Supplementary Material for
"A Cross-Sectional Method for Right-Tailed PANIC Tests under a Moderately Local to Unity Framework"
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Appendix I. Results under the LTU Framework

Becheri and van den Akker (2015) and Westerlund (2015) derived the local asymptotic power of the pooled panel unit root tests where the common factors are extracted using the PANIC method. In doing so, the first-order autoregressive (AR) coefficients are assumed to shrink to one at a fast rate of T^{-1}, i.e. the local to unity (LTU) time series rate.[1] Here, we present the local asymptotic power of the common and the idiosyncratic tests of the PANIC and the CS tests by using the LTU framework:

Assumption A. The AR coefficients satisfy $\alpha_T = 1 + c/T$ and $\rho_{i,T} = 1 + c_i/T$, where both c and c_i are the fixed constants.

The localizing coefficients c and c_i can be either positive or negative, where $c > 0$ and $c_i > 0$ relate to explosive processes and $c < 0$ and $c_i < 0$ pertain to stationary processes. Therefore, the local asymptotic results are valid against either the explosive alternative hypothesis or the stationary alternative hypothesis. The following theorem is obtained.

Theorem SA-1. *Suppose Assumptions 1–4 and A hold. Let $W_c(r)$ and $W_{c,i}(r)$ be independent Ornstein and Uhlenbeck processes defined on $r \in [0,1]$, $\bar{W}_c(r) = W_c(r) - \int W_c(r)dr$, and $\bar{W}_{c,i}(r) = W_{c,i}(r) - \int W_{c,i}(r)dr$. The following hold as $N, T \to \infty$.*

(i-a: The case of no deterministic components, common tests)

$$t_{\hat{F}} \Rightarrow c\left[\int_0^1 W_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_c(r)dW(r)}{\left[\int_0^1 W_c(r)^2 dr\right]^{1/2}}.$$

(i-b: The case of no deterministic components, idiosyncratic tests)

$$t_{\hat{U}}(i) \Rightarrow c_i\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_{c,i}(r)dW_i(r)}{\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2}}.$$

(ii-a: The case of the intercept, common tests)

$$\bar{t}_{\hat{F}} \Rightarrow c\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_c(r)dW(r)}{\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2}}.$$

[1] The rate also depends on N because they consider the pooled tests.

(ii-b: The case of the intercept, idiosyncratic tests)

$$\bar{t}_{\hat{U}}(i) \Rightarrow c_i \left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_{c,i}(r) dW_i(r)}{\left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2}}.$$

For brevity, we present the proof of Theorem SA-1 under the i.i.d. assumptions $C(L) = 1$ and $D_i(L) = 1$ in Appendix II. We can obtain the proof of the tests based on the regression augmented by p lags under $p \to \infty$ and $p^3/\min\{N, T\} \to 0$ by closely following Appendix C of Bai and Ng (2004); hence, it is condensed. This confirms that the factor estimation errors are immaterial in the limit distributions in a locally explosive environment. Note that the independent Ornstein and Uhlenbeck processes reduce to the independent Wiener processes when $c = 0$ in cases (i-a) and (ii-a) and when $c_i = 0$ in cases (i-b) and (ii-b); hence, the results encompass the asymptotic null distributions.

This shows that the size of the common (idiosyncratic) test is robust to the LTU deviations in the idiosyncratic (common and other idiosyncratic) components. As for the power, it ensures that the common (idiosyncratic) test has the standard local power even though the idiosyncratic (common and other idiosyncratic) components have the LTU deviations. Hence, this theorem theoretically confirms Bai and Ng's (2004) Monte Carlo findings in both the left- and right-tailed tests and implies that the PANIC method can disentangle the common and idiosyncratic explosive components.

We next consider the CS tests. Note that the time dimension of the testing sample is now h instead of T; hence, we now denote $\alpha_h = 1 + \frac{c}{h}$ and $\rho_{i,h} = 1 + \frac{c_i}{h}$ in Assumption SA. We present Theorem SA-2 under the LTU framework. For brevity, again, we provide a proof under the i.i.d. assumptions $C(L) = 1$ and $D_i(L) = 1$ in Appendix V.

