Online Appendices to "Testing for a General Class of Functional Inequalities"

Appendix A gives the proofs of Theorems 1-5, and Appendices B and C offer auxiliary results for the proofs of Theorems 1-5. Appendix D contains the proof of Theorem AUC1. In Appendix E, we discuss potential areas of applications of our test.

Appendix A. Proofs of Theorems 1-5

The roadmap of Appendix A is as follows. Appendix A begins with the proofs of Lemma 1 (the representation of $\hat{\theta}$) and Lemma 2 (the uniform convergence of $\hat{v}_{\tau,j}(x)$). Then we establish auxiliary results, Lemmas A1-A4, to prepare for the proofs of Theorems 1-3. The brief descriptions of these auxiliary results are given below.

Lemma A1 establishes asymptotic representation of the location normalizers for the test statistic both in the population and in the bootstrap distribution. The crucial implication is that the difference between the population version and the bootstrap version is of order $o_P(h^{d/2})$, $\mathcal{P}$-uniformly. The result is in fact an immediate consequence of Lemma C12 in Appendix C.

Lemma A2 establishes uniform asymptotic normality of the representation of $\hat{\theta}$ and its bootstrap version. The asymptotic normality results use the method of Poissonization as in Giné, Mason, and Zaitsev (2003) and Lee, Song, and Whang (2013). However, in contrast to the preceding research, the results established here are much more general, and hold uniformly over a wide class of probabilities. The lemma relies on Lemmas B7-B9 in Appendix B and their bootstrap versions in Lemmas C7-C9 in Appendix C. These results are employed to obtain the uniform asymptotic normality of the representation of $\hat{\theta}$ in Lemma A2.

Lemma A3 establishes that the estimated contact sets $\hat{B}_A(\hat{c}_n)$ are covered by its enlarged population version, and covers its shrunk population version with probability approaching one uniformly over $P \in \mathcal{P}$. In fact, this is an immediate consequence of the uniform convergence results for $\hat{v}_{\tau,j}(x)$ and $\hat{\sigma}_{\tau,j}(x)$ in Assumptions 3 and 5. Lemma A3 is used later, when we replace the estimated contact sets by their appropriate population versions, eliminating the nuisance to deal with the estimation errors in $\hat{B}_A(\hat{c}_n)$.

Lemma A4 presents the approximation result of the critical values for the original and bootstrap test statistics in Lemma A2, by critical values from the standard normal distribution uniformly over $P \in \mathcal{P}$. Although we do not propose using the normal critical values, the result is used as an intermediate step for justifying the use of the bootstrap method in this paper. Obviously, Lemma A4 follows as a consequence of Lemma A2.

Equipped with Lemmas A1-A4, we proceed to prove Theorem 1. For this, we first use the representation result of Lemma 1 for $\hat{\theta}$. In doing so, we use $B_A(c_{n,L}, c_{n,U})$ as a population version of $\hat{B}_A(\hat{c}_n)$. This is because

$$B_A(c_{n,L}, c_{n,U}) \subset \hat{B}_A(\hat{c}_n)$$

with probability approaching one by Lemma A3, and thus, makes the bootstrap test statistic $\hat{\theta}^*$ dominate the one that involves $B_A(c_{n,L}, c_{n,U})$ in place of $\hat{B}_A(\hat{c}_n)$. The distribution of the latter bootstrap version with $B_A(c_{n,L}, c_{n,U})$ is asymptotically equivalent to the representation of $\hat{\theta}$ with $B_A(c_{n,L}, c_{n,U})$ after location-scale normalization, as long as the limiting distribution is nondegenerate. When the limiting distribution is degenerate, we use the second component $h^{d/2}\eta + \hat{a}^*$ in the definition of $c^*_{\alpha,\eta}$ to ensure the asymptotic validity of the bootstrap procedure. For both cases of degenerate and nondegenerate limiting distributions, Lemma A1 which enables one to replace $\hat{a}^*$ by an appropriate population version is crucial.

The proof of Theorem 2 that shows the asymptotic exactness of the bootstrap test modifies the proof of Theorem 1 substantially. Instead of using the representation result of Lemma 1 for $\hat{\theta}$ with $B_{n,A}(c_{n,L}, c_{n,U})$, we now use the same version but with $B_{n,A}(c_{n,U}, c_{n,L})$. This is because for asymptotic exactness, we need to approximate the original and bootstrap quantities by versions using $B_{n,A}(q_n)$ for small q_n, and to do this, we need to control the remainder term in the bootstrap statistic with the integral domain $\hat{B}_A(\hat{c}_n) \backslash B_{n,A}(q_n)$. By our choice of $B_{n,A}(c_{n,U}, c_{n,L})$ and by the fact that we have

$$\hat{B}_A(\hat{c}_n) \subset B_{n,A}(c_{n,U}, c_{n,L}),$$

with probability approaching one by Lemma A3, we can bound the remainder term with a version with the integral domain $B_{n,A}(c_{n,U}, c_{n,L}) \backslash B_{n,A}(q_n)$. Thus this remainder term vanishes by the condition for λ_n and q_n in the definition of $\mathcal{P}_n(\lambda_n, q_n)$.

The rest of the proofs are devoted to proving the power properties of the bootstrap procedure. Theorem 3 establishes consistency of the bootstrap test. Theorems 4 and 5 establish local power functions under Pitman local drifts. The proofs of Theorems 4-5 are similar to the proof of Theorem 2, as we need to establish the asymptotically exact form of the rejection probability for the bootstrap test statistic. Nevertheless, we need to employ some delicate arguments to deal with the Pitman local alternatives, and need to expand the rejection probability to obtain the final results. For this, we first establish Lemmas A5-A7. Essentially, Lemma A5 is a version of the representation result of Lemma 1 under local alternatives. Lemma A6 and Lemma A7 parallel Lemma A1 and Lemma 2 under local alternatives.

Let us begin by proving Lemma 1. First, recall the following definitions

$$\text{(A.1)} \quad \hat{\mathbf{s}}_\tau(x) \equiv \left[\frac{r_{n,j}\{\hat{v}_{\tau,j}(x) - v_{n,\tau,j}(x)\}}{\hat{\sigma}_{\tau,j}(x)} \right]_{j \in \mathbb{N}_J} \text{ and } \hat{\mathbf{s}}^*_\tau(x) \equiv \left[\frac{r_{n,j}\{\hat{v}^*_{\tau,j}(x) - \hat{v}_{,\tau,j}(x)\}}{\hat{\sigma}^*_{\tau,j}(x)} \right]_{j \in \mathbb{N}_J}.$$

Also, define

$$\text{(A.2)} \qquad \hat{\mathbf{u}}_\tau(x) \equiv \left[\frac{r_{n,j}\hat{v}_{\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)}\right]_{j\in\mathbb{N}_J} \text{ and } \mathbf{u}_\tau(x;\hat{\sigma}) \equiv \left[\frac{r_{n,j}v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)}\right]_{j\in\mathbb{N}_J}.$$

Proof of Lemma 1. It suffices to show the following two statements:

Step 1: As $n \to \infty$,

$$\inf_{P\in\mathcal{P}_0} P\left\{\int_{\mathcal{S}\backslash B_n(c_{n,1},c_{n,2})} \Lambda_p\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau) = 0\right\} \to 1,$$

where we recall $B_n(c_{n,1},c_{n,2}) \equiv \cup_{A\in\mathcal{N}_J} B_{n,A}(c_{n,1},c_{n,2})$.

Step 2: For each $A \in \mathcal{N}_J$, as $n \to \infty$,

$$\inf_{P\in\mathcal{P}_0} P\left\{\int_{B_{n,A}(c_{n,1},c_{n,2})} \left\{\Lambda_p\left(\hat{\mathbf{u}}_\tau(x)\right) - \Lambda_{A,p}\left(\hat{\mathbf{u}}_\tau(x)\right)\right\} dQ(x,\tau) = 0\right\} \to 1.$$

First, we prove Step 1. We write the integral in the probability as

$$\text{(A.3)} \qquad \int_{\mathcal{S}\backslash B_n(c_{n,1},c_{n,2})} \Lambda_p\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau).$$

Let

$$A_n(x,\tau) \equiv \left\{j \in \mathbb{N}_J : \frac{r_{n,j}v_{n,\tau,j}(x)}{\sigma_{n,\tau,j}(x)} \geq -(c_{n,1}\wedge c_{n,2})\right\}.$$

We first show that when $(x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})$, we have $A_n(x,\tau) = \varnothing$ under the null hypothesis. Suppose that $(x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})$ but to the contrary, $A_n(x,\tau)$ is nonempty. By the definition of $A_n(x,\tau)$, we have $(x,\tau) \in B_{n,A_n(x,\tau)}(c_{n,1},c_{n,2})$. However, since

$$\mathcal{S}\backslash B_n(c_{n,1},c_{n,2}) = \mathcal{S} \cap \left(\cap_{A\in\mathcal{N}_J} B^c_{n,A}(c_{n,1},c_{n,2})\right) \subset B^c_{n,A_n(x,\tau)}(c_{n,1},c_{n,2}),$$

this contradicts the fact that $(x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})$. Hence whenever $(x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})$, we have $A_n(x,\tau) = \varnothing$.

Note that

$$\frac{v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)} = \frac{v_{n,\tau,j}(x)}{\sigma_{n,\tau,j}(x)}\left\{1 + \frac{\sigma_{n,\tau,j}(x) - \hat{\sigma}_{\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)}\right\} = \frac{v_{n,\tau,j}(x)}{\sigma_{n,\tau,j}(x)}\left\{1 + o_P(1)\right\},$$

where $o_P(1)$ is uniform over $(x,\tau) \in \mathcal{S}$ and over $P \in \mathcal{P}$ by Assumption A5. Fix a small $\varepsilon > 0$. We have for all $j \in \mathbb{N}_J$,

$$\begin{aligned} &\inf_{P\in\mathcal{P}_0} P\left\{\frac{r_{n,j}v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)} < -\frac{c_{n,1}\wedge c_{n,2}}{1+\varepsilon} \text{ for all } (x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})\right\} \\ \geq\ &\inf_{P\in\mathcal{P}_0} P\left\{\frac{r_{n,j}v_{n,\tau,j}(x)}{\sigma_{n,\tau,j}(x)} < -\frac{c_{n,1}\wedge c_{n,2}}{(1+\varepsilon)\left\{1+o_P(1)\right\}} \text{ for all } (x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})\right\} \to 1, \end{aligned}$$

as $n \to \infty$, where the last convergence follows because $A_n(x,\tau) = \varnothing$ for all $(x,\tau) \in \mathcal{S}\backslash B_n(c_{n,1},c_{n,2})$. Therefore, with probability approaching one, the term in (A.3) is bounded by

$$\int_{\mathcal{S}\backslash B_n(c_{n,1},c_{n,2})} \Lambda_p\left(\hat{\mathbf{s}}_\tau(x) - \left(\frac{c_{n,1}\wedge c_{n,2}}{1+\varepsilon}\right)\mathbf{1}_J\right) dQ(x,\tau), \tag{A.4}$$

where $\mathbf{1}_J$ is a J-dimensional vector of ones. Using the definition of $\Lambda_p(\mathbf{v})$, bound the above integral by

$$J^{p/2}\left(\sum_{j=1}^{J}\left[r_{n,j}\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{\hat{v}_{\tau,j}(x)-v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)}\right| - \frac{c_{n,1}\wedge c_{n,2}}{1+\varepsilon}\right]_+^2\right)^{p/2}. \tag{A.5}$$

Note that by Assumption A3,

$$r_{n,j}\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{\hat{v}_{\tau,j}(x)-v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)}\right| = O_P\left(\sqrt{\log n}\right).$$

Fix any arbitrarily large $M>0$ and denote by E_n the event that

$$r_{n,j}\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{\hat{v}_{\tau,j}(x)-v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)}\right| \le M\sqrt{\log n}.$$

The term (A.5), when restricted to this event E_n, is bounded by

$$J^{p/2}\left(\sum_{j=1}^{J}\left[M\sqrt{\log n} - \frac{c_{n,1}\wedge c_{n,2}}{1+\varepsilon}\right]_+^2\right)^{p/2}$$

which becomes zero from some large n on, given that $(c_{n,1}\wedge c_{n,2})/\sqrt{\log n}\to\infty$. Since $\sup_{P\in\mathcal{P}_0}PE_n^c\to 0$ as $n\to\infty$ and then $M\to\infty$ by Assumption A3, we obtain the desired result of Step 1.

As for Step 2, we have for any small $\varepsilon>0$, and for all $j\in\mathbb{N}_J\backslash A$,

$$\begin{aligned} &P\left\{\frac{r_{n,j}v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)} < -\frac{c_{n,1}\wedge c_{n,2}}{1+\varepsilon} \text{ for all } (x,\tau)\in B_{n,A}(c_{n,1},c_{n,2})\right\} \\ \ge\ &P\left\{\frac{r_{n,j}v_{n,\tau,j}(x)}{\sigma_{n,\tau,j}(x)} < -\frac{c_{n,1}\wedge c_{n,2}}{(1+\varepsilon)\left\{1+o_P(1)\right\}} \text{ for all } (x,\tau)\in B_{n,A}(c_{n,1},c_{n,2})\right\}\to 1, \end{aligned} \tag{A.6}$$

similarly as before. Let $\bar{\mathbf{s}}_{\tau,A}(x)$ be a J-dimensional vector whose j-th entry is $r_{n,j}\hat{v}_{n,\tau,j}(x)/\hat{\sigma}_{\tau,j}(x)$ if $j\in A$, and $r_{n,j}\{\hat{v}_{n,\tau,j}(x)-v_{n,\tau,j}(x)\}/\hat{\sigma}_{\tau,j}(x)$ if $j\in\mathbb{N}_J\backslash A$. Since by Assumption A5, we have

$$\inf_{P\in\mathcal{P}_0} P\left\{\mathbf{u}_\tau(x;\hat{\sigma})\le 0 \text{ for all } (x,\tau)\in\mathcal{S}\right\}\to 1,$$

as $n \to \infty$, using either definition of $\Lambda_p(\mathbf{v})$ in (3.1), we find that with probability approaching one (uniformly over $P \in \mathcal{P}_0$),

$$\begin{aligned}
\text{(A.7)} \qquad & \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_{A,p}\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau) \\
\le & \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_p\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau) \\
\le & \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_p\left(\bar{\mathbf{s}}_{\tau,A}(x) - \frac{c_{n,1} \wedge c_{n,2}}{1+\varepsilon}\mathbf{1}_{-A}\right) dQ(x,\tau),
\end{aligned}$$

where $\mathbf{1}_{-A}$ is the J-dimensional vector whose j-th entry is zero if $j \in A$ and one if $j \in \mathbb{N}_J \backslash A$, and the last inequality holds with probability approaching one by (A.6). Note that by Assumption A3 and by the assumption that $\sqrt{\log n}\{c_{n,1}^{-1} + c_{n,2}^{-1}\} \to \infty$, we deduce that for any $j \in \mathbb{N}_J$,

$$\inf_{P \in \mathcal{P}_0} P\left\{ r_{n,j} \sup_{(x,\tau)\in\mathcal{S}} \left| \frac{\hat{v}_{\tau,j}(x) - v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)} \right| \le \frac{c_{n,1} \wedge c_{n,2}}{1+\varepsilon} \right\} \to 1,$$

as $n \to \infty$. Hence, as $n \to \infty$,

$$\inf_{P \in \mathcal{P}_0} P\left\{ \begin{array}{c} \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_p\left(\bar{\mathbf{s}}_{\tau,A}(x) - ((c_{n,1} \wedge c_{n,2})/(1+\varepsilon))\mathbf{1}_{-A}\right) dQ(x,\tau) \\ = \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_{A,p}\left(\bar{\mathbf{s}}_{\tau,A}(x)\right) dQ(x,\tau) \end{array} \right\} \to 1.$$

Since

$$\int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_{A,p}\left(\bar{\mathbf{s}}_{\tau,A}(x)\right) dQ(x,\tau) = \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_{A,p}\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau),$$

we obtain the desired result from (A.7). ∎

Now let us turn to the proof of Lemma 2 in Section 4.4.

Proof of Lemma 2. (i) Recall the definition $b_{n,ij}(x,\tau) \equiv \beta_{n,x,\tau,j}\left(Y_{ij},(X_i - x)/h\right)$. Take $M_{n,j} \equiv \sqrt{nh^d}/\sqrt{\log n}$, and let

$$b_{n,ij}^a(x,\tau) \equiv b_{n,ij}(x,\tau)1_{n,ij} \text{ and } b_{n,ij}^b(x,\tau) \equiv b_{n,ij}(x,\tau)\left(1 - 1_{n,ij}\right),$$

where $1_{n,ij} \equiv 1\{\sup_{(x,\tau)\in\mathcal{S}}|b_{n,ij}(x,\tau)| \le M_{n,j}/2\}$. First, note that by Assumption A1,

$$\begin{aligned}
\text{(A.8)} \qquad & r_{n,j}\sqrt{h^d} \sup_{(x,\tau)\in\mathcal{S}} \left| \frac{\hat{v}_{\tau,j}(x) - v_{n,\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)} \right| \\
\le & \sup_{(x,\tau)\in\mathcal{S}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(b_{n,ij}^a(x,\tau) - \mathbf{E}\left[b_{n,ij}^a(x,\tau)\right]\right) \right| \\
\text{(A.9)} \qquad & + \sup_{(x,\tau)\in\mathcal{S}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(b_{n,ij}^b(x,\tau) - \mathbf{E}\left[b_{n,ij}^b(x,\tau)\right]\right) \right| + o_P(1), \ \mathcal{P}\text{-uniformly.}
\end{aligned}$$

We now prove part (i) by proving the following two steps.
Step 1:

$$\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{nh^d}}\sum_{i=1}^{n}\left(b_{n,ij}^b(x,\tau)-\mathbf{E}\left[b_{n,ij}^b(x,\tau)\right]\right)\right| = o_P(\sqrt{\log n}),\ \mathcal{P}\text{-uniformly.}$$

Step 2:

$$\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{nh^d}}\sum_{i=1}^{n}\left(b_{n,ij}^a(x,\tau)-\mathbf{E}\left[b_{n,ij}^a(x,\tau)\right]\right)\right| = O_P(\sqrt{\log n}),\ \mathcal{P}\text{-uniformly.}$$

Step 1 is carried out by some elementary moment calculations, whereas Step 2 is proved using a maximal inequality of Massart (2007, Theorem 6.8).

Proof of Step 1: It is not hard to see that

$$\begin{aligned}
&\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(b_{n,ij}^b(x,\tau)-\mathbf{E}\left[b_{n,ij}^b(x,\tau)\right]\right)\right|\right] \\
\le\ & 2\sqrt{n}\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}|b_{n,ij}(x,\tau)|\left(1-1_{n,ij}\right)\right] \\
\le\ & C\sqrt{n}\left(\frac{M_{n,j}}{2}\right)^{-3}\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}|b_{n,ij}(x,\tau)|^4\right]\le C_1\sqrt{n}\left(\frac{M_{n,j}}{2}\right)^{-3},
\end{aligned}$$

for some $C_1>0$, $C>0$. The last bound follows by the uniform fourth moment bound for $b_{n,ij}(x,\tau)$ assumed in Lemma 2. Note that

$$\sqrt{n}\left(M_{n,j}\right)^{-3} = n^{-1}h^{-3d/2}\left(\log n\right)^{3/2} = o\left(\sqrt{\log n}h^{d/2}\right),$$

by the condition that $n^{-1/2}h^{-d-\nu}\to 0$ for some small $\nu>0$.

Proof of Step 2: For each $j\in\mathbb{N}_J$, let $\mathcal{F}_{n,j}\equiv\{\beta_{n,x,\tau,j}^a(\cdot,(\cdot-x)/h)/M_{n,j} : (x,\tau)\in\mathcal{S}\}$, where $\beta_{n,x,\tau,j}^a(Y_{ij},(X_i-x)/h)\equiv b_{n,ij}^a(x,\tau)$. Note that the indicator function $1_{n,ij}$ in the definition of $\beta_{n,x,\tau,j}^a$ does not depend on (x,τ) of $\beta_{n,x,\tau,j}^a$. Using (3.11) in Lemma 2 and following (part of) the arguments in the proof of Theorem 3 of Chen, Linton, and Van Keilegom (2003), we find that there exist $C_1>0$ and $C_{2,j}>0$ such that for all $\varepsilon>0$,

$$N_{[]}\left(\varepsilon,\mathcal{F}_{n,j},L_2(P)\right)\le N\left(\left(\frac{\varepsilon M_{n,j}}{\delta_{n,j}}\right)^{2/\gamma_j},\mathcal{X}\times\mathcal{T},||\cdot||\right)\le C_1\left(\frac{\varepsilon M_{n,j}}{\delta_{n,j}}\wedge 1\right)^{-C_{2,j}},$$

where $N_{[]}\left(\varepsilon,\mathcal{F}_{n,j},L_2(P)\right)$ denotes the ε-bracketing number of the class $\mathcal{F}_{n,j}$ with respect to the $L_2(P)$-norm and $N\left(\varepsilon,\mathcal{X}\times\mathcal{T},||\cdot||\right)$ denotes the ε-covering number of the space $\mathcal{X}\times\mathcal{T}$ with respect to the Euclidean norm $||\cdot||$. The last inequality follows by the assumption that $\mathcal{X}$ and $\mathcal{T}$ are compact subsets of a Euclidean space. The class $\mathcal{F}_{n,j}$ is uniformly bounded by

1/2.

Let $\{[\beta^a_{n,x_k,\tau_k,j}(\cdot,(\cdot-x_k)/h)/M_{n,j}-\Delta_k(\cdot,\cdot)/M_{n,j},\beta^a_{n,x_k,\tau_k,j}(\cdot,(\cdot-x_k)/h)/M_{n,j}+\Delta_k(\cdot,\cdot)/M_{n,j}]:k=1,...,N_{n,j}\}$ constitutes ε-brackets, where $\Delta_k(Y_{ij},X_i)\equiv\sup|\beta^a_{n,x,\tau,j}(Y_{ij},(X_i-x)/h)-\beta^a_{n,x_k,\tau_k,j}(Y_{ij},(X_i-x_k)/h)|$ and the supremum is over $(x,\tau)\in\mathcal{S}$ such that

$$\sqrt{||x-x_k||^2+||\tau-\tau_k||^2}\leq C_1(\varepsilon M_{n,j}/\delta_{n,j})^{2/\gamma_j}.$$

By the previous covering number bound, we can take $N_{n,j}\leq C_1\left((\varepsilon M_{n,j}/\delta_{n,j})\wedge 1\right)^{-C_{2,j}}$, and

$$\mathbf{E}\Delta_k^2(Y_{ij},X_i)M_{n,j}^{-2}<\varepsilon^2.$$

Note that for any $k\geq 2$,

$$\mathbf{E}\left[|b^a_{n,ij}(x,\tau)/M_{n,j}|^k\right]\leq\mathbf{E}\left[b^2_{n,ij}(x,\tau)\right]/M^2_{n,j}\leq CM_{n,j}^{-2}h^d=C(\log n)/n,$$

by the fact that $|b^a_{n,ij}(x,\tau)/M_{n,j}|\leq 1/2$. Furthermore,

$$\mathbf{E}\left[|\Delta_k(Y_{ij},X_i)/M_{n,j}|^k\right]\leq\mathbf{E}\left[\Delta_k^2(Y_{ij},X_i)/M^2_{n,j}\right]\leq\varepsilon^2,$$

where the first inequality follows because $|\Delta_k(Y_{ij},X_i)/M_{n,j}|\leq 1$. Therefore, by Theorem 6.8 of Massart (2007), we have (from sufficiently large n on)

$$\text{(A.10)}\qquad \mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{M_{n,j}\sqrt{n}}\sum_{i=1}^n\left(b^a_{n,ij}(x,\tau)-\mathbf{E}\left[b^a_{n,ij}(x,\tau)\right]\right)\right|\right]$$

$$\leq C_1\int_0^{\frac{C_2h^{d/2}}{M_{n,j}}}\left\{\left(-C_3\log\left(\frac{\varepsilon M_{n,j}}{\delta_{n,j}}\wedge 1\right)\right)\wedge n\right\}^{1/2}d\varepsilon-\frac{C_4}{\sqrt{n}}\log\left(\frac{\sqrt{\log n}}{\sqrt{n}}\right),$$

where C_1,C_2,C_3, and C_4 are positive constants. (The inequality above follows because $\sqrt{\log n}/\sqrt{n}\to 0$ as $n\to\infty$.) The leading integral has a domain restricted to $[0,\delta_{n,j}/M_{n,j}]$, so that it is equal to

$$C_1\int_0^{\frac{C_2h^{d/2}}{M_{n,j}}\wedge\frac{\delta_{n,j}}{M_{n,j}}}\left\{\left(-C_3\log\left(\frac{\varepsilon M_{n,j}}{\delta_{n,j}}\right)\right)\wedge n\right\}^{1/2}d\varepsilon$$

$$=\frac{C_1\delta_{n,j}}{M_{n,j}}\int_0^{\frac{C_2h^{d/2}}{\delta_{n,j}}\wedge 1}\sqrt{(-C_3\log\varepsilon)\wedge n}\,d\varepsilon$$

$$=O\left(\frac{\delta_{n,j}}{M_{n,j}}\left(\frac{h^{d/2}}{\delta_{n,j}}\wedge 1\right)\sqrt{-\log\left(\frac{h^{d/2}}{\delta_{n,j}}\wedge 1\right)}\right).$$

After multiplying by $M_{n,j}/h^{d/2}$, the last term is of order

$$O\left(\left(1\wedge\frac{\delta_{n,j}}{h^{d/2}}\right)\sqrt{-\log\left(\frac{h^{d/2}}{\delta_{n,j}}\wedge 1\right)}\right)=O\left(\sqrt{-\log\left(\frac{h^{d/2}}{\delta_{n,j}}\wedge 1\right)}\right)=O(\sqrt{\log n}),$$

because $\delta_{n,j} = n^{s_{1,j}}$ and $h = n^{s_2}$ for some $s_{1,j}, s_2 \in \mathbf{R}$.

Also, note that after multiplying by $M_{n,j}/h^{d/2} = \sqrt{n}/\sqrt{\log n}$, the last term in (A.10) (with minus sign) becomes

$$-\frac{C_4}{\sqrt{\log n}} \log\left(\frac{\sqrt{\log n}}{\sqrt{n}}\right) \leq \frac{C_4\sqrt{\log n}}{2} - \frac{C_4 \log\sqrt{\log n}}{\sqrt{\log n}} = O\left(\sqrt{\log n}\right),$$

where the inequality follows because $\sqrt{\log n} \geq 1$ for all $n \geq e \equiv \exp(1)$. Collecting the results for both the terms on the right hand side of (A.10), we obtain the desired result of Step 2.

(ii) Define $b^*_{n,ij}(x,\tau) \equiv \beta_{n,x,\tau,j}(Y^*_{ij}, (X^*_i - x)/h)$. By Assumptions B1 and B3, it suffices to show that

$$\sup_{(x,\tau)\in\mathcal{S}} \left|\frac{1}{\sqrt{nh^d}} \sum_{i=1}^n \left(b^*_{n,ij}(x,\tau) - \mathbf{E}^*\left[b^*_{n,ij}(x,\tau)\right]\right)\right| = O_{P^*}(\sqrt{\log n}),\ \mathcal{P}\text{-uniformly.}$$

Using Le Cam's Poissonization lemma in Giné and Zinn (1990) (Proposition 2.2 on p.855) and following the arguments in the proof of Theorem 2.2 of Giné (1997), we deduce that

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}^*\left(\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{nh^d}}\sum_{i=1}^n\left(b^*_{n,ij}(x,\tau) - \mathbf{E}^*\left[b^*_{n,ij}(x,\tau)\right]\right)\right|\right)\right] \\
\leq\ & \frac{e}{e-1}\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{nh^d}}\sum_{i=1}^n (N_i - 1)\left\{b_{n,ij}(x,\tau) - \frac{1}{n}\sum_{k=1}^n b_{n,kj}(x,\tau)\right\}\right|\right],
\end{aligned}$$

where N_i's are i.i.d. Poisson random variables with mean 1 and independent of $\{(X_i, Y_i)\}_{i=1}^n$. The last expectation is bounded by

$$\begin{aligned}
&\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{nh^d}}\sum_{i=1}^n\left\{(N_i-1)\,b_{n,ij}(x,\tau) - \mathbf{E}\left[(N_i-1)\,b_{n,ij}(x,\tau)\right]\right\}\right|\right] \\
&+\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=1}^n (N_i-1)\right|\left|\frac{1}{\sqrt{nh^d}}\sum_{k=1}^n\left(b_{n,kj}(x,\tau) - \mathbf{E}\left[b_{n,kj}(x,\tau)\right]\right)\right|\right].
\end{aligned}$$

Using the same arguments as in the proof of (i), we find that the first expectation is $O\left(\sqrt{\log n}\right)$ uniformly in $P \in \mathcal{P}$. Using independence, we write the second expectation as

$$\mathbf{E}\left[\left|\frac{1}{n}\sum_{i=1}^n (N_i-1)\right|\right]\cdot\mathbf{E}\left[\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{\sqrt{nh^d}}\sum_{k=1}^n\left(b_{n,kj}(x,\tau) - \mathbf{E}\left[b_{n,kj}(x,\tau)\right]\right)\right|\right]$$

which, as shown in the proof of part (i), is $O(\sqrt{\log n})$, uniformly in $P \in \mathcal{P}$. ∎

For further proofs, we introduce new notation. Define for any positive sequences $c_{n,1}$ and $c_{n,2}$, and any $\mathbf{v} \in \mathbf{R}^J$,

$$\bar{\Lambda}_{x,\tau}(\mathbf{v}) \equiv \sum_{A \in \mathcal{N}_J} \Lambda_{A,p}(\mathbf{v}) 1\{(x,\tau) \in B_{n,A}(c_{n,1}, c_{n,2})\}. \tag{A.11}$$

We let

$$\begin{aligned} a_n^R(c_{n,1}, c_{n,2}) &\equiv \int_{\mathcal{X}\times\mathcal{T}} \mathbf{E}\left[\bar{\Lambda}_{x,\tau}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right] dQ(x,\tau), \text{ and} \\ a_n^{R*}(c_{n,1}, c_{n,2}) &\equiv \int_{\mathcal{X}\times\mathcal{T}} \mathbf{E}^*\left[\bar{\Lambda}_{x,\tau}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x))\right] dQ(x,\tau), \end{aligned} \tag{A.12}$$

where $\mathbf{z}_{N,\tau}(x)$ and $\mathbf{z}^*_{N,\tau}(x)$ are random vectors whose j-th entry is respectively given by

$$\begin{aligned} z_{N,\tau,j}(x) &\equiv \frac{1}{nh^d}\sum_{i=1}^N \left(\beta_{n,x,\tau,j}\left(Y_{ij}, \frac{X_i - x}{h}\right) - \mathbf{E}\left[\beta_{n,x,\tau,j}\left(Y_{ij}, \frac{X_i - x}{h}\right)\right]\right) \text{ and} \\ z^*_{N,\tau,j}(x) &\equiv \frac{1}{nh^d}\sum_{i=1}^N \left(\beta_{n,x,\tau,j}\left(Y^*_{ij}, \frac{X^*_i - x}{h}\right) - \mathbf{E}^*\left[\beta_{n,x,\tau,j}\left(Y^*_{ij}, \frac{X^*_i - x}{h}\right)\right]\right), \end{aligned}$$

and N is a Poisson random variable with mean n and independent of $\{Y_i, X_i\}_{i=1}^\infty$. We also define

$$a_n(c_{n,1}, c_{n,2}) \equiv \int \mathbf{E}\left[\bar{\Lambda}_{x,\tau}(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0))\right] dQ(x,\tau).$$

(See Section 6.3 for the definition of $\mathbb{W}^{(1)}_{n,\tau,\tau}(x,u)$.)

Lemma A1. *Suppose that Assumptions A6(i) and B4 hold and let $c_{n,1}$ and $c_{n,2}$ be any nonnegative sequences. Then*

$$\begin{aligned} \left|a_n^R(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})\right| &= o(h^{d/2}), \text{ uniformly in } P \in \mathcal{P}, \text{ and} \\ \left|a_n^{R*}(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})\right| &= o_P(h^{d/2}), \ \mathcal{P}\text{-uniformly.} \end{aligned}$$

Proof of Lemma A1. The proof is essentially the same as the proof of Lemma C12 in Appendix C. ∎

For any given nonnegative sequences $c_{n,1}, c_{n,2}$, we define

$$\sigma_n^2(c_{n,1}, c_{n,2}) \equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{\mathcal{X}} \bar{C}_{\tau_1,\tau_2}(x) dx d\tau_1 d\tau_2, \tag{A.13}$$

where

$$\bar{C}_{\tau_1,\tau_2}(x) \equiv \int_{\mathcal{U}} Cov\left(\bar{\Lambda}_{n,x,\tau_1}(\mathbb{W}^{(1)}_{n,\tau_1,\tau_2}(x,u)), \bar{\Lambda}_{n,x,\tau_2}(\mathbb{W}^{(2)}_{n,\tau_1,\tau_2}(x,u))\right) du.$$

Let

$$\bar{\theta}_n(c_{n,1}, c_{n,2}) \equiv \int \bar{\Lambda}_{x,\tau}\left(\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau), \tag{A.14}$$

and

$$\bar{\theta}_n^*(c_{n,1}, c_{n,2}) \equiv \int \bar{\Lambda}_{x,\tau}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau). \tag{A.15}$$

From here on, for any sequence of random quantities Z_n and a random vector Z, we write

$$Z_n \overset{d}{\to} N(0,1),\ \mathcal{P}_0\text{-uniformly},$$

if for each $t > 0$,

$$\sup_{P \in \mathcal{P}_0} \left| P\left\{ Z_n \leq t \right\} - \Phi(t) \right| = o(1).$$

And for any sequence of bootstrap quantities Z_n^* and a random vector Z, we write

$$Z_n^* \overset{d^*}{\to} N(0,1),\ \mathcal{P}_0\text{-uniformly},$$

if for each $t > 0$,

$$\left| P^*\left\{ Z_n^* \leq t \right\} - \Phi(t) \right| = o_{P^*}(1),\ \mathcal{P}_0\text{-uniformly}.$$

Lemma A2. *(i) Suppose that Assumptions A1-A3, A4(i), and A5-A6 are satisfied. Then for any sequences* $c_{n,1}, c_{n,2} > 0$ *such that* $\liminf_{n\to\infty} \inf_{P \in \mathcal{P}_0} \sigma_n^2(c_{n,1}, c_{n,2}) > 0$ *and* $\sqrt{\log n}/c_{n,2} \to 0$*, as* $n \to \infty$,

$$h^{-d/2}\left(\frac{\bar{\theta}_n(c_{n,1}, c_{n,2}) - a_n^R(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})} \right) \overset{d}{\to} N(0,1),\ \mathcal{P}_0\text{-uniformly.}$$

(ii) Suppose that Assumptions A1-A3, A4(i), A5-A6, B1 and B4 are satisfied. Then for any sequences $c_{n,1}, c_{n,2} \geq 0$ *such that* $\liminf_{n\to\infty} \inf_{P \in \mathcal{P}_0} \sigma_n^2(c_{n,1}, c_{n,2}) > 0$ *and* $\sqrt{\log n}/c_{n,2} \to 0$*, as* $n \to \infty$,

$$h^{-d/2}\left(\frac{\bar{\theta}_n^*(c_{n,1}, c_{n,2}) - a_n^{R*}(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})} \right) \overset{d^*}{\to} N(0,1),\ \mathcal{P}_0\text{-uniformly.}$$

Proof of Lemma A2. (i) By Lemma 1, we have (with probability approaching one)

$$\bar{\theta}_n(c_{n,1}, c_{n,2}) = \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_p(\hat{\mathbf{s}}_\tau(x)) dQ(x,\tau) = \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(c_{n,1},c_{n,2})} \Lambda_{A,p}(\hat{\mathbf{s}}_\tau(x)) dQ(x,\tau).$$

Note that $a_n^R(c_{n,1}, c_{n,2}) = \sum_{A \in \mathcal{N}_J} a_{n,A}^R(c_{n,1}, c_{n,2})$, where

$$a_{n,A}^R(c_{n,1}, c_{n,2}) \equiv \int_{B_{n,A}(c_{n,1},c_{n,2})} \mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)) \right] dQ(x,\tau).$$

Using Assumption A1, we find that $h^{-d/2}\{\bar{\theta}_n(c_{n,1}, c_{n,2}) - a_n^R(c_{n,1}, c_{n,2})\}$ is equal to

$$h^{-d/2} \sum_{A \in \mathcal{N}_J} \{\zeta_{n,A}(B_{n,A}(c_{n,1}, c_{n,2})) - \mathbf{E}\zeta_{N,A}(B_{n,A}(c_{n,1}, c_{n,2}))\} + o_P(1),$$

where for any Borel set $B \subset \mathcal{S}$,

$$\begin{aligned} \zeta_{n,A}(B) &\equiv \int_B \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x))dQ(x,\tau), \\ \zeta_{N,A}(B) &\equiv \int_B \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))dQ(x,\tau), \end{aligned}$$

and

$$\mathbf{z}_{n,\tau}(x) \equiv \frac{1}{nh^d}\sum_{i=1}^n \beta_{n,x,\tau}(Y_i,(X_i-x)/h) - \frac{1}{h^d}\mathbf{E}\left[\beta_{n,x,\tau}(Y_i,(X_i-x)/h)\right],$$

with

$$\beta_{n,x,\tau}(Y_i,(X_i-x)/h) = (\beta_{n,x,\tau,1}(Y_{i1},(X_i-x)/h),...,\beta_{n,x,\tau,J}(Y_{iJ},(X_i-x)/h))^\top.$$

We take $0 < \bar{\varepsilon}_l \to 0$ as $l \to \infty$ and take $\mathcal{C}_l \subset \mathbf{R}^d$ such that

$$0 < P\left\{X_i \in \mathbf{R}^d \backslash \mathcal{C}_l\right\} \le \bar{\varepsilon}_l,$$

and $Q((\mathcal{X}\backslash\mathcal{C}_l)\times\mathcal{T}) \to 0$ as $l \to \infty$. Such a sequence $\{\bar{\varepsilon}_l\}_{l=1}^\infty$ exists by Assumption A6(ii) by the condition that $\mathcal{S}$ is compact. We write

$$\begin{aligned} \text{(A.16)}\quad & \frac{h^{-d/2}\sum_{A\in\mathcal{N}_J}\{\zeta_{n,A}(B_{n,A}(c_{n,1},c_{n,2})) - \mathbf{E}\zeta_{N,A}(B_{n,A}(c_{n,1},c_{n,2}))\}}{\sigma_n^2(c_{n,1},c_{n,2})} \\ =\ & \frac{h^{-d/2}\sum_{A\in\mathcal{N}_J}\{\zeta_{n,A}(B_{n,A}(c_{n,1},c_{n,2})\cap(\mathcal{C}_l\times\mathcal{T})) - \mathbf{E}\zeta_{N,A}(B_{n,A}(c_{n,1},c_{n,2})\cap(\mathcal{C}_l\times\mathcal{T}))\}}{\sigma_n^2(c_{n,1},c_{n,2})} \\ & + \frac{h^{-d/2}\sum_{A\in\mathcal{N}_J}\{\zeta_{n,A}(B_{n,A}(c_{n,1},c_{n,2})\backslash(\mathcal{C}_l\times\mathcal{T})) - \mathbf{E}\zeta_{N,A}(B_{n,A}(c_{n,1},c_{n,2})\backslash(\mathcal{C}_l\times\mathcal{T}))\}}{\sigma_n^2(c_{n,1},c_{n,2})} \\ =\ & A_{1n} + A_{2n}, \text{ say.} \end{aligned}$$

As for A_{2n}, we apply Lemma B7 in Appendix B, and the condition that $Q((\mathcal{X}\backslash\mathcal{C}_l)\times\mathcal{T}) \to 0$, as $l \to \infty$, and

$$\text{liminf}_{n\to\infty}\text{inf}_{P\in\mathcal{P}_0}\sigma_n(c_{1n},c_{2n}) > 0,$$

to deduce that $A_{2n} = o_P(1)$, as $n \to \infty$ and then $l \to \infty$. As for A_{1n}, first observe that as $n \to \infty$ and then $l \to \infty$,

$$\text{(A.17)}\qquad \left|\sigma_n^2(c_{n,1},c_{n,2}) - \bar{\sigma}_{n,l}^2(c_{n,1},c_{n,2})\right| \to 0,$$

where $\bar{\sigma}_{n,l}^2(c_{n,1},c_{n,2})$ is equal to $\sigma_n^2(c_{n,1},c_{n,2})$ except that $B_{n,A}(c_{n,1},c_{n,2})$'s are replaced by $B_{n,A}(c_{n,1},c_{n,2}) \cap (\mathcal{C}_l \times \mathcal{T})$. The convergence follows by Assumption 6(i). Also by Lemma B9(i) and the convergence in (A.17) and the fact that

$$\liminf_{n\to\infty}\inf_{P\in\mathcal{P}_0}\sigma_n^2(c_{n,1},c_{n,2}) > 0,$$

we have

$$A_{1n} \xrightarrow{d} N(0,1),\ \mathcal{P}_0\text{-uniformly,}$$

as $n \to \infty$ and as $l \to \infty$. Hence we obtain (i).

