Supplementary material for
"Quantile double autoregression"

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Abstract

This supplementary material provides additional results for the simulation and empirical analysis.

1 Self-weighted CQE

In the first experiment in Section 5.1, owing to space limitations, we only reported the results with $F_b(\cdot)$ being the standard normal or Student's t_5 distribution function. This section provides additional results with $F_b(\cdot)$ being the Student's t_3 distribution function in Tables S.1 and S.2. The simulation findings are similar to those in Section 5.1.

2 Self-weighted residual QACFs

In the third experiment in Section 5.2, owing to space limitations, we only reported the results of data process (5.1) with setting (5.2) and $F_b(\cdot)$ being the standard normal or Student's t_5 distribution function. This section presents additional results of data process (5.1) with setting (5.2) and $F_b(\cdot)$ being the Student's t_3 distribution function in Tables S.3 and S.4, and reports all of the results of data process (5.1) with setting (5.3) in Tables S.5 and S.6. The simulation findings are similar to those in Section 5.2.

Table S.1: Biases, empirical standard deviations (ESDs), and asymptotic standard deviations (ASDs) of $\widehat{\boldsymbol{\theta}}_{\tau n}$ at quantile level $\tau = 0.05$ or 0.25 for model (5.1) with coefficient functions (5.2), where ASD_1 and ASD_2 correspond to the bandwidths h_B and h_{HS}, respectively. $F(\cdot)$ is the Student's t_3 distribution function.

		$\tau = 0.05$					$\tau = 0.25$				
n		True	Bias	ESD	ASD_1	ASD_2	True	Bias	ESD	ASD_1	ASD_2
						Student's t_3 distribution					
500	b	-5.538	-0.180	1.818	2.179	2.081	-0.585	-0.012	0.203	0.218	0.232
	ϕ	-0.200	-0.006	0.349	0.436	0.416	-0.200	0.000	0.109	0.126	0.128
	β	-2.215	-0.359	1.618	2.082	2.016	-0.234	-0.024	0.173	0.187	0.226
1000	b	-5.538	-0.053	1.250	1.494	1.368	-0.585	-0.012	0.142	0.152	0.145
	ϕ	-0.200	-0.006	0.235	0.287	0.266	-0.200	0.000	0.081	0.086	0.083
	β	-2.215	-0.212	1.103	1.418	1.261	-0.234	-0.011	0.121	0.128	0.123

3 Three portmanteau tests

In the fourth experiment in Section 5.2, to save space we only reported the results of data process (5.5) with bandwidth h_{HS}. In this section, for ease of comparison, we summarize the rejection rates of data process (5.5) with two bandwidths, h_{HS} and h_B, in Table S.7. We find the two bandwidths perform similarly in terms of both size and power, and the other findings are the same as those in Section 5.2.

Moreover, we also consider the following data generating process,

$$y_t = 0.1y_{t-1} + c_1 y_{t-2} + S_{\mathrm{Q}}\left(b(u_t) + 0.1u_t b(u_t) y_{t-1}^2 + c_2 b(u_t) y_{t-2}^2\right), \qquad \text{(S.1)}$$

where all the other settings are the same as those in the fourth experiment in Section 5.2. Table S.8 gives the rejection rates of $Q_1(6)$, $Q_2(6)$, and their combined version $Q(6)$, where the critical values of the three tests are calculated based on the estimated covariance matrix $\widehat{\Pi}(\tau)$ using two bandwidths, h_{HS} and h_B. The findings are the same as those for data process (5.5) in Section 5.

4 Finite sample comparison of efficiency

In the last experiment in Section 5.3, we only illustrated the boxplots when the sample size is set to 2000. This section provides the boxplots in Figures S.1, S.2, and S.3 with the

Table S.2: Biases, empirical standard deviations (ESDs), and asymptotic standard deviations (ASDs) of $\widehat{\boldsymbol{\theta}}_{\tau n}$ at quantile level $\tau = 0.05$ or 0.25 for model (5.1) with coefficient functions (5.3), where ASD_1 and ASD_2 correspond to the bandwidths h_B and h_{HS}, respectively. $F(\cdot)$ is the Student's t_3 distribution function.

		$\tau = 0.05$					$\tau = 0.25$				
n		True	Bias	ESD	ASD_1	ASD_2	True	Bias	ESD	ASD_1	ASD_2
						Student's t_3 distribution					
500	b	-5.538	-0.030	1.491	2.555	1.836	-0.585	-0.006	0.182	0.197	0.184
	ϕ	0.025	-0.012	0.282	0.425	0.350	0.125	-0.005	0.089	0.104	0.100
	β	-0.138	-0.438	0.848	2.447	1.299	-0.073	-0.021	0.096	0.121	0.117
1000	b	-5.538	0.022	0.998	1.318	1.489	-0.585	-0.007	0.121	0.135	0.128
	ϕ	0.025	-0.012	0.184	0.231	0.274	0.125	-0.003	0.065	0.071	0.067
	β	-0.138	-0.238	0.487	0.875	1.311	-0.073	-0.010	0.066	0.080	0.074

Table S.3: Biases ($\times 100$), empirical standard deviations (ESDs) ($\times 100$), and asymptotic standard deviations (ASDs) ($\times 100$) of $\widehat{\rho}_{k,\tau}$ with $k = 2$, 4, or 6 for model (5.2). The quantile level is $\tau = 0.05$ or 0.25, and ASD_1 and ASD_2 correspond to the bandwidths h_B and h_{HS}, respectively. $F(\cdot)$ is the Student's t_3 distribution function.