Theorem SA-2. *Suppose Assumptions 1–4 and A hold. The following hold as $N, T, h \to \infty$.*

(i-a: The case of no deterministic components, common tests)

$$t^*_{\tilde{F}} \Rightarrow c \left[\int_0^1 W_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_c(r) dW(r)}{\left[\int_0^1 W_c(r)^2 dr\right]^{1/2}}.$$

(i-b: The case of no deterministic components, idiosyncratic tests)

$$t^*_{\tilde{U}}(i) \Rightarrow c_i \left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_{c,i}(r) dW_i(r)}{\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2}}.$$

(ii-a: The case of the intercept, common tests)

$$\bar{t}^*_{\tilde{F}} \Rightarrow c \left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_c(r) dW(r)}{\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2}}.$$

(ii-b: The case of the intercept, idiosyncratic tests)

$$\bar{t}^*_{\widetilde{U}}(i) \Rightarrow c_i \left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_{c,i}(r) dW_i(r)}{\left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2}}.$$

Similar to Theorem SA-1, this theorem shows that the common test asymptotically achieves the correct size and is consistent under the LTU framework. The idiosyncratic test also asymptotically attains the correct size and is consistent under the LTU framework.

Appendix II. Proof of Theorem SA-1 and Theorem 1

Throughout Appendix II, we use the notation $\theta = \min\left\{\sqrt{N}, \sqrt{T}\right\}$ and $H = V^{-1}(\hat{f}'f/T)(\Lambda'\Lambda/N)$, where V is the largest eigenvalue of $xx'/(NT)$. We also let $F_{t-1}^c = F_{t-1} - \bar{F}$, where $\bar{F} = T^{-1}\sum_{t=1}^T F_{t-1}$ and $\hat{F}_{t-1}^c = \hat{F}_{t-1} - \overline{\hat{F}}$, where $\overline{\hat{F}} = T^{-1}\sum_{t=1}^T \hat{F}_{t-1}$. Let also $f_t^c = f_t - \bar{f}$, where $\bar{f} = T^{-1}\sum_{t=1}^T f_t$ and $\hat{f}_t^c = \hat{f}_t - \overline{\hat{f}}$, where $\overline{\hat{f}} = T^{-1}\sum_{t=1}^T \hat{f}_t$. In addition, we let $U_{i,t-1}^c = U_{i,t-1} - \bar{U}_i$ where $\bar{U}_i = T^{-1}\sum_{t=1}^T U_{i,t-1}$ and $\hat{U}_{i,t-1}^c = \hat{U}_{i,t-1} - \overline{\hat{U}}_i$, where $\overline{\hat{U}}_i = T^{-1}\sum_{t=1}^T \hat{U}_{i,t-1}$. Let also $u_{i,t}^c = u_{i,t} - \bar{u}_i$, where $\bar{u}_i = T^{-1}\sum_{t=1}^T u_{i,t}$ and $\hat{u}_{i,t}^c = \hat{u}_{i,t} - \overline{\hat{u}}_i$, where $\overline{\hat{u}}_i = T^{-1}\sum_{t=1}^T \hat{u}_{i,t}$. As we explained in the main text, the proofs are presented under $r = 1$, $C(L) = 1$, and $D_i(L) = 1$ for all i.

Lemma A1. Under Assumptions 1, 3, 4, and A, the following hold.
(a) $T^{-1/2}F_{[Tr]} \Rightarrow \sigma W_c(r)$,
(b) $T^{-3/2}\sum_{t=1}^T F_t \Rightarrow \sigma\int W_c(r)dr$,
(c) $T^{-1}\sum_{t=1}^T F_{t-1}e_t \Rightarrow \sigma^2\int W_c(r)dW(r)$,
(d) $T^{-2}\sum_{t=1}^T F_t^2 \Rightarrow \sigma^2\int W_c(r)^2dr$,
(e) $T^{-1/2}U_{i,[Tr]} \Rightarrow \sigma_i W_{c,i}(r)$,
(f) $T^{-3/2}\sum_{t=1}^T U_{i,t} \Rightarrow \sigma_i\int W_{c,i}(r)dr$,
(g) $T^{-1}\sum_{t=1}^T U_{i,t-1}z_{i,t} \Rightarrow \sigma_i^2\int W_{c,i}(r)dW(r)$,
(h) $T^{-2}\sum_{t=1}^T U_{i,t}^2 \Rightarrow \sigma_i^2\int W_{c,i}(r)^2dr$,
(i) $T^{-1}\sum_{t=1}^T F_{t-1}^c e_t \Rightarrow \sigma^2\int \bar{W}_c(r)dW(r)$,
(j) $T^{-2}\sum_{t=1}^T F_{t-1}^{c2} \Rightarrow \sigma^2\int \bar{W}_c(r)^2dr$,
(k) $T^{-1}\sum_{t=1}^T U_{i,t-1}^c z_{i,t} \Rightarrow \sigma_i^2\int \bar{W}_{c,i}(r)dW(r)$,
(l) $T^{-2}\sum_{t=1}^T U_{i,t-1}^c \Rightarrow \sigma_i^2\int \bar{W}_{c,i}(r)^2dr$,
where $W_c(r)$ and $W_{c,i}(r)$ are independent Ornstein and Uhlenbeck processes defined on $r \in [0,1]$ and $\bar{W}_c(r) \equiv W_c(r) - \int W_c(r)dr$, $\bar{W}_{c,i}(r) \equiv W_{c,i}(r) - \int W_{c,i}(r)dr$.