(ii) The proof can be done in the same way as in the proof of (i), using Lemmas C7 and C9(i) in Appendix C instead of Lemmas B7 and B9(i) in Appendix B. ∎

Lemma A3. *Suppose that Assumptions A1-A5 hold. Then for any sequences* $c_{n,L}, c_{n,U} > 0$ *satisfying Assumption A4(ii), and for each* $A \in \mathcal{N}_J$,

$$\inf_{P\in\mathcal{P}} P\left\{B_{n,A}(c_{n,L}, c_{n,U}) \subset \hat{B}_A(\hat{c}_n) \subset B_{n,A}(c_{n,U}, c_{n,L})\right\} \to 1,\ \textit{as}\ n \to \infty.$$

Proof of Lemma A3. By using Assumptions A3-A5, and following the proof of Theorem 2, Claim 1 in Linton, Song, and Whang (2010), we can complete the proof. Details are omitted. ∎

Define for $c_{n,1}, c_{n,2} > 0$,

$$\begin{aligned} T_n(c_{n,1}, c_{n,2}) &\equiv h^{-d/2}\left(\frac{\bar{\theta}_n(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})}\right) \text{ and} \\ T_n^*(c_{n,1}, c_{n,2}) &\equiv h^{-d/2}\left(\frac{\bar{\theta}_n^*(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})}\right). \end{aligned}$$

We introduce critical values for the finite sample distribution of $\hat{\theta}$ as follows:

$$\gamma_n^\alpha(c_{n,1}, c_{n,2}) \equiv \inf\left\{c \in \mathbf{R} : P\left\{T_n(c_{n,1}, c_{n,2}) \le c\right\} > 1-\alpha\right\}.$$

Similarly, let us introduce bootstrap critical values:

$$\gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) \equiv \inf\left\{c \in \mathbf{R} : P^*\left\{T_n^*(c_{n,1}, c_{n,2}) \le c\right\} > 1-\alpha\right\}. \tag{A.18}$$

Finally, we introduce asymptotic critical values: $\gamma_\infty^\alpha \equiv \Phi^{-1}(1-\alpha)$, where Φ denotes the standard normal CDF.

Lemma A4. *Suppose that Assumptions A1-A3, A4(i), and A5-A6 hold. Then the following holds.*

(i) For any $c_{n,1}, c_{n,2} \to \infty$ *such that*

$$\liminf_{n\to\infty} \inf_{P\in\mathcal{P}} \sigma_n^2(c_{n,1}, c_{n,2}) > 0,$$

it is satisfied that

$$\sup_{P\in\mathcal{P}} |\gamma_n^\alpha(c_{n,1}, c_{n,2}) - \gamma_\infty^\alpha| \to 0,\ \textit{as}\ n \to \infty.$$

(ii) Suppose further that Assumptions B1 and B4 hold. Then for any $c_{n,1}, c_{n,2} \to \infty$ *such that*

$$\liminf_{n\to\infty} \inf_{P\in\mathcal{P}} \sigma_n^2(c_{n,1}, c_{n,2}) > 0,$$

it is satisfied that

$$\sup_{P\in\mathcal{P}} |\gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) - \gamma_\infty^\alpha| \to 0, \ \text{as } n \to \infty.$$

Proof of Lemma A4. (i) The statement immediately follows from the first statement of Lemma A2(i) and Lemma A1.

(ii) We show only the second statement. Fix $a > 0$. Let us introduce two events:

$$E_{n,1} \equiv \{\gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) - \gamma_\infty^\alpha < -a\} \text{ and } E_{n,2} \equiv \{\gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) - \gamma_\infty^\alpha > a\}.$$

On the event $E_{n,1}$, we have

$$\begin{aligned} \alpha &= P^* \left\{ h^{-d/2} \left(\frac{\bar{\theta}_n^*(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})} \right) > \gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) \right\} \\ &\geq P^* \left\{ h^{-d/2} \left(\frac{\bar{\theta}_n^*(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})} \right) > \gamma_\infty^\alpha - a \right\}. \end{aligned}$$

By Lemma A2(ii) and Lemma A1, the last probability is equal to

$$1 - \Phi\left(\gamma_\infty^\alpha - a\right) + o_P(1) > \alpha + o_P(1),$$

where $o_P(1)$ is uniform over $P \in \mathcal{P}$ and the last strict inequality follows by the definition of γ_∞^α and $a > 0$. Hence $\sup_{P\in\mathcal{P}} PE_{n,1} \to 0$ as $n \to \infty$. Similarly, on the event $E_{n,2}$, we have

$$\begin{aligned} \alpha &= P^* \left\{ h^{-d/2} \left(\frac{\bar{\theta}_n^*(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})} \right) > \gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) \right\} \\ &\leq P^* \left\{ h^{-d/2} \left(\frac{\bar{\theta}_n^*(c_{n,1}, c_{n,2}) - a_n(c_{n,1}, c_{n,2})}{\sigma_n(c_{n,1}, c_{n,2})} \right) > \gamma_\infty^\alpha + a \right\}. \end{aligned}$$

By the first statement of Lemma A2(ii) and Lemma A1, the last bootstrap probability is bounded by

$$1 - \Phi\left(\gamma_\infty^\alpha + a\right) + o_P(1) < \alpha + o_P(1),$$

so that we have $\sup_{P\in\mathcal{P}} PE_{n,2} \to 0$ as $n \to \infty$. We conclude that

$$\sup_{P\in\mathcal{P}} P\left\{|\gamma_n^{\alpha*}(c_{n,1}, c_{n,2}) - \gamma_\infty^\alpha| > a\right\} = \sup_{P\in\mathcal{P}} \left(PE_{n,1} + PE_{n,2}\right) \to 0,$$

as $n \to \infty$, obtaining the desired result. ∎

Proof of Theorem 1. By Lemma 1, we have

$$\inf_{P\in\mathcal{P}_0} P\left\{\hat{\theta} = \sum_{A\in\mathcal{N}_J} \int_{B_{n,A}(c_{n,L}, c_{n,U})} \Lambda_{A,p}\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau)\right\} \to 1,$$

as $n \to \infty$. Since under the null hypothesis, we have $v_{n,\tau,j}(\cdot)/\hat{\sigma}_{\tau,j}(\cdot) \leq 0$ for all $j \in \mathbb{N}_J$, with probability approaching one by Assumption A5, we have with probability approaching one (uniformly over $P \in \mathcal{P}_0$),

$$\begin{aligned}&\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(c_{n,L},c_{n,U})}\Lambda_{A,p}\left(\hat{\mathbf{u}}_\tau(x)\right)dQ(x,\tau)\\ \leq\ &\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(c_{n,L},c_{n,U})}\Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)\right)dQ(x,\tau)\equiv\bar{\theta}_n(c_{n,L},c_{n,U}).\end{aligned}$$

Thus, we have as $n\to\infty$,

$$\inf_{P\in\mathcal{P}_0}P\left\{\hat{\theta}\leq\bar{\theta}_n(c_{n,L},c_{n,U})\right\}\to 1. \tag{A.19}$$

Let the $(1-\alpha)$-th percentile of the bootstrap distribution of

$$\bar{\theta}_n^*(c_{n,L},c_{n,U})=\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(c_{n,L},c_{n,U})}\Lambda_{A,p}(\hat{\mathbf{s}}_\tau^*(x))dQ(x,\tau)$$

be denoted by $\bar{c}_{n,L}^{\alpha*}$. By Lemma A3 and Assumption A4(ii), with probability approaching one,

$$\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(c_{n,L},c_{n,U})}\Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right)dQ(x,\tau)\leq\sum_{A\in\mathcal{N}_J}\int_{\hat{B}_A(\hat{c}_n)}\Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right)dQ(x,\tau). \tag{A.20}$$

This implies that as $n\to\infty$,

$$\inf_{P\in\mathcal{P}}P\left\{c_\alpha^*\geq\bar{c}_{n,L}^{\alpha*}\right\}\to 1. \tag{A.21}$$

There exists a sequence of probabilities $\{P_n\}_{n\geq1}\subset\mathcal{P}_0$ such that

$$\begin{aligned}\limsup_{n\to\infty}\sup_{P\in\mathcal{P}_0}P\left\{\hat{\theta}>c_{\alpha,\eta}^*\right\}&=\limsup_{n\to\infty}P_n\left\{\hat{\theta}>c_{\alpha,\eta}^*\right\}\\&=\lim_{n\to\infty}P_{w_n}\left\{\hat{\theta}_{w_n}>c_{w_n,\alpha,\eta}^*\right\},\end{aligned} \tag{A.22}$$

where $\{w_n\}\subset\{n\}$ is a certain subsequence, and $\hat{\theta}_{w_n}$ and $c_{w_n,\alpha,\eta}^*$ are the same as $\hat{\theta}$ and $c_{\alpha,\eta}^*$ except that the sample size n is now replaced by w_n.

By Assumption A6(i), $\{\sigma_n(c_{n,L},c_{n,U})\}_{n\geq1}$ is a bounded sequence. Therefore, there exists a subsequence $\{u_n\}_{n\geq1}\subset\{w_n\}_{n\geq1}$, such that $\sigma_{u_n}(c_{u_n,L},c_{u_n,U})$ converges. We consider two cases:

Case 1: $\lim_{n\to\infty}\sigma_{u_n}(c_{u_n,L},c_{u_n,U})>0$, and

Case 2: $\lim_{n\to\infty}\sigma_{u_n}(c_{u_n,L},c_{u_n,U})=0$.

In both cases, we will show below that

$$\limsup_{n\to\infty}P_{u_n}\{\hat{\theta}_{u_n}>c_{u_n,\alpha,\eta}^*\}\leq\alpha. \tag{A.23}$$

Since along $\{w_n\}$, $P_{w_n}\{\hat{\theta}_{w_n}>c_{w_n,\alpha,\eta}^*\}$ converges, it does so along any subsequence of $\{w_n\}$. Therefore, the above limsup is equal to the last limit in (A.22). This completes the proof.

Proof of (A.23) in Case 1: We write $P_{u_n}\{\hat{\theta}_{u_n} > c^*_{u_n,\alpha,\eta}\}$ as

$$\begin{aligned} & P_{u_n}\left(h^{-d/2}\left(\frac{\hat{\theta}_{u_n} - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right) > h^{-d/2}\left(\frac{c^*_{u_n,\alpha,\eta} - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right)\right) \\ \leq \; & P_{u_n}\left(h^{-d/2}\left(\frac{\hat{\theta}_{u_n} - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right) > h^{-d/2}\left(\frac{\bar{c}^{\alpha*}_{u_n,L} - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right)\right) + o(1), \end{aligned}$$

where the inequality follows by the fact that $c^*_{\alpha,\eta} \geq c^*_\alpha \geq \bar{c}^{\alpha*}_{n,L}$ with probability approaching one by (A.21). Using (A.19), we bound the last probability by

(A.24)
$$P_{u_n}\left\{h^{-d/2}\left(\frac{\bar{\theta}_{u_n}(c_{u_n,L}, c_{u_n,U}) - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right) > h^{-d/2}\left(\frac{\bar{c}^{\alpha*}_{u_n,L} - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right)\right\} + o(1).$$

Therefore, since $\lim_{n\to\infty}\sigma_{u_n}(c_{u_n,L}, c_{u_n,U}) > 0$, by Lemmas A2 and A4, we rewrite the last probability in (A.24) as

$$\begin{aligned} & P_{u_n}\left\{h^{-d/2}\left(\frac{\bar{\theta}_{u_n}(c_{u_n,L}, c_{u_n,U}) - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right) > \gamma^{\alpha*}_{u_n}(c_{u_n,L}, c_{u_n,U})\right\} + o(1) \\ = \; & P_{u_n}\left\{h^{-d/2}\left(\frac{\bar{\theta}_{u_n}(c_{u_n,L}, c_{u_n,U}) - a_{u_n}(c_{u_n,L}, c_{u_n,U})}{\sigma_{u_n}(c_{u_n,L}, c_{u_n,U})}\right) > \gamma^{\alpha}_{\infty}\right\} + o(1) = \alpha + o(1). \end{aligned}$$

This completes the proof of Step 1.

Proof of (A.23) in Case 2: First, observe that

$$a^*_{u_n}(c_{u_n,L}, c_{u_n,U}) \leq a^*_{u_n}(\hat{c}_{u_n}),$$

with probability approaching one by Lemma A3. Hence using this and (A.19),

$$\begin{aligned} P_{u_n}\left\{\hat{\theta}_{u_n} > c^*_{u_n,\alpha,\eta}\right\} &= P_{u_n}\left\{h^{-d/2}\left(\hat{\theta}_{u_n} - a_{u_n}(c_{u_n,L}, c_{u_n,U})\right) > h^{-d/2}\left(c^*_{u_n,\alpha,\eta} - a_{u_n}(c_{u_n,L}, c_{u_n,U})\right)\right\} \\ &\leq P_{u_n}\left\{\begin{array}{c} h^{-d/2}\left(\bar{\theta}_{u_n}(c_{u_n,L}, c_{u_n,U}) - a_{u_n}(c_{u_n,L}, c_{u_n,U})\right) \\ > h^{-d/2}\left(h^{d/2}\eta + a^*_{u_n}(c_{u_n,L}, c_{u_n,U}) - a_{u_n}(c_{u_n,L}, c_{u_n,U})\right) \end{array}\right\} + o(1). \end{aligned}$$

By Lemma A1, the leading probability is equal to

$$P_{u_n}\left\{h^{-d/2}\left(\bar{\theta}_{u_n}(c_{u_n,L}, c_{u_n,U}) - a_{u_n}(c_{u_n,L}, c_{u_n,U})\right) > \eta + o_P(1)\right\} + o(1).$$

Since $\eta > 0$ and $\lim_{n\to\infty}\sigma_{u_n}(c_{u_n,L}, c_{u_n,U}) = 0$, the leading probability vanishes by Lemma B9(ii). ∎

Proof of Theorem 2. We focus on probabilities $P \in \mathcal{P}_n(\lambda_n, q_n) \cap \mathcal{P}_0$. Recalling the definition of $\mathbf{u}_{n,\tau}(x; \hat{\sigma}) \equiv [r_{n,j} v_{n,\tau,j}(x)/\hat{\sigma}_{\tau,j}(x)]_{j\in\mathbb{N}_J}$ and applying Lemma 1 along with the condition that

$$\sqrt{\log n}/c_{n,U} < \sqrt{\log n}/c_{n,L} \to 0,$$

as $n \to \infty$, we find that with probability approaching one,

$$\begin{aligned}\hat{\theta} &= \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(c_{n,U}, c_{n,L})} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_{n,\tau}(x;\hat{\sigma})\right) dQ(x,\tau) \\ &= \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_{n,\tau}(x;\hat{\sigma})\right) dQ(x,\tau) \\ &\quad + \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(c_{n,U}, c_{n,L}) \backslash B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_{n,\tau}(x;\hat{\sigma})\right) dQ(x,\tau).\end{aligned}$$

Since under $P \in \mathcal{P}_0$, $\mathbf{u}_{n,\tau}(x;\hat{\sigma}) \leq 0$ for all $x \in \mathcal{S}$, with probability approaching one by Assumption 5, the last term multiplied by $h^{-d/2}$ is bounded by (from some large n on)

$$\begin{aligned}& h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(c_{n,U}, c_{n,L}) \backslash B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau) \\ \leq\; & h^{-d/2} \sum_{A \in \mathcal{N}_J} \left(\sup_{(x,\tau) \in \mathcal{S}} \|\hat{\mathbf{s}}_\tau(x)\| \right)^p Q\left(B_{n,A}(c_{n,U}, c_{n,L}) \backslash B_{n,A}(q_n)\right) \\ =\; & O_P\left(h^{-d/2} (\log n)^{p/2} \lambda_n\right) = o_P(1),\end{aligned}$$

where the second to last equality follows because $Q\left(B_{n,A}(c_{n,U}, c_{n,L}) \backslash B_{n,A}(q_n)\right) \leq \lambda_n$ by the definition of $\mathcal{P}_n(\lambda_n, q_n)$, and the last equality follows by (3.10).

On the other hand,

$$\begin{aligned}& h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_{n,\tau}(x;\hat{\sigma})\right) dQ(x,\tau) \\ =\; & h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau) \\ & + h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_{n,\tau}(x;\hat{\sigma})\right) dQ(x,\tau) \\ & - h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau).\end{aligned}$$

From the definition of Λ_p in (3.1), the last difference (in absolute value) is bounded by

$$\begin{aligned}& C h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \|[\mathbf{u}_{n,\tau}(x;\hat{\sigma})]_A\| \, \|[\hat{\mathbf{s}}_\tau(x)]_A\|^{p-1} dQ(x,\tau) \\ & + C h^{-d/2} \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \|[\mathbf{u}_{n,\tau}(x;\hat{\sigma})]_A\| \, \|[\mathbf{u}_{n,\tau}(x;\hat{\sigma})]_A\|^{p-1} dQ(x,\tau),\end{aligned}$$

where $[a]_A$ is a vector a with the j-th entry is set to zero for all $j \in \mathbb{N}_J \backslash A$ and $C > 0$ is a constant that does not depend on $n \geq 1$ or $P \in \mathcal{P}$. We have $\sup_{(x,\tau) \in B_{n,A}(q_n)} \|[\mathbf{u}_{n,\tau}(x;\hat{\sigma})]_A\| \leq$

$q_n(1+o_P(1))$, by the null hypothesis and by Assumption A5. Also, by Assumptions A3 and A5,

$$\sup_{(x,\tau)\in B_{n,A}(q_n)} \|[\hat{\mathbf{s}}_\tau(x)]_A\| = O_P\left(\sqrt{\log n}\right).$$

Therefore, we conclude that

$$\begin{aligned} & h^{-d/2}\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)+\mathbf{u}_{n,\tau}(x;\hat{\sigma})\right) dQ(x,\tau) \\ = \;& h^{-d/2}\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau) + O_P\left(h^{-d/2}q_n\{(\log n)^{(p-1)/2}+q_n^{p-1}\}\right). \end{aligned}$$

The last $O_P(1)$ term is $o_P(1)$ by the condition for q_n in (3.10). Thus we find that

$$\hat{\theta} = \bar{\theta}_n(q_n) + o_P(h^{d/2}), \tag{A.25}$$

where $\bar{\theta}_n(q_n) = \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau)$.

Now let us consider the bootstrap statistic. We write

$$\begin{aligned} \hat{\theta}^* &= \sum_{A\in\mathcal{N}_J}\int_{\hat{B}_A(\hat{c}_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau) \\ &= \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau) + \sum_{A\in\mathcal{N}_J}\int_{\hat{B}_A(\hat{c}_n)\backslash B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau). \end{aligned}$$

By Lemma A3, we find that

$$\inf_{P\in\mathcal{P}} P\left\{\hat{B}_{n,A}(\hat{c}_n)\subset B_{n,A}(c_{n,U},c_{n,L})\right\}\to 1, \text{ as } n\to\infty,$$

so that

$$\sum_{A\in\mathcal{N}_J}\int_{\hat{B}_A(\hat{c}_n)\backslash B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau) \le \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}(c_{n,U},c_{n,L})\backslash B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau),$$

with probability approaching one. The last term multiplied by $h^{-d/2}$ is bounded by

$$\begin{aligned} & h^{-d/2}\left(\sup_{(x,\tau)\in\mathcal{S}} ||\hat{\mathbf{s}}_\tau^*(x)||\right)^p \sum_{A\in\mathcal{N}_J} Q\left(B_{n,A}(c_{n,U},c_{n,L})\backslash B_{n,A}(q_n)\right) \\ = \;& O_{P^*}\left(h^{-d/2}(\log n)^{p/2}\lambda_n\right) = o_{P^*}(1), \; \mathcal{P}_n(\lambda_n,q_n)\text{-uniformly}, \end{aligned}$$

where the second to last equality follows by Assumption B2 and the definition of $\mathcal{P}_n(\lambda_n,q_n)$, and the last equality follows by (3.10). Thus, we conclude that

$$\frac{h^{-d/2}(\hat{\theta}^*-a_n(q_n))}{\sigma_n(q_n)} = \frac{h^{-d/2}\left(\bar{\theta}_n^*(q_n)-a_n(q_n)\right)}{\sigma_n(q_n)} + o_{P^*}(1), \; \mathcal{P}_n(\lambda_n,q_n)\text{-uniformly}, \tag{A.26}$$

where

$$\bar{\theta}^*(q_n) \equiv \sum_{A \in \mathcal{N}_J} \int_{B_{n,A}(q_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau).$$

Using the same arguments, we also observe that

$$\hat{a}^* = \hat{a}^*(q_n) + o_P(h^{d/2}) = a_n(q_n) + o_P(h^{d/2}), \tag{A.27}$$

where the last equality uses Lemma A1. Let the $(1-\alpha)$-th percentile of the bootstrap distribution of $\bar{\theta}^*(q_n)$ be denoted by $\bar{c}_n^{\alpha *}(q_n)$. Then by (A.26), we have

$$\frac{h^{-d/2}\left(c_\alpha^* - a_n(q_n)\right)}{\sigma_n(q_n)} = \frac{h^{-d/2}\left(\bar{c}_n^{\alpha *}(q_n) - a_n(q_n)\right)}{\sigma_n(q_n)} + o_{P^*}(1), \ \mathcal{P}_n(\lambda_n, q_n)\text{-uniformly.} \tag{A.28}$$

By Lemma A4(ii) and by the condition that $\sigma_n(q_n) \geq \eta/\Phi^{-1}(1-\alpha)$, the leading term on the right hand side is equal to

$$\Phi^{-1}(1-\alpha) + o_{P^*}(1),\ \mathcal{P}_n(\lambda_n, q_n)\text{-uniformly.}$$

Note that

$$c_\alpha^* \geq h^{d/2}\eta + \hat{a}_n^* + o_P(h^{d/2}), \tag{A.29}$$

by the restriction $\sigma_n(q_n) \geq \eta/\Phi^{-1}(1-\alpha)$ in the definition of $\mathcal{P}_n(\lambda_n, q_n)$ and (A.27). Using this, and following the proof of Step 1 in the proof of Theorem 2, we deduce that

$$\begin{aligned}
& P\left\{h^{-d/2}\left(\frac{\hat{\theta} - a_n(q_n)}{\sigma_n(q_n)}\right) > h^{-d/2}\left(\frac{c_{\alpha,\eta}^* - a_n(q_n)}{\sigma_n(q_n)}\right)\right\} \\
= \ & P\left\{h^{-d/2}\left(\frac{\bar{\theta}_n(q_n) - a_n(q_n)}{\sigma_n(q_n)}\right) > h^{-d/2}\left(\frac{c_\alpha^* - a_n(q_n)}{\sigma_n(q_n)}\right)\right\} + o(1) \\
= \ & P\left\{h^{-d/2}\left(\frac{\bar{\theta}_n(q_n) - a_n(q_n)}{\sigma_n(q_n)}\right) > h^{-d/2}\left(\frac{\bar{c}_n^{\alpha *}(q_n) - a_n(q_n)}{\sigma_n(q_n)}\right)\right\} + o(1),
\end{aligned}$$

where the first equality uses (A.25) and (A.29), and the second equality uses (A.28). Since $\sigma_n(q_n) \geq \eta/\Phi^{-1}(1-\alpha) > 0$ for all $P \in \mathcal{P}_n(\lambda_n, q_n) \cap \mathcal{P}_0$ by definition, using the same arguments in the proof of Lemma A4, we obtain that the last probability is equal to

$$\alpha + o(1),$$

uniformly over $P \in \mathcal{P}_n(\lambda_n, q_n) \cap \mathcal{P}_0$. ∎

Proof of Theorem 3. For any convex nonnegative map f on $\mathbf{R}^J$, we have $2f(b/2) \leq f(a+b) + f(-a)$. Hence we find that

$$\begin{aligned}\hat{\theta} &= \int \Lambda_p\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau) \\ &\geq \frac{1}{2^{p-1}} \int \Lambda_p\left(\mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau) - \int \Lambda_p\left(-\hat{\mathbf{s}}_\tau(x)\right) dQ(x,\tau).\end{aligned}$$

From Assumption A3, the last term is $O_P((\log n)^{p/2})$. Using Assumption A3, we bound the leading integral from below by

$$\min_{j\in\mathbb{N}_J} r_{n,j}^p \left(\int \Lambda_p\left(\tilde{\mathbf{v}}_{n,\tau}(x)\right) dQ(x,\tau) \left\{\frac{\int \Lambda_p\left(\mathbf{v}_{n,\tau}(x)\right) dQ(x,\tau)}{\int \Lambda_p\left(\tilde{\mathbf{v}}_{n,\tau}(x)\right) dQ(x,\tau)} - 1\right\} + o_P(1)\right), \tag{A.30}$$

where $\mathbf{v}_{n,\tau}(x) \equiv [v_{n,\tau,j}(x)/\sigma_{n,\tau,j}(x)]_{j\in\mathbb{N}_J}$ and $\tilde{\mathbf{v}}_{n,\tau}(x) \equiv [v_{\tau,j}(x)/\sigma_{n,\tau,j}(x)]_{j\in\mathbb{N}_J}$. Since

$$\text{liminf}_{n\to\infty} \int \Lambda_p\left(\tilde{\mathbf{v}}_{n,\tau}(x)\right) dQ(x,\tau) > 0,$$

we use Assumption C1 and apply the Dominated Convergence Theorem to write (A.30) as

$$\min_{j\in\mathbb{N}_J} r_{n,j}^p \int \Lambda_p\left(\tilde{\mathbf{v}}_{n,\tau}(x)\right) dQ(x,\tau)\left(1 + o_P(1)\right).$$

Since $\min_{j\in\mathbb{N}_J} r_{n,j} \to \infty$ as $n \to \infty$ and $\text{liminf}_{n\to\infty} \int \Lambda_p\left(\tilde{\mathbf{v}}_{n,\tau}(x)\right) dQ(x,\tau) > 0$, we have for any $M_n \to \infty$ such that $M_n/\min_{j\in\mathbb{N}_J} r_{n,j} \to 0$, and $M_n/\sqrt{\log n} \to \infty$,

$$P\left\{\frac{1}{2^{p-1}} \int \Lambda_p\left(\mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau) > M_n\right\} \to 1,$$

as $n \to \infty$. Also since $\sqrt{\log n}/\min_{j\in\mathbb{N}_J} r_{n,j} \to 0$ (Assumption A4(i)), Assumption A3 implies that

$$P\left\{\hat{\theta} > M_n\right\} \to 1.$$

Also, note that by Lemma A2(ii), $h^{-d/2}(c_\alpha^* - a_n)/\sigma_n = O_P(1)$. Hence

$$c_\alpha^* = a_n + O_P(h^{d/2}) = O_P(1).$$

Given that $c_\alpha^* = O_P(1)$ and $\hat{a}^* = O_P(1)$ by Lemma A1 and Assumption A6(i), we obtain that $P\{\hat{\theta} > c_{\alpha,\eta}^*\} \geq P\{\hat{\theta} > M_n\} + o(1) \to 1$, as $n \to \infty$. ∎

Lemma A5. *Suppose that the conditions of Theorem 4 or Theorem 5 hold. Then as* $n \to \infty$, *the following holds: for any* $c_{n,1}, c_{n,2} > 0$ *such that*

$$\sqrt{\log n}/c_{n,2} \to 0,$$

as $n \to \infty$. Then

$$\inf_{P \in \mathcal{P}_n^0(\lambda_n)} P\left\{ \int_{\mathcal{S} \setminus B_n^0(c_{n,1}, c_{n,2})} \Lambda_p\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau) = 0 \right\} \to 1.$$

Furthermore, we have for any $A \in \mathcal{N}_J$,

$$\inf_{P \in \mathcal{P}_n^0(\lambda_n)} P\left\{ \int_{B_{n,A}^0(c_{n,1}, c_{n,2})} \left\{\Lambda_p\left(\hat{\mathbf{u}}_\tau(x)\right) - \Lambda_{A,p}\left(\hat{\mathbf{u}}_\tau(x)\right)\right\} dQ(x,\tau) = 0 \right\} \to 1.$$

Proof of Lemma A5. Consider the first statement. Let λ be either $d/2$ or $d/4$. We write

$$\begin{aligned}
&\int_{\mathcal{S} \setminus B_n^0(c_{n,1}, c_{n,2})} \Lambda_p\left(\hat{\mathbf{u}}_\tau(x)\right) dQ(x,\tau) \\
&= \int_{\mathcal{S} \setminus B_n^0(c_{n,1}, c_{n,2})} \Lambda_p\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau). \\
&= \int_{\mathcal{S} \setminus B_n^0(c_{n,1}, c_{n,2})} \Lambda_p\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau^0(x;\hat{\sigma}) + h^\lambda \delta_{\tau,\hat{\sigma}}(x)\right) dQ(x,\tau),
\end{aligned}$$

where $\mathbf{u}_\tau^0(x;\hat{\sigma}) \equiv (r_{n,1} v_{n,\tau,1}^0(x)/\hat{\sigma}_{\tau,1}(x), ..., r_{n,J} v_{n,\tau,J}^0(x)/\hat{\sigma}_{\tau,J}(x))$ and

$$\delta_{\tau,\hat{\sigma}}(x) \equiv \left(\frac{\delta_{\tau,1}(x)}{\hat{\sigma}_{\tau,1}(x)}, ..., \frac{\delta_{\tau,J}(x)}{\hat{\sigma}_{\tau,J}(x)} \right). \tag{A.31}$$

Note that $\delta_{\tau,\hat{\sigma}}(x)$ is bounded with probability approaching one by Assumption A3. Also note that for each $j \in \mathbb{N}_J$,

(A.32)

$$\begin{aligned}
\sup_{(x,\tau)\in\mathcal{S}} \left| \frac{r_{n,j}\{\hat{v}_{n,\tau,j}(x) - v_{n,\tau,j}^0(x)\}}{\hat{\sigma}_{\tau,j}(x)} \right| &\leq \sup_{(x,\tau)\in\mathcal{S}} \left| \frac{r_{n,j}\{\hat{v}_{n,\tau,j}(x) - v_{n,\tau,j}(x)\}}{\hat{\sigma}_{\tau,j}(x)} \right| + h^\lambda \sup_{(x,\tau)\in\mathcal{S}} \left| \frac{\delta_{\tau,j}(x)}{\hat{\sigma}_{\tau,j}(x)} \right| \\
&= O_P\left(\sqrt{\log n} + h^\lambda\right) = O_P\left(\sqrt{\log n}\right),
\end{aligned}$$

by Assumption A3. Hence we obtain the desired result, using the same arguments as in the proof of Lemma 1.

Given that we have (A.32), the proof of the second statement can proceed in the same way as the proof of the first statement. ∎

Recall the definitions of $\bar{\Lambda}_{x,\tau}(\mathbf{v})$ in (A.11). We define for $\mathbf{v} \in \mathbf{R}^J$, $\bar{\Lambda}_{x,\tau}^0(\mathbf{v})$ to be $\bar{\Lambda}_{x,\tau}(\mathbf{v})$ except that $B_{n,A}(c_{n,1}, c_{n,2})$ is replaced by $B_{n,A}^0(c_{n,1}, c_{n,2})$. Define for $\lambda \in \{0, d/4, d/2\}$,

$$\hat{\theta}_\delta(c_{n,1}, c_{n,2}; \lambda) \equiv \int \bar{\Lambda}_{x,\tau}^0\left(\hat{\mathbf{s}}_\tau(x) + h^\lambda \delta_{\tau,\sigma}(x)\right) dQ(x,\tau). \tag{A.33}$$

Let

$$a_{n,\delta}^R(c_{n,1}, c_{n,2}; \lambda) \equiv \int \mathbf{E}\left[\bar{\Lambda}_{x,\tau}^0\left(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x) + h^\lambda \delta_{\tau,\sigma}(x)\right)\right] dQ(x,\tau),$$

$$\hat{\theta}^*_\delta(c_{n,1}, c_{n,2}; \lambda) \equiv \int \bar{\Lambda}^0_{x,\tau}\left(\hat{\mathbf{s}}^*_\tau(x) + h^\lambda \delta_{\tau,\sigma}(x)\right) dQ(x,\tau),$$

and

$$a^{R*}_{n,\delta}(c_{n,1}, c_{n,2}; \lambda) \equiv \int \mathbf{E}^*\left[\bar{\Lambda}^0_{x,\tau}\left(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x) + h^\lambda \delta_{\tau,\sigma}(x)\right)\right] dQ(x,\tau). \tag{A.34}$$

We also define

$$a_{n,\delta}(c_{n,1}, c_{n,2}; \lambda) \equiv \int \mathbf{E}\left[\bar{\Lambda}^0_{x,\tau}(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0) + h^\lambda \delta_{\tau,\sigma}(x))\right] dQ(x,\tau).$$

When $c_{n,1} = c_{n,2} = c_n$, we simply write $a^R_{n,\delta}(c_n; \lambda)$, $a^{R*}_{n,\delta}(c_n; \lambda)$, and $a_{n,\delta}(c_n; \lambda)$, instead of writing $a^R_{n,\delta}(c_n, c_n; \lambda)$, $a^{R*}_{n,\delta}(c_n, c_n; \lambda)$, and $a_{n,\delta}(c_n, c_n; \lambda)$.