		$\tau = 0.05$				$\tau = 0.25$			
n	Lag	Bias	ESD	ASD_1	ASD_2	Bias	ESD	ASD_1	ASD_2
					Student's t_3 distribution				
500	2	0.00	1.75	1.61	1.63	-0.09	1.55	1.46	1.47
	4	0.07	2.30	2.29	2.36	-0.10	2.40	2.20	2.21
	6	0.01	2.60	2.56	2.55	-0.14	2.62	2.50	2.52
1000	2	-0.01	1.15	1.06	1.06	-0.04	0.98	0.95	0.95
	4	0.06	1.51	1.58	1.58	-0.02	1.56	1.50	1.50
	6	0.03	1.70	1.78	1.78	-0.10	1.79	1.73	1.73

sample size of 1000. All conclusions remain unchanged.

Table S.4: Biases ($\times$100), empirical standard deviations (ESDs) ($\times$100), and asymptotic standard deviations (ASDs) ($\times$100) of $\widehat{r}_{k,\tau}$ with $k = 2$, 4, or 6 for model (5.2). The quantile level is $\tau = 0.05$ or 0.25, and ASD_1 and ASD_2 correspond to the bandwidths h_B and h_{HS}, respectively. $F(\cdot)$ is the Student's t_3 distribution function.

		$\tau = 0.05$				$\tau = 0.25$			
n	Lag	Bias	ESD	ASD_1	ASD_2	Bias	ESD	ASD_1	ASD_2
				Student's t_3 distribution					
500	2	0.02	1.79	1.64	1.66	-0.04	1.51	1.42	1.49
	4	0.08	2.33	2.30	2.37	-0.05	2.36	2.17	2.18
	6	0.02	2.64	2.55	2.55	-0.01	2.48	2.48	2.54
1000	2	-0.01	1.16	1.07	1.07	-0.02	1.01	0.92	0.92
	4	0.06	1.53	1.58	1.58	0.01	1.59	1.47	1.47
	6	0.04	1.69	1.78	1.77	-0.06	1.72	1.71	1.71

Table S.5: Biases (×100), empirical standard deviations (ESDs) (×100), and asymptotic standard deviations (ASDs) (×100) of $\widehat{\rho}_{k,\tau}$ at quantile level $\tau = 0.05$ or 0.25 and $k = 2$, 4, or 6 for model (5.3), where ASD_1 and ASD_2 correspond to the bandwidths h_B and h_{HS}, respectively. $F(\cdot)$ is the normal, Student's t_5 or Student's t_3 distribution function.

		$\tau = 0.05$				$\tau = 0.25$			
n	Lag	Bias	ESD	ASD_1	ASD_2	Bias	ESD	ASD_1	ASD_2
				Normal distribution					
500	2	-0.07	2.71	2.83	3.29	-0.13	2.72	2.72	2.73
	4	-0.06	3.04	3.18	3.49	-0.14	3.16	3.11	3.11
	6	-0.01	3.12	3.29	3.48	-0.26	3.15	3.14	3.14
1000	2	-0.07	1.96	1.96	1.92	-0.14	1.90	1.92	1.91
	4	-0.05	2.14	2.21	2.23	-0.13	2.21	2.19	2.19
	6	-0.10	2.22	2.26	2.25	-0.18	2.27	2.22	2.22
				Student's t_5 distribution					
500	2	-0.04	2.24	2.71	2.23	-0.07	2.18	2.19	2.22
	4	-0.09	2.86	3.10	2.87	-0.10	2.92	2.84	2.87
	6	-0.02	2.86	3.10	2.96	-0.29	3.01	2.95	2.96
1000	2	-0.04	1.61	1.53	1.64	-0.10	1.52	1.51	1.61
	4	-0.06	1.99	2.02	2.09	-0.10	2.00	1.99	2.04
	6	-0.09	2.06	2.10	2.16	-0.18	2.13	2.09	2.11
				Student's t_3 distribution					
500	2	-0.01	1.82	1.85	1.83	-0.05	1.72	1.69	1.71
	4	-0.05	2.66	2.68	2.58	-0.09	2.67	2.53	2.54
	6	0.03	2.68	2.86	2.78	-0.28	2.87	2.77	2.78
1000	2	-0.04	1.30	1.15	1.19	-0.08	1.18	1.12	1.11
	4	-0.02	1.74	1.77	1.79	-0.10	1.77	1.75	1.74
	6	-0.09	1.92	1.96	2.03	-0.18	2.02	1.96	1.96

Table S.6: Biases ($\times 100$), empirical standard deviations (ESDs) ($\times 100$), and asymptotic standard deviations (ASDs) ($\times 100$) of $\widehat{r}_{k,\tau}$ at quantile level $\tau = 0.05$ or 0.25 and $k = 2$, 4, or 6 for model (5.3), where ASD_1 and ASD_2 correspond to the bandwidths h_B and h_{HS}, respectively. $F(\cdot)$ is the normal, Student's t_5, or Student's t_3 distribution function.