Proof of Lemma A1. See Phillips (1987) for parts (a)–(h). For parts (i)–(l), the results are directly obtained from them.■

Lemma A2. Under Assumptions 1, 3, 4, and A, the following hold.
(a) $T^{-1}\sum_{t=1}^T f_t^2 \xrightarrow{p} \Sigma_f$, a positive constant,
(b) $\mathbb{E}(u_{i,t}) = 0$ and $\mathbb{E}\left|u_{i,t}\right|^8 = O(1)$,
(c) $\left|\gamma_{s,s}^*\right| = O(1)$ for all s and $T^{-1}\sum_{s=1}^T\sum_{t=1}^T\left|\gamma_{s,t}^*\right| = O(1)$, where $\gamma_{st}^* = N^{-1}\sum_{i=1}^N \mathbb{E}(u_{i,s}u_{i,t})$,
(d) $\sum_{i=1}^N\left|\phi_{i,j}^*\right| = O(1)$ for all j and $N^{-1}\sum_{i=1}^N\sum_{j=1}^N\left|\phi_{i,j}^*\right| = O(1)$, where $\phi_{i,j}^* = \mathbb{E}(u_{i,t}u_{j,t})$,
(e) $\zeta_{s,t}^* = O(1)$, where $\zeta_{s,t}^* = \mathbb{E}\left|N^{-1/2}\sum_{i=1}^N[u_{i,s}u_{i,t} - \mathbb{E}(u_{i,s}u_{i,t})]\right|^4$.

Proof of Lemma A2. (a) We start with

$$f_t = \frac{c}{T}F_{t-1} + e_t.$$

Squaring both sides, summing over t, and multiplying both sides by T^{-1} yields

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T f_t^2 &= \frac{c^2}{T^3}\sum_{t=1}^T F_{t-1}^2 + 2\frac{c}{T^2}\sum_{t=1}^T F_{t-1}e_t + \frac{1}{T}\sum_{t=1}^T e_t^2, \\ &= I + II + T^{-1}\sum_{t=1}^T e_t^2 \xrightarrow{p} \sigma^2,\end{aligned}$$

because $I = O_p(T^{-1})$ by using Lemma A1 (d) and $II = o_p(T^{-1})$ by using Lemma A1 (c). The convergence of the third term is implied by the weak law of large numbers in Assumption 1. Hence, the result follows.

(b) It is straightforward that

$$\mathbb{E}(u_{i,t}) = \mathbb{E}(U_{i,t}) - \mathbb{E}(U_{i,t-1}) = 0,$$

from Assumption 3 (a). Next,

$$\begin{aligned}\mathbb{E}\left|u_{i,t}\right|^8 &= \mathbb{E}\left|\frac{c}{T}U_{i,t-1} + z_{i,t}\right|^8, \\ &\leq 2^8 \times \max\left\{\frac{c_i^8}{T^8}\mathbb{E}\left|U_{i,t-1}\right|^8, \mathbb{E}\left|z_{i,t}\right|^8\right\},\end{aligned}$$

but

$$\mathbb{E}\left|U_{i,t-1}\right|^8 \leq T^8 \rho_{i,T}^{8T}\mathbb{E}\left|z_{i,t}\right|^8,$$

so that

$$\frac{c_i^8}{T^8}\mathbb{E}\left|U_{i,t-1}\right|^8 \leq c_i^8 \rho_{i,T}^{8T}\mathbb{E}\left|z_{i,t}\right|^8,$$

where $\rho_{i,T}^{8T} = (1 + \frac{c_i}{T})^{8T} \to \exp(8c_i)$ and $\mathbb{E}\left|z_{i,t}\right|^8 \leq M$ from Assumption 3 (a). Hence, the result follows.