Lemma A6. *Suppose that the conditions of Assumptions A6(i) and B4 hold. Then for each $P \in \mathcal{P}$ such that the local alternatives in (4.2) hold with $b_{n,j} = r_{n,j}h^{-\lambda}$, $j = 1, ..., J$, for some $\lambda \in \{0, d/4, d/2\}$, and for all nonnegative sequences $c_{n,1}, c_{n,2}$,*

$$\begin{aligned} \left|a^R_{n,\delta}(c_{n,1}, c_{n,2}; \lambda) - a_{n,\delta}(c_{n,1}, c_{n,2}; \lambda)\right| &= o(h^{d/2}), \text{ and} \\ \left|a^{R*}_{n,\delta}(c_{n,1}, c_{n,2}; \lambda) - a_{n,\delta}(c_{n,1}, c_{n,2}; \lambda)\right| &= o_P(h^{d/2}). \end{aligned}$$

Proof of Lemma A6. The result follows immediately from Lemma C12 in Appendix C. ∎

Lemma A7. *Suppose that the conditions of Theorem 4 are satisfied. Then for each $\lambda \in \{0, d/4, d/2\}$, for each $P \in \mathcal{P}^0_n(\lambda_n)$ such that the local alternatives in (4.2) hold,*

$$\begin{aligned} h^{-d/2}\left(\frac{\bar{\theta}_{n,\delta}(c_{n,U}, c_{n,L}; \lambda) - a^R_{n,\delta}(c_{n,U}, c_{n,L}; \lambda)}{\sigma_n(c_{n,U}, c_{n,L})}\right) &\xrightarrow{d} N(0,1) \text{ and} \\ h^{-d/2}\left(\frac{\bar{\theta}^*_{n,\delta}(c_{n,U}, c_{n,L}; \lambda) - a^{R*}_{n,\delta}(c_{n,U}, c_{n,L}; \lambda)}{\sigma_n(c_{n,U}, c_{n,L})}\right) &\xrightarrow{d^*} N(0,1), \ \mathcal{P}^0_n(\lambda_n)\text{-uniformly.} \end{aligned}$$

Proof of Lemma A7. Note that by the definition of $\mathcal{P}^0_n(\lambda_n)$, we have

$$\liminf_{n\to\infty} \inf_{P \in \mathcal{P}^0_n(\lambda_n)} \sigma^2_n(c_{n,U}, c_{n,L}) \geq \frac{\eta}{\Phi^{-1}(1-\alpha)}.$$

Hence we can follow the proof of Lemma A2 to obtain the desired results. ∎

Proof of Theorem 4. Using Lemma A5, we find that

$$\hat{\theta} = \sum_{A \in \mathcal{N}_J} \int_{B^0_{n,A}(c_{n,U}, c_{n,L})} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x; \hat{\sigma})\right) dQ(x,\tau)$$

with probability approaching one. We write the leading sum as

$$\sum_{A \in \mathcal{N}_J} \int_{B^0_{n,A}(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x; \hat{\sigma})\right) dQ(x,\tau) + R_n,$$

where

$$R_n \equiv \sum_{A\in\mathcal{N}_J} \int_{B^0_{n,A}(c_{n,U},c_{n,L})\backslash B^0_{n,A}(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau).$$

We write $h^{-d/2}R_n$ as

$$\begin{aligned} & h^{-d/2} \sum_{A\in\mathcal{N}_J} \int_{B^0_{n,A}(c_{n,U},c_{n,L})\backslash B^0_{n,A}(0)} \Lambda_{A,p}\left(\begin{array}{c} \hat{\mathbf{s}}_\tau(x) + \mathbf{u}^0_\tau(x;\hat{\sigma}) \\ +h^{d/2}\delta_{\tau,\hat{\sigma}}(x)(1+o(1)) \end{array}\right) dQ(x,\tau) \\ \leq\ & h^{-d/2} \sum_{A\in\mathcal{N}_J} \int_{B^0_{n,A}(c_{n,U},c_{n,L})\backslash B^0_{n,A}(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + h^{d/2}\delta_{\tau,\hat{\sigma}}(x)(1+o(1))\right) dQ(x,\tau), \end{aligned}$$

by Assumption C2. We bound the last sum as

$$Ch^{-d/2} \sum_{A\in\mathcal{N}_J} \left(\sup_{(x,\tau)\in\mathcal{S}} ||\hat{\mathbf{s}}_\tau(x)||\right)^p Q\left(B^0_{n,A}(c_{n,U},c_{n,L})\backslash B^0_{n,A}(0)\right) = O_P\left(h^{-d/2}\left(\log n\right)^{p/2}\lambda_n\right) = o_P(1)$$

using Assumption A3 and the rate condition in (3.10). We conclude that

$$\begin{aligned} h^{-d/2}\hat{\theta} &= h^{-d/2} \sum_{A\in\mathcal{N}_J} \int_{B^0_{n,A}(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + \mathbf{u}_\tau(x;\hat{\sigma})\right) dQ(x,\tau) + o_P(1) \\ &= h^{-d/2} \sum_{A\in\mathcal{N}_J} \int_{B^0_{n,A}(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x) + h^{d/2}\delta_{\tau,\hat{\sigma}}(x)\right) dQ(x,\tau) + o_P(1), \end{aligned} \tag{A.35}$$

where the second equality follows by Assumption C2 and by the definition of $B^0_{n,A}(0)$.

Fix small $\kappa > 0$ and define

$$\begin{aligned} \delta^L_{\tau,\sigma,\kappa,j}(x) &\equiv \begin{cases} \frac{\delta_{\tau,j}(x)}{(1+\kappa)\sigma_{n,\tau,j}(x)} & \text{if } \delta_{\tau,j}(x) \geq 0 \\ \frac{\delta_{\tau,j}(x)}{(1-\kappa)\sigma_{n,\tau,j}(x)} & \text{if } \delta_{\tau,j}(x) < 0 \end{cases} \quad \text{and} \\ \delta^U_{\tau,\sigma,\kappa,j}(x) &\equiv \begin{cases} \frac{\delta_{\tau,j}(x)}{(1-\kappa)\sigma_{n,\tau,j}(x)} & \text{if } \delta_{\tau,j}(x) \geq 0 \\ \frac{\delta_{\tau,j}(x)}{(1+\kappa)\sigma_{n,\tau,j}(x)} & \text{if } \delta_{\tau,j}(x) < 0 \end{cases}. \end{aligned}$$

Define $\delta^L_{\tau,\sigma,\kappa}(x)$ and $\delta^U_{\tau,\sigma,\kappa}(x)$ to be $\mathbf{R}^J$-valued maps whose j-th entries are given by $\delta^L_{\tau,\sigma,\kappa,j}(x)$ and $\delta^U_{\tau,\sigma,\kappa,j}(x)$ respectively. By construction, Assumptions A3 and C2(ii), we have

$$P\left\{\delta^L_{\tau,\sigma,\kappa}(x) \leq \delta_{\tau,\hat{\sigma}}(x) \leq \delta^U_{\tau,\sigma,\kappa}(x)\right\} \to 1,$$

as $n \to \infty$. Therefore, with probability approaching one,

$$\text{(A.36)}\quad \begin{aligned}\hat{\theta}_{\delta,L}(0;d/2) &\equiv \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)+h^{d/2}\delta_{\tau,\sigma,\kappa}^L(x)\right)dQ(x,\tau)\\ &\le \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)+h^{d/2}\delta_{\tau,\hat{\sigma}}(x)\right)dQ(x,\tau)\\ &\le \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau(x)+h^{d/2}\delta_{\tau,\sigma,\kappa}^U(x)\right)dQ(x,\tau) \equiv \hat{\theta}_{\delta,U}(0;d/2).\end{aligned}$$

We conclude from (A.35) that

$$\text{(A.37)}\qquad \hat{\theta}_{\delta,L}(0;d/2)+o_P(h^{d/2}) \le \hat{\theta} \le \hat{\theta}_{\delta,U}(0;d/2)+o_P(h^{d/2}).$$

As for the bootstrap counterpart, note that since $\delta_{\tau,j}(x)$ is bounded and $\sigma_{n,\tau,j}(x)$ is bounded away from zero uniformly over $(x,\tau)\in\mathcal{S}$ and $n\ge 1$, and hence

$$\text{(A.38)}\qquad \sup_{(x,\tau)\in\mathcal{S}}\left|\frac{1}{h^{-d/2}}\frac{\delta_{\tau,j}(x)}{\sigma_{n,\tau,j}(x)}\right| \le Ch^{d/2}\to 0,$$

as $n\to\infty$. By (A.38), the difference between $r_{n,j}v_{n,\tau,j}(x)/\sigma_{n,\tau,j}(x)$ and $r_{n,j}v_{n,\tau,j}^0(x)/\sigma_{n,\tau,j}(x)$ vanishes uniformly over $(x,\tau)\in\mathcal{S}$. Therefore, combining this with Lemma A3, we find that

$$\text{(A.39)}\qquad P\left\{\hat{B}_n(\hat{c}_n)\subset B_n^0(c_{n,U},c_{n,L})\right\}\to 1,$$

as $n\to\infty$.

Now with probability approaching one,

$$\text{(A.40)}\qquad \begin{aligned}\hat{\theta}^* &= \sum_{A\in\mathcal{N}_J}\int_{\hat{B}_A(\hat{c}_n)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right)dQ(x,\tau)\\ &= \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right)dQ(x,\tau)\\ &\quad+\sum_{A\in\mathcal{N}_J}\int_{\hat{B}_A(\hat{c}_n)\setminus B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right)dQ(x,\tau).\end{aligned}$$

As for the last sum, it is bounded by

$$\sum_{A\in\mathcal{N}_J}\int_{B_{n,A}^0(c_{n,U},c_{n,L})\setminus B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right)dQ(x,\tau),$$

with probability approaching one by (A.39). The above sum multiplied by $h^{-d/2}$ is bounded by

$$\begin{aligned} & h^{-d/2}\left(\sup_{(x,\tau)\in\mathcal{S}}||\hat{\mathbf{s}}_\tau^*(x)||\right)^p \sum_{A\in\mathcal{N}_J} Q\left(B_{n,A}^0(c_{n,U},c_{n,L})\backslash B_{n,A}^0(0)\right) \\ = \;& O_{P^*}\left(h^{-d/2}(\log n)^{p/2}\lambda_n\right) = o_{P^*}(1),\ \mathcal{P}\text{-uniformly}, \end{aligned}$$

by Assumption B2 and the rate condition for λ_n. Thus, we conclude that

$$\hat{\theta}^* = \bar{\theta}^*(0) + o_{P^*}(h^{d/2}),\ \mathcal{P}_n^0(\lambda_n)\text{-uniformly}, \tag{A.41}$$

where

$$\bar{\theta}^*(0) \equiv \sum_{A\in\mathcal{N}_J}\int_{B_{n,A}^0(0)} \Lambda_{A,p}\left(\hat{\mathbf{s}}_\tau^*(x)\right) dQ(x,\tau).$$

Let $\bar{c}_n^{\alpha*}(0)$ be the $(1-\alpha)$-th quantile of the bootstrap distribution of $\bar{\theta}^*(0)$ and let $\gamma_n^{\alpha*}(0)$ be the $(1-\alpha)$-th quantile of the bootstrap distribution of

$$h^{-d/2}\left(\frac{\bar{\theta}^*(0)-a_n^{R*}(0)}{\sigma_n(0)}\right). \tag{A.42}$$

By the definition of $\mathcal{P}_n^0(\lambda_n)$, we have $\sigma_n^2(0) > \eta/\Phi^{-1}(1-\alpha)$. Let $a_{\delta,U}^R(0;d/2)$ and $a_{\delta,L}^R(0;d/2)$ be $a_{n,\delta}^R(0;d/2)$ except that $\delta_{\tau,\sigma}$ is replaced by $\delta_{\tau,\sigma,\kappa}^U$ and $\delta_{\tau,\sigma,\kappa}^L$ respectively. Also, let $a_{\delta,U}(0;d/2)$ and $a_{\delta,L}(0;d/2)$ be $a_{n,\delta}(0;d/2)$ except that $\delta_{\tau,\sigma}$ is replaced by $\delta_{\tau,\sigma,\kappa}^U$ and $\delta_{\tau,\sigma,\kappa}^L$ respectively. We bound $P\{\hat{\theta} > c_{\alpha,\eta}^*\}$ by

$$\begin{aligned} & P\left\{h^{-d/2}\left(\frac{\hat{\theta}_{\delta,U}(0;d/2)-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)}\right) > h^{-d/2}\left(\frac{c_\alpha^*-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)}\right)\right\} + o(1) \\ = \;& P\left\{h^{-d/2}\left(\frac{\hat{\theta}_{\delta,U}(0;d/2)-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)}\right) > h^{-d/2}\left(\frac{\bar{c}_n^{\alpha*}(0)-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)}\right)\right\} + o(1), \end{aligned}$$

where the equality uses (A.41). Then we observe that

$$\begin{aligned} \frac{\bar{c}_n^{\alpha*}(0)-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)} &= \frac{\bar{c}_n^{\alpha*}(0)-a_n^{R*}(0)}{\sigma_n(0)} + \frac{a_n^{R*}(0)-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)} \\ &= h^{d/2}\gamma_n^{\alpha*}(0) + \frac{a_n^{R*}(0)-a_{\delta,U}^R(0;d/2)}{\sigma_n(0)}. \end{aligned}$$

As for the last term, we use Lemmas A1 and A6 to deduce that

$$\begin{aligned} a_n^{R*}(0)-a_{\delta,U}^R(0;d/2) &= a_n^R(0)-a_{\delta,U}^R(0;d/2)+o_P(h^{d/2}) \\ &= a_n(0)-a_{\delta,U}(0;d/2)+o_P(h^{d/2}). \end{aligned}$$

As for $a_n(0) - a_{\delta,U}(0; d/2)$, we observe that

$$
\begin{aligned}
\text{(A.43)} \quad & \sigma_n(0)^{-1} h^{-d/2} \left\{ \mathbf{E}\left[\Lambda_{A,p}\left(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0) + h^{d/2}\delta^U_{\tau,\sigma,\kappa}(x)\right)\right] - \mathbf{E}\left[\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0))\right]\right\} \\
= \; & \sigma_n(0)^{-1} h^{-d/2} \left\{ \mathbf{E}\left[\Lambda_{A,p}\left(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0) + h^{d/2}\delta^U_{\tau,\sigma,\kappa}(x)\right)\right] - \mathbf{E}\left[\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0))\right]\right\} \\
= \; & \psi^{(1)}_{n,A,\tau}(\mathbf{0}; x)^\top \delta^U_{\tau,\sigma,\kappa}(x) + O\left(h^{d/2}\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{h^{-d/2}\left(a_n(0) - a_{\delta,U}(0)\right)}{\sigma_n(0)} &= -\sum_{A \in \mathcal{N}_J} \int \psi^{(1)}_{n,A,\tau}(\mathbf{0}; x)^\top \delta^U_{\tau,\sigma,\kappa}(x) dQ(x,\tau) + o(1) \\
&= -\sum_{A \in \mathcal{N}_J} \int \psi^{(1)}_{A,\tau}(\mathbf{0}; x)^\top \delta^U_{\tau,\sigma,\kappa}(x) dQ(x,\tau) + o(1),
\end{aligned}
$$

where the last equality follows by the Dominated Convergence Theorem. On the other hand, by Lemma A7, we have

$$
h^{-d/2}\left(\frac{\hat{\theta}_{\delta,U}(0; d/2) - a^R_{\delta,U}(0; d/2)}{\sigma_n(0)}\right) \xrightarrow{d} N(0,1).
$$

Since $\gamma^{\alpha*}_n(0) = \gamma_{\alpha,\infty} + o_P(1)$ by Lemma A4, we use this result to deduce that

$$
\begin{aligned}
& \lim_{n\to\infty} P\left\{ h^{-d/2}\left(\frac{\hat{\theta}_{\delta,U}(0; d/2) - a^R_{\delta,U}(0; d/2)}{\sigma_n(0)}\right) > h^{-d/2}\left(\frac{\bar{c}^{\alpha*}_n(0) - a^R_{\delta,U}(0; d/2)}{\sigma_n(0)}\right)\right\} \\
= \; & 1 - \Phi\left(z_{1-\alpha} - \sum_{A \in \mathcal{N}_J} \int \psi^{(1)}_{A,\tau}(\mathbf{0}; x)^\top \delta^U_{\tau,\sigma,\kappa}(x) dQ(x,\tau)\right).
\end{aligned}
$$

Similarly, we also use (A.37) to bound $P\left\{\hat{\theta} > c^*_{\alpha,\eta}\right\}$ from below by

$$
P\left\{ h^{-d/2}\left(\frac{\hat{\theta}_{\delta,L}(0; d/2) - a^R_{\delta,L}(0; d/2)}{\sigma_n(0)}\right) > h^{-d/2}\left(\frac{\bar{c}^{\alpha*}_n(0) - a^R_{\delta,L}(0; d/2)}{\sigma_n(0)}\right)\right\} + o(1),
$$

and using similar arguments as before, we obtain that

$$
\begin{aligned}
& \lim_{n\to\infty} P\left\{ h^{-d/2}\left(\frac{\hat{\theta}_{\delta,L}(0; d/2) - a^R_{\delta,L}(0; d/2)}{\sigma_n(0)}\right) > h^{-d/2}\left(\frac{\bar{c}^{\alpha*}_n(0) - a^R_{\delta,L}(0; d/2)}{\sigma_n(0)}\right)\right\} \\
= \; & 1 - \Phi\left(z_{1-\alpha} - \sum_{A \in \mathcal{N}_J} \int \psi^{(1)}_{A,\tau}(\mathbf{0}; x)^\top \delta^L_{\tau,\sigma,\kappa}(x) dQ(x,\tau)\right).
\end{aligned}
$$

We conclude from this and (A.36) that for any small $\kappa > 0$,

$$
\begin{aligned}
& 1-\Phi\left(z_{1-\alpha}-\sum_{A\in\mathcal{N}_J}\int \psi_{A,\tau}^{(1)}(\mathbf{0};x)^\top \delta_{\tau,\sigma,\kappa}^{L}(x)dQ(x,\tau)\right)+o(1) \\
\leq \;& P\left\{\hat{\theta}>c_{\alpha,\eta}^{*}\right\}\leq 1-\Phi\left(z_{1-\alpha}-\sum_{A\in\mathcal{N}_J}\int \psi_{A,\tau}^{(1)}(\mathbf{0};x)^\top \delta_{\tau,\sigma,\kappa}^{U}(x)dQ(x,\tau)\right)+o(1).
\end{aligned}
$$

Note that $\psi_{A,\tau}^{(1)}(\mathbf{0};x)^\top \delta_{\tau,\sigma,\kappa}^{U}(x)$ and $\psi_{A,\tau}^{(1)}(\mathbf{0};x)^\top \delta_{\tau,\sigma,\kappa}^{L}(x)$ are bounded maps in (x,τ) by the assumption of the theorem, and that

$$
\lim_{\kappa\to 0}\delta_{\tau,\sigma,\kappa}^{L}(x)=\lim_{\kappa\to 0}\delta_{\tau,\sigma,\kappa}^{U}(x)=\delta_{\tau,\sigma}(x),
$$

for each $(x,\tau)\in\mathcal{S}$. Hence by sending $\kappa\to 0$ and applying the Dominated Convergence Theorem to both the bounds above, we obtain the desired result. ∎

Proof of Theorem 5. First, observe that Lemma A5 continues to hold. This can be seen by following the proof of Lemma A5 and noting that (A.32) becomes here

$$
\sup_{(x,\tau)\in\mathcal{S}}\left|\frac{r_{n,j}\{\hat{v}_{n,\tau,j}(x)-v_{n,\tau,j}^{0}(x)\}}{\hat{\sigma}_{\tau,j}(x)}\right| = O_P\left(\sqrt{\log n}+h^{d/4}\right)=O_P\left(\sqrt{\log n}\right),
$$

yielding the same convergence rate. The rest of the proof is the same. Similarly, Lemma A6 continues to hold also under the modified local alternatives of (4.2) with $b_{n,j}=r_{n,j}h^{-d/4}$. We define

$$
\tilde{\delta}_{\tau,\sigma}(x)\equiv h^{-d/4}\delta_{\tau,\sigma}(x). \tag{A.44}
$$

We follow the proof of Theorem 4 and take up arguments from (A.43). Observe that

$$
\begin{aligned}
& \sigma_n(0)^{-1}h^{-d/2}\left\{\mathbf{E}\left[\Lambda_{A,p}\left(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0)+h^{d/2}\tilde{\delta}_{\tau,\sigma}(x)\right)\right]-\mathbf{E}\left[\Lambda_{A,p}(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0))\right]\right\} \\
=\;& \sigma_n(0)^{-1}h^{-d/2}\left\{\mathbf{E}\left[\Lambda_{A,p}\left(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0)+h^{d/2}\tilde{\delta}_{\tau,\sigma}(x)\right)\right]-\mathbf{E}\left[\Lambda_{A,p}(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0))\right]\right\} \\
=\;& \psi_{n,A,\tau}^{(1)}(\mathbf{0};x)^\top\tilde{\delta}_{\tau,\sigma}(x)+h^{d/2}\tilde{\delta}_{\tau,\sigma}(x)^\top\psi_{n,A,\tau}^{(2)}(\mathbf{0};x)\tilde{\delta}_{\tau,\sigma}(x)/2.
\end{aligned}
$$

By the Dominated Convergence Theorem,

$$
\begin{aligned}
\int \psi_{n,A,\tau}^{(1)}(\mathbf{0};x)^\top\tilde{\delta}_{\tau,\sigma}(x)dQ(x,\tau) &= \int \psi_{A,\tau}^{(1)}(\mathbf{0};x)^\top\tilde{\delta}_{\tau,\sigma}(x)dQ(x,\tau)+o(1) \text{ and} \\
\int \psi_{n,A,\tau}^{(2)}(\mathbf{0};x)^\top\tilde{\delta}_{\tau,\sigma}(x)dQ(x,\tau) &= \int \psi_{A,\tau}^{(2)}(\mathbf{0};x)^\top\tilde{\delta}_{\tau,\sigma}(x)dQ(x,\tau)+o(1).
\end{aligned}
$$

Since $\sum_{A\in\mathcal{N}_J}\int\psi_{A,\tau}^{(1)}(\mathbf{0};x)^\top\tilde{\delta}_{\tau,\sigma}(x)dQ(x,\tau)=0$, by the condition for $\delta_{\tau,\sigma}(x)$ in the theorem,

$$\begin{aligned}&\sum_{A\in\mathcal{N}_J}\int h^{-d/2}\left\{\begin{array}{c}\mathbf{E}\left[\Lambda_{A,p}\left(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0)+h^{d/2}\tilde{\delta}_{\tau,\sigma}(x)\right)\right]\\-\mathbf{E}\left[\Lambda_{A,p}(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0))\right]\end{array}\right\}dQ(x,\tau)\\=\;&\frac{1}{2}\sum_{A\in\mathcal{N}_J}\int\delta_{\tau,\sigma}(x)^\top\psi_{A,\tau}^{(2)}(\mathbf{0};x)\delta_{\tau,\sigma}(x)dQ(x,\tau)+o(1).\end{aligned}$$

Now we can use the above result by replacing $\delta_{\tau,\sigma}(x)$ by $\delta_{\tau,\sigma,\kappa}^U(x)$ and $\delta_{\tau,\sigma,\kappa}^L(x)$ and follow the proof of Theorem 4 to obtain the desired result. ∎

Appendix B. Proofs of Auxiliary Results for Lemmas A2(i), Lemma A4(i), and Theorem 1

The eventual result in this appendix is Lemma B9 which is used to show the asymptotic normality of the location-scale normalized representation of $\hat{\theta}$ and its bootstrap version, and to establish its asymptotic behavior in the degenerate case. For this, we first prepare Lemmas B1-B3. To obtain uniformity that covers the case of degeneracy, this paper uses a method of regularization, where the covariance matrix of random quantities is added by a diagonal matrix of small diagonal elements. The regularized random quantities having this covariance matrix do not suffer from degeneracy in the limit, even when the original quantities have covariate matrix that is degenerate in the limit. Thus, for these regularized quantities, we can obtain uniform asymptotic theory using an appropriate Berry-Esseen-type bound. Then, we need to deal with the difference between the regularized covariance matrix and the original one. Lemma B1 is a simple result of linear algebra that is used to control this discrepancy.

Lemma B2 has two sub-results from which one can deduce a uniform version of Levy's continuity theorem. We have not seen any such results in the literature or monographs, so we provide its full proof. The result has two functions. First, the result enables one to deduce convergence in distribution in terms of convergence of cumulative distribution functions and convergence in distribution in terms of convergence of characteristic functions in a manner that is uniform over a given collection of probabilities. The original form of convergence in distribution due to the Poissonization method in Giné, Mason, and Zaitsev (2003) is convergence of characteristic functions. Certainly pointwise in P, this convergence implies convergence of cumulative distribution functions, but it is not clear under what conditions this implication is uniform over a given class of probabilities. Lemma B2 essentially clarifies this issue.

Lemma B3 is an extension of the de-Poissonization lemma that appeared in Beirlant and Mason (1995). The proof is based on the proof of their same result in Giné, Mason, and

Zaitsev (2003), but involves a substantial modification, because unlike their results, we need a version that holds uniformly over $P \in \mathcal{P}$. This de-Poissonization lemma is used to transform the asymptotic distribution theory for the Poissonized version of the test statistic into that for the original test statistic.

Lemmas B4-B5 establish some moment bounds for a normalized sum of independent quantities. This moment bound is later used to control a Berry-Esseen-type bound, when we approximate those sums by corresponding centered normal random vectors.

Lemma B6 obtains an approximate version for the scale normalizer σ_n. The approximate version involves a functional of a Gaussian random vector, which stems from approximating a normalized sum of independent random vectors by a Gaussian random vector through using a Berry-Esseen-type bound. For this result, we use the regularization method that we mentioned before. Due to the regularization, we are able to cover the degenerate case eventually.

Lemma B7 is an auxiliary result that is used to establish Lemma B9 in combination with the de-Poissonization lemma (Lemma B3). And Lemma B8 establishes asymptotic normality of the Poissonized version of the test statistics. The asymptotic normality for the Poissonized statistic involves the discretization of the integrals, thereby, reducing the integral to a sum of 1-dependent random variables, and then applies the Berry-Esseen-type bound in Shergin (1993). Note that by the moment bound in Lemmas B4-B5 that is uniform over $P \in \mathcal{P}$, we obtain the asymptotic approximation that is uniform over $P \in \mathcal{P}$. The lemma also presents a corresponding result for the degenerate case.

Finally, Lemma B9 combines the asymptotic distribution theory for the Poissonized test statistic in Lemma B7 with the de-Poissonization lemma (Lemma B3) to obtain the asymptotic distribution theory for the original test statistic. The result of Lemma B9 is used to establish the asymptotic normality result in Lemma A7.

The following lemma provides some inequality of matrix algebra.

Lemma B1. *For any $J \times J$ positive semidefinite symmetric matrix Σ and any $\varepsilon > 0$,*

$$\left\| (\Sigma + \varepsilon I)^{1/2} - \Sigma^{1/2} \right\| \leq \sqrt{J\varepsilon},$$

where $\|A\| = \sqrt{tr(AA')}$ for any square matrix A.

Remark 1. The main point of Lemma B1 is that the bound $\sqrt{J\varepsilon}$ is independent of the matrix Σ. Such a uniform bound is crucially used for the derivation of asymptotic validity of the test uniform in $P \in \mathcal{P}$.

Proof of Lemma B1. First observe that

$$tr\{(\Sigma+\varepsilon I)^{1/2}-\Sigma^{1/2}\}^2 = tr\,(2\Sigma+\varepsilon I)-2tr((\Sigma+\varepsilon I)^{1/2}\,\Sigma^{1/2}). \tag{B.1}$$

Since $\Sigma \leq \Sigma+\varepsilon I$, we have $\Sigma^{1/2} \leq (\Sigma+\varepsilon I)^{1/2}$. For any positive semidefinite matrices A and B, $tr(AB) \geq 0$ (see e.g. Abadir and Magnus (2005)). Therefore, $tr(\Sigma) \leq tr((\Sigma+\varepsilon I)^{1/2}\,\Sigma^{1/2})$. From (B.1), we find that

$$\begin{aligned} & tr\,(2\Sigma+\varepsilon I)-2tr((\Sigma+\varepsilon I)^{1/2}\,\Sigma^{1/2}) \\ \leq\ & tr\,(2\Sigma+\varepsilon I)-2tr(\Sigma)=\varepsilon J. \end{aligned}$$

∎

The following lemma can be used to derive a version of Levy's Continuity Theorem that is uniform in $P \in \mathcal{P}$.

Lemma B2. *Suppose that $V_n \in \mathbf{R}^d$ is a sequence of random vectors and $V \in \mathbf{R}^d$ is a random vector. We assume without loss of generality that V_n and V live on the same measure space $(\Omega, \mathcal{F})$, and $\mathcal{P}$ is a given collection of probabilities on $(\Omega,\mathcal{F})$. Furthermore define*

$$\begin{aligned} \varphi_n(t) &\equiv \mathbf{E}\left[\exp(it^\top V_n)\right],\ \varphi(t)\equiv \mathbf{E}\left[\exp(it^\top V)\right], \\ F_n(t) &\equiv P\left\{V_n \leq t\right\},\ \textit{and}\ F(t)\equiv P\left\{V\leq t\right\}. \end{aligned}$$

(i) Suppose that the distribution $P\circ V^{-1}$ is uniformly tight in $\{P\circ V^{-1}: P\in\mathcal{P}\}$. Then for any continuous function f on $\mathbf{R}^d$ taking values in $[-1,1]$ and for any $\varepsilon\in(0,1]$, we have

$$\sup_{P\in\mathcal{P}}|\mathbf{E}f(V_n)-\mathbf{E}f(V)| \leq \varepsilon^{-d}C_d \sup_{P\in\mathcal{P}}\sup_{t\in\mathbf{R}^d}|F_n(t)-F(t)|+4\varepsilon,$$

where $C_d>0$ is a constant that depends only on d.
(ii) Suppose that $\sup_{P\in\mathcal{P}}\mathbf{E}||V||^2<\infty$. If

$$\sup_{P\in\mathcal{P}}\sup_{u\in\mathbf{R}^d}|\varphi_n(u)-\varphi(u)|\to 0,\ \textit{as}\ n\to\infty,$$

then for each $t\in\mathbf{R}^d$,

$$\sup_{P\in\mathcal{P}}|F_n(t)-F(t)|\to 0,\ \textit{as}\ n\to\infty.$$

On the other hand, if for each $t\in\mathbf{R}^d$,

$$\sup_{P\in\mathcal{P}}|F_n(t)-F(t)|\to 0,\ \textit{as}\ n\to\infty,$$

then for each $u\in\mathbf{R}^d$,

$$\sup_{P\in\mathcal{P}}|\varphi_n(u)-\varphi(u)|\to 0,\ \textit{as}\ n\to\infty.$$

Proof of Lemma B2. (i) The proof uses arguments in the proof of Lemma 2.2 of van der Vaart (1998). Take a large compact rectangle $B \subset \mathbf{R}^d$ such that $P\{V \notin B\} < \varepsilon$. Since the distribution of V is tight uniformly over $P \in \mathcal{P}$, we can take such B independently of $P \in \mathcal{P}$. Take a partition $B = \cup_{j=1}^{J_\varepsilon} B_j$ and points $x_j \in B_j$ such that $J_\varepsilon \leq C_{d,1}\varepsilon^{-d}$, and $|f(x) - f_\varepsilon(x)| < \varepsilon$ for all $x \in B$, where $C_{d,1} > 0$ is a constant that depends only on d, and

$$f_\varepsilon(x) \equiv \sum_{j=1}^{J_\varepsilon} f(x_j) 1\{x \in B_j\}.$$

Thus we have

$$\begin{aligned}
|\mathbf{E}f(V_n) - \mathbf{E}f(V)| &\leq |\mathbf{E}f(V_n) - \mathbf{E}f_\varepsilon(V_n)| + |\mathbf{E}f_\varepsilon(V_n) - \mathbf{E}f_\varepsilon(V)| + |\mathbf{E}f_\varepsilon(V) - \mathbf{E}f(V)| \\
&\leq 2\varepsilon + P\{V_n \notin B\} + P\{V \notin B\} + |\mathbf{E}f_\varepsilon(V_n) - \mathbf{E}f_\varepsilon(V)| \\
&\leq 4\varepsilon + |P\{V_n \notin B\} - P\{V \notin B\}| + |\mathbf{E}f_\varepsilon(V_n) - \mathbf{E}f_\varepsilon(V)| \\
&= 4\varepsilon + |P\{V_n \in B\} - P\{V \in B\}| + |\mathbf{E}f_\varepsilon(V_n) - \mathbf{E}f_\varepsilon(V)|.
\end{aligned}$$

The second inequality following by $P\{V \notin B\} < \varepsilon$. As for the last term, we let

$$b_n \equiv \sup_{P \in \mathcal{P}} \sup_{t \in \mathbf{R}^d} |F_n(t) - F(t)|,$$

and observe that

$$\begin{aligned}
|\mathbf{E}f_\varepsilon(V_n) - \mathbf{E}f_\varepsilon(V)| &\leq \sum_{j=1}^{J_\varepsilon} |P\{V_n \in B_j\} - P\{V \in B_j\}| \, |f(x_j)| \\
&\leq \sum_{j=1}^{J_\varepsilon} |P\{V_n \in B_j\} - P\{V \in B_j\}| \leq C_{d,2} b_n J_\varepsilon,
\end{aligned}$$

where $C_{d,2} > 0$ is a constant that depends only on d. The last inequality follows because for any rectangle B_j, we have $|P\{V_n \in B_j\} - P\{V \in B_j\}| \leq C_{d,2} b_n$ for some $C_{d,2} > 0$. We conclude that

$$|\mathbf{E}f(V_n) - \mathbf{E}f(V)| \leq 4\varepsilon + C_{d,2}\left(C_{d,1}\varepsilon^{-d} + 1\right) b_n \leq 4\varepsilon + C_d \varepsilon^{-d} b_n,$$

where $C_d = C_{d,2}\{C_{d,1} + 1\}$. The last inequality follows because $\varepsilon \leq 1$.

(ii) We show the first statement. We first show that under the stated condition, the sequence $\{P \circ V_n^{-1}\}_{n=1}^{\infty}$ is uniformly tight over $P \in \mathcal{P}$. That is, for any $\varepsilon > 0$, we show there exists a compact set $B \subset \mathbf{R}^d$ such that for all $n \geq 1$,

$$\sup_{P \in \mathcal{P}} P\left\{V_n \in \mathbf{R}^d \backslash B\right\} < \varepsilon.$$

For this, we assume $d = 1$ without loss of generality, let P_n denote the distribution of V_n and consider the following: (using arguments in the proof of Theorem 3.3.6 of Durrett (2010))

$$\begin{aligned} P\left\{|V_n| > \frac{2}{u}\right\} &\leq 2\int_{|x|>2/u}\left(1-\frac{1}{|ux|}\right)dP_n(x) \\ &\leq 2\int\left(1-\frac{\sin ux}{ux}\right)dP_n(x) \\ &= \frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt. \end{aligned}$$

Define $\bar{e}_n \equiv \sup_{P\in\mathcal{P}}\sup_{t\in\mathbf{R}}|\varphi_n(t)-\varphi(t)|$. Using Theorem 3.3.8 of Durrett (2010), we bound the last term by

$$\begin{aligned} 2\bar{e}_n + \frac{1}{u}\int_{-u}^{u}(1-\varphi(t))\,dt &\leq 2\bar{e}_n + \left|\frac{1}{u}\int_{-u}^{u}\left(-it\mathbf{E}V+\frac{t^2\mathbf{E}V^2}{2}\right)dt\right| \\ &\quad +2\left|\frac{1}{u}\int_{-u}^{u}t^2\mathbf{E}V^2dt\right|. \end{aligned}$$

The supremum of the right hand side terms over $P \in \mathcal{P}$ vanishes as we send $n \to \infty$ and then $u \downarrow 0$, by the assumption that $\sup_{P\in\mathcal{P}}\mathbf{E}|V|^2 < \infty$. Hence the sequence $\{P \circ V_n^{-1}\}_{n=1}^{\infty}$ is uniformly tight over $P \in \mathcal{P}$.