		$\tau = 0.05$				$\tau = 0.25$			
n	Lag	Bias	ESD	ASD_1	ASD_2	Bias	ESD	ASD_1	ASD_2
				Normal distribution					
500	2	-0.06	2.70	2.81	3.30	-0.01	2.59	2.64	2.66
	4	-0.06	3.03	3.17	3.49	-0.14	3.13	3.09	3.09
	6	0.06	3.13	3.29	3.48	-0.15	3.16	3.13	3.13
1000	2	-0.07	1.95	1.96	1.92	-0.01	1.83	1.85	1.84
	4	-0.06	2.14	2.20	2.22	-0.13	2.21	2.18	2.18
	6	-0.06	2.21	2.26	2.25	-0.11	2.25	2.22	2.21
				Student's t_5 distribution					
500	2	-0.02	2.24	2.60	2.23	0.07	2.06	2.10	2.12
	4	-0.08	2.87	3.08	2.86	-0.10	2.84	2.81	2.85
	6	0.08	2.87	3.10	2.95	-0.17	3.05	2.93	2.96
1000	2	-0.04	1.59	1.54	1.63	0.00	1.43	1.44	1.45
	4	-0.07	1.99	2.01	2.08	-0.10	2.01	1.97	2.01
	6	-0.05	2.02	2.10	2.16	-0.11	2.13	2.08	2.12
				Student's t_3 distribution					
500	2	-0.03	1.80	1.88	1.83	0.05	1.61	1.60	1.60
	4	-0.05	2.64	2.66	2.56	-0.10	2.59	2.49	2.50
	6	0.14	2.71	2.84	2.77	-0.15	2.91	2.76	2.77
1000	2	-0.07	1.27	1.15	1.19	-0.01	1.09	1.06	1.04
	4	-0.03	1.71	1.75	1.78	-0.10	1.80	1.71	1.71
	6	-0.02	1.89	1.96	2.02	-0.11	2.02	1.94	1.94

Table S.7: Rejection rate (%) of test statistics $Q_1(6)$, $Q_2(6)$, and $Q(6)$ for setting (5.5), where Q_i^B (or Q^B) and Q_i^{HS} (or Q^{HS}) correspond to the bandwidths h_B and h_{HS}, respectively. The significance level is 5%, and the quantile level is $\tau = 0.05$ or 0.25. $F(\cdot)$ is the normal, Student's t_5, or Student's t_3 distribution function.

			$\tau = 0.05$						$\tau = 0.25$					
n	c_1	c_2	Q_1^B	Q_1^{HS}	Q_2^B	Q_2^{HS}	Q^B	Q^{HS}	Q_1^B	Q_1^{HS}	Q_2^B	Q_2^{HS}	Q^B	Q^{HS}
							Normal distribution							
500	0.0	0.0	3.6	3.5	3.7	3.7	3.7	3.6	5.4	5.5	4.6	4.4	5.7	5.8
	0.1	0.0	8.4	8.3	7.4	7.3	7.5	7.4	15.7	15.7	9.1	9.0	13.3	13.3
	0.3	0.0	47.8	47.3	42.2	41.8	46.1	45.6	92.7	92.4	53.5	53.4	89.7	89.5
	0.0	0.1	6.2	6.3	6.1	6.2	6.6	6.6	6.5	6.5	6.3	6.4	6.5	6.5
	0.0	0.3	14.1	13.7	14.4	14.1	13.9	13.7	7.5	7.4	11.2	11.2	9.5	9.5
1000	0.0	0.0	5.3	5.2	5.2	5.2	5.2	5.3	4.7	4.7	6.3	6.2	5.8	5.7
	0.1	0.0	13.1	12.9	11.6	11.4	12.1	12.2	28.4	28.1	16.0	16.0	25.1	25.1
	0.3	0.0	84.0	83.6	78.6	78.3	82.0	81.4	99.9	99.9	90.6	90.2	99.8	99.8
	0.0	0.1	7.7	7.4	7.1	7.2	7.6	7.4	5.6	5.7	6.5	6.5	5.8	5.7
	0.0	0.3	16.9	16.5	19.2	19.2	18.2	18.1	6.3	6.2	16.2	16.3	11.8	11.8
							Student's t_5 distribution							
500	0.0	0.0	5.6	5.6	5.5	5.5	5.4	5.6	5.7	5.7	5.2	5.2	5.4	5.3
	0.1	0.0	7.4	7.8	6.2	6.1	6.9	6.8	16.0	15.7	6.7	6.6	11.5	11.5
	0.3	0.0	32.9	32.4	19.2	19.1	26.9	26.9	93.6	93.4	28.4	27.8	86.6	86.3
	0.0	0.1	11.9	11.8	12.3	12.3	12.4	12.2	6.8	6.8	8.1	8.2	7.7	7.7
	0.0	0.3	30.7	31.1	31.7	32.0	32.8	32.9	7.8	7.6	18.1	17.9	16.2	16.0
1000	0.0	0.0	5.6	5.6	5.8	5.6	5.5	5.6	4.6	4.6	6.6	6.5	5.2	5.2
	0.1	0.0	10.0	10.3	6.8	7.0	7.7	7.8	30.3	30.0	11.0	11.0	23.5	23.4
	0.3	0.0	61.6	60.7	38.2	37.9	54.1	53.5	99.9	99.8	57.4	56.8	99.7	99.7
	0.0	0.1	13.9	13.9	15.2	15.2	14.0	14.1	6.4	6.4	10.7	10.5	9.3	9.1
	0.0	0.3	42.3	42.7	51.1	52.0	49.6	49.9	8.7	8.6	30.0	30.0	23.4	23.2
							Student's t_3 distribution							
500	0.0	0.0	7.1	6.9	6.4	6.3	6.7	6.6	5.4	5.3	4.6	4.6	5.3	5.2
	0.1	0.0	8.8	8.9	7.1	7.4	8.1	8.3	15.0	14.8	6.2	6.1	10.9	10.7
	0.3	0.0	27.9	28.1	13.8	14.1	21.9	22.1	91.9	91.9	14.7	14.4	82.9	82.8
	0.0	0.1	16.9	16.8	17.4	17.2	18.1	18.1	7.5	7.5	9.2	9.0	8.3	8.5
	0.0	0.3	42.0	41.8	43.1	43.0	43.6	43.3	7.1	7.2	20.3	20.6	17.4	17.5
1000	0.0	0.0	5.6	5.8	6.3	6.3	6.5	6.5	4.5	4.5	5.5	5.4	5.3	5.2
	0.1	0.0	9.9	10.0	7.3	7.2	9.1	9.0	29.4	29.3	6.3	6.3	21.2	21.0
	0.3	0.0	50.6	50.5	22.5	22.3	40.0	39.8	99.8	99.7	25.1	25.0	99.4	99.3
	0.0	0.1	24.7	24.5	28.0	27.4	27.2	27.0	7.7	7.7	13.4	13.2	12.2	12.2
	0.0	0.3	60.0	60.3	64.1	63.8	63.8	63.6	8.3	8.3	35.0	34.7	28.3	28.4