(c) Without loss of generality, let $s \geq t$. Consider

$$\begin{aligned}\mathbb{E}(u_{i,s}u_{i,t}) &= \mathbb{E}\left[\left(\frac{c_i}{T}U_{i,s-1} + z_{i,s}\right)\left(\frac{c_i}{T}U_{i,t-1} + z_{i,t}\right)\right], \\ &= \frac{c_i^2}{T^2}\mathbb{E}(U_{i,s-1}U_{i,t-1}) + \frac{c_i}{T}\mathbb{E}\left(U_{i,s-1}z_{i,t}\right) + \frac{c_i}{T}\mathbb{E}\left(U_{i,t-1}z_{i,s}\right) + \mathbb{E}(z_{i,s}z_{i,t}), \\ &= I + II + III + IV.\end{aligned}$$

However,

$$I \leq \frac{c_i^2}{T}\mathbb{E}(T^{-1}U_{i,s-1}^2) = O(T^{-1}),$$

by using Lemma A1 (e). For II,

$$\begin{aligned} II &= \frac{c_i}{T}\mathbb{E}\left(U_{i,s-1}z_{i,t}\right) = \frac{c_i}{T}\mathbb{E}[(U_{i,s-1}-U_{i,t})z_{i,t} + u_{i,t}z_{i,t} + U_{i,t-1}z_{i,t}], \\ &= \frac{c_i}{T}\mathbb{E}[(U_{i,s-1}-U_{i,t})z_{i,t}] + \frac{c_i}{T}\mathbb{E}(u_{i,t}z_{i,t}) + \frac{c_i}{T}\mathbb{E}(U_{i,t-1}z_{i,t}), \\ &= IIa + IIb + IIc. \end{aligned}$$

However, since $U_{i,s-1} = z_{i,s-1} + \rho_{i,T}z_{i,s-2} + \cdots + \rho_{i,T}^{s-t-1}U_{i,t}$,

$$\begin{aligned} IIa &= \frac{c_i}{T}\mathbb{E}[\left\{z_{i,s-1} + \rho_{i,T}z_{i,s-2} + \rho_{i,T}^2 z_{i,s-3} + \cdots + (\rho_{i,T}^{s-t-1}-1)U_{i,t}\right\} z_{i,t}], \\ &= \frac{c_i}{T}\mathbb{E}[\left\{z_{i,s-1} + \rho_{i,T}z_{i,s-2} + \rho_{i,T}^2 z_{i,s-3} + \cdots + (\rho_{i,T}^{s-t-1}-1)z_{i,t} + \rho_{i,T}(\rho_{i,T}^{s-t-1}-1)U_{i,t-1}\right\} z_{i,t}], \\ &= \frac{c_i}{T}(\rho_{i,T}^{s-t-1}-1)\mathbb{E}(z_{i,t}^2) = O(T^{-1}), \end{aligned}$$

from Assumption 3 (a),

$$\begin{aligned} IIb &= \frac{c_i}{T}\mathbb{E}[(z_{i,t} + \frac{c_i}{T}z_{i,t-1} + \frac{c_i}{T}\rho_{i,T}z_{i,t-2} + \cdots + \frac{c_i}{T}\rho_{i,T}^t U_{i,0})z_{i,t}], \\ &= \frac{c_i}{T}\mathbb{E}(z_{i,t}^2) = O(T^{-1}), \end{aligned}$$

from Assumption 3 (a), and

$$IIc = \frac{c_i}{T^{1/2}} \underbrace{\mathbb{E}(T^{-1/2}U_{i,t-1})}_{=O(1) \text{ by Lemma A1 (e)}} \mathbb{E}(z_{i,t}) = 0,$$

so that $II = O(T^{-1})$. For III,

$$III = \frac{c_i}{T}\mathbb{E}\left(U_{i,t-1}z_{i,s}\right) = \frac{c_i}{T}\mathbb{E}\left(U_{i,t-1}\right)\mathbb{E}(z_{i,s}) = 0,$$

since $U_{i,t-1}$ and $z_{i,s}$ are independent as long as $s \geq t$. Therefore,

$$\begin{aligned} \mathbb{E}(u_{i,s}u_{i,t}) &= O(T^{-1}) + O(T^{-1}) + O(T^{-1}) + 0 + \mathbb{E}(z_{i,s}z_{i,t}), \\ &= \begin{cases} \sigma_i^2 + O(T^{-1}) & \text{if } s = t \\ O(T^{-1}) & \text{if } s \neq t \end{cases}. \end{aligned}$$

We now consider

$$\begin{aligned} \gamma_{s,t}^* &= \mathbb{E}\left[N^{-1}\textstyle\sum_{i=1}^N u_{i,s}u_{i,t}\right], \\ &= \begin{cases} N^{-1}\sum_{i=1}^N \sigma_i^2 + O(T^{-1}) & \text{if } s = t \\ O(T^{-1}) & \text{if } s \neq t \end{cases}. \end{aligned}$$