Now, for each $t \in \mathbf{R}^d$, there exists a subsequence $\{n'\} \subset \{n\}$ and $\{P_{n'}\} \subset \mathcal{P}$ such that

(B.2) $$\limsup_{n\to\infty}\sup_{P\in\mathcal{P}}|F_n(t)-F(t)| = \lim_{n'\to\infty}|F_{n'}(t;P_{n'})-F(t;P_{n'})|,$$

where

$$F_n(t;P_n) = P_n\{V_n \leq t\} \text{ and } F(t;P_n) = P_n\{V \leq t\}.$$

Since $\{P_{n'} \circ V_{n'}^{-1}\}_{n'=1}^{\infty}$ is uniformly tight (as shown above), there exists a subsequence $\{n'_k\} \subset \{n'\}$ such that

(B.3) $$F_{n'_k}(t;P_{n'_k}) \to F^*(t), \text{ as } k \to \infty,$$

for some cdf F^*. Also $\{P_{n'} \circ V^{-1}\}_{n'=1}^{\infty}$ is uniformly tight (because $\sup_{P\in\mathcal{P}}\mathbf{E}||V||^2 < \infty$), $\{P_{n'_k} \circ V^{-1}\}_{k=1}^{\infty}$ is uniformly tight and hence there exists a further subsequence $\{n'_{k_j}\} \subset \{n'_k\}$ such that

(B.4) $$F(t;P_{n'_{k_j}}) \to F^{**}(t), \text{ as } j \to \infty,$$

for some cdf F^{**}. Since $\{n'_{k_j}\} \subset \{n'_k\}$, we have from (B.3),

(B.5) $$F_{n'_{k_j}}(t;P_{n'_{k_j}}) \to F^*(t), \text{ as } j \to \infty.$$

By the condition of (ii), we have

$$\left|\varphi_{n'_{k_j}}(u;P_{n'_{k_j}})-\varphi(u;P_{n'_{k_j}})\right|\to 0,\text{ as } j\to\infty, \tag{B.6}$$

where

$$\varphi_n\left(u;P_n\right)=\mathbf{E}_{P_n}\left(\exp\left(iuV_n\right)\right)\text{ and }\varphi\left(u;P_n\right)=\mathbf{E}_{P_n}\left(\exp\left(iuV\right)\right),$$

and $\mathbf{E}_{P_n}$ represents expectation with respect to the probability measure P_n. Furthermore, by (B.4) and (B.5), and Levy's Continuity Theorem,

$$\lim_{j\to\infty}\varphi_{n'_{k_j}}(u;P_{n'_{k_j}})\text{ and }\lim_{j\to\infty}\varphi(u;P_{n'_{k_j}})$$

exist and coincide by (B.6). Therefore, for all $t\in\mathbf{R}^d$,

$$F^{**}(t)=F^*(t).$$

In other words,

$$\lim_{n'\to\infty}\left|F_{n'}\left(t;P_{n'}\right)-F\left(t;P_{n'}\right)\right|=\lim_{n'\to\infty}\left|F_{n'_{k_j}}\left(t;P_{n'_{k_j}}\right)-F\left(t;P_{n'_{k_j}}\right)\right|=0.$$

Therefore, the first statement of (ii) follows by the last limit applied to (B.2).

Let us turn to the second statement. Again, we show that $\{P\circ V_n^{-1}\}_{n=1}^{\infty}$ is uniformly tight over $P\in\mathcal{P}$. Note that given a large rectangle B,

$$P\left\{V_n\in\mathbf{R}^d\backslash B\right\}\le\left|P\left\{V_n\in\mathbf{R}^d\backslash B\right\}-P\left\{V\in\mathbf{R}^d\backslash B\right\}\right|+P\left\{V\in\mathbf{R}^d\backslash B\right\}.$$

There exists N such that for all $n\ge N$, the first difference vanishes as $n\to\infty$, uniformly in $P\in\mathcal{P}$, by the condition of the lemma. As for the second term, we bound it by

$$P\left\{V_j>a_j,\ j=1,...,d\right\}\le\sum_{j=1}^{d}\frac{\mathbf{E}V_j^2}{a_j},$$

where V_j is the j-th entry of V and $B=\times_{j=1}^{d}[a_j,b_j]$, $b_j<0<a_j$. By taking a_j's large enough, we make the last bound arbitrarily small independently of $P\in\mathcal{P}$, because $\sup_{P\in\mathcal{P}}\mathbf{E}V_j^2<\infty$ for each $j=1,...,d$. Therefore, $\{P\circ V_n^{-1}\}_{n=1}^{\infty}$ is uniformly tight over $P\in\mathcal{P}$.

Now, we turn to the proof of the second statement of (ii). For each $u\in\mathbf{R}^d$, there exists a subsequence $\{n'\}\subset\{n\}$ and $\{P_{n'}\}\subset\mathcal{P}$ such that

$$\operatorname*{limsup}_{n\to\infty}\sup_{P\in\mathcal{P}}\left|\varphi_n(u)-\varphi(u)\right|=\lim_{n'\to\infty}\left|\varphi_{n'}(u;P_{n'})-\varphi(u;P_{n'})\right|,$$

where $\varphi_n(u;P_n)=\mathbf{E}_{P_n}\exp(iu^{\top}V_n)$ and $\varphi(u;P_n)=\mathbf{E}_{P_n}\exp(iu^{\top}V)$. By the condition in the second statement of (ii), for each $t\in\mathbf{R}^d$,

$$\lim_{n'\to\infty}\left|F_{n'}\left(t;P_{n'}\right)-F\left(t;P_{n'}\right)\right|=0. \tag{B.7}$$

Since $\{P_{n'} \circ V_{n'}^{-1}\}_{n'=1}^{\infty}$ is uniformly tight (as shown above), there exists a subsequence $\{n'_k\} \subset \{n'\}$ such that $F_{n'_k}(t; P_{n'_k}) \to F^*(t)$, as $k \to \infty$, and hence by Levy's Continuity Theorem, we have $\varphi_{n'_k}(u; P_{n'_k}) \to \varphi^*(u)$, as $k \to \infty$. Similarly, we also have $\varphi(u; P_{n'_k}) \to \varphi^{**}(u)$, as $k \to \infty$. By (B.7), we have $F^*(t) = F^{**}(t)$ and $\varphi^*(u) = \varphi^{**}(u)$. Therefore,

$$\lim_{n' \to \infty} |\varphi_{n'}(u; P_{n'}) - \varphi(u; P_{n'})| = \lim_{n' \to \infty} \left| \varphi_{n'_{k_j}}\left(u; P_{n'_{k_j}}\right) - \varphi\left(u; P_{n'_{k_j}}\right) \right| = 0.$$

Thus we arrive at the desired result. ∎

The following lemma offers a version of the de-Poissonization lemma of Beirlant and Mason (1995) (see Theorem 2.1 on page 5). In contrast to the result of Beirlant and Mason (1995), the version here is uniform in $P \in \mathcal{P}$.

Lemma B3. *Let $N_{1,n}(\alpha)$ and $N_{2,n}(\alpha)$ be independent Poisson random variables with $N_{1,n}(\alpha)$ being Poisson $(n(1-\alpha))$ and $N_{2,n}(\alpha)$ being Poisson $(n\alpha)$, where $\alpha \in (0,1)$. Denote $N_n(\alpha) = N_{1,n}(\alpha) + N_{2,n}(\alpha)$ and set*

$$U_n(\alpha) = \frac{N_{1,n}(\alpha) - n(1-\alpha)}{\sqrt{n}} \text{ and } V_n(\alpha) = \frac{N_{2,n}(\alpha) - n\alpha}{\sqrt{n}}.$$

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of random variables and $\mathcal{P}$ be a given set of probabilities P on a measure space on which $(S_n, U_n(\alpha_P), V_n(\alpha_P))$ lives, where $\alpha_P \in (0,1)$ is a quantity that may depend on $P \in \mathcal{P}$ and for some $\varepsilon > 0$,

$$\text{(B.8)} \qquad \varepsilon \leq \inf_{P \in \mathcal{P}} \alpha_P \leq \sup_{P \in \mathcal{P}} \alpha_P \leq 1 - \varepsilon.$$

Furthermore, assume that for each $n \geq 1$, the random vector $(S_n, U_n(\alpha_P))$ is independent of $V_n(\alpha_P)$ with respect to each $P \in \mathcal{P}$. Let for $t_1, t_2 \in \mathbf{R}^2$,

$$b_{n,P}(t_1, t_2; \sigma_P) \equiv \left| P\{S_n \leq t_1, U_n(\alpha_P) \leq t_2\} - P\{\sigma_P \mathbb{Z}_1 \leq t_1, \sqrt{1-\alpha_P}\mathbb{Z}_2 \leq t_2\} \right|,$$

where $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are independent standard normal random variables and $\sigma_P^2 > 0$ for each $P \in \mathcal{P}$. (Note that $\inf_{P \in \mathcal{P}} \sigma_P^2$ is allowed to be zero.)
(i) As $n \to \infty$,

$$\begin{aligned} & \sup_{P \in \mathcal{P}} \sup_{t \in \mathbf{R}} \left| \mathbf{E}[\exp(itS_n) | N_n(\alpha_P) = n] - \exp\left(-\frac{\sigma_P^2 t^2}{2}\right) \right| \\ \leq \;& 2\varepsilon + \left(4C_d \sup_{P \in \mathcal{P}} a_{n,P}(\varepsilon)\right) \sqrt{\frac{2\pi}{\varepsilon}}, \end{aligned}$$

where $a_{n,P}(\varepsilon) \equiv \varepsilon^{-d} b_{n,P} + \varepsilon, b_{n,P} \equiv \sup_{t_1, t_2 \in \mathbf{R}} b_{n,P}(t_1, t_2; \sigma_P)$, and ε is the constant in (B.8).
(ii) Suppose further that for all $t_1, t_2 \in \mathbf{R}$, as $n \to \infty$,

$$\sup_{P \in \mathcal{P}} b_{n,P}(t_1, t_2; 0) \to 0.$$

Then, for all $t \in \mathbf{R}$, we have as $n \to \infty$,

$$\sup_{P \in \mathcal{P}} |\mathbf{E}[\exp(itS_n)|N_n(\alpha_P) = n] - 1| \to 0.$$

Remark 2. While the proof of Lemma B3 follows that of Lemma 2.4 of Giné, Mason, and Zaitsev (2003), it is worth noting that in contrast to Lemma 2.4 of Giné, Mason, and Zaitsev (2003) or Theorem 2.1 of Beirlant and Mason (1995), Lemma B3 gives an explicit bound for the difference between the conditional characteristic function of S_n given $N_n(\alpha_P) = n$ and the characteristic function of $N(0, \sigma_P^2)$. Under the stated conditions, (in particular (B.8)), the explicit bound is shown to depend on $P \in \mathcal{P}$ only through $b_{n,P}$. Thus in order to obtain a bound uniform in $P \in \mathcal{P}$, it suffices to control α_P and $b_{n,P}$ uniformly in $P \in \mathcal{P}$.

Proof of Lemma B3. (i) Let $\phi_{n,P}(t,u) = \mathbf{E}[\exp(itS_n + iuU_n(\alpha_P))]$ and

$$\phi_P(t,u) = \exp(-(\sigma_P^2 t^2 + (1-\alpha_P)u^2)/2).$$

By the condition of the lemma and Lemma B2(i), we have for any $\varepsilon > 0$,

$$\begin{aligned} |\phi_{n,P}(t,u) - \phi_P(t,u)| &\leq (\varepsilon^{-d} C_d b_{n,P} + 4\varepsilon) \\ &\leq 4\varepsilon^{-d} C_d b_{n,P} + 4\varepsilon = 4C_d a_{n,P}(\varepsilon). \end{aligned} \tag{B.9}$$

Note that $a_{n,P}(\varepsilon)$ depends on $P \in \mathcal{P}$ only through $b_{n,P}$.

Following the proof of Lemma 2.4 of Giné, Mason, and Zaitsev (2003), we have

$$\begin{aligned} \psi_{n,P}(t) &= \mathbf{E}[\exp(itS_n)|N_n(\alpha_P) = n] \\ &= \frac{1}{\sqrt{2\pi}}(1+o(1)) \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \phi_{n,P}(t,v) \mathbf{E}\left[\exp(ivV_n(\alpha_P))\right] dv, \end{aligned}$$

uniformly over $P \in \mathcal{P}$. Note that the equality comes after applying Sterling's formula to $2\pi P\{N_n(\alpha_P) = n\}$ and change of variables from u to $v/\sqrt{n}$. (See the proof of Lemma 2.4 of Giné, Mason, and Zaitsev (2003).) The distribution of $N_n(\alpha_P)$, being Poisson (n), does not depend on the particular choice of $\alpha_P \in (0,1)$, and hence the $o(1)$ term is $o(1)$ uniformly over $t \in \mathbf{R}$ and over $P \in \mathcal{P}$. We follow the proof of Theorem 3 of Feller (1966, p.517) to observe that there exists $n_0 > 0$ such that uniformly over $\alpha \in [\varepsilon, 1-\varepsilon]$,

$$\left\{ \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \left| \mathbf{E}\exp(ivV_n(\alpha)) - \exp(-\alpha v^2/2) \right| dv + \int_{|v|>\pi\sqrt{n}} \exp\left(-\alpha v^2/2\right) dv \right\} < \varepsilon,$$

for all $n > n_0$. Note that the distribution of $V_n(\alpha_P)$ depends on $P \in \mathcal{P}$ only through $\alpha_P \in [\varepsilon, 1-\varepsilon]$ and ε does not depend on P. Since there exists n_1 such that for all $n > n_1$,

$$\sup_{P \in \mathcal{P}} \int_{|v|>\pi\sqrt{n}} \exp\left(-\alpha_P v^2/2\right) dv < \varepsilon,$$

the previous inequality implies that for all $n > \max\{n_0, n_1\}$,

$$\text{(B.10)}\qquad \sup_{P\in\mathcal{P}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \left|\phi_{n,P}(t,u)\left(\mathbf{E}\exp(iuV_n(\alpha_P)) - \exp(-\alpha_P u^2/2)\right)\right| du$$

$$\leq \sup_{P\in\mathcal{P}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \left(\sup_{P\in\mathcal{P}} |\phi_{n,P}(t,u)|\right) |\mathbf{E}\exp(iuV_n(\alpha_P)) - \exp(-\alpha_P u^2/2)| du$$

$$\leq \sup_{P\in\mathcal{P}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} |\mathbf{E}\exp(iuV_n(\alpha_P)) - \exp(-\alpha_P u^2/2)| du \leq \varepsilon.$$

By (B.9) and (B.10),

$$\sup_{P\in\mathcal{P}} \left| \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \phi_{n,P}(t,u)\mathbf{E}\left[\exp(iuV_n(\alpha_P))\right] du - \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \phi_P(t,u)\exp\left(-\alpha_P u^2/2\right) du \right|$$

$$\leq \sup_{P\in\mathcal{P}} \sup_{\alpha\in[\varepsilon,1-\varepsilon]} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \left|\phi_{n,P}(t,u)\left(\mathbf{E}\exp(iuV_n(\alpha)) - \exp(-\alpha u^2/2)\right)\right| du$$

$$+ \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \sup_{P\in\mathcal{P}} \sup_{\alpha\in[\varepsilon,1-\varepsilon]} |\phi_{n,P}(t,u) - \phi_P(t,u)| \exp(-\alpha u^2/2) du$$

$$\leq \varepsilon + \left(4C_d \sup_{P\in\mathcal{P}} a_{n,P}(\varepsilon)\right) \sup_{\alpha\in[\varepsilon,1-\varepsilon]} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \exp(-\alpha u^2/2) du$$

$$\leq \varepsilon + \left(4C_d \sup_{P\in\mathcal{P}} a_{n,P}(\varepsilon)\right) \sup_{\alpha\in[\varepsilon,1-\varepsilon]} \sqrt{\frac{2\pi}{\alpha}} = \varepsilon + \left(4C_d \sup_{P\in\mathcal{P}} a_{n,P}(\varepsilon)\right) \sqrt{\frac{2\pi}{\varepsilon}}$$

as $n \to \infty$. Since

$$\exp\left(-\frac{\sigma_P^2 t^2}{2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_P(t,u)\exp\left(-\frac{\alpha_P u^2}{2}\right) du,$$

and from some large n that does not depend on $P \in \mathcal{P}$,

$$\left| \int_{-\infty}^{\infty} \phi_P(t,u)\exp\left(-\frac{\alpha_P u^2}{2}\right) du - \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \phi_P(t,u)\exp\left(-\frac{\alpha_P u^2}{2}\right) du \right|$$

$$= \exp\left(-\frac{\sigma_P^2 t^2}{2}\right) \left| \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du - \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \exp\left(-\frac{u^2}{2}\right) du \right| < \varepsilon,$$

we conclude that for each $t \in \mathbf{R}$,

$$\left| \psi_{n,P}(t) - \exp\left(-\frac{\sigma_P^2 t^2}{2}\right) \right| \leq 2\varepsilon + \left(4C_d \sup_{P\in\mathcal{P}} a_{n,P}(\varepsilon)\right) \sqrt{\frac{2\pi}{\varepsilon}},$$

as $n \to \infty$. Since the right hand side does not depend on $t \in \mathbf{R}$ and $P \in \mathcal{P}$, we obtain the desired result.

(ii) By the condition of the lemma and Lemma B2(ii), we have for any $t, u \in \mathbf{R}$,

$$\sup_{P \in \mathcal{P}} |\phi_{n,P}(t,u) - \phi_P(0,u)| \to 0,$$

as $n \to \infty$. The rest of the proof is similar to that of (i). We omit the details. ∎

Define for $x \in \mathcal{X}$, $\tau_1, \tau_2 \in \mathcal{T}$, and $j, k \in \mathbb{N}_J$,

$$k_{n,\tau,j,m}(x) \equiv \frac{1}{h^d} \mathbf{E}\left[\left|\beta_{n,x,\tau,j}\left(Y_{ij}, \frac{X_i - x}{h}\right)\right|^m\right].$$

Lemma B4. *Suppose that Assumption A6(i) holds. Then for all $m \in [2, M]$, (with $M > 2$ being the constant that appears in Assumption A6(i)), there exists $C_1 \in (0, \infty)$ that does not depend on n such that for each $j \in \mathbb{N}_J$,*

$$\sup_{\tau \in \mathcal{T}, x \in \mathcal{S}_\tau(\varepsilon)} \sup_{P \in \mathcal{P}} k_{n,\tau,j,m}(x) \leq C_1.$$

Proof of Lemma B4. The proof can proceed by using Assumption A6(i) and following the proof of Lemma 4 of Lee, Song, and Whang (2013). ∎

Let N be a Poisson random variable with mean n and independent of $(Y_i^\top, X_i^\top)_{i=1}^\infty$. Also, let $\beta_{n,x,\tau}(Y_i, (X_i - x)/h)$ be the J-dimensional vector whose j-th entry is equal to $\beta_{n,x,\tau,j}(Y_{ij}, (X_i - x)/h)$. We define

$$\begin{aligned} \mathbf{z}_{N,\tau}(x) &\equiv \frac{1}{nh^d} \sum_{i=1}^{N} \beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right) - \frac{1}{h^d} \mathbf{E}\beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right) \text{ and} \\ \mathbf{z}_{n,\tau}(x) &\equiv \frac{1}{nh^d} \sum_{i=1}^{n} \beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right) - \frac{1}{h^d} \mathbf{E}\beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right). \end{aligned}$$

Let N_1 be a Poisson random variable with mean 1, independent of $(Y_i^\top, X_i^\top)_{i=1}^\infty$. Define

$$\begin{aligned} q_{n,\tau}(x) &\equiv \frac{1}{\sqrt{h^d}} \sum_{1 \leq i \leq N_1} \left\{\beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right) - \mathbf{E}\beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right)\right\} \text{ and} \\ \bar{q}_{n,\tau}(x) &\equiv \frac{1}{\sqrt{h^d}} \left\{\beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right) - \mathbf{E}\beta_{n,x,\tau}\left(Y_i, \frac{X_i - x}{h}\right)\right\}. \end{aligned}$$

Lemma B5. *Suppose that Assumption A6(i) holds. Then for any $m \in [2, M]$ (with $M > 2$ being the constant in Assumption A6(i))*

$$\begin{aligned} \sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E}\left[||q_{n,\tau}(x)||^m\right] &\leq \bar{C}_1 h^{d(1-(m/2))} \text{ and} \\ \sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E}\left[||\bar{q}_{n,\tau}(x)||^m\right] &\leq \bar{C}_2 h^{d(1-(m/2))}, \end{aligned} \tag{B.11}$$

where $\bar{C}_1, \bar{C}_2 > 0$ are constants that depend only on m.

If furthermore, $\limsup_{n\to\infty} n^{-(m/2)+1}h^{d(1-(m/2))} < C$ *for some constant* $C > 0$*, then*

$$\text{(B.12)}\qquad \begin{aligned} \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}} \mathbf{E}\left[||n^{1/2}h^{d/2}\mathbf{z}_{N,\tau}(x)||^m\right] &\le \left(\frac{15m}{\log m}\right)^m \max\left\{\bar{C}_1, 2\bar{C}_1 C\right\} \text{ and} \\ \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}} \mathbf{E}\left[||n^{1/2}h^{d/2}\mathbf{z}_{n,\tau}(x)||^m\right] &\le \left(\frac{15m}{\log m}\right)^m \max\left\{\bar{C}_2, 2\bar{C}_2 C\right\}, \end{aligned}$$

where $\bar{C}_1, \bar{C}_2 > 0$ *are the constants that appear in (B.11).*

Proof of Lemma B5. Let $q_{n,\tau,j}(x)$ be the j-th entry of $q_{n,\tau}(x)$. For the first statement of the lemma, it suffices to observe that for some positive constants C_1 and $\bar{C}$,

$$\text{(B.13)}\qquad \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}} \mathbf{E}\left[|q_{n,\tau,j}(x)|^m\right] \le \frac{C_1 h^d k_{n,\tau,j,m}}{h^{dm/2}} \le \bar{C}h^{d(1-(m/2))},$$

where the first inequality uses the definition of $k_{n,\tau,j,m}$, and the last inequality uses Lemma B4 and the fact that $m \in [2, M]$. The second statement in (B.11) follows similarly.

We consider the statements in (B.12). We consider the first inequality in (B.12). Let $z_{N,\tau,j}(x)$ be the j-th entry of $\mathbf{z}_{N,\tau}(x)$. Then using Rosenthal's inequality (e.g. (2.3) of Giné, Mason, and Zaitsev (2003)), we find that

$$\begin{aligned} &\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}} \mathbf{E}[|\sqrt{nh^d} z_{N,\tau,j}(x)|^m] \\ \le\ & \left(\frac{15m}{\log m}\right)^m \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}} \max\left\{\left(\mathbf{E}q_{n,\tau,j}^2(x)\right)^{m/2}, n^{-m/2+1}\mathbf{E}|q_{n,\tau,j}(x)|^m\right\}. \end{aligned}$$

Since $\mathbf{E}q_{n,\tau,j}^2(x) \le (\mathbf{E}|q_{n,\tau,j}(x)|^m)^{2/m}$, by (B.13), the last term is bounded by

$$\begin{aligned} &\left(\frac{15m}{\log m}\right)^m \max\left\{\bar{C}, \bar{C}n^{-(m/2)+1}h^{d(1-(m/2))}\right\} \\ \le\ &\left(\frac{15m}{\log m}\right)^m \max\left\{\bar{C}, 2\bar{C}C\right\}, \end{aligned}$$

from some large n by the condition $\text{limsup}_{n\to\infty} n^{-(m/2)+1}h^{d(1-(m/2))} < C$.

As for the second inequality in (B.12), for some $C > 0$, we use the second inequality in (B.11) and use Rosenthal's inequality in the same way as before, to obtain the inequality. ∎

The following lemma offers a characterization of the scale normalizer of our test statistic. For $A, A' \subset \mathbb{N}_J$, define $\zeta_{n,\tau}(x) \equiv \sqrt{nh^d}\mathbf{z}_{N,\tau}(x)$,

$$\text{(B.14)}\qquad \begin{aligned} C^R_{n,\tau,\tau',A,A'}(x,x') &\equiv h^{-d}Cov\left(\Lambda_{A,p}\left(\zeta_{n,\tau}(x)\right), \Lambda_{A',p}\left(\zeta_{n,\tau'}(x')\right)\right), \text{ and} \\ C_{n,\tau,\tau',A,A'}(x,u) &\equiv Cov\left(\Lambda_{A,p}\left(\mathbb{W}^{(1)}_{n,\tau,\tau'}(x,u)\right), \Lambda_{A',p}\left(\mathbb{W}^{(2)}_{n,\tau,\tau'}(x,u)\right)\right), \end{aligned}$$

where we recall that $[\mathbb{W}^{(1)}_{n,\tau_1,\tau_2}(x,u)^\top, \mathbb{W}^{(2)}_{n,\tau_1,\tau_2}(x,u)^\top]^\top$ is a mean zero $\mathbf{R}^{2J}$-valued Gaussian random vector whose covariance matrix is given by (4.9).

Then for Borel sets $B, B' \subset \mathcal{S}$ and $A, A' \subset \mathbb{N}_J$, let

$$\sigma^R_{n,A,A'}(B,B') \equiv \int_{B'}\int_{B} C^R_{n,\tau,\tau',A,A'}(x,x')dQ(x,\tau)dQ(x',\tau')$$

and

$$\sigma_{n,A,A'}(B,B') \equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_\tau \cap B'_{\tau'}}\int_{\mathcal{U}} C_{n,\tau,\tau',A,A'}(x,u)dudxd\tau d\tau', \tag{B.15}$$

where $B_\tau \equiv \{x \in \mathcal{X} : (x,\tau) \in B\}$ and $B'_{\tau'} \equiv \{x \in \mathcal{X} : (x,\tau') \in B'\}$.

The lemma below shows that $\sigma^R_{n,A,A'}(B,B')$ and $\sigma_{n,A,A'}(B,B')$ are asymptotically equivalent uniformly in $P \in \mathcal{P}$. We introduce some notation. Recall the definition of $\Sigma_{n,\tau_1,\tau_2}(x,u)$, which is found below (4.7). Define for $\bar{\varepsilon} > 0$,

$$\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \equiv \begin{bmatrix} \Sigma_{n,\tau_1,\tau_1}(x,0) + \bar{\varepsilon} I_J & \Sigma_{n,\tau_1,\tau_2}(x,u) \\ \Sigma_{n,\tau_1,\tau_2}(x,u) & \Sigma_{n,\tau_2,\tau_2}(x+uh,0) + \bar{\varepsilon} I_J \end{bmatrix},$$

where I_J is the J dimensional identity matrix. Certainly $\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)$ is positive definite. We define

$$\xi_{N,\tau_1,\tau_2}(x,u;\eta_1,\eta_2) \equiv \sqrt{nh^d}\,\tilde{\Sigma}^{-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \begin{bmatrix} \mathbf{z}_{N,\tau_1}(x;\eta_1) \\ \mathbf{z}_{N,\tau_2}(x+uh;\eta_2) \end{bmatrix},$$

where $\eta_1 \in \mathbf{R}^J$ and $\eta_2 \in \mathbf{R}^J$ are random vectors that are independent, and independent of $(Y_i^\top, X_i^\top)_{i=1}^\infty$, each following $N(0,\bar{\varepsilon} I_J)$, and $\mathbf{z}_{N,\tau}(x;\eta_1) \equiv \mathbf{z}_{N,\tau}(x) + \eta_1/\sqrt{nh^d}$. We are prepared to state the lemma.

Lemma B6. *Suppose that Assumption A6(i) holds and that $nh^d \to \infty$, as $n \to \infty$, and*

$$\limsup_{n\to\infty} n^{-(m/2)+1} h^{d(1-(m/2))} < C,$$

for some constant $C > 0$ and some $m \in [2(p+1), M]$.

Then for any sequences of Borel sets B_n, $B'_n \subset \mathcal{S}$ and for any $A, A' \subset \mathbb{N}_J$,

$$\sigma^R_{n,A,A'}(B_n,B'_n) = \sigma_{n,A,A'}(B_n,B'_n) + o(1),$$

where $o(1)$ vanishes uniformly in $P \in \mathcal{P}$ as $n \to \infty$.

Remark 3. The main innovative element of Lemma B6 is that the result does not require that $\sigma_{n,A,A'}(B_n,B'_n)$ be positive for each finite n or positive in the limit. Hence the result can be applied to the case where the scale normalizer $\sigma^R_{n,A,A'}(B_n,B'_n)$ is degenerate (either in finite samples or asymptotically).

Proof of Lemma B6. Define $B_{n,\tau} \equiv \{x \in \mathcal{X} : (x,\tau) \in B_n\}$, $w_{\tau,B_n}(x) \equiv 1_{B_{n,\tau}}(x)$. For a given $\bar{\varepsilon} > 0$, let

$$\begin{aligned} g_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) &\equiv h^{-d} Cov(\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_1}(x;\eta_1)), \Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_2}(x+uh;\eta_2))), \\ g_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) &\equiv Cov(\Lambda_{A,p}(\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x)), \Lambda_{A',p}(\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x+uh))), \end{aligned}$$

and $\left(\mathbb{Z}^\top_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x), \mathbb{Z}^\top_{n,\tau_1,\tau_2,\bar{\varepsilon}}(v)\right)^\top$ is a centered normal $\mathbf{R}^{2J}$-valued random vector with the same covariance matrix as that of $[\sqrt{nh^d}\mathbf{z}^\top_{N,\tau_1}(x;\eta_1), \sqrt{nh^d}\mathbf{z}^\top_{N,\tau_2}(v;\eta_2)]^\top$. Then we define

$$\sigma^R_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) \equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}}\int_{\mathcal{U}} g_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) w_{\tau_1,B_n}(x) w_{\tau_2,B'_n}(x+uh) du dx d\tau_1 d\tau_2,$$

and

$$\sigma_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) \equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}\cap B'_{n,\tau_2}}\int_{\mathcal{U}} C_{n,\tau_1,\tau_2,A,A',\bar{\varepsilon}}(x,u) du dx d\tau_1 d\tau_2,$$

where

$$C_{n,\tau_1,\tau_2,A,A',\bar{\varepsilon}}(x,u) \equiv Cov\left(\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)), \Lambda_{A',p}(\mathbb{W}^{(2)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u))\right), \tag{B.16}$$

and, with $\mathbb{Z} \sim N(0, I_{2J})$,

$$\begin{bmatrix} \mathbb{W}^{(1)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \\ \mathbb{W}^{(2)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \end{bmatrix} \equiv \tilde{\Sigma}^{1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\mathbb{Z}. \tag{B.17}$$

Thus, $\sigma^R_{n,A,A',\bar{\varepsilon}}(B_n, B'_n)$ and $\sigma_{n,A,A',\bar{\varepsilon}}(B_n, B'_n)$ are "regularized" versions of $\sigma^R_{n,A,A'}(B_n, B'_n)$ and $\sigma_{n,A,A'}(B_n, B'_n)$. We also define

$$\tau_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) \equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}}\int_{\mathcal{U}} g_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) w_{\tau_1,B_n}(x) w_{\tau_2,B'_n}(x+uh) du dx d\tau_1 d\tau_2.$$

Then it suffices for the lemma to show the following two statements.

Step 1: As $n \to \infty$,

$$\begin{aligned} \sup_{P\in\mathcal{P}} \left|\sigma^R_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) - \tau_{n,A,A',\bar{\varepsilon}}(B_n, B'_n)\right| &\to 0, \text{ and} \\ \sup_{P\in\mathcal{P}} \left|\tau_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) - \sigma_{n,A,A',\bar{\varepsilon}}(B_n, B'_n)\right| &\to 0. \end{aligned}$$

Step 2: For some $C > 0$ that does not depend on $\bar{\varepsilon}$ or n,

$$\begin{aligned} \sup_{P\in\mathcal{P}} |\sigma^R_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) - \sigma^R_{n,A,A'}(B_n, B'_n)| &\le C\sqrt{\bar{\varepsilon}}, \text{ and} \\ \sup_{P\in\mathcal{P}} |\sigma_{n,A,A',\bar{\varepsilon}}(B_n, B'_n) - \sigma_{n,A,A'}(B_n, B'_n)| &\le C\sqrt{\bar{\varepsilon}}. \end{aligned}$$

Then the desired result follows by sending $n \to \infty$ and then $\bar{\varepsilon} \downarrow 0$, while chaining Steps 1 and 2.

Proof of Step 1: We first focus on the first statement. For any vector $\mathbf{v} = [\mathbf{v}_1^\top, \mathbf{v}_2^\top]^\top \in \mathbf{R}^{2J}$, we define

$$\begin{aligned}\tilde{\Lambda}_{A,p,1}(\mathbf{v}) &\equiv \Lambda_{A,p}\left(\left[\tilde{\Sigma}^{1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\mathbf{v}\right]_1\right),\\ \tilde{\Lambda}_{A',p,2}(\mathbf{v}) &\equiv \Lambda_{A',p}\left(\left[\tilde{\Sigma}^{1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\mathbf{v}\right]_2\right),\end{aligned}$$

and

$$C_{n,p}(\mathbf{v}) \equiv \tilde{\Lambda}_{A,p,1}(\mathbf{v})\,\tilde{\Lambda}_{A',p,2}(\mathbf{v}), \tag{B.18}$$

where $[a]_1$ of a vector $a \in \mathbf{R}^{2J}$ indicates the vector of the first J entries of a, and $[a]_2$ the vector of the remaining J entries of a. By Theorem 9 of Magnus and Neudecker (2001, p. 208),

$$\begin{aligned}\lambda_{\min}\left(\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right) &\geq \lambda_{\min}\left(\left[\begin{array}{cc}\Sigma_{n,\tau_1,\tau_2}(x,0) & \Sigma_{n,\tau_1,\tau_2}(x,u)\\ \Sigma^\top_{n,\tau_1,\tau_2}(x,u) & \Sigma_{n,\tau_2,\tau_2}(x+uh,0)\end{array}\right]\right)\\ &\quad +\lambda_{\min}\left(\left[\begin{array}{cc}\bar{\varepsilon}I_J & 0\\ 0 & \bar{\varepsilon}I_J\end{array}\right]\right)\\ &\geq \lambda_{\min}\left(\left[\begin{array}{cc}\bar{\varepsilon}I_J & 0\\ 0 & \bar{\varepsilon}I_J\end{array}\right]\right) = \bar{\varepsilon}.\end{aligned} \tag{B.19}$$

Let $q_{n,\tau,j}(x;\eta_{1j}) \equiv p_{n,\tau,j}(x) + \eta_{1j}$, where

$$p_{n,\tau,j}(x) \equiv \frac{1}{\sqrt{h^d}}\sum_{1\leq i\leq N_1}\left\{\beta_{n,x,\tau,j}\left(Y_{ij},\frac{X_i - x}{h}\right) - \mathbf{E}\left[\beta_{n,x,\tau,j}\left(Y_{ij},\frac{X_i-x}{h}\right)\right]\right\},$$

η_{1j} is the j-th entry of η_1, and N_1 is a Poisson random variable with mean 1 and $((\eta_{1j})_{j\in\mathbb{N}_J}, N_1)$ is independent of $\{(Y_i^\top, X_i^\top)\}_{i=1}^\infty$. Let $p_{n,\tau}(x)$ be the column vector of entries $p_{n,\tau,j}(x)$ with j running in the set $\mathbb{N}_J$. Let $[p^{(i)}_{n,\tau_1}(x), p^{(i)}_{n,\tau_2}(x+uh)]$ be i.i.d. copies of $[p_{n,\tau_1}(x), p_{n,\tau_2}(x+uh)]$ and $\eta_1^{(i)}$ and $\eta_2^{(i)}$ be also i.i.d. copies of η_1 and η_2. Define

$$q^{(i)}_{n,\tau,1}(x) \equiv p^{(i)}_{n,\tau}(x) + \eta_1^{(i)} \text{ and } q^{(i)}_{n,\tau,2}(x+uh) \equiv p^{(i)}_{n,\tau}(x+uh) + \eta_2^{(i)}.$$

Note that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n\left[\begin{array}{c}q^{(i)}_{n,\tau_1,1}(x)\\ q^{(i)}_{n,\tau_2,2}(x+uh)\end{array}\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^n\left[\begin{array}{c}p^{(i)}_{n,\tau_1}(x)\\ p^{(i)}_{n,\tau_2}(x+uh)\end{array}\right] + \frac{1}{\sqrt{n}}\sum_{i=1}^n\left[\begin{array}{c}\eta_1^{(i)}\\ \eta_2^{(i)}\end{array}\right].$$

The last sum has the same distribution as $[\eta_1^\top, \eta_2^\top]^\top$ and the leading sum on the right-hand side has the same distribution as that of $[\mathbf{z}^\top_{N,\tau_1}(x), \mathbf{z}^\top_{N,\tau_2}(x+uh)]^\top$. Therefore, we conclude that

$$\xi_{N,\tau_1,\tau_2}(x,u;\eta_1,\eta_2) \stackrel{d}{=} \frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u),$$

where

$$\tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u) \equiv \tilde{\Sigma}^{-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \left[\begin{array}{c} q^{(i)}_{n,\tau_1,1}(x) \\ q^{(i)}_{n,\tau_2,2}(x+uh) \end{array} \right].$$

Now we invoke the Berry-Esseen-type bound of Sweeting (1977, Theorem 1) to prove Step 1. By Lemma B5, we deduce that

$$\text{(B.20)} \qquad \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}||q^{(i)}_{n,\tau,1}(x)||^3 \leq Ch^{-d/2},$$

for some $C > 0$. Also, recall the definition of $\rho_{n,\tau_1,\tau_1,j,j}(x,0)$ in (4.7) and note that

$$\text{(B.21)} \qquad \sup_{\tau\in\mathcal{T}} \sup_{(x,u)\in\mathcal{S}_\tau(\varepsilon)\times\mathcal{U}} \sup_{P\in\mathcal{P}} tr\left(\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right)$$
$$\leq \sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \sum_{j\in J} \left(\rho_{n,\tau_1,\tau_1,j,j}(x,0) + \rho_{n,\tau_2,\tau_2,j,j}(x,0) + 2\bar{\varepsilon}\right) \leq C,$$

for some $C > 0$ that depends only on J and $\bar{\varepsilon}$ by Lemma B4. Observe that by the definition of $C_{n,p}$ in (B.18), and (B.21),

$$\sup_{\mathbf{v}\in\mathbf{R}^{2J}} \frac{|C_{n,p}(\mathbf{v}) - C_{n,p}(0)|}{1+||\mathbf{v}||^{2p+2}\min\{||\mathbf{v}||,1\}} \leq C.$$

We find that for each $u \in \mathcal{U}$, $||\tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u)||^2$ is equal to

$$\text{(B.22)} \qquad tr\left(\tilde{\Sigma}^{-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\left[\begin{array}{c} q^{(i)}_{n,\tau_1,1}(x) \\ q^{(i)}_{n,\tau_1,2}(x+uh)\end{array}\right]\left[\begin{array}{c} q^{(i)}_{n,\tau_2,1}(x) \\ q^{(i)}_{n,\tau_2,2}(x+uh)\end{array}\right]^\top \tilde{\Sigma}^{-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right)$$
$$\leq \lambda_{\max}\left(\tilde{\Sigma}^{-1}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right) tr\left(\left[\begin{array}{c} q^{(i)}_{n,\tau_1,1}(x) \\ q^{(i)}_{n,\tau_1,2}(x+uh)\end{array}\right]\left[\begin{array}{c} q^{(i)}_{n,\tau_2,1}(x) \\ q^{(i)}_{n,\tau_2,2}(x+uh)\end{array}\right]^\top\right).$$

Therefore, $\mathbf{E}||\tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u)||^3$ is bounded by

$$\lambda^{3/2}_{\max}\left(\tilde{\Sigma}^{-1}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right)\mathbf{E}\left\|\left[\begin{array}{c} q^{(i)}_{n,\tau_1,1}(x) \\ q^{(i)}_{n,\tau_2,2}(x+uh)\end{array}\right]\right\|^3.$$

From (B.19),

$$\lambda^{3/2}_{\max}(\tilde{\Sigma}^{-1}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)) = \lambda^{-3/2}_{\min}(\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)) \leq \bar{\varepsilon}^{-3/2}.$$

Therefore, we conclude that

$$\sup_{\tau\in\mathcal{T}} \sup_{(x,u)\in\mathcal{S}_\tau(\varepsilon)\times\mathcal{U}} \sup_{P\in\mathcal{P}} \mathbf{E}||\tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u)||^3$$
$$\leq C_1\bar{\varepsilon}^{-3/2}\cdot \sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}||q^{(i)}_{n,\tau_1,1}(x)||^3$$
$$+C_1\bar{\varepsilon}^{-3/2}\cdot \sup_{\tau\in\mathcal{T}} \sup_{(x,u)\in\mathcal{S}_\tau(\varepsilon)\times\mathcal{U}} \sup_{P\in\mathcal{P}} \mathbf{E}||q^{(i)}_{n,\tau_2,2}(x+uh)||^3 \leq C_2\bar{\varepsilon}^{-3/2}/\sqrt{h^d},$$

where $C_1 > 0$ and $C_2 > 0$ are constants depending only on J, and the last bound follows by (B.20). Therefore, by Theorem 1 of Sweeting (1977), we find that with $\bar{\varepsilon} > 0$ fixed and $n \to \infty$,

$$\text{(B.23)} \quad \sup_{\tau \in \mathcal{T}} \sup_{(x,u) \in \mathcal{S}_\tau(\varepsilon) \times \mathcal{U}} \sup_{P \in \mathcal{P}} \left| \mathbf{E} C_{n,p} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{W}_{n,\tau_1,\tau_2}^{(i)}(x,u) \right) - \mathbf{E} C_{n,p} \left(\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2}(x,u) \right) \right|$$
$$= O\left(n^{-1/2} h^{-d/2}\right) = o(1),$$

where $\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2}(x,u) = [\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x)^\top, \mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x+uh)^\top]^\top$.

Using similar arguments, we also deduce that for $j = 1, 2$, and $A \subset \mathbb{N}_J$,

$$\sup_{\tau \in \mathcal{T}} \sup_{(x,u) \in \mathcal{S}_\tau(\varepsilon) \times \mathcal{U}} \sup_{P \in \mathcal{P}} \left| \mathbf{E} \tilde{\Lambda}_{A,p,j} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{W}_{n,\tau_1,\tau_2}^{(i)}(x,u) \right) - \mathbf{E} \tilde{\Lambda}_{A,p,j} \left(\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2}(x,u) \right) \right| = o(1).$$

For some $C > 0$,

$$\sup_{\tau \in \mathcal{T}} \sup_{(x,u) \in \mathcal{S}_\tau(\varepsilon) \times \mathcal{U}} \sup_{P \in \mathcal{P}} Cov\left(\Lambda_p(\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x)), \Lambda_p(\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x+uh))\right)$$
$$\leq \sup_{\tau \in \mathcal{T}} \sup_{(x,u) \in \mathcal{S}_\tau(\varepsilon) \times \mathcal{U}} \sup_{P \in \mathcal{P}} \sqrt{\mathbf{E} \left\| \mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x) \right\|^{2p}} \sqrt{\mathbf{E} \left\| \mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x+uh) \right\|^{2p}} < C.$$

The last inequality follows because $\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x)$ and $\mathbb{Z}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x+uh)$ are centered normal random vectors with a covariance matrix that has a finite Euclidean norm by Lemma B4. Hence we apply the Dominated Convergence Theorem to deduce the first statement of Step 1 from (B.23).

We turn to the second statement of Step 1. The statement immediately follows because for each $u \in \mathcal{U}$, the covariance matrix of $\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^{-1/2}(x,u)\xi_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)$ is equal to the covariance matrix of $[\mathbb{W}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^{(1)\top}(x,u), \mathbb{W}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^{(2)\top}(x,u)]^\top and$

$$\left| w_{\tau_1,B_n}(x) w_{\tau_2,B_n'}(x+uh) - w_{\tau_1,B_n}(x) w_{\tau_2,B_n'}(x) \right| \to 0,$$

as $n \to \infty$, for each $u \in \mathcal{U}$, and for almost every $x \in \mathcal{X}$ (with respect to Lebesgue measure.)

Proof of Step 2: We consider the first statement. First, we write

$$\text{(B.24)} \quad \left| \left(\sigma_{n,A,A',\bar{\varepsilon}}^R(B_n, B_n')\right)^2 - \left(\sigma_{n,A,A'}^R(B_n, B_n')\right)^2 \right|$$
$$\leq \int_{\mathcal{T}} \int_{\mathcal{T}} \int_{B_n} \int_{\mathcal{U}} \left| \Delta_{n,\tau_1,\tau_2,1}^\eta(x,u) \right| w_{\tau_1,B_n}(x) w_{\tau_2,B_n'}(x+uh) du dx d\tau_1 d\tau_2$$
$$+ \int_{\mathcal{T}} \int_{\mathcal{T}} \int_{B_n} \int_{\mathcal{U}} \left| \Delta_{n,\tau_1,\tau_2,2}^\eta(x,u) \right| w_{\tau_1,B_n}(x) w_{\tau_2,B_n'}(x+uh) du dx d\tau_1 d\tau_2,$$

where

$$\begin{aligned}\Delta^{\eta}_{n,\tau_1,\tau_2,1}(x,u) &= \mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_1}(x))\mathbf{E}\Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_2}(x+uh)) \\ &\quad -\mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_1}(x;\eta_1))\mathbf{E}\Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_2}(x+uh;\eta_2)),\end{aligned}$$

and

$$\begin{aligned}\Delta^{\eta}_{n,\tau_1,\tau_2,2}(x,u) &= \mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_1}(x))\Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_2}(x+uh)) \\ &\quad -\mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_1}(x;\eta_1))\Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}_{N,\tau_2}(x+uh;\eta_2)).\end{aligned}$$

By Hölder inequality, for $C > 0$ that depends only on P,

$$\left|\Delta^{\eta}_{n,\tau_1,\tau_2,2}(x,u)\right| \leq CA_{1n}(x,u) + CA_{2n}(x,u),$$

where, if $p = 1$ then we set $s = 2$, and $q = 1$, and if $p > 1$, we set $s = (p+1)/(p-1)$ and $q = (1-1/s)^{-1}$,

$$\begin{aligned}A_{1n}(x,u) &= (nh^d)^p \left\{\mathbf{E}\left\|\mathbf{z}_{N,\tau_1}(x) - \mathbf{z}_{N,\tau_1}(x;\eta_1)\right\|^{2q}\right\}^{\frac{1}{2q}} \\ &\quad \times \left(\left\{\mathbf{E}\left\|\mathbf{z}_{N,\tau_1}(x)\right\|^{2s(p-1)}\right\}^{\frac{1}{2s}} + \left\{\mathbf{E}\left\|\mathbf{z}_{N,\tau_1}(x;\eta_1)\right\|^{2s(p-1)}\right\}^{\frac{1}{2s}}\right) \\ &\quad \times \sqrt{\mathbf{E}\left(\left\|\mathbf{z}_{N,\tau_2}(x+uh)\right\|^{2p}\right)},\end{aligned}$$

and

$$\begin{aligned}A_{2n}(x,u) &= (nh^d)^p \left\{\mathbf{E}\left\|\mathbf{z}_{N,\tau_2}(x+uh) - \mathbf{z}_{N,\tau_2}(x+uh;\eta_2)\right\|^{2q}\right\}^{\frac{1}{2q}} \\ &\quad \times \left(\left\{\mathbf{E}\left\|\mathbf{z}_{N,\tau_2}(x+uh)\right\|^{2s(p-1)}\right\}^{\frac{1}{2s}} + \left\{\mathbf{E}\left\|\mathbf{z}_{N,\tau_2}(x+uh;\eta_2)\right\|^{2s(p-1)}\right\}^{\frac{1}{2s}}\right) \\ &\quad \times \sqrt{\mathbf{E}\left(\left\|\mathbf{z}_{N,\tau_1}(x;\eta_1)\right\|^{2p}\right)}.\end{aligned}$$

Now,

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left\|\sqrt{nh^d}\{\mathbf{z}_{N,\tau}(x) - \mathbf{z}_{N,\tau}(x;\eta_1)\}\right\|^{2q} = \mathbf{E}\left\|\sqrt{\bar{\varepsilon}}\mathbb{Z}\right\|^{2q} = C\bar{\varepsilon}^q,$$

where $\mathbb{Z} \in \mathbf{R}^J$ is a centered normal random vector with identity covariance matrix I_J. Also, we deduce that for some $C > 0$ that does not depend on n,

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)\right\|^{2s(p-1)} \leq C,$$

by (B.12) of Lemma B5 and by the fact that $2s(p-1) = 2(p+1) \leq M$. Similarly, from some large n on,

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left(\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}(x+uh;\eta_2)\right\|^{2p}\right) \leq \sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}\left(\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}(x;\eta_2)\right\|^{2p}\right) < C,$$

for some $C > 0$. Thus we conclude that for some $C > 0$,

$$\sup_{(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}} \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} (A_{1n}(x,u) + A_{2n}(x,u)) \leq C\sqrt{\bar{\varepsilon}},$$

and that for some $C > 0$,

$$\sup_{(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}} \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} \left|\Delta^\eta_{n,\tau_1,\tau_2,2}(x,u)\right| \leq C\sqrt{\bar{\varepsilon}}.$$

Using similar arguments, we also find that for some $C > 0$,

$$\sup_{(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}} \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} \left|\Delta^\eta_{n,\tau_1,\tau_2,1}(x,u)\right| \leq C\sqrt{\bar{\varepsilon}}.$$

Therefore, there exist $C_1 > 0$ and $C_2 > 0$ such that from some large n on,

$$\sup_{P\in\mathcal{P}} \left|\sigma^2_{n,A,A',\bar{\varepsilon}}(B_n,B'_n) - \sigma^2_{n,A,A'}(B_n,B'_n)\right| \leq C_1\sqrt{\bar{\varepsilon}} \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_n}\int_{\mathcal{U}} w_{\tau_1,B_n}(x)w_{\tau_2,B'_n}(x+uh)dudxd\tau_1 d\tau_2.$$

Since the last multiple integral is finite, we obtain the first statement of Step 2.

We turn to the second statement of Step 2. Similarly as before, we write

$$\begin{aligned} &\left|\sigma^2_{n,A,A',\bar{\varepsilon}}(B_n,B'_n) - \sigma^2_{n,A,A'}(B_n,B'_n)\right| \\ \leq\ & \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_n}\int_{\mathcal{U}} \left|\Delta^\eta_{1,\tau_1,\tau_2}(x,u)\right| w_{\tau_1,B_n}(x)w_{\tau_2,B'_n}(x+uh)dudxd\tau_1 d\tau_2 \\ &+ \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_n}\int_{\mathcal{U}} \left|\Delta^\eta_{2,\tau_1,\tau_2}(x,u)\right| w_{\tau_1,B_n}(x)w_{\tau_2,B'_n}(x+uh)dudxd\tau_1 d\tau_2, \end{aligned}$$

where

$$\begin{aligned} \Delta^\eta_{1,\tau_1,\tau_2}(x,u) &= \mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau_1,\tau_2}(x,u))\mathbf{E}\Lambda_{A',p}(\mathbb{W}^{(2)}_{n,\tau_1,\tau_2}(x,u)) \\ &\quad -\mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u))\mathbf{E}\Lambda_{A',p}(\mathbb{W}^{(2)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)), \end{aligned}$$

and

$$\begin{aligned} \Delta^\eta_{2,\tau_1,\tau_2}(x,u) &= \mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau_1,\tau_2}(x,u))\Lambda_{A',p}(\mathbb{W}^{(2)}_{n,\tau_1,\tau_2}(x,u)) \\ &\quad -\mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u))\Lambda_{A',p}(\mathbb{W}^{(2)}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)). \end{aligned}$$

Now, observe that for $C > 0$ that does not depend on $\bar{\varepsilon}$, we have by Lemma B1(i),

$$\sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} \left\| \tilde{\Sigma}^{1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) - \left[\begin{array}{cc} \Sigma_{n,\tau_1}(x,0) & \Sigma_{n,\tau_1,\tau_2}(x,u) \\ \Sigma_{n,\tau_1,\tau_2}(x,u) & \Sigma_{n,\tau_2}(x+uh) \end{array}\right]^{1/2} \right\| \le C\sqrt{\bar{\varepsilon}}.$$

Using this, recalling the definitions of $\mathbb{W}^{(1)}_{n,\tau_1,\tau_2}(x,u)$ and $\mathbb{W}^{(2)}_{n,\tau_1,\tau_2}(x,u)$ in (B.17), and following the previous arguments in the proof of Step 1, we obtain the second statement of Step 2. ∎

Lemma B7. *Suppose that for some small $\nu_1 > 0$, $n^{-1/2}h^{-d-\nu_1} \to 0$, as $n \to \infty$ and the conditions of Lemma B6 hold. Then there exists $C > 0$ such that for any sequence of Borel sets $B_n \subset \mathcal{S}$, and $A \subset \mathbb{N}_J$, from some large n on,*

$$\begin{aligned} &\sup_{P\in\mathcal{P}} \mathbf{E}\left[\left| h^{-d/2} \int_{B_n} \left\{ \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x)) - \mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right]\right\} dQ(x,\tau) \right|\right] \\ \le\ & C\sqrt{Q(B_n)}. \end{aligned}$$

Remark 4. The result is in the same spirit as Lemma 6.2 of Giné, Mason, and Zaitsev (2003). (Also see Lemma A8 of Lee, Song and Whang (2013).) However, unlike these results, the location normalization here involves $\mathbf{E}[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))]$ instead of $\mathbf{E}[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x))]$. We can obtain the same result with $\mathbf{E}[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))]$ replaced by $\mathbf{E}[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x))]$, but with a stronger bandwidth condition.

Like Lemma B6, the result of Lemma B7 does not require that the quantities $\sqrt{nh^d}\mathbf{z}_{n,\tau}(x)$ and $\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)$ have a (pointwise in x) nondegenerate limit distribution.

Proof of Lemma B7. As in the proof of Lemma A8 of Lee, Song, and Whang (2013), it suffices to show that there exists $C > 0$ such that C does not depend on n and for any Borel set $B_n \subset \mathbf{R}$,

Step 1:

$$\sup_{P\in\mathcal{P}} \mathbf{E}\left[\left| h^{-d/2} \int_{B_n} \left\{ \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x)) - \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right\} dQ(x,\tau) \right|\right] \le CQ(B_n), \text{ and}$$

Step 2:

$$\sup_{P\in\mathcal{P}} \mathbf{E}\left[\left| h^{-d/2} \int_{B_n} \left\{ \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)) - \mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right]\right\} dQ(x,\tau) \right|\right] \le C\sqrt{Q(B_n)}.$$

By chaining Steps 1 and 2, we obtain the desired result.

Proof of Step 1: Similarly as in (2.13) of Horváth (1991), we first write

$$\mathbf{z}_{n,\tau}(x) = \mathbf{z}_{N,\tau}(x) + \mathbf{v}_{n,\tau}(x) + \mathbf{s}_{n,\tau}(x), \tag{B.25}$$

where, for $\beta_{n,x,\tau}(Y_i,(X_i-x)/h)$ defined prior to Lemma B5,

$$\begin{aligned}\mathbf{v}_{n,\tau}(x) &\equiv \left(\frac{n-N}{n}\right)\cdot\frac{1}{h^d}\mathbf{E}\left[\beta_{n,x,\tau}\left(Y_i,\frac{X_i-x}{h}\right)\right] \text{ and}\\ \mathbf{s}_{n,\tau}(x) &\equiv \frac{1}{nh^d}\sum_{i=N+1}^{n}\left\{\beta_{n,x,\tau}\left(Y_i,\frac{X_i-x}{h}\right)-\mathbf{E}\left[\beta_{n,x,\tau}\left(Y_i,\frac{X_i-x}{h}\right)\right]\right\},\end{aligned}$$

and we write $\sum_{i=N+1}^{n}=0$ if $N=n$, and $\sum_{i=N+1}^{n}=-\sum_{i=n+1}^{N}$ if $N>n$.

Using (B.25), we deduce that for some $C_1,C_2>0$ that depend only on p,

$$\begin{aligned}&\int_{B_n}\left|\Lambda_{A,p}\left(\mathbf{z}_{n,\tau}(x)\right)-\Lambda_{A,p}\left(\mathbf{z}_{N,\tau}(x)\right)\right|dQ(x,\tau)\\ \le\;& C_1\int_{B_n}\|\mathbf{v}_{n,\tau}(x)\|\left(\|\mathbf{z}_{n,\tau}(x)\|^{p-1}+\|\mathbf{z}_{N,\tau}(x)\|^{p-1}\right)dQ(x,\tau)\\ &+C_2\int_{B_n}\|\mathbf{s}_{n,\tau}(x)\|\left(\|\mathbf{z}_{n,\tau}(x)\|^{p-1}+\|\mathbf{z}_{N,\tau}(x)\|^{p-1}\right)dQ(x,\tau)\\ \equiv\;& D_{1n}+D_{2n},\text{ say.}\end{aligned} \tag{B.26}$$

To deal with D_{1n} and D_{2n}, we first show the following:

Claim 1: $\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}[||\mathbf{v}_{n,\tau}(x)||^2]=O(n^{-1})$, and

Claim 2: $\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}[||\mathbf{s}_{n,\tau}(x)||^2]=O(n^{-3/2}h^{-d})$.

Proof of Claim 1: First, note that

$$\sup_{(x,\tau)\in\mathcal{S}}\mathbf{E}\left[||\mathbf{v}_{n,\tau}(x)||^2\right]\le\mathbf{E}\left|\frac{n-N}{n}\right|^2\cdot\sup_{(x,\tau)\in\mathcal{S}}\left\|\frac{1}{h^d}\mathbf{E}\left[\beta_{n,x,\tau}\left(Y_i,\frac{X_i-x}{h}\right)\right]\right\|^2.$$

Since $\mathbf{E}|n^{-1/2}(n-N)|^2$ does not depend on the joint distribution of (Y_i,X_i), $\mathbf{E}|n^{-1/2}(n-N)|^2\le O(1)$ uniformly over $P\in\mathcal{P}$. Combining this with the second statement of (B.12), the product on the right hand side becomes $O(n^{-1})$ uniformly over $P\in\mathcal{P}$.

Proof of Claim 2: Let $\eta_1\in\mathbf{R}^J$ be the random vector defined prior to Lemma B6, and define

$$\mathbf{s}_{n,\tau}(x;\eta_1)\equiv\mathbf{s}_{n,\tau}(x)+\frac{(N-n)\eta_1}{n^{3/2}h^{d/2}}.$$

Note that

$$\mathbf{E}\left\|\mathbf{s}_{n,\tau}(x)\right\|^2\le 2\mathbf{E}\left\|\mathbf{s}_{n,\tau}(x;\eta_1)\right\|^2+\frac{2}{n^2h^d}\mathbf{E}\left\|\frac{(N-n)\eta_1}{\sqrt{n}}\right\|^2. \tag{B.27}$$

As for the last term, since N and η_1 are independent, it is bounded by

$$\frac{1}{n^2h^d}\left(\mathbf{E}\left|\frac{N-n}{\sqrt{n}}\right|^2\right)\cdot\mathbf{E}\left\|\eta_1\right\|^2\le\frac{C\bar{\varepsilon}}{n^2h^d}=O(n^{-2}h^{-d-\nu_1}),$$

from some large n on.

As for the leading expectation on the right hand side of (B.27), we write

$$\begin{aligned}\mathbf{E}\left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}(x;\eta_1)\right\|^2 &= \mathbf{E}\left\|\frac{1}{\sqrt{n}}\sum_{i=N+1}^{n} q_{n,\tau,1}^{(i)}(x)\right\|^2 \\ &= \frac{1}{n}\sum_{j=1}^{J}\bar{\sigma}_{n,\tau,j}^2(x)\mathbf{E}\left(\sum_{i=N+1}^{n}\frac{q_{n,\tau,1,j}^{(i)}(x)}{\bar{\sigma}_{n,\tau,j}(x)}\right)^2,\end{aligned}$$

where $q_{n,\tau,1}^{(i)}(x)$'s $(i = 1, 2, ...)$ are i.i.d. copies of $q_{n,\tau}(x) + \eta_1$ and $q_{n,\tau,1,j}^{(i)}(x)$ is the j-th entry of $q_{n,\tau,1}^{(i)}(x)$, and $\bar{\sigma}_{n,\tau,j}^2(x) \equiv Var(q_{n,\tau,1,j}^{(i)}(x))$. Recall that $q_{n,\tau}(x)$ was defined prior to Lemma B5. Now we apply Lemma 1(i) of Horváth (1991) to deduce that

$$\begin{aligned}&\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left(\sum_{i=N+1}^{n}\frac{q_{n,\tau,1,j}^{(i)}(x)}{\bar{\sigma}_{n,\tau,j}(x)}\right)^2 \\ \leq\ & \mathbf{E}|N-n|\cdot\mathbf{E}|\mathbb{Z}_1|^2 + C\mathbf{E}\left|N-n\right|^{1/2}\cdot\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left|\frac{q_{n,\tau,1,j}^{(i)}(x)}{\bar{\sigma}_{n,\tau,j}(x)}\right|^3 \\ &+C\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left|\frac{q_{n,\tau,1,j}^{(i)}(x)}{\bar{\sigma}_{n,\tau,j}(x)}\right|^4,\end{aligned}$$

for some $C > 0$, where $\mathbb{Z}_1 \sim N(0, 1)$.

First, observe that $\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\bar{\sigma}_{n,\tau,j}(x) < \infty$ by Lemma B5, and

$$\inf_{(x,\tau)\in\mathcal{S}}\inf_{P\in\mathcal{P}}\bar{\sigma}_{n,\tau,j}(x) > \bar{\varepsilon} > 0, \tag{B.28}$$

due to the additive term η_1 in $q_{n,\tau}(x)+\eta_1$. Let η_{1j} be the j-th entry of η_1. We apply Lemma B5 to deduce that for some $C > 0$, from some large n on,

$$\begin{aligned}\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}|(q_{n,\tau,j}(x)+\eta_{1j})/\bar{\sigma}_{n,\tau,j}(x)|^3 &\leq Ch^{-(d/2)-(\nu_1/2)} \text{ and} \\ \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}|(q_{n,\tau,j}(x)+\eta_{1j})/\bar{\sigma}_{n,\tau,j}(x)|^4 &\leq Ch^{-d-\nu_1}.\end{aligned} \tag{B.29}$$

Since $\mathbf{E}|N-n| = O(n^{1/2})$ and $\mathbf{E}\left|N-n\right|^{1/2} = O(n^{1/4})$ (e.g. (2.21) and (2.22) of Horváth (1991)), there exists $C > 0$ such that

$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left(\sum_{i=N+1}^{n}\frac{q_{n,\tau,1,j}^{(i)}(x)}{\bar{\sigma}_{n,\tau,j}(x)}\right)^2 \leq \frac{C}{\bar{\varepsilon}^4}\left\{n^{1/2}+n^{1/4}h^{-(d/2)-(\nu_1/2)}+h^{-d-\nu_1}\right\}. \tag{B.30}$$

This implies that for some $C > 0$, (with $\bar{\varepsilon} > 0$ fixed while $n \to \infty$)

$$\begin{aligned}
\text{(B.31)} \qquad & \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}(x)\right\|^2 \\
\leq \;& O\left(n^{-1}h^{-\nu_1}\right) + O\left(n^{-1/2} + n^{-3/4}h^{-(d/2)-(\nu_1/2)} + n^{-1}h^{-d-\nu_1}\right) \\
= \;& O\left(n^{-1}h^{-\nu_1}\right) + O(n^{-1/2}) = O(n^{-1/2}),
\end{aligned}$$

since $n^{-1/2}h^{-d-\nu_1} \to 0$. Hence, we obtain Claim 2.

Using Claim 1 and the second statement of Lemma B5, we deduce that

$$\begin{aligned}
\sup_{P\in\mathcal{P}} \mathbf{E}\left[n^{p/2}h^{d(p-1)/2}D_{1n}\right] \;\leq\;& C_1 Q(B_n) \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \sqrt{\mathbf{E}\left\|\sqrt{n}\mathbf{v}_{n,\tau}(x)\right\|^2} \\
& \times\sqrt{\mathbf{E}\left\|\sqrt{nh^d}\mathbf{z}_{n,\tau}(x)\right\|^{2p-2} + \mathbf{E}\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)\right\|^{2p-2}} \\
\leq\;& C_2 Q(B_n),
\end{aligned}$$

for $C_1, C_2 > 0$. Similarly, we can see that

$$\sup_{P\in\mathcal{P}} \mathbf{E}\left[n^{p/2}h^{d(p-1)/2}D_{2n}\right] = O(n^{-1/2}h^{-d}) = o(1),$$

using Claim 2 and the second statement of Lemma B5. Thus, we obtain Step 1.

Proof of Step 2: We can follow the proof of Lemma B6 to show that

$$\begin{aligned}
& \mathbf{E}\left[h^{-d/2}\int_{B_n}\left(\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)) - \mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right]\right)dQ(x,\tau)\right]^2 \\
=\;& \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}\cap B_{n,\tau_2}}\int_{\mathcal{U}} C_{n,\tau_1,\tau_2,A,A'}(x,u)dudxd\tau_1 d\tau_2 + o(1),
\end{aligned}$$

where $B_{n,\tau}$ is the τ-section of B_n defined at the beginning of the proof of Lemma B6, $C_{n,\tau_1,\tau_2,A,A'}(x,u)$ is defined in (B.14), and the last $o(1)$ term is $o(1)$ uniform over $P \in \mathcal{P}$. Now, observe that

$$\begin{aligned}
& \sup_{(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}} \sup_{u\in\mathcal{U}} \sup_{x\in\mathcal{X}} \sup_{P\in\mathcal{P}} |C_{n,\tau_1,\tau_2,A,A'}(x,u)| \\
\leq\;& \sup_{(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}} \sup_{u\in\mathcal{U}} \sup_{x\in\mathcal{X}} \sup_{P\in\mathcal{P}} \sqrt{\mathbf{E}||\mathbb{W}^{(1)}_{n,\tau_1,\tau_2}(x,u)||^{2p}\mathbf{E}||\mathbb{W}^{(2)}_{n,\tau_1,\tau_2}(x,u)||^{2p}} < \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\left[\left|h^{-d/2}\int_{B_n}\left(\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))-\mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right]\right)dQ(x,\tau)\right|\right]\\
&\leq \sqrt{\int_{\mathcal{T}}\int_{\mathcal{T}}\int_{\mathcal{U}}\int_{B_{n,\tau_1}\cap B_{n,\tau_2}} C_{n,\tau_1,\tau_2,A,A'}(x,u)dxdud\tau_1 d\tau_2}+o(1)\\
&\leq C\sqrt{\int_{\mathcal{T}}\int_{\mathcal{T}}\int_{\mathcal{U}}\int_{B_{n,\tau_1}\cap B_{n,\tau_2}} dxdud\tau_1 d\tau_2}+o(1),
\end{aligned}$$

for some $C>0$. Now, observe that

$$\int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}\cap B_{n,\tau_2}} dxd\tau_1 d\tau_2 \leq \int_{\mathcal{T}} d\tau_2\cdot\left(\int_{\mathcal{T}}\int_{B_{n,\tau_1}} dxd\tau_1\right)\leq CQ(B_n),$$

because $\mathcal{T}$ is a bounded set. Thus the proof of Step 2 is completed. ∎

The next lemma shows the joint asymptotic normality of a Poissonized version of a normalized test statistic and a Poisson random variable. Using this result, we can apply the de-Poissonization lemma in Lemma B3. To define a Poissonized version of a normalized test statistic, we introduce some notation.

Let $\mathcal{C}\subset\mathbf{R}^d$ be a compact set such that $\mathcal{C}$ does not depend on $P\in\mathcal{P}$ and $\alpha_P\equiv P\{X\in\mathbf{R}^d\backslash\mathcal{C}\}$ satisfies that $0<\inf_{P\in\mathcal{P}}\alpha_P\leq\sup_{P\in\mathcal{P}}\alpha_P<1$. Existence of such $\mathcal{C}$ is assumed in Assumption A6(ii). For $c_n\to\infty$, we let $B_{n,A}(c_n;\mathcal{C})\equiv B_{n,A}(c_n)\cap(\mathcal{C}\times\mathcal{T})$, where we recall the definition of $B_{n,A}(c_n)=B_{n,A}(c_n,c_n)$. (Recall the definition of $B_{n,A}(c_{n,1},c_{n,2})$ before Lemma 1.) Define

$$\begin{aligned}
\zeta_{n,A} &\equiv \int_{B_{n,A}(c_n;\mathcal{C})}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x))dQ(x,\tau),\text{ and}\\
\zeta_{N,A} &\equiv \int_{B_{n,A}(c_n;\mathcal{C})}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))dQ(x,\tau).
\end{aligned}$$

Let μ_A's be real numbers indexed by $A\in\mathcal{N}_J$, and define

$$\sigma_n^2(\mathcal{C})\equiv\sum_{A\in\mathcal{N}_J}\sum_{A'\in\mathcal{N}_J}\mu_A\mu_{A'}\sigma_{n,A,A'}(B_{n,A}(c_n;\mathcal{C}),B_{n,A'}(c_n;\mathcal{C})),$$

where we recall the definition of $\sigma_{n,A,A'}(\cdot,\cdot)$ prior to Lemma B6. Define

$$S_n\equiv h^{-d/2}\sum_{A\in\mathcal{N}_J}\mu_A\left\{\zeta_{N,A}-\mathbf{E}\zeta_{N,A}\right\}.$$

Also define

$$U_n \equiv \frac{1}{\sqrt{n}}\left\{\sum_{i=1}^{N} 1\{X_i \in \mathcal{C}\} - nP\left\{X_i \in \mathcal{C}\right\}\right\}, \text{ and}$$
$$V_n \equiv \frac{1}{\sqrt{n}}\left\{\sum_{i=1}^{N} 1\{X_i \in \mathbf{R}^d \backslash \mathcal{C}\} - nP\left\{X_i \in \mathbf{R}^d \backslash \mathcal{C}\right\}\right\}.$$

Let

$$H_n \equiv \left[\frac{S_n}{\sigma_n(\mathcal{C})}, \frac{U_n}{\sqrt{1-\alpha_P}}\right]^\top.$$

The following lemma establishes the joint convergence of H_n. In doing so, we need to be careful in dealing with uniformity in $P \in \mathcal{P}$, and potential degeneracy of the normalized test statistic S_n.

Lemma B8. *Suppose that the conditions of Lemma B7 hold and that $c_n \to \infty$ as $n \to \infty$.*
(i) If $\liminf_{n\to\infty} \inf_{P\in\mathcal{P}} \sigma_n^2(\mathcal{C}) > 0$, then

$$\sup_{P\in\mathcal{P}} \sup_{t\in\mathbf{R}^2} \left|P\left\{H_n \le t\right\} - P\left\{\mathbb{Z} \le t\right\}\right| \to 0,$$

where $\mathbb{Z} \sim N(0, I_2)$.
(ii) If $\limsup_{n\to\infty} \sigma_n^2(\mathcal{C}) = 0$, then for each $(t_1, t_2) \in \mathbf{R}^2$,

$$\left|P\left\{S_n \le t_1 \text{ and } \frac{U_n}{\sqrt{1-\alpha_P}} \le t_2\right\} - 1\{0 \le t_1\}P\left\{\mathbb{Z}_1 \le t_2\right\}\right| \to 0,$$

where $\mathbb{Z}_1 \sim N(0, 1)$.

Remark 5. The joint convergence result in Lemma B8 is divided into two separate results. The first case is a situation where S_n is asymptotically nondegenerate uniformly in $P \in \mathcal{P}$. The second case deals with a situation where S_n is asymptotically degenerate for some $P \in \mathcal{P}$.

Proof of Lemma B8. (i) Choose any small $\bar{\varepsilon} > 0$ and let

$$H_{n,\bar{\varepsilon}} \equiv \left[\frac{S_{n,\bar{\varepsilon}}}{\sigma_{n,\bar{\varepsilon}}(\mathcal{C})}, \frac{U_n}{\sqrt{1-\alpha_P}}\right]^\top,$$

where $S_{n,\bar{\varepsilon}}$ is equal to S_n, except that $\zeta_{N,A}$ is replaced by

$$\zeta_{N,A,\bar{\varepsilon}} \equiv \int_{B_{n,A}(c_n;\mathcal{C})} \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x;\eta_1))dQ(x,\tau),$$

and $\mathbf{z}_{N,\tau}(x;\eta_1)$ is as defined prior to Lemma B6, and $\sigma_{n,\bar{\varepsilon}}(\mathcal{C})$ is $\sigma_n(\mathcal{C})$ except that $\tilde{\Sigma}_{n,\tau_1,\tau_2}(x,u)$ is replaced by $\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)$. Also let

$$C_n \equiv \mathbf{E}H_nH_n^\top \text{ and } C_{n,\bar{\varepsilon}} \equiv \mathbf{E}H_{n,\bar{\varepsilon}}H_{n,\bar{\varepsilon}}^\top.$$

First, we show the following statements.

Step 1: For some $C > 0$, $\sup_{P\in\mathcal{P}} |Cov(S_{n,\bar{\varepsilon}} - S_n, U_n)| \leq C\sqrt{\bar{\varepsilon}}$, for each fixed $\bar{\varepsilon} > 0$.

Step 2: $\sup_{P\in\mathcal{P}} |Cov(S_{n,\bar{\varepsilon}}, U_n)| = o(h^{d/2})$, as $n \to \infty$.

Step 3: There exists $c > 0$ such that from some large n on,

$$\inf_{P\in\mathcal{P}} \lambda_{\min}(C_n) > c.$$

Step 4: As $n \to \infty$,

$$\sup_{P\in\mathcal{P}} \sup_{t\in\mathbf{R}^2} \left| P\left\{ C_n^{-1/2} H_n \leq t \right\} \to P\left\{ \mathbb{Z} \leq t \right\} \right| \to 0.$$

From Steps 1-3, we find that $\sup_{P\in\mathcal{P}} \|C_n - I_2\| \to 0$, as $n \to \infty$ and as $\bar{\varepsilon} \to 0$. By Step 4, we obtain (i) of Lemma B8.

Proof of Step 1: Observe that from an inequality similar to (B.26) in the proof of Lemma B7,

$$|\zeta_{N,A,\bar{\varepsilon}} - \zeta_{N,A}| \leq C||\eta_1|| \int_{B_{n,A}(c_n;\mathcal{C})} \left\| \sqrt{nh^d} \mathbf{z}_{N,\tau}(x) \right\|^{p-1} dQ(x,\tau).$$

Using the fact that $\mathcal{S}$ is compact and does not depend on $P \in \mathcal{P}$, for some constants $C_1, C_2, C_3 > 0$ that do not depend on $P \in \mathcal{P}$ or n,

$$\begin{aligned} \mathbf{E}\left|\zeta_{N,A,\bar{\varepsilon}} - \zeta_{N,A}\right|^2 &\leq C_1 \mathbf{E}\left[||\eta_1||^2\right] \cdot \int_{B_{n,A}(c_n;\mathcal{C})} \mathbf{E}\left\| \sqrt{nh^d} \mathbf{z}_{N,\tau}(x) \right\|^{2p-2} dQ(x,\tau) \\ &\leq C_2 \bar{\varepsilon} \cdot \int_{B_{n,A}(c_n;\mathcal{C})} \mathbf{E}\left\| \sqrt{nh^d} \mathbf{z}_{N,\tau}(x) \right\|^{2p-2} dQ(x,\tau) \leq C_3 \bar{\varepsilon}, \end{aligned}$$

by the independence between η_1 and $\{\mathbf{z}_{N,\tau}(x) : (x,\tau) \in \mathcal{S}\}$, and by the second statement of Lemma B5. From the fact that

$$\sup_{P\in\mathcal{P}} \mathbf{E}U_n^2 \leq \sup_{P\in\mathcal{P}} (1 - \alpha_P) \leq 1,$$

we obtain the desired result.

Proof of Step 2: Let $\Sigma_{2n,\tau,\bar{\varepsilon}}$ be the covariance matrix of $[(q_{n,\tau}(x) + \eta_1)^\top, \tilde{U}_n]^\top$, where $\tilde{U}_n = U_n/\sqrt{P\{X \in \mathcal{C}\}}$ and $q_{n,\tau}(x)$ was defined prior to Lemma B5. We can write $\Sigma_{2n,\tau,\bar{\varepsilon}}$ as

$$\begin{aligned} &\left[\begin{array}{cc} \Sigma_{n,\tau,\tau}(x,0) + \bar{\varepsilon} I_J & \mathbf{E}[(q_{n,\tau}(x) + \eta_1)\tilde{U}_n] \\ \mathbf{E}[(q_{n,\tau}(x) + \eta_1)^\top \tilde{U}_n] & 1 \end{array} \right] \\ = &\left[\begin{array}{cc} \Sigma_{n,\tau,\tau}(x,0) & \sqrt{1-\bar{\varepsilon}}\mathbf{E}[q_{n,\tau}(x)\tilde{U}_n] \\ \sqrt{1-\bar{\varepsilon}}\mathbf{E}[q_{n,\tau}^\top(x)\tilde{U}_n] & 1 - \bar{\varepsilon} \end{array} \right] + \left[\begin{array}{cc} \bar{\varepsilon} I_J & \mathbf{0} \\ \mathbf{0}^\top & \bar{\varepsilon} \end{array} \right] + A_{n,\tau}(x), \end{aligned}$$

where

$$A_{n,\tau}(x) \equiv \begin{bmatrix} \mathbf{0} & \left(1-\sqrt{1-\bar{\varepsilon}}\right)\mathbf{E}[q_{n,\tau}(x)\tilde{U}_n] \\ \left(1-\sqrt{1-\bar{\varepsilon}}\right)\mathbf{E}[q_{n,\tau}^\top(x)\tilde{U}_n] & \mathbf{0} \end{bmatrix}.$$

The first matrix on the right hand side is certainly positive semidefinite. Note that

$$\left(q_{n,\tau,j}(x), \tilde{U}_n\right) \overset{d}{=} \left(\frac{1}{\sqrt{n}}\sum_{k=1}^n q_{n,\tau,j}^{(k)}(x), \frac{1}{\sqrt{n}}\sum_{k=1}^n \tilde{U}_n^{(k)}\right),$$

where $(q_{n,\tau,j}^{(k)}(x), \tilde{U}_n^{(k)})$'s with $k = 1, ..., n$ are i.i.d. copies of $(q_{n,\tau,j}(x), \bar{U}_n)$, where

$$\bar{U}_n \equiv \frac{1}{\sqrt{P\{X \in \mathcal{C}\}}}\left\{\sum_{1\le i\le N_1} 1\{X_i \in \mathcal{C}\} - P\left\{X_i \in \mathcal{C}\right\}\right\},$$

where N_1 is the Poisson random variable with mean 1 that is involved in the definition of $q_{n,\tau,j}(x)$. Hence as for $A_{n,\tau}(x)$, note that for $C_1, C_2 > 0$,

$$\text{(B.32)} \qquad \begin{aligned} \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\left|\mathbf{E}\left[q_{n,\tau,j}(x)\tilde{U}_n\right]\right| &\le \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\left|\mathbf{E}\left[q_{n,\tau,j}^{(k)}(x)\tilde{U}_n^{(k)}\right]\right| \\ &\le \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\frac{\mathbf{E}\left[|q_{n,\tau,j}(x)|\right]}{\sqrt{P\{X_i \in \mathcal{C}\}}} \\ &\le \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\frac{C_1 h^d k_{n,\tau,j,1}(x)}{h^{d/2}\left(1-\alpha_P\right)} \le C_2 h^{d/2}, \end{aligned}$$

where $k_{n,\tau,j,1}(x)$ was defined prior to Lemma B4. We conclude that

$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}||A_{n,\tau}(x)|| = O(h^{d/2}).$$

Therefore, from some large n on,

$$\text{(B.33)} \qquad \inf_{(x,\tau)\in\mathcal{S}}\inf_{P\in\mathcal{P}}\lambda_{\min}\left(\Sigma_{2n,\tau,\bar{\varepsilon}}\right) \ge \bar{\varepsilon}/2.$$

Let

$$W_{n,\tau}(x;\eta_1) \equiv \Sigma_{2n,\tau,\bar{\varepsilon}}^{-1/2}\begin{bmatrix} q_{n,\tau}(x)+\eta_1 \\ \tilde{U}_n \end{bmatrix}.$$

Similarly as in (B.22), we find that for some $C > 0$, from some large n on,

$$\begin{aligned} &\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left\|W_{n,\tau}(x;\eta_1)\right\|^3 \\ \le\ & C\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\lambda_{\max}^{3/2}\left(\Sigma_{2n,\tau,\bar{\varepsilon}}^{-1}\right)\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\left\{\mathbf{E}\left[||q_{n,\tau}(x)+\eta_1||^3\right] + \mathbf{E}\left[|\tilde{U}_n|^3\right]\right\} \\ \le\ & C\left(\frac{\bar{\varepsilon}}{2}\right)^{-3/2}\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\left\{\mathbf{E}\left[||q_{n,\tau}(x)+\eta_1||^3\right] + \mathbf{E}\left[|\tilde{U}_n|^3\right]\right\}, \end{aligned}$$

where the last inequality uses (B.33). As for the last expectation, note that by Rosenthal's inequality, we have

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left[|\tilde{U}_n|^3\right] \le C$$

for some $C > 0$. We apply the first statement of Lemma B5 to conclude that

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left\|W_{n,\tau}(x;\eta_1)\right\|^3 \le C\bar{\varepsilon}^{-3/2}h^{-d/2},$$

for some $C > 0$. For any vector $\mathbf{v} = [\mathbf{v}_1^\top, v_2]^\top \in \mathbf{R}^{J+1}$, we define

$$D_{n,\tau,p}(\mathbf{v}) \equiv \Lambda_{A,p}\left(\left[\Sigma_{2n,\tau,\bar{\varepsilon}}^{1/2}\mathbf{v}\right]_1\right)\left[\Sigma_{2n,\tau,\bar{\varepsilon}}^{1/2}\mathbf{v}\right]_2,$$

where $[a]_1$ of a vector $a \in \mathbf{R}^{J+1}$ indicates the vector of the first J entries of a, and $[a]_2$ the last entry of a. By Theorem 1 of Sweeting (1977), we find that (with $\bar{\varepsilon} > 0$ fixed)

$$\mathbf{E}\left[D_{n,\tau,p}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{n,\tau}^{(i)}(x;\eta_1)\right)\right] = \mathbf{E}\left[D_{n,\tau,p}\left(\mathbb{Z}_{J+1}\right)\right] + O(n^{-1/2}h^{-d/2}) = o(n^{d/2}),$$

where $\mathbb{Z}_{J+1} \sim N(0, I_{J+1})$ and $W_{n,\tau}^{(i)}(x;\eta_1)$'s are i.i.d. copies of $W_{n,\tau}(x;\eta_1)$. The last equality follows because $n^{-1/2}h^{-d/2} = o(h^{d/2})$ (by the condition that $n^{-1/2}h^{-d-\nu_1} \to 0$ for some small $\nu_1 > 0$ as $n \to \infty$) and $\mathbf{E}[D_{n,\tau,p}\left(\mathbb{Z}_{J+1}\right)] = 0$. Since

$$Cov\left(\Lambda_{A,p}\left(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x;\eta_1)\right), U_n\right) = \mathbf{E}\left[D_{n,\tau,p}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{n,\tau}^{(i)}(x;\eta_1)\right)\right],$$

we conclude that

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \left|Cov\left(\Lambda_{A,p}\left(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x;\eta_1)\right), U_n\right)\right| = o(h^{d/2}).$$

By applying the Dominated Convergence Theorem, we obtain Step 2.

Proof of Step 3: First, we show that

$$Var\left(S_n\right) = \sigma_n^2(\mathcal{C}) + o(1), \tag{B.34}$$

where $o(1)$ is an asymptotically negligible term uniformly over $P \in \mathcal{P}$. Note that

$$Var\left(S_n\right) = \sum_{A\in\mathcal{N}_J}\sum_{A'\in\mathcal{N}_J} \mu_A\mu_{A'}Cov(\psi_{n,A}, \psi_{n,A'}),$$

where $\psi_{n,A} \equiv h^{-d/2}(\zeta_{N,A} - \mathbf{E}\zeta_{N,A})$. By Lemma B6, we find that for $A, A' \in \mathcal{N}_J$,

$$Cov(\psi_{n,A}, \psi_{n,A'}) = \sigma_{n,A,A'}(B_{n,A}(c_n;\mathcal{C}), B_{n,A'}(c_n;\mathcal{C})) + o(1),$$

uniformly in $P \in \mathcal{P}$, yielding the desired result.

Combining Steps 1 and 2, we deduce that

$$\text{(B.35)} \qquad \sup_{P\in\mathcal{P}} |Cov(S_n, U_n)| \le C\sqrt{\bar{\varepsilon}} + o(h^{d/2}).$$

Let $\bar{\sigma}_1^2 \equiv \inf_{P\in\mathcal{P}} \sigma_n^2(\mathcal{C})$ and $\bar{\sigma}_2^2 \equiv \inf_{P\in\mathcal{P}}(1-\alpha_P)$. Note that for some $C_1 > 0$,

$$\text{(B.36)} \qquad \inf_{P\in\mathcal{P}} \bar{\sigma}_1^2\bar{\sigma}_2^2 > C_1,$$

by the condition of the lemma. A simple calculation gives us

$$\begin{aligned}
\text{(B.37)} \quad \lambda_{\min}(C_n) &= \frac{\bar{\sigma}_1^2+\bar{\sigma}_2^2}{2} - \frac{1}{2}\left(\sqrt{(\bar{\sigma}_1^2+\bar{\sigma}_2^2)^2 - 4\left\{\bar{\sigma}_1^2\bar{\sigma}_2^2 - Cov(S_n,U_n)^2\right\}}\right) \\
&\ge \frac{1}{2}\left\{\sqrt{(\bar{\sigma}_1^2+\bar{\sigma}_2^2)^2} - \left(\sqrt{(\bar{\sigma}_1^2+\bar{\sigma}_2^2)^2 - 4\bar{\sigma}_1^2\bar{\sigma}_2^2}\right)\right\} - |Cov(S_n,U_n)| \\
&\ge \bar{\sigma}_1^2\bar{\sigma}_2^2 - |Cov(S_n,U_n)| \ge C_1 - C\sqrt{\bar{\varepsilon}} + o(h^{d/2}),
\end{aligned}$$

where the last inequality follows by (B.35) and (B.36). Taking $\bar{\varepsilon}$ small enough, we obtain the desired result.

Proof of Step 4: Suppose that $\liminf_{n\to\infty}\inf_{P\in\mathcal{P}}\sigma_n^2(\mathcal{C}) > 0$. Let κ be the diameter of the compact set $\mathcal{K}_0$ introduced in Assumption A2. Let $\mathcal{C}$ be given as in the lemma. Let $\mathbb{Z}^d$ be the set of d-tuples of integers, and let $\{R_{n,\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^d\}$ be the collection of rectangles in $\mathbf{R}^d$ such that $R_{n,\mathbf{i}} = [a_{n,\mathbf{i}_1}, b_{n,\mathbf{i}_1}] \times ... \times [a_{n,\mathbf{i}_d}, b_{n,\mathbf{i}_d}]$, where $\mathbf{i}_j$ is the j-th entry of $\mathbf{i}$, and $h\kappa \le b_{n,\mathbf{i}_j} - a_{n,\mathbf{i}_j} \le 2h\kappa$, for all $j = 1, ..., d$, and two different rectangles $R_{n,\mathbf{i}}$ and $R_{n,\mathbf{j}}$ do not have intersection with nonempty interior, and the union of the rectangles $R_{n,\mathbf{i}}$, $\mathbf{i} \in \mathbb{Z}_n^d$, cover $\mathcal{X}$ from some sufficiently large n. Here, $\mathbb{Z}_n^d$ is the set of d-tuples of integers whose absolute values are less than or equal to n.

We let

$$\begin{aligned}
B_{n,A,x}(c_n) &\equiv \{\tau \in \mathcal{T} : (x,\tau) \in B_A(c_n)\}, \\
B_{n,\mathbf{i}} &\equiv R_{n,\mathbf{i}} \cap \mathcal{C},
\end{aligned}$$

and $\mathcal{I}_n \equiv \{\mathbf{i} \in \mathbb{Z}_n^d : B_{n,\mathbf{i}} \neq \varnothing\}$. Then $B_{n,\mathbf{i}}$ has Lebesgue measure $m(B_{n,\mathbf{i}})$ bounded by C_1h^d and the cardinality of the set $\mathcal{I}_n$ is bounded by C_2h^{-d} for some positive constants C_1 and C_2. Now let us define

$$\Delta_{n,A,\mathbf{i}} \equiv h^{-d/2}\int_{B_{n,\mathbf{i}}}\int_{B_{n,A,x}(c_n)}\left\{\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x)) - \mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right]\right\}d\tau dx.$$

And also define $B_{n,A,\mathbf{i}}(c_n) \equiv (B_{n,\mathbf{i}} \times \mathcal{T}) \cap B_{n,A}(c_n)$,

$$\begin{aligned} \alpha_{n,\mathbf{i}} &\equiv \frac{\sum_{A \in \mathcal{N}_J} \mu_A \Delta_{n,A,\mathbf{i}}}{\sigma_n(\mathcal{C})} \text{ and} \\ u_{n,\mathbf{i}} &\equiv \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{N} 1\{X_i \in B_{n,\mathbf{i}}\} - nP\{X_i \in B_{n,\mathbf{i}}\} \right\}. \end{aligned}$$

Then, we can write

$$\frac{S_n}{\sigma_n(\mathcal{C})} = \sum_{\mathbf{i} \in \mathcal{I}_n} \alpha_{n,\mathbf{i}} \text{ and } U_n = \sum_{\mathbf{i} \in \mathcal{I}_n} u_{n,\mathbf{i}}.$$

By the definition of $\mathcal{K}_0$ in Assumption A2, by the definition of $R_{n,\mathbf{i}}$ and by the properties of Poisson processes, one can see that the array $\{(\alpha_{n,\mathbf{i}}, u_{n,\mathbf{i}})\}_{\mathbf{i} \in \mathcal{I}_n}$ is an array of 1-dependent random field. (See Mason and Polonik (2009) for details.) For any $q_1, q_2 \in \mathbf{R}$, let $y_{n,\mathbf{i}} \equiv q_1 \alpha_{n,\mathbf{i}} + q_2 u_{n,\mathbf{i}}$. The focus is on the convergence in distribution of $\sum_{\mathbf{i} \in \mathcal{I}_n} y_{n,\mathbf{i}}$ uniform over $P \in \mathcal{P}$. Without loss of generality, we choose $q_1, q_2 \in \mathbf{R} \backslash \{0\}$. Define

$$Var_P \left(\sum_{\mathbf{i} \in \mathcal{I}_n} y_{n,\mathbf{i}} \right) = q_1^2 + q_2^2 (1 - \alpha_P) + 2 q_1 q_2 c_{n,P},$$

uniformly over $P \in \mathcal{P}$, where $c_{n,P} = Cov(S_n, U_n)$. On the other hand, using Lemma B4 and following the proof of Lemma A8 of Lee, Song, and Whang (2013), we deduce that

$$\sup_{P \in \mathcal{P}} \sum_{\mathbf{i} \in \mathcal{I}_n} \mathbf{E} |y_{n,\mathbf{i}}|^r = o(1) \tag{B.38}$$

as $n \to \infty$, for any $r \in (2, (2p+2)/p]$. By Theorem 1 of Shergin (1993), we have

$$\begin{aligned} & \sup_{P \in \mathcal{P}} \sup_{t \in \mathbf{R}} \left| P \left\{ \frac{1}{\sqrt{q_1^2 + q_2^2(1 - \alpha_P) + 2 q_1 q_2 c_{n,P}}} \sum_{\mathbf{i} \in \mathcal{I}_n} y_{n,\mathbf{i}} \leq t \right\} - \Phi(t) \right| \\ \leq \; & \sup_{P \in \mathcal{P}} \frac{C}{\{q_1^2 + q_2^2(1 - \alpha_P) + 2 q_1 q_2 c_{n,P}\}^{r/2}} \left\{ \sum_{\mathbf{i} \in \mathcal{I}_n} \mathbf{E} |y_{n,\mathbf{i}}|^r \right\}^{1/2} = o(1), \end{aligned}$$

for some $C > 0$, by (B.38). Therefore, by Lemma B2(i), we have for each $t \in \mathbf{R}$, and each $q \in \mathbf{R}^2 \backslash \{0\}$, as $n \to \infty$,

$$\sup_{P \in \mathcal{P}} \left| \mathbf{E} \left[\exp \left(it \frac{q^\top H_n}{\sqrt{q^\top C_n q}} \right) \right] - \exp \left(- \frac{t^2}{2} \right) \right| \to 0.$$

Thus by Lemma B2(ii), for each $t \in \mathbf{R}^2$, we have

$$\sup_{P \in \mathcal{P}} \left| P \left\{ C_n^{-1/2} H_n \leq t \right\} - P \{ \mathbb{Z} \leq t \} \right| \to 0.$$

Since the limit distribution of $C_n^{-1/2}H_n$ is continuous, the convergence above is uniform in $t \in \mathbf{R}^2$.

(ii) We fix $P \in \mathcal{P}$ such that $\limsup_{n\to\infty}\sigma_n^2(\mathcal{C}) = 0$. Then by (B.34) above,

$$Var\,(S_n) = \sigma_n^2(\mathcal{C}) + o(1) = o(1).$$

Hence, we find that $S_n = o_P(1)$. The desired result follows by applying Theorem 1 of Shergin (1993) to the sum $U_n = \sum_{\mathbf{i}\in\mathcal{I}_n} u_{n,\mathbf{i}}$, and then applying Lemma B2(ii). ∎

Lemma B9. *Let $\mathcal{C}$ be the Borel set in Lemma B8.*
(i) Suppose that the conditions of Lemma B8(i) are satisfied. Then as $n \to \infty$,

$$\sup_{P\in\mathcal{P}}\sup_{t\in\mathbf{R}}\left|P\left\{\frac{h^{-d/2}\sum_{A\in\mathcal{N}_J}\mu_A\left\{\zeta_{n,A} - \mathbf{E}\zeta_{N,A}\right\}}{\sigma_n(\mathcal{C})} \le t\right\} - \Phi(t)\right| \to 0.$$

(ii) Suppose that the conditions of Lemma B8(ii) are satisfied. Then as $n \to \infty$,

$$h^{-d/2}\sum_{A\in\mathcal{N}_J}\mu_A\left\{\zeta_{n,A} - \mathbf{E}\zeta_{N,A}\right\} \overset{p}{\to} 0.$$

Note that in both statements, the location normalization has $\mathbf{E}\zeta_{N,A}$ instead of $\mathbf{E}\zeta_{n,A}$.

Proof of Lemma B9. (i) The conditional distribution of $S_n/\sigma_n(\mathcal{C})$ given $N = n$ is equal to that of

$$\frac{\sum_{A\in\mathcal{N}_J}\mu_A\int_{B_{n,A}(c_n;\mathcal{C})\cap\mathcal{C}}\left\{\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{n,\tau}(x)) - \mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x))\right\}dQ(x,\tau)}{h^{d/2}\sigma_n(\mathcal{C})}.$$

Using Lemmas B3(i) and B8(i), we find that

$$\frac{h^{-d/2}\sum_{A\in\mathcal{N}_J}\mu_A\left\{\zeta_{n,A} - \mathbf{E}\zeta_{N,A}\right\}}{\sigma_n(\mathcal{C})} \overset{d}{\to} N(0,1).$$

Since the limit distribution $N(0,1)$ is continuous and the convergence is uniform in $P \in \mathcal{P}$, we obtain the desired result.
(ii) Similarly as before, the result follows from Lemmas B3(ii), B2(ii), and B8(ii). ∎

Appendix C. Proofs of Auxiliary Results for Lemmas A2(ii), Lemma A4(ii), and Theorem 1

The auxiliary results in this section are mostly bootstrap versions of the results in Appendix B. To facilitate comparison, we name the first lemma to be Lemma C3, which is used to control the discrepancy between the sample version of the scale normalizer σ_n, and its population version. Then we proceed to prove Lemmas C4-C9 which run in parallel with Lemmas B4-B9 as their bootstrap counterparts. We finish this subsection with Lemmas

C10-C12 which are crucial for dealing with the bootstrap test statistic's location normalization. More specifically, Lemmas C10 and C11 are auxiliary moment bound results that are used for proving Lemma C12. Lemma C12 essentially delivers the result of Lemma A1 in Appendix A. This lemma is used to deal with the discrepancy between the population location normalizer and the sample location normalizer. Controlling this discrepancy to the rate $o_P(h^{d/2})$ is crucial for our purpose, because the bootstrap test statistic that is proposed here does not involve the sample version of the location normalizer a_n for the sake of computational expediency. Lemmas C10 and C11 provide necessary moment bounds to achieve this convergence rate.

Let the random variables N and N_1 represent Poisson random variables with mean n and 1 respectively. These random variables are taken to be independent of $((Y_i^*, X_i^*)_{i=1}^{\infty}, (Y_i, X_i)_{i=1}^{\infty})$. Let η_1 and η_2 be centered normal random vectors that are independent of each other and independent of

$$((Y_i^*, X_i^*)_{i=1}^{\infty}, (Y_i, X_i)_{i=1}^{\infty}, N, N_1) .$$

We will specify their covariance matrices in the proofs below. Throughout the proofs, the bootstrap distribution P^* and expectations $\mathbf{E}^*$ are viewed as the distribution of

$$((Y_i^*, X_i^*)_{i=1}^{n}, N, N_1, \eta_1, \eta_2) ,$$

conditional on $(Y_i, X_i)_{i=1}^{n}$.

Define

$$\begin{aligned} \tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u) &\equiv \frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau_1,j}\left(Y_{ij}^*, \frac{X_i^*-x}{h}\right)\beta_{n,x,\tau_2,k}\left(Y_{ik}^*, \frac{X_i^*-x}{h}+u\right)\right] \text{ and} \\ \tilde{k}_{n,\tau,j,m}(x) &\equiv \frac{1}{h^d}\mathbf{E}^*\left[\left|\beta_{n,x,\tau,j}\left(Y_{ij}^*, \frac{X_i^*-x}{h}\right)\right|^m\right]. \end{aligned}$$

Note that $\tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u)$ and $\tilde{k}_{n,\tau,j,m}(x)$ are bootstrap versions of $\rho_{n,\tau_1,\tau_2,j,k}(x,u)$ and $\tilde{k}_{n,\tau,j,m}(x)$. The lemma below establishes that the bootstrap version $\tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u)$ is consistent for $\rho_{n,\tau_1,\tau_2,j,k}(x,u)$.

Lemma C3. *Suppose that Assumption A6(i) holds and that $n^{-1/2}h^{-d/2} \to 0$, as $n \to \infty$. Then for each $\varepsilon \in (0, \varepsilon_1)$, with $\varepsilon_1 > 0$ as in Assumption A6(i), as $n \to \infty$,*

$$\sup_{(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}} \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} \mathbf{E}\left(|\tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u) - \rho_{n,\tau_1,\tau_2,j,k}(x,u)|^2\right) \to 0.$$

Proof of Lemma C3. Define $\pi_{n,x,u,\tau_1,\tau_2}(y,z) = \beta_{n,x,\tau_1,j}(y_j,(z-x)/h)\beta_{n,x,\tau_2,k}(y_k,(z-x)/h+u)$ for $y = (y_1, ..., y_J)^\top \in \mathbf{R}^J$, and write

$$\tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u) - \rho_{n,\tau_1,\tau_2,j,k}(x,u) = \frac{1}{nh^d}\sum_{i=1}^{n}\left\{\pi_{n,x,u,\tau_1,\tau_2}(Y_i,X_i) - \mathbf{E}\left[\pi_{n,x,u,\tau_1,\tau_2}(Y_i,X_i)\right]\right\}.$$

First, we note that

$$\mathbf{E}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\pi_{n,x,u,\tau_1,\tau_2}(Y_i,X_i)-\mathbf{E}\left[\pi_{n,x,u,\tau_1,\tau_2}(Y_i,X_i)\right]\right\}\right)^2 \leq \mathbf{E}\left[\pi^2_{n,x,u,\tau_1,\tau_2}(Y_i,X_i)\right].$$

By change of variables and Assumption A6(i), we have $\mathbf{E}\left[\pi^2_{n,x,u,\tau_1,\tau_2}(Y_i,X_i)\right] = O(h^d)$ uniformly over $(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}$, $(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}$ and over $P\in\mathcal{P}$. Hence

$$\mathbf{E}\left(|\tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u)-\rho_{n,\tau_1,\tau_2,j,k}(x,u)|^2\right) = O\left(n^{-1}h^{-d}\right),$$

uniformly over $(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}$, $(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}$ and over $P\in\mathcal{P}$. Since we have assumed that $n^{-1/2}h^{-d/2}\to 0$ as $n\to\infty$, we obtain the desired result. ∎

Lemma C4. *Suppose that Assumption A6(i) holds and that for some $C>0$,*

$$\limsup_{n\to\infty} n^{-1/2}h^{-d/2} \leq C.$$

Then for all $m\in[2,M]$ and all $\varepsilon\in(0,\varepsilon_1)$, with $M>2$ and $\varepsilon_1>0$ being the constants that appear in Assumption A6(i)), there exists $C_1\in(0,\infty)$ that does not depend on n such that for each $j\in\mathbb{N}_J$,

$$\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\tilde{k}^2_{n,\tau,j,m}(x)\right]\leq C_1.$$

Proof of Lemma C4. Since $\mathbf{E}^*[|\beta_{n,x,\tau,j}(Y^*_{ij},(X^*_i-x)/h)|^m]=\frac{1}{n}\sum_{i=1}^n|\beta_{n,x,\tau,j}(Y_{ij},(X_i-x)/h)|^m$, we find that

$$\tilde{k}^2_{n,\tau,j,m}(x)\leq 2k^2_{n,\tau,j,m}(x)+2e^2_{n,\tau,j,m}(x),$$

where

$$e_{n,\tau,j,m}(x)\equiv\left|\frac{1}{nh^d}\sum_{i=1}^n\left|\beta_{n,x,\tau,j}\left(Y_{ij},\frac{X_i-x}{h}\right)\right|^m-\frac{1}{h^d}\mathbf{E}\left(\left|\beta_{n,x,\tau,j}\left(Y_{ij},\frac{X_i-x}{h}\right)\right|^m\right)\right|.$$

Similarly as in the proof of Lemma C3, we note that

$$\begin{aligned}&\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\left|e^2_{n,\tau,j,m}(x)\right|\right]\\ \leq\ &\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\sup_{P\in\mathcal{P}}\frac{1}{nh^{2d}}\mathbf{E}\left[\left|\beta_{n,x,\tau,j}\left(Y_{ij},\frac{X_i-x}{h}\right)\right|^{2m}\right]=O(n^{-1}h^{-d})=o(1),\ \text{as } n\to\infty.\end{aligned}$$

Hence the desired statement follows from Lemma B4. ∎

Let

$$\mathbf{z}^*_{n,\tau}(x) \equiv \frac{1}{nh^d}\sum_{i=1}^{n}\beta_{n,x,\tau}\left(Y_i^*,\frac{X_i^*-x}{h}\right)-\frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau}\left(Y_i^*,\frac{X_i^*-x}{h}\right)\right], \text{ and}$$

$$\mathbf{z}^*_{N,\tau}(x) \equiv \frac{1}{nh^d}\sum_{i=1}^{N}\beta_{n,x,\tau}\left(Y_i^*,\frac{X_i^*-x}{h}\right)-\frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau}\left(Y_i^*,\frac{X_i^*-x}{h}\right)\right].$$

We also let

$$q^*_{n,\tau}(x) \equiv \frac{1}{\sqrt{h^d}}\sum_{i\le N_1}\left\{\beta_{n,x,\tau}(Y_i^*,(X_i^*-x)/h)-\mathbf{E}^*\beta_{n,x,\tau}(Y_i^*,(X_i^*-x)/h)\right\} \text{ and}$$

$$\bar{q}^*_{n,\tau}(x) \equiv \frac{1}{\sqrt{h^d}}\left\{\beta_{n,x,\tau}(Y_i^*,(X_i^*-x)/h)-\mathbf{E}^*\beta_{n,x,\tau}(Y_i^*,(X_i^*-x)/h)\right\}.$$

Lemma C5. *Suppose that Assumption A6(i) holds and that for some $C>0$,*

$$\textit{limsup}_{n\to\infty}n^{-1/2}h^{-d/2}\le C.$$

Then for any $m\in[2,M]$ (with M being the constant M in Assumption A6(i)),

(C.1)
$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\sqrt{\mathbf{E}\left[\left(\mathbf{E}^*\left[||q^*_{n,\tau}(x)||^m\right]\right)^2\right]} \le \bar{C}_1h^{d(1-(m/2))}, \text{ and}$$

$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\sqrt{\mathbf{E}\left[\left(\mathbf{E}^*\left[||\bar{q}^*_{n,\tau}(x)||^m\right]\right)^2\right]} \le \bar{C}_2h^{d(1-(m/2))},$$

where $\bar{C}_1,\bar{C}_2>0$ are constants that depend only on m. If furthermore,

$$\limsup_{n\to\infty}n^{-(m/2)+1}h^{d(1-(m/2))}<C,$$

for some constant $C>0$, then

(C.2)
$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\mathbf{E}^*\left[||\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x)||^m\right]\right] \le \left(\frac{15m}{\log m}\right)^m\max\left\{\bar{C}_1,2\bar{C}_1C\right\}, \text{ and}$$

$$\sup_{(x,\tau)\in\mathcal{X}^{\varepsilon/2}\times\mathcal{T}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\mathbf{E}^*\left[||\sqrt{nh^d}\mathbf{z}^*_{n,\tau}(x)||^m\right]\right] \le \left(\frac{15m}{\log m}\right)^m\max\left\{\bar{C}_2,2\bar{C}_2C\right\},$$

where $\bar{C}_1,\bar{C}_2>0$ are the constants that appear in (C.1).

Proof of Lemma C5. Let $q^*_{n,\tau,j}(x)$ be the j-th entry of $q^*_{n,\tau}(x)$. For the first statement of the lemma, it suffices to observe that for each $\varepsilon\in(0,\varepsilon_1)$, there exist $C_1>0$ and $\bar{C}_1>0$ such that

$$\begin{aligned}&\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\mathbf{E}\left[\left(\mathbf{E}^*\left[|q^*_{n,\tau,j}(x)|^m\right]\right)^2\right]\\ \le\; &\frac{C_1h^{2d}\sum_{j=1}^{J}\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\tilde{k}^2_{n,\tau,j,m}(x)\right]}{h^{dm}}\le\bar{C}_1h^{2d(1-(m/2))},\end{aligned}$$

where the last inequality uses Lemma C4. The second inequality in (C.1) follows similarly.

Let us consider (C.2). Let $z^*_{N,\tau,j}(x)$ be the j-th entry of $\mathbf{z}^*_{N,\tau}(x)$. Then using Rosenthal's inequality (e.g. (2.3) of Giné, Mason, and Zaitsev (2003)), for some constant $C_1 > 0$,

$$\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\mathbf{E}^*[|\sqrt{nh^d}z^*_{N,\tau,j}(x)|^m]\right]$$
$$\leq \left(\frac{15m}{\log m}\right)^{2m}\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)}\sup_{P\in\mathcal{P}}\left\{\left(\mathbf{E}\left[\mathbf{E}^*\left(q^{*2}_{n,\tau,j}(x)\right)\right]\right)^{m/2}+\mathbf{E}\left[n^{-(m/2)+1}\mathbf{E}^*|q^*_{n,\tau,j}(x)|^m\right]\right\}.$$

The first expectation is bounded by $\bar{C}_1$ by (C.1).

The second expectation is bounded by $\bar{C}_1 n^{-(m/2)+1}h^{d(1-(m/2))}$. This gives the first bound in (C.2). The second bound in (C.2) can be obtained similarly. ∎

For any Borel sets $B, B' \subset \mathcal{S}$ and $A, A' \subset \mathbb{N}_J$, let

$$\tilde{\sigma}^R_{n,A,A'}(B,B') \equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B'_{\tau_2}}\int_{B_{\tau_1}} C^*_{n,\tau_1,\tau_2,A,A'}(x,v)dxdvd\tau_1 d\tau_2,$$

where $B_\tau \equiv \{x \in \mathcal{X} : (x,\tau) \in B\}$,

$$C^*_{n,\tau_1,\tau_2,A,A'}(x,v) \equiv h^{-d}Cov^*\left(\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau_1}(x)),\Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau_2}(v))\right), \tag{C.3}$$

and Cov^* represents covariance under P^*. We also define

$$\tilde{\sigma}^R_{n,A}(B) \equiv \tilde{\sigma}^R_{n,A,A}(B,B), \tag{C.4}$$

for brevity. Also, let $\Sigma^*_{n,\tau_1,\tau_2}(x,u)$ be the $J \times J$ matrix whose (j,k)-th entry is given by $\tilde{\rho}_{n,\tau_1,\tau_2,j,k}(x,u)$. Fix $\bar{\varepsilon} > 0$ and define

$$\tilde{\Sigma}^*_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \equiv \left[\begin{array}{cc}\Sigma^*_{n,\tau_1,\tau_1}(x,0)+\bar{\varepsilon}I_J & \Sigma^*_{n,\tau_1,\tau_2}(x,u)\\ \Sigma^*_{n,\tau_1,\tau_2}(x,u) & \Sigma^*_{n,\tau_2,\tau_2}(x,0)+\bar{\varepsilon}I_J\end{array}\right].$$

We also define

$$\xi^*_{N,\tau_1,\tau_2}(x,u;\eta_1,\eta_2) \equiv \sqrt{nh^d}\Sigma^{*-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\left[\begin{array}{c}\mathbf{z}^*_{N,\tau_1}(x;\eta_1)\\ \mathbf{z}^*_{N,\tau_2}(x+uh;\eta_2)\end{array}\right],$$

where $\eta_1 \in \mathbf{R}^J$ and $\eta_2 \in \mathbf{R}^J$ are random vectors that are independent, and independent of $((Y^*_i, X^*_i)^\infty_{i=1}, (Y_i, X_i)^\infty_{i=1}, N, N_1)$, each following $N(0, \bar{\varepsilon}I_J)$, and define $\mathbf{z}^*_{N,\tau}(x;\eta_1) \equiv \mathbf{z}^*_{N,\tau}(x) + \eta_1/\sqrt{nh^d}$.

Lemma C6. *Suppose that Assumption A6(i) holds and that $nh^d \to \infty$, and*

$$\limsup_{n\to\infty} n^{-(m/2)+1}h^{d(1-(m/2))} < C,$$

for some $C > 0$ and some $m \in [2(p+1), M]$.

Then for any sequences of Borel sets $B_n, B_n' \subset \mathcal{S}$ and for any $A, A' \subset \mathbb{N}_J$,

$$\sup_{P \in \mathcal{P}} \mathbf{E}\left(\left|\tilde{\sigma}^R_{n,A,A'}(B_n, B_n' - \sigma_{n,A,A'}(B_n, B_n')\right|\right) \to 0,$$

where $\sigma_{n,A,A'}(B_n, B_n')$ is as defined in (B.15).

Proof of Lemma C6. The proof is very similar to that of Lemma B6. For brevity, we sketch the proof here. Define for $\bar{\varepsilon} > 0$,

$$\begin{aligned}
\tilde{\sigma}^R_{n,A,A',\bar{\varepsilon}}(B_n, B_n') &\equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}}\int_{\mathcal{U}} \tilde{g}_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) w_{\tau_1,B_n}(x) w_{\tau_2,B_n'}(x+uh) du dx d\tau_1 d\tau_2, \\
\tilde{\tau}_{n,A,A',\bar{\varepsilon}}(B_n, B_n') &\equiv \int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}}\int_{\mathcal{U}} \tilde{g}_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) w_{\tau_1,B_n}(x) w_{\tau_2,B_n'}(x+uh) du dx d\tau_1 d\tau_2,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{g}_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) &\equiv h^{-d} Cov^*(\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau_1}(x;\eta_1)), \Lambda_{A',p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau_2}(x+uh;\eta_2))), \text{ and} \\
\tilde{g}_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) &\equiv Cov^*(\Lambda_{A,p}(\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x)), \Lambda_{A',p}(\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x+uh))),
\end{aligned}$$

and $[\tilde{\mathbb{Z}}^\top_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x), \tilde{\mathbb{Z}}^\top_{n,\tau_1,\tau_2,\bar{\varepsilon}}(z)]^\top$ is a centered normal $\mathbf{R}^{2J}$-valued random vector with the same covariance matrix as the covariance matrix of $[\sqrt{nh^d}\mathbf{z}^{*\top}_{N,\tau_1}(x;\eta_1), \sqrt{nh^d}\mathbf{z}^{*\top}_{N,\tau_2}(z;\eta_2)]^\top$ under the product measure of the bootstrap distribution P^* and the distribution of $(\eta_1^\top, \eta_2^\top)^\top$. As in the proof of Lemma B6, it suffices for the lemma to show the following two statements.
(*Step 1*): As $n \to \infty$,

$$\begin{aligned}
\sup_{P \in \mathcal{P}} \mathbf{E}\left(\left|\tilde{\sigma}^R_{n,A,A',\bar{\varepsilon}}(B_n, B_n') - \tilde{\tau}_{n,A,A',\bar{\varepsilon}}(B_n, B_n')\right|\right) &\to 0, \text{ and} \\
\sup_{P \in \mathcal{P}} \mathbf{E}\left(\left|\tilde{\tau}_{n,A,A',\bar{\varepsilon}}(B_n, B_n') - \sigma_{n,A,A',\bar{\varepsilon}}(B_n, B_n')\right|\right) &\to 0.
\end{aligned}$$

(*Step 2*): For some $C > 0$ that does not depend on $\bar{\varepsilon}$ or n,

$$\sup_{P \in \mathcal{P}} |\tilde{\sigma}^R_{n,A,A',\bar{\varepsilon}}(B_n, B_n') - \tilde{\sigma}^R_{n,A,A'}(B_n, B_n')| \le C\sqrt{\bar{\varepsilon}}.$$

Then the desired result follows by sending $n \to \infty$ and $\bar{\varepsilon} \downarrow 0$, while chaining Steps 1 and 2 and the second convergence in Step 2 in the proof of Lemma B6.

We first focus on the first statement of Step 1. For any vector $\mathbf{v} = [\mathbf{v}_1^\top, \mathbf{v}_2^\top]^\top \in \mathbf{R}^{2J}$, we define

$$\tilde{C}_{n,p}(\mathbf{v}) \equiv \Lambda_{A,p}\left(\left[\tilde{\Sigma}^{*1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\mathbf{v}\right]_1\right)\Lambda_{A',p}\left(\left[\tilde{\Sigma}^{*1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\mathbf{v}\right]_2\right), \tag{C.5}$$

where $[a]_1$ of a vector $a \in \mathbf{R}^{2J}$ indicates the vector of the first J entries of a, and $[a]_2$ the vector of the remaining J entries of a. Also, similarly as in (B.19),

$$\lambda_{\min}\left(\tilde{\Sigma}^*_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right) \ge \bar{\varepsilon}. \tag{C.6}$$

Let $\bar{q}^*_{n,\tau}(x;\eta_1)$ be the column vector of entries $\bar{q}^*_{n,\tau,j}(x;\eta_{1j})$ with j running in the set $\mathbb{N}_J$, and with

$$\bar{q}^*_{n,\tau,j}(x;\eta_{1j}) \equiv p^*_{n,\tau,j}(x) + \eta_{1j},$$

where

$$p^*_{n,\tau,j}(x) = \frac{1}{\sqrt{h^d}} \sum_{1 \le i \le N_1} \left\{ \beta_{n,x,\tau,j}\left(Y^*_{ij}, \frac{X^*_i - x}{h}\right) - \mathbf{E}\left[\beta_{n,x,\tau,j}\left(Y^*_{ij}, \frac{X^*_i - x}{h}\right)\right]\right\},$$

η_{1j} is the j-th entry of η_1, and N_1 is a Poisson random variable with mean 1 and $((\eta_{1j})_{j\in A}, N_1)$ is independent of $\{(Y_i^\top, X_i^\top, Y_i^{*\top}, X_i^{*\top})\}_{i=1}^n$. Let $[p^{*(i)}_{n,\tau_1}(x), p^{*(i)}_{n,\tau_2}(x+uh)]$ be the i.i.d. copies of $[p^*_{n,\tau_1}(x), p^*_{n,\tau_2}(x+uh)]$ conditional on the observations $\{(Y_i, X_i)\}_{i=1}^n$, and $\eta_1^{(i)}$ and $\eta_2^{(i)}$ be i.i.d. copies of η_1 and η_2. Define

$$q^{*(i)}_{n,\tau_1}(x;\eta_1^{(i)}) = p^{*(i)}_{n,\tau_1}(x) + \eta_1^{(i)} \text{ and } q^{*(i)}_{n,\tau_2}(x+uh;\eta_2^{(i)}) = p^{*(i)}_{n,\tau_2}(x+uh) + \eta_2^{(i)}.$$

Note that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\begin{array}{c} q^{*(i)}_{n,\tau_1}(x;\eta_1^{(i)}) \\ q^{*(i)}_{n,\tau_2}(x+uh;\eta_2^{(i)}) \end{array}\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\begin{array}{c} p^{*(i)}_{n,\tau_1}(x) \\ p^{*(i)}_{n,\tau_2}(x+uh) \end{array}\right] + \frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\begin{array}{c} \eta_1^{(i)} \\ \eta_2^{(i)} \end{array}\right].$$

The last sum has the same distribution as $[\eta_1^\top, \eta_2^\top]^\top$ and the leading sum on the right-hand side has the same bootstrap distribution as that of $[\mathbf{z}^{*\top}_{N,\tau_1}(x), \mathbf{z}^{*\top}_{N,\tau_2}(x+uh)]^\top$, P-a.e. Therefore, we conclude that

$$\xi^*_{N,\tau_1,\tau_2}(x,u;\eta_1^{(i)},\eta_2^{(i)}) \overset{d^*}{=} \frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u;\eta_1^{(i)},\eta_2^{(i)}),$$

where $\overset{d^*}{=}$ indicates the distributional equivalence with respect to the product measure of the bootstrap distribution P^* and the joint distribution of $(\eta_1^{(i)}, \eta_2^{(i)})$, P-a.e, and

$$\tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u;\eta_1^{(i)},\eta_2^{(i)}) \equiv \tilde{\Sigma}^{*-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u) \left[\begin{array}{c} q_n^{(i)}(x;\eta_1^{(i)}) \\ q_n^{(i)}(x+uh;\eta_2^{(i)}) \end{array}\right].$$

Following the arguments in the proof of Lemma B6, we find that for each $u \in \mathcal{U}$, and for $\varepsilon \in (0,\varepsilon_1)$ with ε_1 as in Assumption A6(i),

$$\begin{aligned}
& \sup_{(x,u)\in(\mathcal{S}_{\tau_1}\cup\mathcal{S}_{\tau_2})\times\mathcal{U}} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\mathbf{E}^*||\tilde{W}^{(i)}_{n,\tau_1,\tau_2}(x,u;\eta_1^{(i)},\eta_2^{(i)})||^3\right] \\
\le\ & C_1 \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\lambda^3_{\min}\left(\tilde{\Sigma}^{*-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right)\mathbf{E}^*||q^{*(i)}_{n,\tau_1}(x;\eta_1^{(i)})||^3\right] \\
& + C_1 \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\lambda^3_{\min}\left(\tilde{\Sigma}^{*-1/2}_{n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right)\mathbf{E}^*||q^{*(i)}_{n,\tau_2}(x+uh;\eta_2^{(i)})||^3\right],
\end{aligned}$$

for some $C_1 > 0$. As for the leading term,

$$\begin{aligned}
&\sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\lambda_{\min}^3\left(\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^{*-1/2}(x,u)\right)\mathbf{E}^*||q_{n,\tau_1}^{*(i)}(x;\eta_1^{(i)})||^3\right]\\
\le\;& \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\sqrt{\mathbf{E}\left[\left(\mathbf{E}^*||q_{n,\tau_1}^{*(i)}(x;\eta_1^{(i)})||^3\right)^2\right]}\\
&\times \sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\sqrt{\mathbf{E}\left[\lambda_{\min}^6\left(\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^{*-1/2}(x,u)\right)\right]}\le\frac{C_2\bar{\varepsilon}^{-3}}{\sqrt{h^d}},
\end{aligned}$$

by Lemma C5 and (C.6). Similarly, we observe that

$$\sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\lambda_{\min}^3\left(\tilde{\Sigma}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^{*-1/2}(x,u)\right)\mathbf{E}^*||q_{n,\tau_2}^{*(i)}(x+uh;\eta_2^{(i)})||^3\right]\le\frac{C_2\bar{\varepsilon}^{-3}}{\sqrt{h^d}}.$$

Define

$$c_{n,\tau_1,\tau_2}(x,u)=\tilde{C}_{n,p}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n\tilde{W}_{n,\tau_1,\tau_2}^{(i)}(x,u;\eta_1^{(i)},\eta_2^{(i)})\right).$$

Let $\Phi_{n,\tau_1,\tau_2}(\cdot;x,u)$ be the joint CDF of the random vector $(\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^\top(x),\tilde{\mathbb{Z}}_{n,\tau_1,\tau_2,\bar{\varepsilon}}^\top(x+uh))^\top$. By Theorem 1 of Sweeting (1977),

$$\begin{aligned}
\text{(C.7)}\quad&\sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\left|c_{n,\tau_1,\tau_2}(x,u)-\int\tilde{C}_{n,p}(\zeta)d\Phi_{n,\tau_1,\tau_2}(\zeta;x,u)\right|\right]\\
\le\;&\frac{C_1}{\sqrt{n}}\sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\mathbf{E}^*||\tilde{W}_{n,\tau_1,\tau_2}^{(i)}(x,u;\eta_1^{(i)},\eta_2^{(i)})||^3\right]\le\frac{C_2\bar{\varepsilon}^{-3}}{\sqrt{nh^d}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbf{E}\left[\left|\int_{B_{\tau_1}}\int_{\mathcal{U}}\left\{\tilde{g}_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)-\tilde{g}_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right\}w_{\tau_1,B}(x)w_{\tau_2,B'}(x+uh)dudx\right|\right]\\
\le\;&\int_{B_{\tau_1}}\int_{\mathcal{U}}\mathbf{E}\left|\tilde{g}_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)-\tilde{g}_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right|w_{\tau_1,B}(x)w_{\tau_2,B'}(x+uh)dudx\\
\le\;&\int_{B_{\tau_1}}w_{\tau_1,B}(x)w_{\tau_2,B'}(x)dx\\
&\times\sup_{(x,u)\in(\mathcal{S}_{\tau_1}(\varepsilon)\cup\mathcal{S}_{\tau_2}(\varepsilon))\times\mathcal{U}}\sup_{P\in\mathcal{P}}\mathbf{E}\left|\tilde{g}_{1n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)-\tilde{g}_{2n,\tau_1,\tau_2,\bar{\varepsilon}}(x,u)\right|\\
\to\;&0,
\end{aligned}$$

as $n\to\infty$. The last convergence is due to (C.7) and hence uniform over $(\tau_1,\tau_2)\in\mathcal{T}\times\mathcal{T}$. The proof of Step 1 is thus complete.

We turn to the second statement of Step 1. Similarly as in the proof of Step 1 in the proof of Lemma B6, the second statement of Step 1 follows by Lemma C4.

Now we turn to Step 2. In view of the proof of Step 2 in the proof of Lemma B6, it suffices to show that with $s = (p+1)/(p-1)$ if $p > 1$ and $s = 2$ if $p = 1$,

$$\text{(C.8)} \qquad \sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\mathbf{E}^*\left\|\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x)\right\|^{2s(p-1)}\right] < C \text{ and}$$

$$\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\mathbf{E}^*\left\|\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x;\eta_1)\right\|^{2s(p-1)}\right] < C,$$

for some $C > 0$. First note that for any $q > 0$,

$$\begin{aligned} &\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\mathbf{E}^*\left\|\sqrt{nh^d}\{\mathbf{z}^*_{N,\tau}(x) - \mathbf{z}^*_{N,\tau}(x;\eta_1)\}\right\|^{2q}\right] \\ &= \mathbf{E}\left\|\sqrt{\bar{\varepsilon}}\mathbb{Z}\right\|^{2q} = C\bar{\varepsilon}^q, \end{aligned}$$

where $\mathbb{Z} \in \mathbf{R}^J$ is a centered normal random vector with covariance matrix I_J. Also, we deduce that for some constants $C_1, C_2 > 0$,

$$\begin{aligned} &\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\mathbf{E}^*\left\|\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x)\right\|^{2s(p-1)}\right] \\ \leq &\sup_{\tau\in\mathcal{T},x\in\mathcal{S}_\tau(\varepsilon)} \sup_{P\in\mathcal{P}} \mathbf{E}\left[\mathbf{E}^*\left\|\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x;\eta_1)\right\|^{2s(p-1)}\right] + C_1\bar{\varepsilon}^{s(p-1)} \leq C_1 + C_2\bar{\varepsilon}^{s(p-1)}, \end{aligned}$$

by the third statement of Lemma C5. This leads to the first and second statements of (C.8). Thus the proof of the lemma is complete. ∎

Lemma C7. *Suppose that for some small $\nu_1 > 0$, $n^{-1/2}h^{-d-\nu_1} \to 0$, as $n \to \infty$ and the conditions of Lemma B6 hold. Then there exists $C > 0$ such that for any sequence of Borel sets $B_n \subset \mathcal{S}$, and $A \subset \mathbb{N}_J$, from some large n on,*

$$\begin{aligned} &\sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^*\left[\left|h^{-d/2}\int_{B_n}\left\{\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{n,\tau}(x)) - \mathbf{E}^*\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x))\right]\right\}dQ(x,\tau)\right|\right]\right) \\ &\leq C\sqrt{Q(B_n)}. \end{aligned}$$

Proof of Lemma C7. We follow the proof of Lemma B7 and show that for some $C > 0$,

Step 1: $\sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^*\left[\left|h^{-d/2}\int_{B_n}\left\{\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{n,\tau}(x)) - \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x))\right\}dQ(x,\tau)\right|\right]\right) \leq CQ(B_n)$, and

Step 2:

$$\begin{aligned} &\sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^*\left[\left|h^{-d/2}\int_{B_n}\left\{\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x)) - \mathbf{E}^*[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x))]\right\}dQ(x,\tau)\right|\right]\right) \\ \leq\ &C\sqrt{Q(B_n)}. \end{aligned}$$

Proof of Step 1: Similarly as in the proof of Step 1 in the proof of Lemma B7, we first write

$$\mathbf{z}_{n,\tau}^*(x) = \mathbf{z}_{N,\tau}^*(x) + \mathbf{v}_{n,\tau}^*(x) + \mathbf{s}_{n,\tau}^*(x),$$

where

$$\begin{aligned}\mathbf{v}_{n,\tau}^*(x) &\equiv \left(\frac{n-N}{n}\right) \cdot \frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau}\left(Y_i^*, \frac{X_i^*-x}{h}\right)\right] \text{ and} \\ \mathbf{s}_{n,\tau}^*(x) &\equiv \frac{1}{nh^d}\sum_{i=N+1}^{n}\left\{\beta_{n,x,\tau}\left(Y_i^*, \frac{X_i^*-x}{h}\right) - \mathbf{E}^*\left[\beta_{n,x,\tau}\left(Y_i^*, \frac{X_i^*-x}{h}\right)\right]\right\}.\end{aligned}$$

Similarly as in the proof of Lemma B7, we deduce that for some $C_1, C_2 > 0$,

$$\begin{aligned}&\left|\int_{B_n}\left\{\Lambda_{A,p}\left(\mathbf{z}_{n,\tau}^*(x)\right) - \Lambda_{A,p}\left(\mathbf{z}_{N,\tau}^*(x)\right)\right\}dQ(x,\tau)\right| \\ &\le C_1\int_{B_n}\left\|\mathbf{v}_{n,\tau}^*(x)\right\|\left(\left\|\mathbf{z}_{n,\tau}^*(x)\right\|^{p-1} + \left\|\mathbf{z}_{N,\tau}^*(x)\right\|^{p-1}\right)dQ(x,\tau) \\ &\quad + C_2\int_{B_n}\left\|\mathbf{s}_{n,\tau}^*(x)\right\|\left(\left\|\mathbf{z}_{n,\tau}^*(x)\right\|^{p-1} + \left\|\mathbf{z}_{N,\tau}^*(x)\right\|^{p-1}\right)dQ(x,\tau) \\ &= D_{1n}^* + D_{2n}^*, \text{ say.}\end{aligned}$$

To deal with D_{1n}^* and D_{2n}^*, we first show the following:

Claim 1: $\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left(\mathbf{E}^*[||\mathbf{v}_{n,\tau}^*(x)||^2]\right) = O(n^{-1})$.

Claim 2: $\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left(\mathbf{E}^*[||\mathbf{s}_{n,\tau}^*(x)||^2]\right) = O(n^{-3/2}h^{-d})$.

Proof of Claim 1: Similarly as in the proof of Lemma B7, we note that

$$\mathbf{E}\left(\mathbf{E}^*[||\mathbf{v}_{n,\tau}^*(x)||^2]\right) \le \mathbf{E}\left|\left(\frac{n-N}{n}\right)\right|^2\mathbf{E}\left[\left\|\frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau}\left(Y_i^*, \frac{X_i^*-x}{h}\right)\right]\right\|^2\right].$$

By the first statement of Lemma C5, we have

$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}\left[\left\|\frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau}\left(Y_i^*, \frac{X_i^*-x}{h}\right)\right]\right\|^2\right] = O(1).$$

Since $\mathbf{E}\left|(n-N)/n\right|^2 = O(n^{-1})$, we obtain Claim 1.

Proof of Claim 2: Let

$$\mathbf{s}_{n,\tau}^*(x;\eta_1) = \mathbf{s}_{n,\tau}^*(x) + \frac{(N-n)\eta_1}{n^{3/2}h^{d/2}},$$

where η_1 is a random vector independent of $((Y_i^*, X_i^*)_{i=1}^n, (Y_i, X_i)_{i=1}^n, N)$ and follows $N(0, \bar{\varepsilon} I_J)$. Note that

$$\begin{aligned}
&\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^* \left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}^*(x)\right\|^2\right) \\
&\leq 2 \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^* \left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}^*(x;\eta_1)\right\|^2\right) + \frac{2}{n}\mathbf{E}\left\|\frac{(N-n)\eta_1}{\sqrt{n}}\right\|^2 \\
&\leq 2 \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^* \left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}^*(x;\eta_1)\right\|^2\right) + \frac{C\bar{\varepsilon}^2}{n},
\end{aligned}$$

as in the proof of Lemma B7. As for the leading expectation on the right hand side of (B.27), we let $C_1 > 0$ be as in Lemma C4 and note that

$$\begin{aligned}
\mathbf{E}\left(\mathbf{E}^* \left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}^*(x;\eta_1)\right\|^2\right) &= \sum_{j\in\mathbb{N}_J} \mathbf{E}\left(\mathbf{E}^*\left(\frac{1}{\sqrt{n}}\sum_{i=N+1}^n q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})\right)^2\right) \\
&= \frac{1}{n}\sum_{j\in\mathbb{N}_J} \mathbf{E}\left(\tilde{\sigma}_{n,\tau,j}^2(x)\mathbf{E}^*\left(\sum_{i=N+1}^n \frac{q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})}{\tilde{\sigma}_{n,\tau,j}(x)}\right)^2\right),
\end{aligned}$$

where $q_{n,\tau}^{*(i)}(x;\eta_1^{(i)})$'s $(i = 1, 2, \ldots)$ are as defined in the proof of Lemma C6 and $q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})$ is the j-th entry of $q_{n,\tau}^{*(i)}(x;\eta_1^{(i)})$ and $\tilde{\sigma}_{n,\tau,j}^2(x) = Var^*(q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})) > 0$ and Var^* denotes the variance with respect to the joint distribution of $((Y_i^*, X_i^*)_{i=1}^n, \eta_{1j}^{(i)})$ conditional on $(Y_i, X_i)_{i=1}^n$. We apply Lemma 1(i) of Horváth (1991) to deduce that

$$\text{(C.9)} \quad \begin{aligned}
\mathbf{E}^*\left(\sum_{i=N+1}^n \frac{q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})}{\tilde{\sigma}_{n,\tau,j}(x)}\right)^2 &\leq C\sqrt{n} + C\mathbf{E}^*\left(\left|\frac{q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})}{\tilde{\sigma}_{n,\tau,j}(x)}\right|^3\right) \\
&\quad + C\mathbf{E}^*\left(\left|\frac{q_{n,\tau,j}^{*(i)}(x;\eta_{1j}^{(i)})}{\tilde{\sigma}_{n,\tau,j}(x)}\right|^4\right),
\end{aligned}$$

for some $C > 0$. Using this, Lemma C5, and following arguments similarly as in (B.29), (B.30) and (B.31), we conclude that

$$\begin{aligned}
\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\left(\mathbf{E}^* \left\|\sqrt{nh^d}\mathbf{s}_{n,\tau}^*(x)\right\|^2\right) &\leq O\left(n^{-1}h^{-\nu_1}\right) + O\left(n^{-1/2} + n^{-3/4}h^{-d/2-\nu_1} + n^{-1}h^{-d-\nu_1}\right) \\
&= O\left(n^{-1}h^{-\nu_1}\right) + O\left(n^{-1/2}\right),
\end{aligned}$$

since $n^{-1/2}h^{-d-\nu_1} \to 0$. This delivers Claim 2.

Using Claims 1 and 2, and following the arguments in the proof of Lemma B7, we obtain Step 1.

Proof of Step 2: We can follow the proof of Lemma B6 to show that

$$\begin{aligned} & \mathbf{E}\left[\mathbf{E}^*\left[h^{-d/2}\int_{B_n}\left(\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x)) - \mathbf{E}^*\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x))\right]\right)dQ(x,\tau)\right]^2\right] \\ = & \mathbf{E}\left[\int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}\cap B_{n,\tau_2}}\int_{\mathcal{U}} C^*_{n,\tau_1,\tau_2,A,A'}(x,u)dudxd\tau_1 d\tau_2\right] + o(1) \\ \leq & C\int_{\mathcal{T}}\int_{\mathcal{T}}\int_{B_{n,\tau_1}\cap B_{n,\tau_2}} dxd\tau_1 d\tau_2 + o(1) \leq CQ(B_n), \end{aligned}$$

where $C^*_{n,\tau_1,\tau_2,A,A'}(x,v)$ is as defined in (C.3). We obtain the desired result of Step 2. ∎

Let $\mathcal{C} \subset \mathbf{R}^d$, $\alpha_P \equiv P\{X \in \mathbf{R}^d \backslash \mathcal{C}\}$ and $B_{n,A}(c_n;\mathcal{C})$ be as introduced prior to Lemma B8. Define

$$\begin{aligned} \zeta^*_{n,A} & \equiv \int_{B_{n,A}(c_n;\mathcal{C})} \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{n,\tau}(x))dQ(x,\tau), \text{ and} \\ \zeta^*_{N,A} & \equiv \int_{B_{n,A}(c_n;\mathcal{C})} \Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x))dQ(x,\tau). \end{aligned}$$

Let μ_A's be real numbers indexed by $A \subset \mathbb{N}_J$. We also define $B_{n,A}(c_n;\mathcal{C})$ as prior to Lemma B8 and let

$$\begin{aligned} S^*_n & \equiv h^{-d/2}\sum_{A\in\mathcal{N}_J}\mu_A\left\{\zeta^*_{N,A} - \mathbf{E}^*\zeta^*_{N,A}\right\}, \\ U^*_n & \equiv \frac{1}{\sqrt{n}}\left\{\sum_{i=1}^N 1\{X^*_i \in \mathcal{C}\} - nP^*\left\{X^*_i \in \mathcal{C}\right\}\right\}, \text{ and} \\ V^*_n & \equiv \frac{1}{\sqrt{n}}\left\{\sum_{i=1}^N 1\{X^*_i \in \mathbf{R}^d\backslash\mathcal{C}\} - nP^*\left\{X^*_i \in \mathbf{R}^d\backslash\mathcal{C}\right\}\right\}. \end{aligned}$$

We let

$$H^*_n \equiv \left[\frac{S^*_n}{\sigma_n(\mathcal{C})}, \frac{U^*_n}{\sqrt{1-\alpha_P}}\right].$$

The following lemma is a bootstrap counterpart of Lemma B8.

Lemma C8. *Suppose that the conditions of Lemma C6 hold and that $c_n \to \infty$, as $n \to \infty$.* *(i) If* $\liminf_{n\to\infty}\inf_{P\in\mathcal{P}}\sigma^2_n(\mathcal{C}) > 0$, *then for all $a > 0$,*

$$\sup_{P\in\mathcal{P}} P\left\{\sup_{t\in\mathbf{R}^2}\left|P^*\left\{H^*_n \leq t\right\} - P\left\{\mathbb{Z} \leq t\right\}\right| > a\right\} \to 0,$$

where $\mathbb{Z} \sim N(0, I_2)$.

(ii) If $\limsup_{n\to\infty}\sigma_n^2(\mathcal{C})=0$*, then, for each* $(t_1,t_2)\in\mathbf{R}^2$ *and* $a>0$*,*

$$\sup_{P\in\mathcal{P}} P\left\{\left|P^*\left\{S_n^*\leq t_1 \text{ and } \frac{U_n^*}{\sqrt{1-\alpha_P}}\leq t_2\right\}-1\{0\leq t_1\}P\{\mathbb{Z}_1\leq t_2\}\right|>a\right\}\to 0.$$

Proof of Lemma C8. Similarly as in the proof of Lemma C8, we fix $\bar{\varepsilon}>0$ and let

$$H_{n,\bar{\varepsilon}}^*\equiv\left[\frac{S_{n,\bar{\varepsilon}}^*}{\sigma_{n,\bar{\varepsilon}}(\mathcal{C})},\frac{U_n^*}{\sqrt{1-\alpha_P}}\right]^\top,$$

where $S_{n,\bar{\varepsilon}}^*$ is equal to S_n^*, except that $\zeta_{N,A}^*$ is replaced by

$$\zeta_{N,A,\bar{\varepsilon}}^*\equiv\int_{B_{n,A}(c_n;\mathcal{C})}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x;\eta_1))dQ(x,\tau),$$

and $\mathbf{z}_{N,\tau}^*(x;\eta_1)$ is as defined prior to Lemma C6. Also let

$$\tilde{C}_n\equiv\mathbf{E}^*H_n^*H_n^{*\top}\text{ and }\tilde{C}_{n,\bar{\varepsilon}}\equiv\mathbf{E}^*H_{n,\bar{\varepsilon}}^*H_{n,\bar{\varepsilon}}^{*\top}.$$

First, we show the following statements.

Step 1: $\sup_{P\in\mathcal{P}}P\left\{|Cov^*(S_{n,\bar{\varepsilon}}^*-S_n^*,U_n^*)|>M\sqrt{\bar{\varepsilon}}\right\}\to 0$, as $n\to\infty$ and $M\to\infty$.

Step 2: For any $a>0$, $\sup_{P\in\mathcal{P}}P\left\{\left|Cov(S_{n,\bar{\varepsilon}}^*,U_n^*)\right|>ah^{d/2}\right\}\to 0$, as $n\to\infty$.

Step 3: There exists $c>0$ such that from some large n on,

$$\inf_{P\in\mathcal{P}}\lambda_{\min}(\tilde{C}_n)>c.$$

Step 4: For any $a>0$, as $n\to\infty$,

$$\sup_{P\in\mathcal{P}}P\left\{\sup_{t\in\mathbf{R}^2}\left|P^*\left\{\tilde{C}_n^{-1/2}H_n^*\leq t\right\}\to P\{\mathbb{Z}\leq t\}\right|>a\right\}\to 0.$$

Combining Steps 1-4, we obtain (i) of Lemma B8.

Proof of Step 1: Observe that

$$\left|\zeta_{N,A,\bar{\varepsilon}}^*-\zeta_{N,A}^*\right|\leq C||\eta_1||\int_{B_{n,A}(c_n;\mathcal{C})}\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x)\right\|^{p-1}dQ(x,\tau).$$

As in the proof of Step 1 in the proof of Lemma B8, we deduce that

$$\mathbf{E}^*\left[\left|\zeta_{N,A,\bar{\varepsilon}}^*-\zeta_{N,A}^*\right|^2\right]\leq C\bar{\varepsilon}\int_{B_{n,A}(c_n;\mathcal{C})}\mathbf{E}^*\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x)\right\|^{2p-2}dQ(x,\tau).$$

Hence for some $C_1,C_2>0$,

$$\text{(C.10)}\qquad \begin{aligned}&\mathbf{E}\left(\mathbf{E}^*\left[\left|\zeta_{N,A,\bar{\varepsilon}}^*-\zeta_{N,A}^*\right|^2\right]\right)\\ \leq\ & C\bar{\varepsilon}\int_{B_{n,A}(c_n;\mathcal{C})}\mathbf{E}\left(\mathbf{E}^*\left\|\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x)\right\|^{2p-2}\right)dQ(x,\tau)\leq C_2\bar{\varepsilon}\end{aligned}$$

by the second statement of Lemma C5.

On the other hand, observe that $\mathbf{E}^* U_n^{*2} \leq 1$. Hence

$$P\left\{|Cov^*(S_{n,\bar{\varepsilon}}^* - S_n^*, U_n^*)| > M\sqrt{\bar{\varepsilon}}\right\} \leq |\mathcal{N}_J| \cdot P\left\{\max_{A \in \mathcal{N}_J} \mathbf{E}^*\left[\left|\zeta_{N,A,\bar{\varepsilon}}^* - \zeta_{N,A}^*\right|^2\right] > M^2 \bar{\varepsilon}\right\}.$$

By Markov's inequality, the last probability is bounded by (for some $C > 0$ that does not depend on $P \in \mathcal{P}$)

$$M^{-2}\bar{\varepsilon}^{-1} \sum_{A \in \mathcal{N}_J} \mathbf{E}\left(\mathbf{E}^*\left[\left|\zeta_{N,A,\bar{\varepsilon}}^* - \zeta_{N,A}^*\right|^2\right]\right) \leq CM^{-2},$$

by (C.10). Hence we obtain the desired result.

Proof of Step 2: Let $\tilde{\Sigma}_{2n,\tau,\bar{\varepsilon}}^*$ be the covariance matrix of $[(q_{n,\tau}^*(x) + \eta_1)^\top, \tilde{U}_n^*]^\top$ under P^*, where $\tilde{U}_n^* = U_n^*/\sqrt{P\{X \in \mathcal{C}\}}$. Using Lemma C4 and following the same arguments in (B.32), we find that

$$\sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E}\left[\mathbf{E}^*\left[q_{n,\tau,j}^*(x)\tilde{U}_n^*\right]\right] \leq C_2 h^{d/2},$$

for some $C_2 > 0$. Therefore, using this result and following the proof of Step 3 in the proof of Lemma B8, we deduce that (everywhere)

$$\lambda_{\min}\left(\tilde{\Sigma}_{2n,\tau,\bar{\varepsilon}}^*\right) \geq \bar{\varepsilon} - \left\|A_{n,\tau}^*(x)\right\|, \tag{C.11}$$

for some random matrix $A_{n,\tau}^*(x)$ such that

$$\sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E}\left[\left\|A_{n,\tau}^*(x)\right\|\right] = O(h^{d/2}).$$

Hence by (C.11),

$$\begin{aligned} &\inf_{(x,\tau) \in \mathcal{S}} \inf_{P \in \mathcal{P}} P\left\{\lambda_{\min}\left(\tilde{\Sigma}_{2n,\tau,\bar{\varepsilon}}^*\right) \geq \bar{\varepsilon}/2\right\} \\ \geq\ &\inf_{(x,\tau) \in \mathcal{S}} \inf_{P \in \mathcal{P}} P\left\{\left\|A_{n,\tau}^*(x)\right\| \leq \bar{\varepsilon}/2\right\} \\ \geq\ &1 - \frac{2}{\bar{\varepsilon}} \sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E}\left[\left\|A_{n,\tau}^*(x)\right\|\right] \to 1, \end{aligned} \tag{C.12}$$

as $n \to \infty$.

Now note that

$$\left(q_{n,\tau,j}^*(x), \tilde{U}_n^*\right) \stackrel{d^*}{=} \left(\frac{1}{\sqrt{n}}\sum_{k=1}^n q_{n,\tau,j}^{(k)*}(x), \frac{1}{\sqrt{n}}\sum_{k=1}^n \tilde{U}_n^{(k)*}\right),$$

where $(q_{n,\tau,j}^{(k)*}(x), \tilde{U}_n^{(k)*})$'s with $k = 1, ..., n$ are i.i.d. copies of $(q_{n,\tau,j}^*(x), \bar{U}_n^*)$, and

$$\bar{U}_n^* \equiv \frac{1}{\sqrt{nP\{X \in \mathcal{C}\}}}\left\{\sum_{1 \leq i \leq N_1} 1\{X_i^* \in \mathcal{C}\} - P^*\left\{X_i^* \in \mathcal{C}\right\}\right\}.$$

Note also that by Rosenthal's inequality,

$$\text{limsup}_{n\to\infty} \sup_{P\in\mathcal{P}} P\left\{\mathbf{E}^*\left[|\tilde{U}_n^{(k)*}|^3\right] > M\right\} \to 0,$$

as $M \to \infty$. Define

$$W_{n,\tau}^*(x;\eta_1) \equiv \tilde{\Sigma}_{2n,\tau,\bar{\varepsilon}}^{*-1/2} \left[\begin{array}{c} q_{n,\tau}^*(x) + \eta_1 \\ \tilde{U}_n^* \end{array}\right].$$

Using (C.12) and Lemma C5, and following the same arguments in the proof of Step 2 in the proof of Lemma B8, we deduce that

$$\text{limsup}_{n\to\infty} \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} P\left\{\mathbf{E}^*\left\|W_{n,\tau}^*(x;\eta_1)\right\|^3 > M\bar{\varepsilon}^{-3/2}h^{-d/2}\right\} \to 0,$$

as $M \to \infty$. For any vector $\mathbf{v} = [\mathbf{v}_1^\top, v_2]^\top \in \mathbf{R}^{J+1}$, we define

$$\tilde{D}_{n,\tau,p}(\mathbf{v}) \equiv \Lambda_p\left(\left[\tilde{\Sigma}_{2n,\tau,\bar{\varepsilon}}^{*1/2}\mathbf{v}\right]_1\right)\left[\tilde{\Sigma}_{2n,\tau,\bar{\varepsilon}}^{*1/2}\mathbf{v}\right]_2,$$

where $[a]_1$ of a vector $a \in \mathbf{R}^{J+1}$ indicates the vector of the first J entries of a, and $[a]_2$ the last entry of a. By Theorem 1 of Sweeting (1977), we find that (with $\bar{\varepsilon} > 0$ fixed)

$$\mathbf{E}^*\left[\tilde{D}_{n,\tau,p}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n W_{n,\tau}^{(i)*}(x;\eta_1)\right)\right] = \mathbf{E}\left[\tilde{D}_{n,\tau,p}\left(\mathbb{Z}_{J+1}\right)\right] + O_P(n^{-1/2}h^{-d/2}) = o_P(n^{d/2}),$$

$\mathcal{P}$-uniformly, where $\mathbb{Z}_{J+1} \sim N(0, I_{J+1})$ and $W_{n,\tau}^{(i)*}(x;\eta_1)$'s are i.i.d. copies of $W_{n,\tau}^*(x;\eta_1)$ under P^*. The last equality follows because $n^{-1/2}h^{-d/2} = o(h^{d/2})$ and $\mathbf{E}[\tilde{D}_{n,\tau,p}(\mathbb{Z}_{J+1})] = 0$. Since

$$Cov^*\left(\Lambda_{A,p}\left(\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x;\eta_1)\right), U_n^*\right) = \mathbf{E}^*\left[\tilde{D}_{n,\tau,p}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n W_{n,\tau}^{(i)*}(x)\right)\right],$$

we conclude that

$$\text{(C.13)} \qquad \sup_{(x,\tau)\in\mathcal{S}}\left|Cov^*\left(\Lambda_{A,p}\left(\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x;\eta_1)\right), U_n^*\right)\right| = o_P(h^{d/2}),$$

uniformly in $P \in \mathcal{P}$.

Now for some $C > 0$,

$$P\left\{\left|Cov(S_{n,\bar{\varepsilon}}^*, U_n^*)\right| > ah^{d/2}\right\} \le P\left\{C \sup_{(x,\tau)\in\mathcal{S}}\left|Cov^*\left(\Lambda_{A,p}\left(\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x;\eta_1)\right), U_n^*\right)\right| > ah^{d/2}\right\}.$$

The last probability vanishes uniformly in $P \in \mathcal{P}$ by (C.13). By applying the Dominated Convergence Theorem, we obtain Step 2.

Proof of Step 3: First, we show that

$$\text{(C.14)} \qquad Var^*\left(S_n^*\right) = \sigma_n^2(\mathcal{C}) + o_P(1),$$

where $o_P(1)$ is uniform over $P \in \mathcal{P}$. Note that

$$Var^*(S_n^*) = \sum_{A \in \mathcal{N}_J} \sum_{A' \in \mathcal{N}_J} \mu_A \mu_{A'} Cov^*(\psi_{n,A}^*, \psi_{n,A'}^*),$$

where $\psi_{n,A}^* \equiv h^{-d/2}(\zeta_{N,A}^* - \mathbf{E}^* \zeta_{N,A}^*)$. By Lemma C6, we find that for $A, A \in \mathcal{N}_J$,

$$Cov^*(\psi_{n,A}^*, \psi_{n,A'}^*) = \sigma_{n,A,A'}(B_{n,A}(c_n; \mathcal{C}), B_{n,A'}(c_n; \mathcal{C})) + o_P(1),$$

uniformly in $P \in \mathcal{P}$, yielding the desired result of (C.14).

Combining Steps 1 and 2, we deduce that for some $C > 0$,

$$\sup_{P \in \mathcal{P}} |Cov^*(S_n^*, U_n^*)| \leq \sqrt{\bar{\varepsilon}} \cdot O_P(1) + o_P(h^{d/2}).$$

Let $\tilde{\sigma}_1^2 \equiv Var^*(S_n^*)$ and $\tilde{\sigma}_2^2 \equiv 1 - \tilde{\alpha}_P$, where $\tilde{\alpha}_P \equiv P^* \left\{ X_i^* \in \mathbf{R}^d \backslash \mathcal{C} \right\}$. Observe that

$$\tilde{\sigma}_1^2 = \sigma_n(\mathcal{C}) + o_P(1) > C_1 + o_P(1), \ \mathcal{P}\text{-uniformly},$$

for some $C_1 > 0$ that does not depend on n or P by the assumption of the lemma. Also note that

$$\tilde{\alpha}_P = \alpha_P + o_P(1) < 1 - C_2 + o_P(1), \ \mathcal{P}\text{-uniformly},$$

for some $C_2 > 0$. Therefore, following the same arguments as in (B.37), we obtain the desired result.

Proof of Step 4: We take $\{R_{n,\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^d\}$, and define

$$\begin{aligned} B_{A,x}(c_n) &\equiv \{\tau \in \mathcal{T} : (x, \tau) \in B_A(c_n)\}, \\ B_{n,\mathbf{i}} &\equiv R_{n,\mathbf{i}} \cap \mathcal{C}, \\ B_{n,A,\mathbf{i}}(c_n) &\equiv (B_{n,\mathbf{i}} \times \mathcal{T}) \cap B_A(c_n), \end{aligned}$$

and $\mathcal{I}_n \equiv \{\mathbf{i} \in \mathbb{Z}_n^d : B_{n,\mathbf{i}} \neq \varnothing\}$ as in the proof of Step 4 in the proof of Lemma B8. Also, define

$$\Delta_{n,A,\mathbf{i}}^* \equiv h^{-d/2} \int_{B_{n,\mathbf{i}}} \int_{B_{A,x}(c_n)} \left\{ \Lambda_{A,p}(\mathbf{z}_{N,\tau}^*(x)) - \mathbf{E}^* \left[\Lambda_{A,p}(\mathbf{z}_{N,\tau}^*(x)) \right] \right\} d\tau dx.$$

Also, define

$$\begin{aligned} \alpha_{n,\mathbf{i}}^* &\equiv \frac{\sum_{A \in \mathcal{N}_J} \mu_A \Delta_{n,A,\mathbf{i}}^*}{\sqrt{Var^*(S_n^*)}} \text{ and} \\ u_{n,\mathbf{i}}^* &\equiv \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^N 1\{X_i^* \in B_{n,\mathbf{i}}\} - nP^*\{X_i^* \in B_{n,\mathbf{i}}\} \right\} \end{aligned}$$

and write

$$\frac{S_n^*}{\sqrt{Var^*(S_n^*)}} = \sum_{\mathbf{i} \in \mathcal{I}_n} \alpha_{n,\mathbf{i}}^* \text{ and } U_n^* = \sum_{\mathbf{i} \in \mathcal{I}_n} u_{n,\mathbf{i}}^*.$$

By the properties of Poisson processes, one can see that the array $\{(\alpha^*_{n,\mathbf{i}}, u^*_{n,\mathbf{i}})\}_{\mathbf{i}\in\mathcal{I}_n}$ is an array of 1-dependent random field under P^*. For any $q = (q_1, q_2) \in \mathbf{R}^2\backslash\{0\}$, let $y^*_{n,\mathbf{i}} \equiv q_1\alpha^*_{n,\mathbf{i}} + q_2 u^*_{n,\mathbf{i}}$ and write

$$Var^*\left(\sum_{\mathbf{i}\in\mathcal{I}_n} y^*_{n,\mathbf{i}}\right) = q_1^2 + q_2^2(1-\tilde{\alpha}_P) + 2q_1q_2\tilde{c}_{n,P},$$

uniformly over $P \in \mathcal{P}$, where $\tilde{c}_{n,P} = Cov^*(S^*_n, U^*_n)$. On the other hand, following the proof of Lemma A8 of Lee, Song, and Whang (2013) using Lemma C4, we deduce that

$$\sum_{\mathbf{i}\in\mathcal{I}_n} \mathbf{E}^*|y^*_{n,\mathbf{i}}|^r = o_P(1), \ \mathcal{P}\text{-uniformly}, \tag{C.15}$$

as $n \to \infty$, for any $r \in (2, (2p+2)/p]$, uniformly over $P \in \mathcal{P}$. By Theorem 1 of Shergin (1993), we have

$$\begin{aligned}
&\sup_{t\in\mathbf{R}}\left|P^*\left\{\frac{1}{\sqrt{q_1^2 + q_2^2(1-\tilde{\alpha}_P) + 2q_1q_2\tilde{c}_{n,P}}}\sum_{\mathbf{i}\in\mathcal{I}_n} y^*_{n,\mathbf{i}} \le t\right\} - \Phi(t)\right| \\
&\le \frac{C}{\{q_1^2 + q_2^2(1-\tilde{\alpha}_P) + 2q_1q_2\tilde{c}_{n,P}\}^{r/2}}\left\{\sum_{\mathbf{i}\in\mathcal{I}_n}\mathbf{E}^*|y^*_{n,\mathbf{i}}|^r\right\}^{1/2} = o_P(1),
\end{aligned}$$

for some $C > 0$ uniformly in $P \in \mathcal{P}$, by (C.15). By Lemma B2(i), we have for each $t \in \mathbf{R}$ and $q \in \mathbf{R}^2\backslash\{\mathbf{0}\}$ as $n \to \infty$,

$$\left|\mathbf{E}^*\left[\exp\left(it\frac{q^\top H^*_n}{\sqrt{q^\top \tilde{C}_n q}}\right)\right] - \exp\left(-\frac{t^2}{2}\right)\right| = o_P(1),$$

uniformly in $P \in \mathcal{P}$. Thus by Lemma B2(ii), for each $t \in \mathbf{R}^2$, we have

$$\left|P^*\left\{\tilde{C}_n^{-1/2}H^*_n \le t\right\} - P\left\{\mathbb{Z} \le t\right\}\right| = o_P(1).$$

Since the limit distribution of $\tilde{C}_n^{-1/2}H^*_n$ is continuous, the convergence above is uniform in $t \in \mathbf{R}^2$.

(ii) We fix $P \in \mathcal{P}$ such that $\text{limsup}_{n\to\infty}\sigma_n^2(\mathcal{C}) = 0$. Then by (C.14) above and Lemma C6,

$$Var^*(S^*_n) = \sigma^2_n(\mathcal{C}) + o_P(1) = o_P(1).$$

Hence, we find that $S^*_n = o_{P^*}(1)$ in P. The desired result follows by applying Theorem 1 of Shergin (1993) to the sum $U^*_n = \sum_{\mathbf{i}\in\mathcal{I}_n} u^*_{n,\mathbf{i}}$, and then applying Lemma B2. ∎

Lemma C9. *Let $\mathcal{C}$ be the Borel set in Lemma C8.*

(i) Suppose that the conditions of Lemma C8(i) are satisfied. Then for each $a > 0$, as $n \to \infty$,

$$\sup_{P \in \mathcal{P}} P \left\{ \sup_{t \in \mathbf{R}} \left| P \left\{ \frac{h^{-d/2} \sum_{A \in \mathcal{N}_J} \mu_A \left\{ \zeta_{n,A}^* - \mathbf{E}^* \zeta_{N,A}^* \right\}}{\sigma_n(\mathcal{C})} \leq t \right\} - \Phi(t) \right| > a \right\} \to 0.$$

(ii) Suppose that the conditions of Lemma C8(ii) are satisfied. Then for each $a > 0$, as $n \to \infty$,

$$\sup_{P \in \mathcal{P}} P \left\{ \left| h^{-d/2} \sum_{A \in \mathcal{N}_J} \mu_A \left\{ \zeta_{n,A}^* - \mathbf{E}^* \zeta_{N,A}^* \right\} \right| > a \right\} \to 0.$$

Proof of Lemma C9. The proofs are precisely the same as those of Lemma B9, except that we use Lemma C8 instead of Lemma B8 here. ∎

Lemma C10. *Suppose that the conditions of Lemma B5 hold. Then for any small $\nu > 0$, there exists a positive sequence $\varepsilon_n = o(h^d)$ such that for all $r \in [2, M/2]$ (with $M \geq 4$ being as in Assumption A6(i)),*

$$\sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E} ||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x) q_{n,\tau}(x;\eta_n)||^r = O\left(h^{-(r-2)\left(\frac{M-1}{M-2}\right)d - \nu} \right),$$

where $\eta_n \in \mathbf{R}^J$ is distributed as $N(0, \varepsilon_n I_J)$ and independent of $((Y_i^\top, X_i^\top)_{i=1}^\infty, N)$ in the definition of $q_{n,\tau}(x)$, and

$$\Sigma_{n,\tau,\varepsilon_n}(x) \equiv \Sigma_{n,\tau,\tau}(x,0) + \varepsilon_n I_J \text{ and } q_{n,\tau}(x;\eta_n) \equiv q_{n,\tau}(x) + \eta_n. \tag{C.16}$$

Suppose furthermore that $\lambda_{\min}(\Sigma_{n,\tau,\tau}(x,0)) > c > 0$ for some $c > 0$ that does not depend on n or $P \in \mathcal{P}$. Then

$$\sup_{(x,\tau) \in \mathcal{S}} \sup_{P \in \mathcal{P}} \mathbf{E} ||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x) q_{n,\tau}(x;\eta_n)||^r = O\left(h^{-(r-2)d/2} \right).$$

Proof of Lemma C10. We first establish the following fact.

Fact: Suppose that W is a random vector such that $\mathbf{E}||W||^2 \leq c_W$ for some constant $c_W > 0$. Then, for any $r \geq 2$ and a positive integer $m \geq 1$,

$$\mathbf{E}\left[||W||^r \right] \leq C_m \left(\mathbf{E}\left[||W||^{a_m(r)} \right] \right)^{1/(2^m)},$$

where $a_m(r) = 2^m(r-2) + 2$, and $C_m > 0$ is a constant that depends only on m and c_W.

Proof of Fact: The result follows by repeated application of Cauchy-Schwarz inequality:

$$\mathbf{E}||W||^r \leq \left(\mathbf{E}||W||^{2(r-1)} \right)^{1/2} \left(\mathbf{E}||W||^2 \right)^{1/2} \leq c_W^{1/2} \left(\mathbf{E}||W||^{2(r-1)} \right)^{1/2},$$

where we replace r on the left hand side by $2(r-1)$, and repeat the procedure to obtain Fact.

Let us consider the first statement of the lemma. Using Fact, we take a small $\nu_1 > 0$ and

$\varepsilon_n = h^{d+\nu_1}$, and choose a largest integer $m \geq 1$ such that $a_m(r) \leq M$. Such an m exists because $2 \leq r \leq M/2$. We bound

$$\mathbf{E}||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}(x;\eta_n)||^r \leq C_m \left(\mathbf{E}||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}(x;\eta_n)||^{a_m(r)}\right)^{1/(2^m)}.$$

By Lemma B5, we find that

$$\begin{aligned}
&\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\mathbf{E}||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}(x;\eta_n)||^{a_m(r)} \\
\leq\ & \sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}}\lambda_{\max}^{a_m(r)/2}\left(\Sigma_{n,\tau,\varepsilon_n}^{-1}(x)\right)\mathbf{E}||q_{n,\tau}(x;\eta_n)||^{a_m(r)} \\
\leq\ & \lambda_{\min}^{-a_m(r)/2}\left(\varepsilon_n I_J\right)h^{(1-(a_m(r)/2))d}.
\end{aligned} \tag{C.17}$$

By the definition of $\varepsilon_n = h^{d+\nu_1}$,

$$\varepsilon_n^{-a_m(r)/2}h^{(1-(a_m(r)/2))d} = h^{(1-a_m(r))d-a_m(r)\nu_1/2}.$$

We conclude that

$$\begin{aligned}
\mathbf{E}||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}(x;\eta_n)||^r &\leq C_m\left(h^{(1-a_m(r))d-a_m(r)\nu_1/2}\right)^{1/2^m} \\
&= C_m\left(h^{(-1-2^m(r-2))d-(2^m(r-2)+2)\nu_1/2}\right)^{1/2^m} \\
&= C_m h^{(-2^{-m}-(r-2))d-((r-2)+2^{-m+1})\nu_1/2}.
\end{aligned}$$

Since $a_m(r) \leq M$, or $2^{-m} \geq (r-2)/(M-2)$, the last term is bounded by

$$C_m h^{-(r-2)\left(\frac{M-1}{M-2}\right)d-\left((r-2)+\frac{2(r-2)}{M-2}\right)\nu_1/2}.$$

By taking ν_1 small enough, we obtain the desired result.

Now, let us turn to the second statement of the lemma. Since

$$\lambda_{\max}^{a_m(r)/2}\left(\Sigma_{n,\tau,\varepsilon_n}^{-1}(x)\right) < c^{-a_m(r)/2},$$

the last bound in (C.17) turns out to be

$$c^{-a_m(r)/2}h^{(1-(a_m(r)/2))d}.$$

Therefore, we conclude that

$$\begin{aligned}
\mathbf{E}||\Sigma_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}(x;\eta_n)||^r &\leq C_m\left(c^{-a_m(r)/2}h^{(1-(a_m(r)/2))d}\right)^{1/2^m} \\
&= C_m c^{-\{(r-2)+2^{1-m}\}/2}h^{(2^{-m}-\{(r-2)+2^{1-m}\}/2)d} \\
&= C_m c^{-\{(r-2)+2^{1-m}\}/2}h^{-(r-2)d/2}.
\end{aligned}$$

Again, using the inequality $2^{-m} \geq (r-2)/(M-2)$, we obtain the desired result. ∎

Lemma C11. *Suppose that the conditions of Lemma C5 hold. Then for any small $\nu > 0$, there exists a positive sequence $\varepsilon_n = o(h^d)$ such that for all $r \in [2, M/2]$ (with $M \geq 4$ being* *as in Assumption A6(i)),*

$$\sup_{(x,\tau)\in\mathcal{S}} \mathbf{E}^*||\tilde{\Sigma}_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}^*(x;\eta_n)||^r = O_P\left(h^{-(r-2)\left(\frac{M-1}{M-2}\right)d-\nu}\right), \text{ uniformly in } P\in\mathcal{P},$$

where $\eta_n \in \mathbf{R}^J$ *is distributed as* $N(0,\varepsilon_n I_J)$ *and independent of* $((Y_i^{*\top}, X_i^{*\top})_{i=1}^n, (Y_i^\top, X_i^\top)_{i=1}^n, N)$ *in the definition of* $q_{n,\tau}^*(x)$*, and*

$$\tilde{\Sigma}_{n,\tau,\varepsilon_n}(x) \equiv \tilde{\Sigma}_{n,\tau,\tau}(x,0) + \varepsilon_n I_J.$$

Suppose furthermore that

$$\sup_{(x,\tau)\in\mathcal{S}}\sup_{P\in\mathcal{P}} P\left\{\lambda_{\min}(\tilde{\Sigma}_{n,\tau,\tau}(x,0)) > c\right\} \to 0,$$

for some $c > 0$ *that does not depend on* n *or* $P \in \mathcal{P}$*. Then*

$$\sup_{(x,\tau)\in\mathcal{S}} \mathbf{E}^*||\tilde{\Sigma}_{n,\tau,\varepsilon_n}^{-1/2}(x)q_{n,\tau}^*(x;\eta_n)||^r = O_P\left(h^{-(r-2)d/2}\right), \text{ uniformly in } P\in\mathcal{P}.$$

Proof of Lemma C11. The proof is precisely the same as that of Lemma C10, where we use Lemma C5 instead of Lemma B5. ∎

We let for a sequence of Borel sets B_n in $\mathcal{S}$ and $\lambda \in \{0, d/4, d/2\}$, $A \subset \mathbb{N}_J$, and a fixed bounded function δ on $\mathcal{S}$,

$$\begin{aligned}
a_n^R(B_n) &\equiv \int_{B_n} \mathbf{E}\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x) + h^\lambda\delta(x,\tau))\right] dQ(x,\tau) \\
a_n^{R*}(B_n) &\equiv \int_{B_n} \mathbf{E}^*\left[\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}^*(x) + h^\lambda\delta(x,\tau))\right] dQ(x,\tau), \text{ and} \\
a_n(B_n) &\equiv \int_{B_n} \mathbf{E}\left[\Lambda_{A,p}(\mathbb{W}_{n,\tau,\tau}^{(1)}(x,0) + h^\lambda\delta(x,\tau))\right] dQ(x,\tau),
\end{aligned}$$

where $\mathbf{z}_{N,\tau}^*(x)$ is a random vector whose j-th entry is given by

$$z_{N,\tau,j}^*(x) \equiv \frac{1}{nh^d}\sum_{i=1}^N \beta_{n,x,\tau,j}(Y_{ij}^*, (X_i^*-x)/h) - \frac{1}{h^d}\mathbf{E}^*\left[\beta_{n,x,\tau,j}(Y_{ij}^*, (X_i^*-x)/h)\right].$$

Lemma C12. *Suppose that the conditions of Lemmas C10 and C11 hold and that*

$$n^{-1/2}h^{-\left(\frac{3M-4}{2M-4}\right)d-\nu} \to 0,$$

as $n\to\infty$*, for some small* $\nu > 0$*. Then for any sequence of Borel sets* B_n *in* $\mathcal{S}$*,*

$$\begin{aligned}
\sup_{P\in\mathcal{P}} \left|a_n^R(B_n) - a_n(B_n)\right| &= o(h^{d/2}) \text{ and} \\
\sup_{P\in\mathcal{P}} P\left\{\left|a_n^{R*}(B_n) - a_n(B_n)\right| > ah^{d/2}\right\} &= o(1).
\end{aligned}$$

Proof of Lemma C12. For the statement, it suffices to show that uniformly in $P \in \mathcal{P}$,

$$\text{(C.18)} \qquad \sup_{(x,\tau)\in\mathcal{S}} \left| \begin{array}{c} \mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x) + h^\lambda\delta(x,\tau)) \\ -\mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0) + h^\lambda\delta(x,\tau)) \end{array} \right| = o(h^{d/2}), \text{ and}$$

$$\sup_{(x,\tau)\in\mathcal{S}} \left| \begin{array}{c} \mathbf{E}^*\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}^*_{N,\tau}(x) + h^\lambda\delta(x,\tau)) \\ -\mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau,\tau}(x,0) + h^\lambda\delta(x,\tau)) \end{array} \right| = o_P(h^{d/2}).$$

We prove the first statement of (C.18). The proof of the second statement of (C.18) can be done in a similar way.

Take small $\nu > 0$. We apply Lemma C10 by choosing a positive sequence $\varepsilon_n = o(h^d)$ such that for any $r \in [2, M/2]$,

$$\text{(C.19)} \qquad \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}||\Sigma^{-1/2}_{n,\tau,\varepsilon_n}(x) q_{n,\tau}(x;\eta_n)||^r = O\left(h^{-(r-2)\left(\frac{M-1}{M-2}\right)d-\nu}\right),$$

where $q_{n,\tau}(x;\eta_n)$ and $\Sigma_{n,\tau,\varepsilon_n}(x)$ are as in Lemma C10. We follow the arguments in the proof of Step 2 in Lemma B6 to bound the left-hand side in the first supremum in (C.18) by

$$\sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \left| \mathbf{E}\Lambda_{A,p}(\sqrt{nh^d}\mathbf{z}_{N,\tau}(x;\eta_n) + h^\lambda\delta(x,\tau)) - \mathbf{E}\Lambda_{A,p}(\mathbb{W}^{(1)}_{n,\tau,\tau,\varepsilon_n}(x,0) + h^\lambda\delta(x,\tau)) \right| + C\sqrt{\varepsilon_n},$$

for some $C > 0$, where

$$\mathbf{z}_{N,\tau}(x;\eta_n) \equiv \mathbf{z}_{N,\tau}(x) + \eta_n/\sqrt{nh^d},$$

and $\mathbb{W}^{(1)}_{n,\tau,\tau,\varepsilon_n}(x,0)$ is as defined in (B.17). Let

$$\begin{aligned} \xi_{N,\tau}(x;\eta_n) &\equiv \sqrt{nh^d}\Sigma^{-1/2}_{n,\tau,\varepsilon_n}(x) \cdot \mathbf{z}_{N,\tau}(x;\eta_n) \text{ and} \\ \mathbb{Z}^{(1)}_{n,\tau,\tau,\varepsilon_n}(x,0) &\equiv \Sigma^{-1/2}_{n,\tau,\varepsilon_n}(x) \cdot \mathbb{W}^{(1)}_{n,\tau,\tau,\varepsilon_n}(x,0). \end{aligned}$$

We rewrite the previous absolute value as

$$\text{(C.20)} \qquad \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \left| \mathbf{E}\Lambda^\Sigma_{A,n,p}(\sqrt{nh^d}\xi_{N,\tau}(x;\eta_n)) - \mathbf{E}\Lambda^\Sigma_{n,p}(\mathbb{Z}^{(1)}_{n,\tau,\tau,\varepsilon_n}(x,0)) \right|,$$

where $\Lambda^\Sigma_{A,n,p}(\mathbf{v}) \equiv \Lambda_{A,p}(\Sigma^{1/2}_{n,\tau,\varepsilon_n}(x)\mathbf{v} + h^\lambda\delta(x,\tau))$. Note that the condition for M in Assumption A6(i) that $M \geq 2(p+2)$, we can choose $r = \max\{p, 3\}$. Then $r \in [2, M/2]$ as required. Using Theorem 1 of Sweeting (1977), we bound the above supremum by (with $r = \max\{p, 3\}$)

$$\begin{aligned} &\frac{C_1}{\sqrt{n}} \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}||\Sigma^{-1/2}_{n,\tau,\varepsilon_n}(x) q_{n,\tau}(x;\eta_n)||^3 \\ &+ \frac{C_2}{\sqrt{n^{r-2}}} \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}||\Sigma^{-1/2}_{n,\tau,\varepsilon_n}(x) q_{n,\tau}(x;\eta_n)||^r \\ &+ C_3 \sup_{(x,\tau)\in\mathcal{S}} \sup_{P\in\mathcal{P}} \mathbf{E}\omega_{n,p}\left(\mathbb{Z}^{(1)}_{n,\tau,\tau,\varepsilon_n}(x,0); \frac{C_4}{\sqrt{n}}\mathbf{E}||\Sigma^{-1/2}_{n,\tau,\varepsilon_n}(x) q_{n,\tau}(x;\eta_n)||^3\right), \end{aligned}$$

for some positive constants C_1, C_2, C_3, and C_4, where

$$\omega_{n,p}(\mathbf{v}; c) \equiv \sup\left\{|\Lambda^{\Sigma}_{A,n,p}(\mathbf{v}) - \Lambda^{\Sigma}_{A,n,p}(\mathbf{y})| : \mathbf{y} \in \mathbf{R}^{|A|}, ||\mathbf{v}-\mathbf{y}|| \leq c\right\}.$$

The proof is complete by (C.19) and by the condition $n^{-1/2}h^{-\left(\frac{3M-4}{2M-4}\right)d-\nu} \to 0$. ∎

Appendix D. Proof of Theorem AUC1

The conclusion of Theorem AUC1 follows immediately from Theorem 1, provided that all the regularity conditions in Theorem 1 are satisfied. The following lemma shows that Assumptions AUC1-AUC4 are sufficient conditions for that purpose. One key condition to check the regularity condition of Theorem 1 is to establish asymptotic linear representations in Assumptions A1 and B1. We borrow the results from Lee, Song, and Whang (2015).

Lemma AUC 1. *Suppose that Assumptions AUC1-AUC4 hold. Then Assumptions A1-A6 and B1-B4 hold with the following definitions:* $J = 2,\ r_{n,j} \equiv \sqrt{nh^d}$,

$$\begin{aligned} v_{n,\tau,1}(x) &\equiv \mathbf{e}_1^\top\{\gamma_{\tau,2}(x) - \gamma_{\tau,3}(x)\}, \\ v_{n,\tau,2}(x) &\equiv \underline{b} - \mathbf{e}_1^\top\{2\gamma_{\tau,2}(x) - \gamma_{\tau,3}(x)\}, \\ \beta_{n,x,\tau,1}(Y_i, z) &\equiv \alpha_{n,x,\tau,2}(Y_i, z) - \alpha_{n,x,\tau,3}(Y_i, z),\ \textit{and} \\ \beta_{n,x,\tau,2}(Y_i, z) &\equiv -2\alpha_{n,x,\tau,2}(Y_i, z) + \alpha_{n,x,\tau,3}(Y_i, z), \end{aligned}$$

where $\tilde{l}_\tau(u) \equiv \tau - 1\{u \leq 0\}$, $Y_i = \{(B_{\ell i}, L_i) : \ell = 1, \ldots, L_i\}$, *and*

$$\alpha_{n,x,\tau,k}(Y_i, z) \equiv -1\{L_i = k\} \sum_{l=1}^{k} \tilde{l}_\tau\left(B_{\ell i} - \gamma_{\tau,k}^\top(x) \cdot H \cdot c(z)\right) \mathbf{e}_1^\top M_{n,\tau,k}^{-1}(x) c(z) K(z).$$

Proof of Lemma AUC1. First, let us turn to Assumption A1. By Assumptions AUC2 and AUC3, it suffices to consider $\hat{v}_{\tau,2}(x)$ that uses $\underline{b}$ instead of $\hat{\underline{b}}$. The asymptotic linear representation in Assumption A1 follows from Theorem 1 of Lee, Song, and Whang (2015). The error rate $o_P(\sqrt{h^d})$ in Assumption A1 is satisfied, because

$$h^{-d/2}\left(\frac{\log^{1/2} n}{n^{1/4}h^{d/4}}\right) = n^{-1/4}h^{-3d/4}\log^{1/2} n \to 0, \tag{D.1}$$

by Assumption AUC2(ii) and the condition $r > 3d/2 - 1$. Assumption A2 follows because both $\beta_{n,x,\tau,1}(Y_i, z)$ and $\beta_{n,x,\tau,2}(Y_i, z)$ have a multiplicative component of $K(z)$ which has a compact support by Assumption AUC2(i). As for Assumption A3, we use Lemma 2. First define

$$\begin{aligned} e_{x,\tau,k,li} &\equiv 1\left\{L_i = k\right\} \tilde{l}_\tau \left(B_{li} - \gamma_{\tau,k}^\top(x) \cdot H \cdot c\left(\frac{X_i - x}{h}\right)\right) \text{ and} \\ \xi_{x,\tau,k,i} &\equiv \mathbf{e}_1^\top M_{n,\tau,k}^{-1}(x) c\left(\frac{X_i - x}{h}\right) K\left(\frac{X_i - x}{h}\right) \end{aligned}$$

First observe that for each fixed $x_2 \in \mathbf{R}^d$, $\tau_2 \in \mathcal{T}$, and $\lambda > 0$,

$$\begin{aligned} &\mathbf{E}\left[\sup_{||x_2 - x_3|| + ||\tau_2 - \tau_3|| \leq \lambda} \left(\alpha_{n,x_2,\tau_2,2}\left(Y_i, \frac{X_i - x_2}{h}\right) - \alpha_{n,x_3,\tau_3,2}\left(Y_i, \frac{X_i - x_3}{h}\right)\right)^2\right] \quad \text{(D.2)} \\ &\leq 2\sum_{l=1}^{k} \mathbf{E}\left[\mathbf{E}\left[\sup_{||x_2 - x_3|| + ||\tau_2 - \tau_3|| \leq \lambda} \left(e_{x_2,\tau_2,k,li} - e_{x_3,\tau_3,k,li}\right)^2 |X_i\right] \xi_{x_2,\tau_2,k,i}^2\right] \\ &\quad + 2\sum_{l=1}^{k} \mathbf{E}\left[\sup_{||x_2 - x_3|| + ||\tau_2 - \tau_3|| \leq \lambda} \left(\xi_{x_2,\tau_2,k,i} - \xi_{x_3,\tau_3,k,i}\right)^2\right]. \end{aligned}$$

Using Lipschitz continuity of the conditional density of B_{li} given $L_i = k$ and $X_i = x$ in (x, τ) and Lipschitz continuity of $\gamma_{\tau,k}(x)$ in (x, τ) (Assumption AUC1), we find that the first term is bounded by $Ch^{-s_1}\lambda$ for some $C > 0$ and $s_1 > 0$. Since

$$M_{n,\tau,k}(x) = kP\left\{L_i = k|X_i = x\right\} f_{\tau,k}(0|x) f(x) \int K(t)c(t)c(t)^\top dt + o(1),$$

we find that $M_{n,\tau,k}^{-1}(x)$ is Lipschitz continuous in (x, τ) by Assumptions AUC1. Hence the last term in (D.2) is also bounded by $Ch^{-s_2}\lambda^2$ for some $C > 0$ and $s_2 > 0$. Therefore, if we take

$$b_{n,ij}(x, \tau) = \alpha_{n,x,\tau,2}\left(Y_i, \frac{X_i - x}{h}\right),$$

this function satisfies the condition in Lemma 2. Also, observe that

$$\mathbf{E}\left[\left|\alpha_{n,x,\tau,2}\left(Y_i, \frac{X_i - x}{h}\right)\right|^4\right] \leq C,$$

because $\alpha_{n,x,\tau,2}(\cdot,\cdot)$ is uniformly bounded. We also obtain the same result for $\alpha_{n,x,\tau,3}(\cdot,\cdot)$. Thus the conditions of Lemma 2 are satisfied with $b_{n,ij}(x, \tau)$ taken to be $\beta_{n,x,\tau,1}(Y_i, (X_i - x)/h)$ or $\beta_{n,x,\tau,2}(Y_i, (X_i - x)/h)$. Now Assumption A3 follows from Lemma 2(i). The rate condition in Assumption A4(i) is satisfied by Assumption AUC2(ii). Assumption A4(ii) is imposed directly by Assumption AUC4(i). Since we are taking $\hat{\sigma}_{\tau,j}(x) = \hat{\sigma}_{\tau,j}^*(x) = 1$, it suffices to take $\sigma_{n,\tau,j}(x) = 1$ in Assumption A5 and Assumption B3. Assumption A6(i) is satisfied because $\beta_{n,x,\tau,j}$ is bounded. Assumption A6(ii) is imposed directly by Assumption AUC4(ii). Assumption B1 follows by Lemma QR2 of Lee, Song, and Whang (2015). Assumption B2 follows from Lemma 2(ii). Assumption B4 follows from the rate condition in Assumption

AUC2(ii). In fact, when $\beta_{n,x,\tau,j}$ is bounded, the rate condition in Assumption B4 is reduced to $n^{-1/2}h^{-3d/2-\nu} \to 0$, as $n \to \infty$, for some small number $\nu > 0$. ∎

Appendix E. Potential Areas of Applications

Econometric models of games belong to a related but distinct branch of the literature, compared to the auction models. In this literature, inference on many game theoretic models are recently based on partial identification or functional inequalities. For example, see Tamer (2003), Andrews, Berry, and Jia (2004), Berry and Tamer (2007), Aradillas-López and Tamer (2008), Ciliberto and Tamer (2009), Beresteanu, Molchanov, and Molinari (2011), Galichon and Henry (2011), Chesher and Rosen (2012), and Aradillas-López and Rosen (2013), among others. See de Paula (2013) and references therein for a broad recent development in this literature. Our general method provides researchers in this field with a new inference tool when they have continuous covariates.

Inequality restrictions also arise in testing revealed preferences. Blundell, Browning, and Crawford (2008) used revealed preference inequalities to provide the nonparametric bounds on average consumer responses to price changes. In addition, Blundell, Kristensen, and Matzkin (2014) used the same inequalities to bound quantile demand functions. It would be possible to use our framework to test revealed preference inequalities either in average demand functions or in quantile demand functions. See also Hoderlein and Stoye (2014) and Kitamura and Stoye (2013) for related issues of testing revealed preference inequalities.

In addition to the literature mentioned above, many results on partial identification can be written as functional inequalities. See, e.g., Imbens and Manski (2004), Manski (2003), Manski (2007), Manski and Pepper (2000), Tamer (2010), Chesher and Rosen (2017), and references therein.

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