Table S.8: Rejection rate (%) of test statistics $Q_1(6)$, $Q_2(6)$, and $Q(6)$ for setting (S.1), where Q_i^B (or Q^B) and Q_i^{HS} (or Q^{HS}) correspond to the bandwidths h_B and h_{HS}, respectively. The significance level is 5%, and the quantile level is $\tau = 0.05$ or 0.25. $F(\cdot)$ is the normal, Student's t_5, or Student's t_3 distribution function.

			$\tau = 0.05$						$\tau = 0.25$					
n	c_1	c_2	Q_1^B	Q_1^{HS}	Q_2^B	Q_2^{HS}	Q^B	Q^{HS}	Q_1^B	Q_1^{HS}	Q_2^B	Q_2^{HS}	Q^B	Q^{HS}
			Normal distribution											
500	0.0	0.0	4.5	4.4	4.7	4.6	4.7	4.7	5.7	5.6	5.5	5.2	6.0	5.9
	0.1	0.0	8.3	8.3	7.3	7.1	7.3	7.6	16.2	16.1	9.1	9.1	14.2	14.1
	0.3	0.0	50.6	50.5	45.2	44.9	48.5	48.8	93.5	93.6	56.2	56.3	89.9	89.7
	0.0	0.1	7.4	7.2	7.4	7.1	7.0	7.1	6.1	6.3	7.0	6.9	6.8	7.1
	0.0	0.3	14.2	14.1	14.4	14.5	13.7	13.6	5.9	5.9	12.5	12.4	10.7	10.6
1000	0.0	0.0	5.0	5.2	5.3	5.4	5.4	5.6	4.7	4.6	4.6	4.5	4.5	4.4
	0.1	0.0	12.9	12.5	12.3	12.1	13.1	12.6	29.3	29.5	16.8	16.8	26.0	25.7
	0.3	0.0	87.3	86.8	82.8	82.2	85.5	85.0	99.9	100.0	91.9	91.7	99.9	99.9
	0.0	0.1	8.2	8.2	7.7	7.7	7.7	7.5	5.3	5.3	6.6	6.6	6.4	6.5
	0.0	0.3	13.9	13.7	18.1	18.1	17.1	16.8	6.4	6.3	16.3	16.0	12.6	12.8
			Student's t_5 distribution											
500	0.0	0.0	5.3	5.5	5.3	5.7	5.0	5.2	6.2	6.2	5.1	4.9	5.3	5.0
	0.1	0.0	7.8	7.6	5.9	5.8	6.6	6.5	16.1	16.3	7.1	7.4	13.3	13.2
	0.3	0.0	29.9	29.9	17.8	17.5	25.9	25.1	94.4	94.3	31.0	30.8	88.0	87.7
	0.0	0.1	10.8	10.8	11.6	11.4	10.9	10.7	5.9	5.7	8.8	8.6	7.8	7.7
	0.0	0.3	31.5	31.5	33.3	33.2	33.9	33.5	6.6	6.8	19.7	19.7	15.7	15.7
1000	0.0	0.0	5.9	5.8	4.6	4.8	5.1	5.1	4.6	4.5	5.3	5.2	4.6	4.6
	0.1	0.0	10.7	10.6	7.8	7.9	8.0	8.0	32.1	31.9	11.3	11.4	26.1	25.9
	0.3	0.0	65.3	64.7	40.9	41.0	58.4	57.9	99.8	99.8	61.5	61.2	99.4	99.3
	0.0	0.1	12.4	12.1	15.8	15.8	14.2	14.1	5.6	5.5	11.2	10.8	8.6	8.7
	0.0	0.3	37.8	37.6	48.9	48.6	46.6	46.3	7.4	7.5	33.1	33.0	25.5	25.5
			Student's t_3 distribution											
500	0.0	0.0	7.6	7.6	7.0	6.9	7.4	7.5	6.0	6.0	4.8	4.9	4.6	4.5
	0.1	0.0	7.4	7.7	5.9	5.8	6.9	7.0	16.6	16.1	5.7	5.8	12.3	12.0
	0.3	0.0	29.0	27.9	12.8	12.6	21.8	21.1	94.5	94.0	15.8	15.6	87.1	86.7
	0.0	0.1	18.1	17.9	19.0	18.9	19.3	18.8	7.9	7.7	10.9	10.9	10.6	10.5
	0.0	0.3	41.6	41.3	44.9	44.6	44.9	44.7	7.5	7.4	24.1	23.7	21.4	21.1
1000	0.0	0.0	5.5	5.5	5.6	5.5	5.9	5.7	5.9	5.9	6.1	6.2	5.3	5.3
	0.1	0.0	8.1	8.3	6.7	6.7	7.0	7.1	34.5	34.3	8.5	8.5	24.9	25.1
	0.3	0.0	54.8	55.4	19.7	19.4	43.4	44.1	99.6	99.5	30.6	30.8	99.3	99.2
	0.0	0.1	22.3	22.1	27.5	27.2	26.8	26.5	7.5	7.5	14.9	15.0	13.4	13.3
	0.0	0.3	55.9	56.1	65.0	65.1	64.5	64.8	9.9	10.1	40.0	39.5	33.4	33.5