We also have

$$\sum_{s=1}^{T} \left|\gamma_{s,t}^{*}\right| = N^{-1} \sum_{i=1}^{N} \sigma_i^2 + O(1),$$

so that

$$T^{-1} \sum_{s=1}^{T} \sum_{t=1}^{T} \left|\gamma_{s,t}^{*}\right| = N^{-1} \sum_{i=1}^{N} \sigma_i^2 + O(1) = O(1).$$

(d) Consider

$$\begin{aligned}
\phi_{i,j}^{*} &= \mathbb{E}(u_{i,t}u_{j,t}) = \mathbb{E}\left[\left(\frac{c_i}{T}U_{i,t-1} + z_{i,t}\right)\left(\frac{c_j}{T}U_{j,t-1} + z_{j,t}\right)\right], \\
&= \frac{c_i c_j}{T^2}\mathbb{E}(U_{i,t-1}U_{j,t-1}) + \frac{c_i}{T}\mathbb{E}(U_{i,t-1}z_{j,t}) + \frac{c_j}{T}\mathbb{E}(U_{j,t-1}z_{i,t}) + \mathbb{E}(z_{i,t}z_{j,t}), \\
&= I + II + III + IV.
\end{aligned}$$

For I,

$$I = \frac{c_i c_j}{T^2}\mathbb{E}(U_{i,t-1}U_{j,t-1}) = \frac{c_i c_j}{T^2}\phi_{i,j}\left[\sum_{l=0}^{t-1}(1+\frac{c_i}{T})^l(1+\frac{c_j}{T})^l\right],$$

and, assuming $c_i \geq c_j$ without loss of generality, we obtain

$$\left[\sum_{l=0}^{t-1}(1+\frac{c_i}{T})^{2l}\right] \leq T(1+\frac{c_i}{T})^{2T} = O(T),$$

so that $I = \phi_{i,j} \times O(T^{-1})$. For II,

$$II = \frac{c_i}{T^{1/2}}\mathbb{E}(T^{-1/2}U_{i,t-1})\mathbb{E}(z_{j,t}) = 0,$$

from Assumption 3 (a), and similarly, $III = 0$. $IV = \phi_{i,j}$ by definition. Therefore,

$$\phi_{i,j}^{*} = \phi_{i,j}[1 + O(T^{-1})],$$

so that

$$\sum_{i=1}^{N} \left|\phi_{i,j}^{*}\right| = [1 + O(T^{-1})] \sum_{i=1}^{N} \left|\phi_{i,j}\right| = O(1),$$

from Assumption 3 (b) and

$$N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \left|\phi_{i,j}^{*}\right| = [1 + O(T^{-1})]N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \left|\phi_{i,j}\right| = O(1),$$

from Assumption 3 (b) as well. Hence, the result follows.

(e) Since

$$u_{i,s}u_{i,t} = \frac{c^2}{T^2}U_{i,s-1}U_{i,t-1} + \frac{c}{T}U_{i,s-1}z_{i,t} + \frac{c}{T}U_{i,t-1}z_{i,s} + z_{i,s}z_{i,t},$$

$$
\begin{aligned}
\zeta^*_{s,t} &= \mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[u_{i,s}u_{i,t}-\mathbb{E}(u_{i,s}u_{i,t})\right]\right|^4, \\
&= \mathbb{E}\left|\frac{c^2}{T^2N^{1/2}}\textstyle\sum_{i=1}^N\left[U_{i,s-1}U_{i,t-1}-\mathbb{E}(U_{i,s-1}U_{i,t-1})\right]\right. \\
&\quad +\frac{c}{TN^{1/2}}\textstyle\sum_{i=1}^N\left[U_{i,s-1}z_{i,t}-\mathbb{E}(U_{i,s-1}z_{i,t})\right] \\
&\quad +\frac{c}{TN^{1/2}}\textstyle\sum_{i=1}^N\left[U_{i,t-1}z_{i,s}-\mathbb{E}(U_{i,t-1}z_{i,s})\right] \\
&\quad \left.+\frac{1}{N^{1/2}}\textstyle\sum_{i=1}^N\left[z_{i,s}z_{i,t}-\mathbb{E}(z_{i,s}z_{i,t})\right]\right|^4, \\
&= \mathbb{E}\left|\Phi_1+\Phi_2+\Phi_3+\Phi_4\right|^4, \\
&\leq 4^4\times\max\left\{\mathbb{E}\left|\Phi_1\right|^4,\mathbb{E}\left|\Phi_2\right|^4,\mathbb{E}\left|\Phi_3\right|^4,\zeta_{s,t}\right\}.
\end{aligned}
$$

Consider $\mathbb{E}\left|\Phi_1\right|^4$. Since $U_{i,s-1}=\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}z_{i,l}$ and $U_{i,t-1}=\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}z_{i,m}$,

$$
\begin{aligned}
\mathbb{E}\left|\Phi_1\right|^4 &= \frac{c_i^8}{T^8}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[U_{i,s-1}U_{i,t-1}-\mathbb{E}(U_{i,s-1}U_{i,t-1})\right]\right|^4, \\
&= \frac{c_i^8}{T^8}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}z_{i,l}z_{i,m}\right.\right. \\
&\quad \left.\left.-\textstyle\sum_{l=1}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}\mathbb{E}(z_{i,l}z_{i,m})\right]\right|^4, \\
&= \frac{c_i^8}{T^8}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}(z_{i,l}z_{i,m}-\mathbb{E}(z_{i,l}z_{i,m}))\right|^4, \\
&\leq \frac{c_i^8}{T^8}\mathbb{E}\left|\textstyle\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}\left|N^{-1/2}\sum_{i=1}^N(z_{i,l}z_{i,m}-\mathbb{E}(z_{i,l}z_{i,m}))\right|\right|^4, \\
&\leq \frac{c_i^8}{T^8}T^8\rho_{i,T}^{8T}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N(z_{i,l}z_{i,m}-\mathbb{E}(z_{i,l}z_{i,m}))\right|^4, \\
&= c_i^8\rho_{i,T}^{8T}M=O(1),
\end{aligned}
$$

from Assumption 3 (c). Next,

$$
\begin{aligned}
\mathbb{E}\left|\Phi_2\right|^4 &= \frac{c_i^4}{T^4}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[U_{i,s-1}z_{i,t}-\mathbb{E}(U_{i,s-1}z_{i,t})\right]\right|^4, \\
&= \frac{c_i^4}{T^4}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}z_{i,l}z_{i,t}-\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\mathbb{E}(z_{i,l}z_{i,t})\right]\right|^4, \\
&= \frac{c_i^4}{T^4}\mathbb{E}\left|\textstyle\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}N^{-1/2}\sum_{i=1}^N\left[z_{i,l}z_{i,t}-\mathbb{E}(z_{i,l}z_{i,t})\right]\right|^4, \\
&\leq \frac{c_i^4}{T^4}T^4\rho_{i,T}^{4T}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[z_{i,l}z_{i,t}-\mathbb{E}(z_{i,l}z_{i,t})\right]\right|^4, \\
&= c_i^4\rho_{i,T}^{4T}M=O(1),
\end{aligned}
$$

and $\mathbb{E}\left|\Phi_3\right|^4=O(1)$ is similarly shown. Therefore,

$$
\zeta^*_{s,t}\leq 4^4\times\max\left\{O(1),\zeta_{s,t}\right\}=O(1),
$$

from Assumption 3 (c). Hence, the result follows.■

Lemma A3. Under Assumptions 1–4 and A, the following hold.
(a) $T^{-1/2}\sum_{t=1}^{T}(\hat{f}_t - Hf_t) = O_p(\theta^{-1})$,
(b) $T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)^2 = O_p(\theta^{-2})$,
(c) $T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t} = O_p(\theta^{-2})$,
(d) $T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)f_t = O_p(\theta^{-2})$,
(e) $T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)\hat{f}_t = O_p(\theta^{-2})$,
(f) $\hat{\lambda}_i - H^{-1}\lambda_i = O_p\left(\frac{1}{\min\left\{N,T^{1/2}\right\}}\right)$.

Proof of Lemma A3. Part (a) follows Theorem 1 of Bai (2003). For part (b), the proof is straightforward from Theorem 1 of Bai and Ng (2002) if their assumptions are replaced with our Lemma A2. For parts (c), (d), and (e), the proof follows Lemmas B1, B2, and B3 of Bai (2003), respectively if their assumptions are replaced with our Lemma A2. For part (f), we have

$$\begin{aligned}
\hat{\lambda}_i - \lambda_i H^{-1} &= T^{-1}H\sum_{t=1}^{T} f_t u_{i,t} \\
&\quad + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)\hat{f}_t\lambda_i + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t}, \\
&= T^{-1}H\sum_{t=1}^{T} f_t u_{i,t} + O_p(\theta^{-2}),
\end{aligned}$$

by using Lemma A3 (e) and (c). Now,

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T} f_t u_{i,t} &= T^{-1}\sum_{t=1}^{T}\left(\frac{c}{T}F_{t-1} + e_t\right)\left(\frac{c_i}{T}U_{i,t-1} + z_{i,t}\right), \\
&= cc_iT^{-3}\sum_{t=1}^{T} F_{t-1}U_{i,t-1} + cT^{-2}\sum_{t=1}^{T} F_{t-1}z_{i,t} \\
&\quad + c_iT^{-2}\sum_{t=1}^{T} U_{i,t-1}e_t + T^{-1}\sum_{t=1}^{T} e_t z_{i,t}, \\
&= I + II + III + IV.
\end{aligned}$$

However, by using the Cauchy–Schwarz inequality, Lemma A1 (d) and (h), and Assumptions 1 and 3 (a), we obtain

$$\begin{aligned}
I &\leq cc_iT^{-1}\left(T^{-2}\sum_{t=1}^{T} F_{t-1}^2\right)^{1/2}\left(T^{-2}\sum_{t=1}^{T} U_{i,t-1}^2\right)^{1/2} = O_p(T^{-1}), \\
II &\leq cT^{-1/2}\left(T^{-2}\sum_{t=1}^{T} F_{t-1}^2\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T} z_{i,t-1}^2\right)^{1/2} = O_p(T^{-1/2}), \\
III &\leq c_iT^{-1/2}\left(T^{-2}\sum_{t=1}^{T} U_{i,t-1}^2\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T} e_{t-1}^2\right)^{1/2} = O_p(T^{-1/2}).
\end{aligned}$$

For IV, Assumptions 1, 3 (a), and 4 imply that $\{e_t z_{i,t}\}_{t=2}^{T}$ is a white noise sequence so that $IV = O_p(T^{-1/2})$. Therefore,

$$\hat{\lambda}_i - H^{-1}\lambda_i = O_p(T^{-1/2}) + O_p(\theta^{-2}) = O_p\left(\frac{1}{\min\left\{N, T^{1/2}\right\}}\right). \tag{1}$$

Hence, the result follows.■

Lemma A4. Under Assumptions 1–4 and A, the following hold.
(a) $T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 = T^{-1}H^2\sum_{t=1}^{T}f_t^2 + O_p(\theta^{-2})$,
(b) $T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2 = T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + O_p(\theta^{-1})$,
(c) $T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t = T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t + O_p(\theta^{-1})$,
(d) $T^{-1}\sum_{t=1}^{T}\hat{u}_{i,t}^2 = T^{-1}\sum_{t=1}^{T}u_{i,t}^2 + O_p(\theta^{-2})$,
(e) $T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t-1}^2 = T^{-2}\sum_{t=1}^{T}U_{i,t-1}^2 + O_p(\theta^{-1})$,
(f) $T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}\hat{u}_{i,t} = T^{-1}\sum_{t=1}^{T}U_{i,t-1}u_{i,t} + O_p(\theta^{-1})$,
(g) $T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}^c\hat{f}_t = T^{-1}H^2\sum_{t=1}^{T}F_{t-1}^c f_t + O_p(\theta^{-1})$,
(h) $T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{c2} = T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^{c2} + O_p(\theta^{-1})$,
(i) $T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}^c\hat{u}_{i,t} = T^{-1}\sum_{t=1}^{T}U_{i,t-1}^c u_{i,t} + O_p(\theta^{-1})$,
(j) $T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t-1}^{c2} = T^{-2}\sum_{t=1}^{T}U_{i,t-1}^{c2} + O_p(\theta^{-1})$.

Proof of Lemma A4. Note that $\hat{F}_0 = 0$ and $\hat{U}_{i,0} = 0$ for all i by definition. (a) We start with the identity

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 &= T^{-1}\sum_{t=1}^{T}\left[Hf_t + (\hat{f}_t - Hf_t)\right]^2, \\
&= T^{-1}H^2\sum_{t=1}^{T}f_t^2 + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)^2 \\
&\quad +2T^{-1}H\sum_{t=1}^{T}f_t(\hat{f}_t - Hf_t), \\
&= T^{-1}H^2\sum_{t=1}^{T}f_t^2 + I + II.
\end{aligned}$$

However, $I = O_p(\theta^{-2})$ by using Lemma A3 (b) and $II = O_p(\theta^{-2})$ by using Lemma A3 (d). Hence, the result follows.

(b) This part closely follows Bai and Ng's (2004) Lemma B2. Since

$$\hat{F}_{t-1} = HF_{t-1} + \sum_{s=1}^{t-1}(\hat{f}_s - Hf_s), \tag{2}$$

squaring both sides, summing over t, and multiplying by T^{-2} yields

$$\begin{aligned}
T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2 &= T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + T^{-1}\sum_{t=1}^{T}\left[T^{-1/2}\sum_{s=1}^{t-1}(\hat{f}_s - Hf_s)\right]^2 \\
&\quad +2T^{-1}H\sum_{t=1}^{T}F_{t-1}\left[T^{-1/2}\sum_{s=1}^{t-1}(\hat{f}_s - Hf_s)\right], \\
&= T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + I + II.
\end{aligned}$$

However, $I = O_p(\theta^{-1})$ by using Lemma A3 (a). For term II, we use the Cauchy–Schwarz inequality to get

$$\begin{aligned}
II &\leq 2\left(T^{-2}\sum_{t=1}^{T}F_{t-1}^2\right)^{1/2}\left[T^{-1}\sum_{t=1}^{T}\left(T^{-1/2}\sum_{s=1}^{t-1}(\hat{f}_s - Hf_s)\right)^2\right]^{1/2}, \\
&= O_p(1)\times O_p(\theta^{-1}),
\end{aligned}$$

by using Lemma A1 (d) for the first term and Lemma A3 (a) for the second term. Hence, the result follows.

(c) Since $F_t^2 = (F_{t-1} + f_t)^2 = F_{t-1}^2 + f_t^2 + 2F_{t-1}f_t$ by construction, we obtain

$$F_{t-1}f_t = \frac{1}{2}(F_t^2 - F_{t-1}^2 - f_t^2).$$

Summing over t and multiplying by T^{-1} yields

$$T^{-1}\sum_{t=1}^{T} F_{t-1}f_t = \frac{1}{2}(T^{-1}F_T^2 - T^{-1}F_0^2 - T^{-1}\sum_{t=1}^{T} f_t^2). \quad (3)$$

We also have by construction

$$\hat{F}_{t-1}\hat{f}_t = \frac{1}{2}(\hat{F}_t^2 - \hat{F}_{t-1}^2 - \hat{f}_t^2),$$

so that

$$T^{-1}\sum_{t=1}^{T} \hat{F}_{t-1}\hat{f}_t = \frac{1}{2}(T^{-1}\hat{F}_T^2 - T^{-1}\hat{F}_0^2 - T^{-1}\sum_{t=1}^{T} \hat{f}_t^2). \quad (4)$$

Subtracting (3) multiplied by H^2 from (4) yields

$$\begin{aligned} T^{-1}\sum_{t=1}^{T} \hat{F}_{t-1}\hat{f}_t &= T^{-1}H^2\sum_{t=1}^{T} F_{t-1}f_t + \frac{1}{2T}(\hat{F}_T^2 - H^2F_T^2) - \frac{1}{2T}(\hat{F}_0^2 - H^2F_0^2) \\ &\quad - \left(T^{-1}\sum_{t=1}^{T} \hat{f}_t^2 - T^{-1}H^2\sum_{t=1}^{T} f_t^2\right), \\ &= T^{-1}H^2\sum_{t=1}^{T} F_{t-1}f_t + I + II + III. \end{aligned}$$

For I, updating (2) to the period T, squaring both sides, and multiplying by T^{-1} yields

$$\begin{aligned} T^{-1}\hat{F}_T^2 &= T^{-1}H^2F_T^2 + \underbrace{\left[T^{-1/2}\sum_{s=1}^{T}(\hat{f}_s - Hf_s)\right]^2}_{=O_p(\theta^{-2}) \text{ by Lemma A3(a)}} \\ &\quad +2 \underbrace{T^{-1/2}F_T}_{=O_p(1) \text{ by Lemma A1(a)}} \underbrace{\left[T^{-1/2}\sum_{s=1}^{T}(\hat{f}_s - Hf_s)\right]}_{=O_p(\theta^{-1}) \text{ by Lemma A3(a)}}, \end{aligned}$$

so that $I = O_p(\theta^{-1})$. For II,

$$\hat{F}_0^2 - H^2F_0^2 = -H^2(\alpha_T^2F_0^2 + e_1^2 + 2\alpha_T F_0 e_1),$$

is bounded as $T \to \infty$ so that $II = O_p(T^{-1})$. Term III is $O_p(\theta^{-2})$ by using Lemma A4 (a). