \\ &\left. + \sum_{j=T_e+1}^{T_c+1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\tilde{X}_j^2\right]\left\{1+o_p(1)\right\} \end{aligned}$$

Since

$$\begin{aligned} &\left(\hat{\rho}_{g_1,g_2}-1\right)^2\left[\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2 + \sum_{j=T_1+1}^{T_e}\tilde{X}_j^2\right] \\ &= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T^{-2\alpha}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = o_p\left(\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T^{-2\beta}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = o_p\left(\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}, \\ &\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_c+2}^{T_r+1}\tilde{X}_j^2 \end{aligned}$$

$$= \begin{cases} O_p\left(T^{-2\beta}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\alpha-2\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases},$$

$$\left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)^2 \sum_{j=T_e+1}^{T_c+1} \tilde{X}_j^2$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ o_p\left(T^{-2\alpha}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\beta-2\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Therefore,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_r+1} \tilde{X}_j^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_r+1} \tilde{X}_j^2 & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)^2 \sum_{j=T_e+1}^{T_c+1} \tilde{X}_j^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)^2 \sum_{j=T_e+1}^{T_c+1} \tilde{X}_j^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

$$\sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

(4)When $\tau_1 \in C$ and $\tau_2 \in B$,

$$\begin{aligned} &\hat{\sigma}_{g_1,g_2}^2 \\ &= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j^* - \hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2 \\ &= \tau_w^{-1}\left[\sum_{j=\tau_1}^{\tau_c-1}\left[\gamma_T^{-1}v_j - \left(\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2 + \sum_{j=\tau_c}^{\tau_2}\left[\delta_T^{-1}v_j - \left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2\right] \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}\right)^2 \sum_{j=\tau_1}^{\tau_c-1} \tilde{X}_{j-1}^{*2} + \left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)^2 \sum_{j=\tau_c}^{\tau_2} \tilde{X}_{j-1}^{*2}\right]\left\{1 + o_p(1)\right\} \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_2} \tilde{X}_j^2 + \left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)^2 \sum_{j=T_1+1}^{T_c+1} \tilde{X}_j^2\right]\left\{1 + o_p(1)\right\} \end{aligned}$$

Since

$$\left(\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_2} \tilde{X}_j^2$$

$$= \begin{cases} O_p\left(T^{-2\beta}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\alpha-2\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases},$$

$$\left(\hat{\rho}_{g_1,g_2} - \delta_T^{-1}\right)^2 \sum_{j=T_1+1}^{T_c+1} \tilde{X}_j^2$$

$$= \begin{cases} o_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ o_p\left(T^{-2\alpha}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = o_p\left(T^{2\beta-2\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\hat{\sigma}^2_{g_1,g_2} = \begin{cases} \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_2}\tilde{X}_j^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2 \sum_{j=T_c+2}^{T_2}\tilde{X}_j^2 & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2 \sum_{j=T_1+1}^{T_c+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2 \sum_{j=T_1+1}^{T_c+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(5) When $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$\begin{aligned} &\hat{\sigma}^2_{g_1,g_2} \\ &= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j^* - \hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2 \\ &= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_e-1}\left[v_j - \left(\hat{\rho}_{g_1,g_2}-1\right)\tilde{X}_{j-1}^*\right]^2 + \tau_w^{-1}\sum_{j=\tau_e}^{\tau_c-1}\left[\gamma_T^{-1}v_j - \left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2 \\ &+ \tau_w^{-1}\sum_{j=\tau_c}^{\tau_2}\left[\delta_T^{-1}v_j - \left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)\tilde{X}_{j-1}^*\right] \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^{*2} + \left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\sum_{j=\tau_e}^{\tau_c-1}\tilde{X}_{j-1}^{*2} + \left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=\tau_c}^{\tau_2}\tilde{X}_{j-1}^{*2}\right]\{1+o_p(1)\} \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2 + \left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\sum_{j=T_c+2}^{T_r+1}\tilde{X}_j^2 + \left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_1+1}^{T_c+1}\tilde{X}_j^2\right]\{1+o_p(1)\} \end{aligned}$$

Since

$$\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\alpha-2\beta-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{4\beta-2\alpha-2}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases},$$

$$\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\sum_{j=T_c+2}^{T_r+1}\tilde{X}_j^2$$

$$= \begin{cases} O_p\left(T^{-2\beta}\right) O_p\left(T^{2\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{2\alpha-2\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\beta}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2(\beta-\alpha-1)}\right) O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{3\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases},$$

$$\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_1+1}^{T_c+1}\tilde{X}_j^2$$

$$= \begin{cases} O_p\left(T^{2(\alpha-\beta-1)}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = o_p\left(T^{3\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{1+\alpha}\delta_T^{2(T_c-T_e)}\right) = O_p\left(T^{1-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{-2\alpha}\right) O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\right) = o_p\left(T^{2\beta-2\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

Therefore,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\sum_{j=T_c+2}^{T_r+1}\tilde{X}_j^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\sum_{j=T_c+2}^{T_r+1}\tilde{X}_j^2 & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_1+1}^{T_c+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\delta_T^{-1}\right)^2\sum_{j=T_1+1}^{T_c+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

$$\sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{f } \alpha>\beta \text{ and } 1+\beta>2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(6) When $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$\begin{aligned} &\hat{\sigma}_{g_1,g_2}^2 \\ &= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j-\hat{\rho}_{g_1,g_2}\tilde{X}_{j-1}^*\right)^2 \\ &= \tau_w^{-1}\left\{\sum_{j=\tau_1}^{\tau_e-1}\left[v_j-\left(\hat{\rho}_{g_1,g_2}-1\right)\tilde{X}_{j-1}^*\right]^2+\sum_{j=\tau_e}^{\tau_2}\left[\gamma_T^{-1}v_j-\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)\tilde{X}_{j-1}^*\right]^2\right\} \\ &= \tau_w^{-1}\left[\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^{*2}+\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=\tau_e}^{\tau_2}\tilde{X}_{j-1}^{*2}\right]\left\{1+o_p(1)\right\} \\ &= \left[\tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2+\tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_1+1}^{T_r+1}\tilde{X}_j^2\right]\left\{1+o_p(1)\right\} \end{aligned}$$

Since we have

$$\begin{aligned} &\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2 \\ &= \begin{cases} O_p\left(T^{-2\beta}\right)O_p\left(T^2\right)=O_p\left(T^{2-2\beta}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-2\beta}\right)O_p\left(T^{2\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right)=O_p\left(\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha<\beta \end{cases} \end{aligned}$$

$$\begin{aligned} &\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_1+1}^{T_r+1}\tilde{X}_j^2 \\ &= \begin{cases} O_p\left(T^{-2\beta}\right)O_p\left(T^2\right)=O_p\left(T^{2-2\beta}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-2}\right)O_p\left(T^{1+\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right)=O_p\left(T^{\beta-1}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha<\beta \end{cases} \end{aligned}$$

Therefore,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2+\tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-\gamma_T^{-1}\right)^2\sum_{j=T_1+1}^{T_r+1}\tilde{X}_j^2 & \text{if } \alpha>\beta \\ \tau_w^{-1}\left(\hat{\rho}_{g_1,g_2}-1\right)^2\sum_{j=T_r+2}^{T_2+1}\tilde{X}_j^2 & \text{if } \alpha<\beta \end{cases}$$

$$= \begin{cases} O_p\left(T^{1-2\beta}\right) & \text{if } \alpha>\beta \\ O_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha<\beta \end{cases}$$

The asymptotic distribution of the Dickey-Fuller t statistic

$$DF_{g_1,g_2}^t = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^{*2}}{\hat{\sigma}^2}\right)^{1/2}\left(\hat{\rho}_{g_1,g_2}-1\right)$$

can be calculated as follows. Notice that the sign of the DF statistic is determined by the quantity $\hat{\rho}_{g_1,g_2} - 1$. When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2} \right)^{1/2} \left(\hat{\rho}_{g_1,g_2} - 1 \right) \\ = O_p \left(T^{1-\alpha/2} \right) \to +\infty$$

when $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2} \right)^{1/2} \left(\hat{\rho}_{g_1,g_2} - 1 \right) \\ = \begin{cases} O_p \left(T^{1/2} \right) \to +\infty & \text{if } \alpha > \beta \\ O_p \left(T^{1-\beta/2} \right) \to +\infty & \text{if } \alpha < \beta \end{cases} .$$

when $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2} \right)^{1/2} \left(\hat{\rho}_{g_1,g_2} - 1 \right) \\ \sim_a \begin{cases} O_p \left(T^{\alpha/2} \right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p \left(T^{(1-\alpha+\beta)/2} \right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p \left(T^{(1-\beta+\alpha)/2} \right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p \left(T^{\beta/2} \right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

for all other cases

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2} \right)^{1/2} \left(\hat{\rho}_{g_1,g_2} - 1 \right) \\ \sim_a \begin{cases} O_p \left(T^{\alpha/2} \right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p \left(T^{(1-\alpha+\beta)/2} \right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p \left(T^{(1-\beta+\alpha)/2} \right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p \left(T^{\beta/2} \right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

2.3.2. The Consistency of f_c and f_r Given that $g_2 = g$ and $g_1 \in [0, g - g_0]$, the asymptotic distributions of the backward sup DF statistic under the alternative hypothesis are:

$$BSDF^*_g(g_0) \sim_a \begin{cases} F_g(W, g_0) & \text{if } g \in N_1 \\ \begin{cases} O_p \left(T^{1/2} \right) \to +\infty & \text{if } \alpha > \beta \\ O_p \left(T^{1-\beta/2} \right) \to +\infty & \text{if } \alpha < \beta \end{cases} & \text{if } g \in C \\ \begin{cases} O_p \left(T^{\alpha/2} \right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p \left(T^{(1-\alpha+\beta)/2} \right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p \left(T^{(1-\beta+\alpha)/2} \right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p \left(T^{\beta/2} \right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} & \text{if } g \in B \end{cases} ;$$

This proves Theorem 3.4.

The origination and termination of the bubble implosion are calculated as

$$\hat{f}_r = 1 - \hat{g}_e, \text{ where } \hat{g}_e = \inf_{g \in [g_0, 1]} \left\{ g : BSDF^*_g(g_0) > scv^*(\beta_T) \right\}$$
$$\hat{f}_c = 1 - \hat{g}_c, \text{ where } \hat{g}_c = \inf_{g \in [\hat{g}_e, 1]} \left\{ g : BSDF^*_g(g_0) < scv^*(\beta_T) \right\}.$$

We know that when $\beta_T \to 0$, $scv^*(\beta_T) \to \infty$.

It is obvious that if $g \in N_1$,

$$\lim_{T \to \infty} \Pr \left\{ BSDF^*_g(g_0) > scv^*(\beta_T) \right\} = \Pr \left\{ F_g(W, g_0) = \infty \right\} = 0.$$

If $g \in C$,

$$\lim_{T\to\infty} \Pr\left\{BSDF_g^*(g_0) > scv^*(\beta_T)\right\} = 1$$

provided that

$$\begin{cases} \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \\ \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \end{cases}.$$

If $g \in B$,

$$\lim_{T\to\infty} \Pr\left\{BSDF_g^*(g_0) < scv^*(\beta_T)\right\} = 1$$

provided that

$$\begin{cases} \frac{T^{\alpha/2}}{scv^*(\beta_T)} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^*(\beta_T)} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^*(\beta_T)} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^*(\beta_T)} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{g}_e > g_e + \eta\} \to 0 \text{ and } \Pr\{\hat{g}_c < g_c - \gamma\} \to 0,$$

since $\Pr\left\{BSDF^*_{g_e+a_\eta}(g_0) > scv^c(\beta_T)\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BSDF^*_{g_c-a_\gamma}(g_0) > scv^c(\beta_T)\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary, $\Pr\{\hat{g}_e < g_e\} \to 0$ and $\Pr\{\hat{g}_c > g_c\} \to 0$, we deduce that $\Pr\{|\hat{g}_e - g_e| > \eta\} \to 0$ and $\Pr\{|\hat{g}_c - g_c| > \gamma\} \to 0$ as $T \to \infty$, provided that

$$\begin{cases} \frac{T^{\alpha/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Therefore, $\hat{f}_r = 1 - \hat{g}_e$ and $\hat{f}_c = 1 - \hat{g}_c$ are consistent estimators of f_r and f_c. This proves Theorem 3.5.

3. SIMULATIONS: THE $BSDF^*$ TEST FOR CRISIS ORIGINATION AND MARKET RECOVERY DATES

For further investigation, we extend the parameter specification in the case of disturbing collapses by varying β from 0.3 to 0.7 and d_{CT} from 5% of the total sample to 15%. Consistent with expectations, the $BSDF^*$ strategy provides higher crisis detection rates when bubbles collapse faster (Table 1). For instance, with a collapse duration of $\lfloor 0.05T \rfloor$, the detection rate increases from 80% to 98% when the value of β declines from 0.7 to 0.3 (corresponding to the faster collapse in the mildly integrated process over this period). Moreover, the crisis termination date is more accurately estimated when there is shorter collapse duration. For example, assuming $\beta = 0.3$, the estimated recovery date is one observation earlier than the true recovery date when $d_{CT} = \lfloor 0.05T \rfloor$. However, it increases to seven observations when the duration increases to 15% of the total sample. The bias direction in these estimates is consistent with the reverse regression scenario underlying the $BSDF^*$ detection strategy.

4. SENSITIVITY ANALYSES

4.1. *Heteroskedasticity*

Conditional heteroskedasticity is a widely recognized feature of financial data and there is increasing evidence of non-stationary volatility including volatility shifts in many financial time series. Harvey et al. (2013) demonstrate by simulations that in the presence of non-stationary volatility, the size of the PWY procedure is substantially above the nominal level, indicating serious size distortion, arising from the assumption of *iid* errors in the DF regression when error heterogeneity is present. A wild bootstrap procedure is shown to be asymptotically valid in this case and is able to effectively control finite sample size under non-stationary volatility.

To evaluate the potential impact of conditional heteroskedasticity or nonstationary volatility on our test outcomes for the Nasdaq stock market, we simulate the 95% finite sample critical value using a wild bootstrapping procedure. A brief outline of the wild bootstrapping procedure for the PSY test is as follows. (i) Estimate the DF model under the null hypothesis using the whole sample period

$$\Delta y_t = \psi_0 + \sum_{i=1}^{k} \psi_i \Delta y_{t-i} + e_t,$$

Table 1: Crisis detection rate and collapse and recovery date estimation (for different collapse rates and collapse durations). Parameters are set to: $X_0 = 100, \sigma = 6.79, c = c_1 = c_2 = 1, \eta = 1, \alpha = 0.6, d_{BT} = \lfloor 0.20T \rfloor, f_e = 0.4, T = 100$. Figures in parentheses are standard deviations.

	$d_{CT} = \lfloor 0.05T \rfloor$	$d_{CT} = \lfloor 0.10T \rfloor$	$d_{CT} = \lfloor 0.15T \rfloor$
$\beta = 0.3$			
Crisis Detection Rate	0.98	0.99	0.98
$\hat{r}_c - r_c$	-0.03 (0.01)	-0.04 (0.01)	-0.04 (0.02)
$\hat{r}_r - r_r$	-0.01 (0.02)	-0.03 (0.02)	-0.07 (0.03)
$\beta = 0.5$			
Crisis Detection Rate	0.95	0.97	0.98
$\hat{r}_c - r_c$	-0.02 (0.03)	-0.03 (0.01)	-0.03 (0.01)
$\hat{r}_r - r_r$	-0.02 (0.04)	-0.03 (0.03)	-0.04 (0.03)
$\beta = 0.7$			
Crisis Detection Rate	0.80	0.94	0.96
$\hat{r}_c - r_c$	-0.01 (0.04)	-0.02 (0.04)	-0.02 (0.03)
$\hat{r}_r - r_r$	-0.03 (0.04)	-0.04 (0.05)	-0.06 (0.05)

Note: Calculations are based on 2,000 replications. The minimum window has 19 observations.

where $t = k+2, ..., T$. Denote the OLS parameter estimates by $\hat{\psi}_0$ and $\hat{\psi}_i$ and the estimation residuals by e_t. (ii) Generate bootstrap residuals e_t^b according to the device $e_t^b = \tilde{e}_t w_t$ and $\tilde{e}_t := e_j$, where $t = k+2, ..., T$, $\{w_t\}_{t=1}^{T-k-1}$ denotes an independent $N(0,1)$ scalar sequence, and j is a random integer generated from a uniform distribution running between $k+2$ and T. (iii) Use initializations $\Delta y_t^b = \Delta y_t$ for $t = 2, ..., k+1$ and let Δy_t^b be obtained from the recursion

$$\Delta y_t^b = \hat{\psi}_0 + \sum_{i=1}^{k} \hat{\psi}_i \Delta y_{t-i}^b + e_{t-k}^b, \text{ for } t = k+2, ..., T.$$

The bootstrap sample y_t^b is then calculated by partial summation using

$$y_t^b = y_1 + \sum_{j=1}^{t} \Delta y_t^b, \text{ for } t = 1, 2, ..., T.$$

(iv) Calculate the $BSDF_t$ and $BSDF_t^*$ statistic sequences using the bootstrapped data series. (v) Repeat steps (i)-(iv) 2,000 times. The 95% bootstrapped critical value sequences are calculated as the 95% percentiles of the corresponding test statistic sequences.

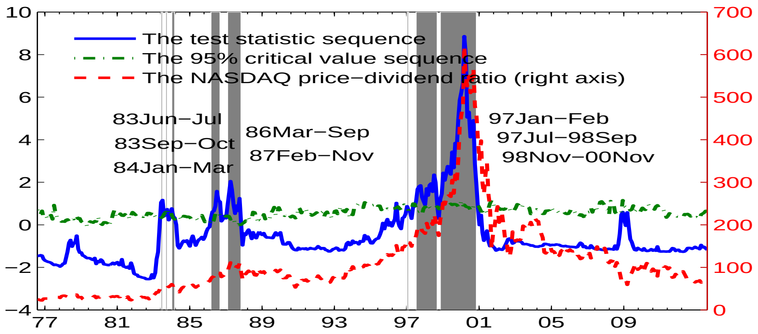
Figure 1 plots the $BSDF_t$ and $BSDF_t^*$ statistic sequences of the Nasdaq price-dividend ratio against their corresponding 95% wild bootstrapped critical value sequences in panels (a) and (b) respectively. We compare panel (a) with Figure 4b for bubble identification and panel (b) to Figure 5 for crisis detection. It is evident from the graphs that the two sets of results are almost identical. There is a maximum of a single observation difference in the estimated starting or ending dates of the bubble episodes. For example, for the Dot Com bubble period, the estimated starting (ending) date is January 1997 (November 2000), compared with December 1996 (December 2000) with critical values obtained from Monte Carlo wild bootstrap simulations. For crisis episodes, there is one observation difference in the estimated Dot Com bubble-led crisis period and no difference in the dates for the 2004 market correction. The market recovery date of the 1973 episode estimated by using the wild bootstrapped critical values is four months later than that from the finite sample Monte Carlo simulation.

4.2. Lag order selection

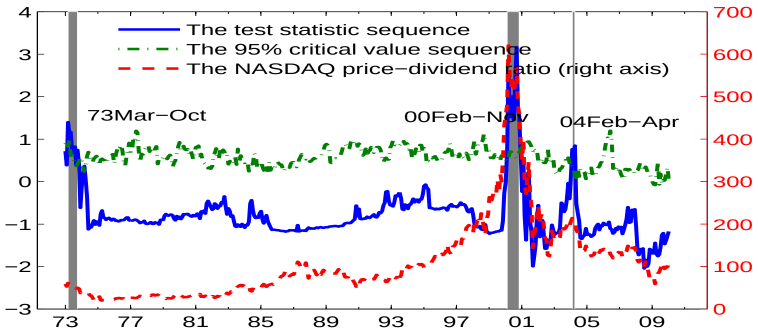
Next, we conduct a sensitivity analysis with respect to lag order selection methods. As an alternative, we consider BIC with a maximum lag order of four for each subsample regression. Figure 2 provides a very similar story for both bubble and crisis episodes in the market. In particular, the procedure using BIC identifies the same crisis episodes (Figure 3b)– 1973 stock market crash and the Dot Com bubble collapse in the early 2000s, although the estimated durations of these two episodes tend to be longer than those obtained from the fixed lag order method. For bubble identification, in addition to the 1983, 1986-87 and the Dot Com bubble episodes observed in Figure 4b, two new episodes emerge – 1978M02-M10 and 2008M10-2009M04. The episode in 1978 is associated with mild upturn in the price-dividend ratio, whereas the 2008 episode is linked to a dramatic downturn in the price-dividend ratio series. The second episode is a falsely identified shift period given that the procedure is designed for dating upward explosive

Figure 1: The identified bubble and crisis episodes based on the price-dividend ratio using the PSY procedure with lag order of one and minimum window size of 44 observations. The finite sample critical value sequences are obtained from the wild bootstrap procedure with 2,000 replications.

(a) Bubble



(b) Crisis



processes. For the three common bubble periods, the procedure based on BIC lag selection criteria tends to find earlier origination dates. For instance, the explosive dynamic in the series becomes evident in November 1995 using BIC selection, whereas the first signal of explosiveness becomes significant in December 1996 using a fixed lag order of one. These results suggest that the procedure based on BIC selection tends to provide a timelier signal for the existence of speculative bubble behavior but this potential advantage comes at the cost of greater false positive discovery. The PSY procedure based on the use of a fixed lag order of unity gives more conservative findings.

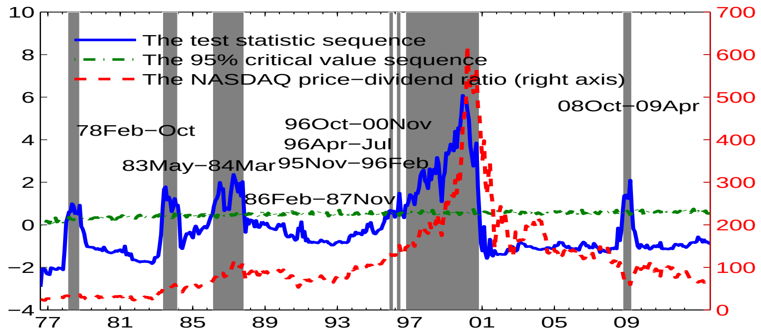
4.3. Data choice

Finally, we apply the PSY procedure for both bubble and crisis dating to the log real Nasdaq price index as in PWY (2011). The lag order is set to one and the 95% critical value sequences are obtained by Monte Carlo simulation. We compare results with those obtained from the price-dividend ratio. As in Figure 4b, the procedure based on the log price series finds explosive dynamics in 1983 and the second half of the 1990s. We observe some minor differences in the estimated origination and termination dates of these two episodes from using the price-dividend ratio and the log price series. From the latter, the identified period of explosiveness in 1983 is shorter (1983M04-M06) and the Dot Com bubble is found to start as early as July 1995. While these results for the log price series show that the bubble signal emerges in July 1995, the signal is intermittent and does not become a stable bubble signal until November 1998. The major difference in the bubble identification results is that the PSY procedure based on the log price series finds a new explosive episode towards the end of 1980 (1980M10-M12) and does not detect any explosive dynamic around 1986/1987.

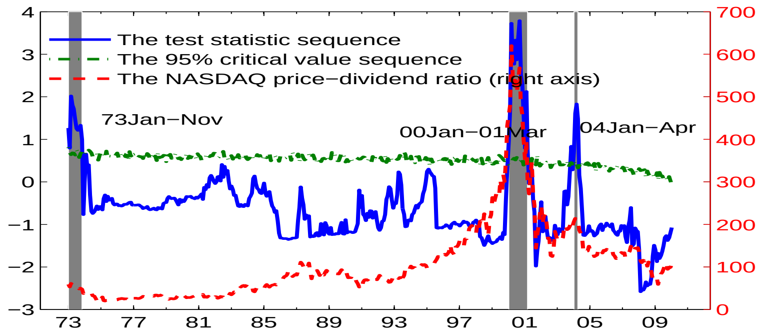
For crisis dating the log price series provides a different story. The major crisis identified based on the price-dividend ratio series is the Dot Com bubble collapse. Using the log price series we observe two (discontinuing) significant observations after 2000 (February 2000 and August 2000). This signal does not become persistent until the period July 2002 to September 2002 after which it returns in the period 2003M01-M04. No explosive dynamic is detected in the reversed log price series during 1973 but weak crisis signals are observed in 1974M08-M12, 1975M09, and 1975M11. Evidence of crisis episodes are also indicated in 1990-91, 1994-95, and 2009. The first and last episodes are likely to be related to the early 1990s recession and the 2008 subprime mortgage crisis. It is interesting to note that the procedure based on the log price series is more likely to detect non-bubble-led crises. This is not surprising

Figure 2: The identified bubble and crisis episodes based on the price-dividend ratio using the PSY procedure with BIC (maximum lag order 4) and minimum window size of 44 observations. The finite sample critical value sequences are obtained from Monte Carlo simulation with 2,000 replications.

(a) Bubble



(b) Crisis



as by conducting the test on the log price series alone there is no control for (or implied connection with) the market fundamentals.

In sum, applying the PSY and reverse PSY tests to the log real Nasdaq price index, instead of the price-dividend ratio tends to provide an earlier but noisier signal of bubble existence. The approach can also lead to the identification of different bubble and crisis episodes, including crises that are not pre-dated by bubble behavior.

4.4. The PWY procedure

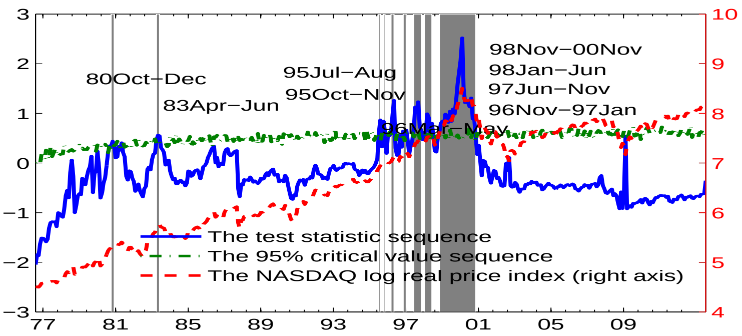
We apply the PWY and the PSY procedures to the Nasdaq price-dividend ratio. The right panel of Figure 4 plots the BSDF statistic sequence against its 95% corresponding critical value sequence. The left panel reports the bubble dating procedure of PWY on the same data series for comparison. The finite sample critical value sequences are obtained by Monte Carlo simulation with 2,000 replications.

Notice that in PWY (2011), the testing procedure is applied to the logarithmic real Nasdaq stock price index with a minimum window size of 38 observations and lag order selected by sequential significance testing as in Campbell and Perron (1991). PSY (2015a) show that sequential significance testing for lag order selection can lead to significant size distortion in both the PWY and PSY bubble identification procedures and they recommend using a fixed small lag order. In addition, the rule $0.01+1.8/\sqrt{T}$ is recommended for setting the minimum window size fraction to maintain satisfactory sizes when the sample size T is large. Therefore, we set the smallest window size according to this rule giving 44 observations and consider a lag order of one for the ADF regression.

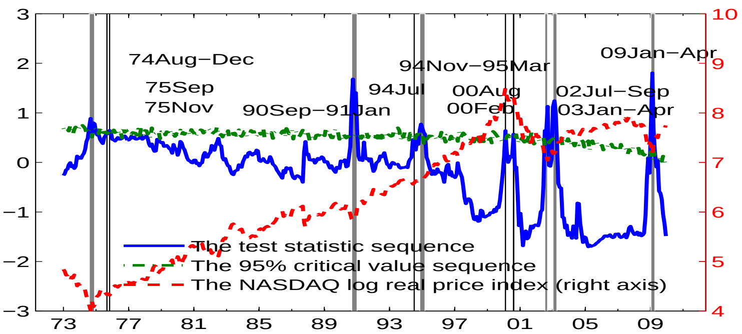
Similar results are observed from these two testing procedures. There is speculative bubble behavior in the stock market in 1986-1987 and 1996-2000, which led successively to the Black Monday crash in October 1987 and the Dot Com bubble crash in early 2000. The origination and termination dates identified by these two procedures for the 1986-1987 episode are almost identical, namely 1986M04-M09 and 1987M02-M10/11. The identified bubble period for the Dot Com episode using PSY is 1996M12-2000M12 (with some small breaks in between). In comparison, the PWY algorithm identifies the starting of the Dot Com episode eight months earlier (i.e. 1996M04) but sets the termination date three months later. The PSY procedure detects an additional bubble period in 1983 (1983M06-M07 and 1983M9-M10) which the PWY algorithm does not.

Figure 3: The identified bubble and crisis episodes based on the log real Nasdaq stock price index using the PSY procedure with lag order of one and minimum window size of 44 observations. The finite sample critical value sequences are obtained from Monte Carlo simulation with 2,000 replications.

(a) Bubble



(b) Crisis



4.5. Real-time Monitoring of Market Crisis and Recovery

We conduct a pseudo-real-time monitoring exercise to assess market collapse and recovery for the Dot Com bubble episode. Specifically, we start implementing the reverse procedure repeatedly for each observation backwards from March 2000 (the peak of the episode). We apply the reverse dating technique first for the window running from January 1973 to March 2000 to see whether any market correction has occurred. If affirmative, we calculate the date of market recovery and record March 2000 as the date for this correction. Otherwise, we expand the sample by one observation forward and apply the same reverse regression technique to the new sample period until a market correction is corrected. We stop the pseudo-real-time investigation upon detection of a market correction. Obviously, this procedure can be implemented in real time as new observations arise. It may be regarded as a form of econometric technical analysis for financial markets.

By construction, the delay in detecting the market recovery date is bounded by the minimum window size as shown in the simulation section. We set the minimum window size here to be 6 months and the lag length is again fixed at one. Using this technique, we identify an episode of market correction after the peak of the Dot Com bubble episode in October 2000. The identified market recovery date by this method is two months earlier than the date of December 2000 that is obtained from the ex post identification strategy. From the perspective of real-time monitoring, there is a delay in this detection (*DD* in earlier notation), which means that the recovery is not affirmed in the data until March 2001 (or $DD + \hat{f}_r$ as a sample fraction) when a 6-month minimum window size is used.

REFERENCES

Phillips, P.C.B., 1987, Time series regression with a unit root, *Econometrica* 55, 277-301.

Phillips, P. C. B. and Hansen, B. E., 1990. "Statistical inference in instrumental variables regression with I(1) processes," *Review of Economic Studies* 57, 99–125.

Phillips, P.C.B., and Perron, P., 1988, Testing for a unit root in time series regression. *Biometrika*, 75(2):335–346.

Phillips, P.C.B., and Magdalinos, T., 2007a, Limit theory for moderate deviations from a unit root. *Journal of Econometrics*, 136:115–130.

Phillips, P. C. B. and T. Magdalinos., 2007b, Limit Theory for Moderate Deviations from Unity under Weak Dependence" in G. D. A. Phillips and E. Tzavalis (Eds.) *The Refinement of Econometric Estimation and Test Procedures: Finite Sample and Asymptotic Analysis*. Cambridge: Cambridge University Press, 2007, pp.123-162.

Phillips, P. C. B., Shi, S., and Yu, J., 2015a, Testing for multiple bubbles: Historical episodes of exuberance and collapse in the S&P 500. *International Economic Review*, 56(4):1043–1078.

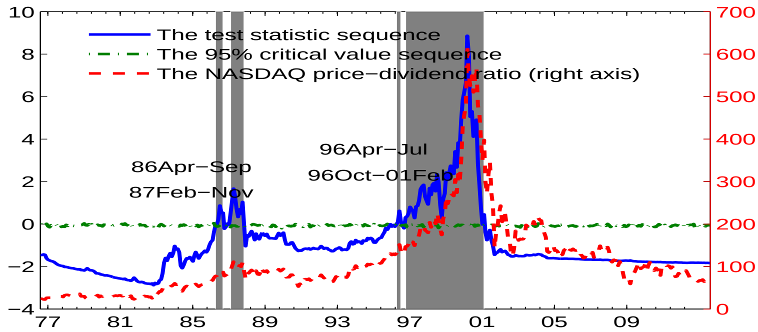
Phillips, P. C. B., Shi, S., and Yu, J., 2015b, Testing for multiple bubbles: Limit Theory of Real Time Detectors. *International Economic Review*, 56(4):1079–1134.

Phillips, P.C.B., and Solo, V., 1992, Asymptotics for linear processes. *The Annals of Statistics*, 20:971–1001.

Phillips, P.C.B., and Yu, J., 2009, Limit theory for dating the origination and collapse of mildly explosive periods in time series data. *Singapore Management University, Unpublished Working Paper.*

Figure 4: The identified bubble episodes using the PWY and PSY procedures with lag order of one and minimum window size of 44 observations. The finite sample critical value sequences are obtained by Monte Carlo simulation with 2,000 replications.

(a) PWY



(b) PSY