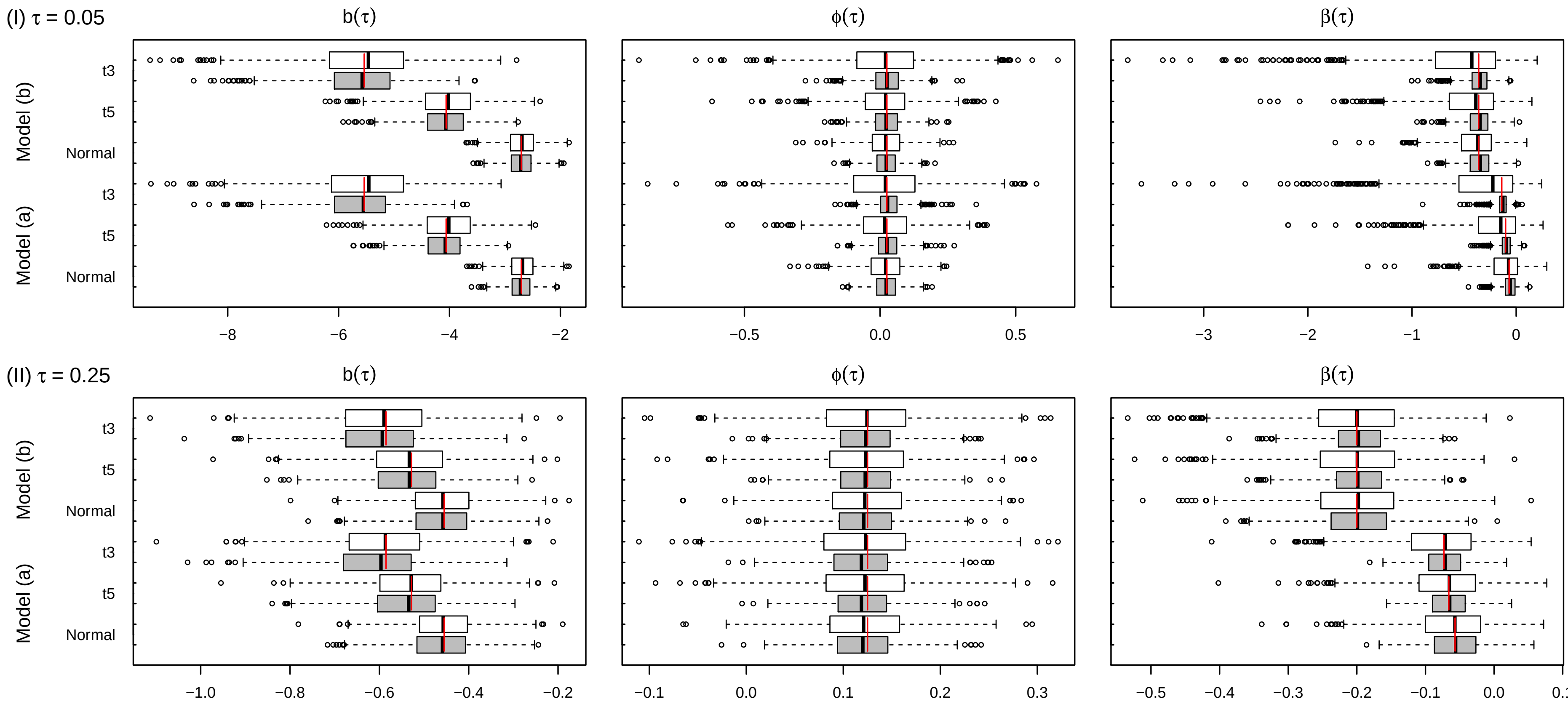

Figure S.1: Box plots for the self-weighted estimator $\widehat{\boldsymbol{\theta}}_{\tau n}$ (white boxes) and unweighted estimator $\widetilde{\boldsymbol{\theta}}_{\tau n}$ (gray boxes), at $\tau = 0.05$ or 0.25, for the two models with $F_b(\cdot)$ being the normal, Student's t_5, or Student's t_3 distribution function. Model (a): $b(\tau) = S_Q^{-1}(F_b^{-1}(\tau)), \phi(\tau) = 0.5\tau, \beta(\tau) = 0.5\tau b(\tau)$; Model (b): $b(\tau) = S_Q^{-1}(F_b^{-1}(\tau)), \phi(\tau) = 0.5\tau, \beta(\tau) = 0.8(\tau - 0.5)$. The thick black line in the center of the box indicates the sample median, and the thin red line indicates the value of the corresponding element of the true parameter vector $\boldsymbol{\theta}_{\tau 0}$. The notations $b(\tau)$, $\phi(\tau)$, and $\beta(\tau)$ represent the corresponding elements of $\widehat{\boldsymbol{\theta}}_{\tau n}$ and $\widetilde{\boldsymbol{\theta}}_{\tau n}$.

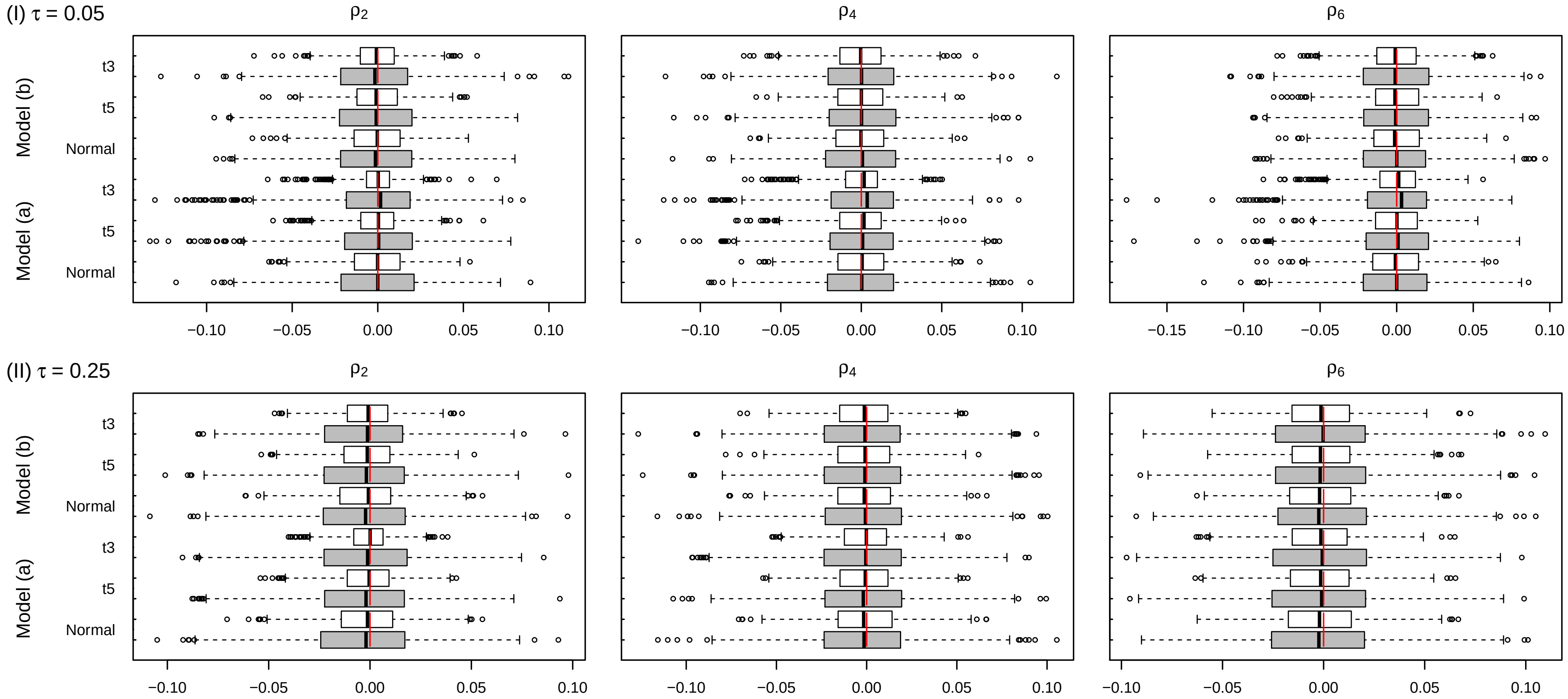

Figure S.2: Box plots for the self-weighted residual QACFs $\widehat{\rho}_{k,\tau}$ (white boxes) and unweighted residual QACFs $\widetilde{\rho}_{k,\tau}$ (gray boxes), at $\tau = 0.05$ or 0.25, $k = 2$, 4, or 6, for the two models with $F_b(\cdot)$ being the normal, Student's t_5, or Student's t_3 distribution function. Model (a): $b(\tau) = S_Q^{-1}(F_b^{-1}(\tau)), \phi(\tau) = 0.5\tau, \beta(\tau) = 0.5\tau b(\tau)$; Model (b): $b(\tau) = S_Q^{-1}(F_b^{-1}(\tau)), \phi(\tau) = 0.5\tau, \beta(\tau) = 0.8(\tau - 0.5)$. The thick black line in the center of the box indicates the sample median, and the thin red line indicates the true value zero of $\rho_{k,\tau}$ if $Q_\tau(y_t|\mathcal{F}_{t-1})$ is correctly specified. The notations ρ_2, ρ_4, and ρ_6 represent $\widehat{\rho}_{k,\tau}$ and $\widetilde{\rho}_{k,\tau}$ at $k = 2$, 4 and 6.

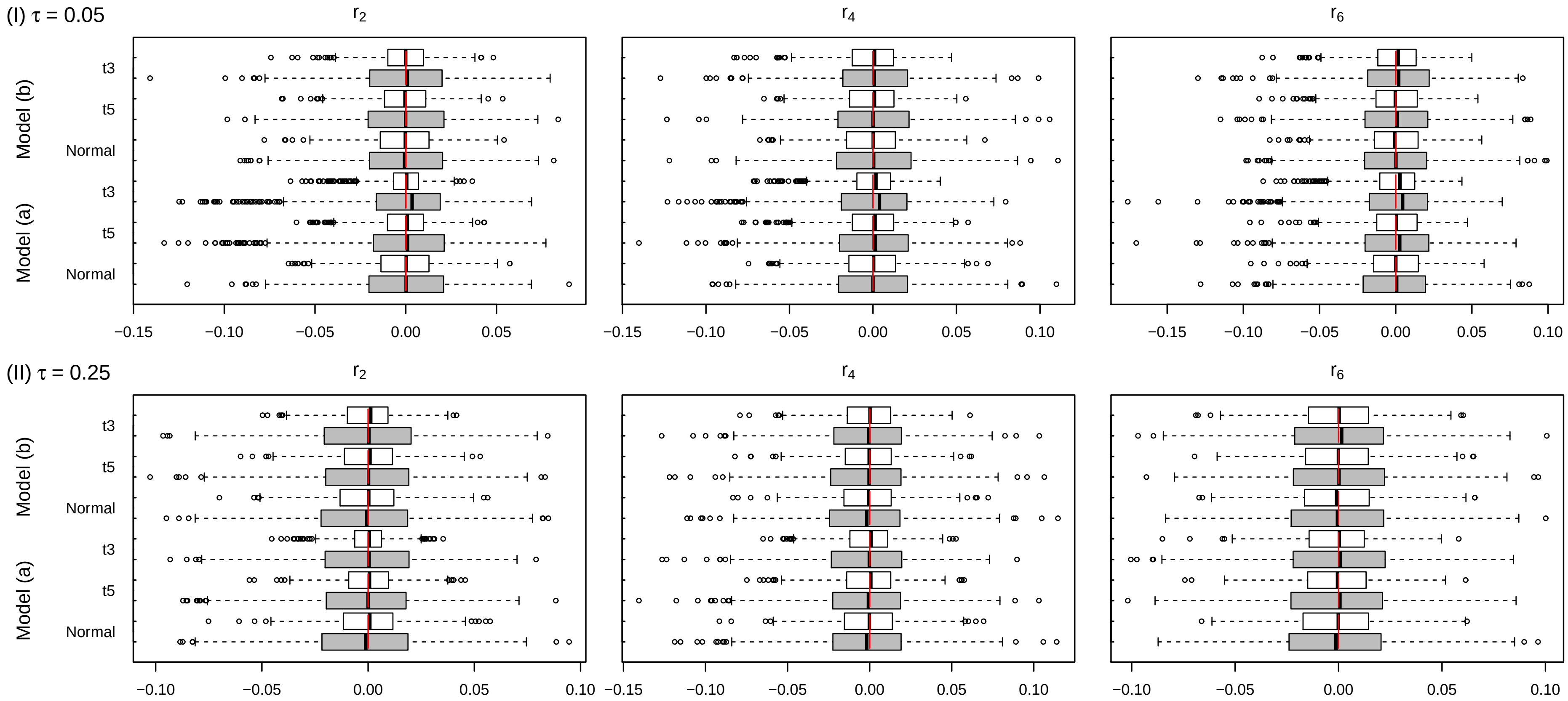

Figure S.3: Box plots for the self-weighted residual QACFs $\widehat{r}_{k,\tau}$ (white boxes) and unweighted residual QACFs $\widetilde{r}_{k,\tau}$ (gray boxes), at $\tau = 0.05$ or 0.25, $k = 2$, 4, or 6, for the two models with $F_b(\cdot)$ being the normal, Student's t_5, or Student's t_3 distribution function. Model (a): $b(\tau) = S_Q^{-1}(F_b^{-1}(\tau)), \phi(\tau) = 0.5\tau, \beta(\tau) = 0.5\tau b(\tau)$; Model (b): $b(\tau) = S_Q^{-1}(F_b^{-1}(\tau)), \phi(\tau) = 0.5\tau, \beta(\tau) = 0.8(\tau - 0.5)$. The thick black line in the center of the box indicates the sample median, and the thin red line indicates the true value zero of $r_{k,\tau}$ if $Q_\tau(y_t|\mathcal{F}_{t-1})$ is correctly specified. The notations r_2, r_4, and r_6 represent $\widehat{r}_{k,\tau}$ and $\widetilde{r}_{k,\tau}$ at $k = 2$, 4, and 6.

5 Forecast performance after quantile rearrangements

The quantile crossing problem can be observed in the fitted conditional quantiles by the QDAR, QAR and TQAR models at the levels of $\tau = 5\%$, 10%, 90%, and 95%. The quantile rearranging method (Chernozhukov et al., 2010) hence is conducted.

Table S.9 reports the empirical coverage rates (ECRs) and p-values of the two VaR backtests after quantile rearrangement, and the results before rearrangement in Table 8 in the paper are also given for easy comparison. It can be seen that ECRs do not change, and p-values of CC and DQ tests change only slightly. Moreover, the negative 5% and 10% VaR forecasts after quantile rearrangement are displayed against the time plot in Figure S.4, and no quantile crossing is observed between these two VaR sequences.

Table S.9: Empirical coverage rates (%) and p-values of the two VaR backtests of the three models before and after rearrangement at the 5%, 10%, 90%, and 95% conditional quantiles. The changed numbers are shown in bold.

	$\tau = 5\%$			$\tau = 10\%$			$\tau = 90\%$			$\tau = 95\%$		
	ECR	CC	DQ	ECR	CC	DQ	ECR	CC	DQ	ECR	CC	DQ
Before rearrangement												
QDAR	5.34	0.88	0.33	9.02	0.34	0.22	91.53	0.25	0.11	95.95	0.23	0.51
QAR	5.16	0.17	0.00	9.58	0.03	0.00	92.45	0.08	0.00	95.95	0.33	0.00
TQAR	6.45	0.20	0.01	10.13	0.81	0.02	91.34	0.19	0.12	95.03	0.42	0.02
After rearrangement												
QDAR	5.34	0.88	**0.34**	9.02	0.34	**0.20**	91.53	0.25	0.11	95.95	0.23	0.51
QAR	5.16	0.17	0.00	9.58	0.03	0.00	92.45	0.08	0.00	95.95	0.33	0.00
TQAR	6.45	0.20	0.01	10.13	0.81	**0.03**	91.34	0.19	0.12	95.03	0.42	0.02

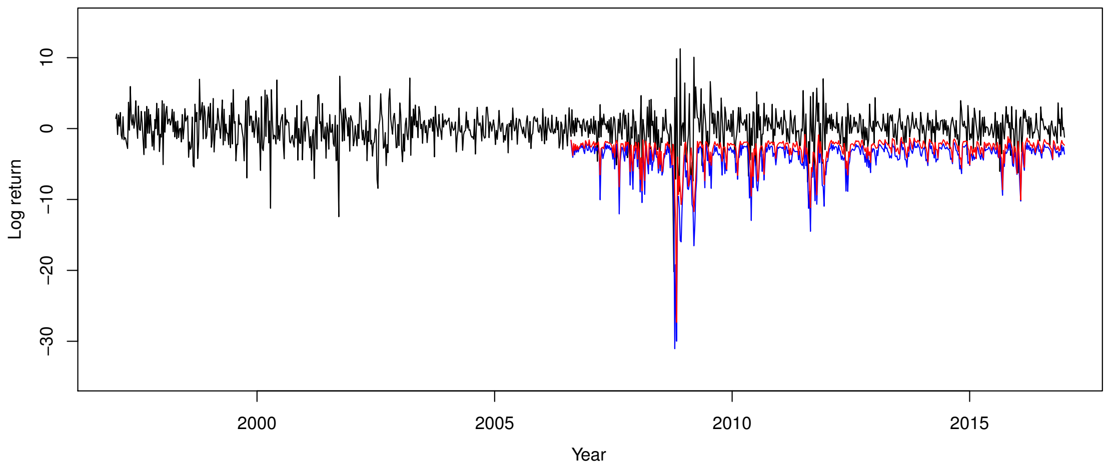

Figure S.4: Time plot of weekly log returns (black line) of the S&P500 Index from January 10, 1997 to December 30, 2016, together with negative 5% VaR forecasts (blue line) and negative 10% VaR forecasts (red line) from August 11, 2006 to December 30, 2016 after quantile rearrangement.

References

Chernozhukov, V., I. Fernández-Val, and A. Galichon (2010). Quantile and probability curves without crossing. *Econometrica 78*, 1093–1125.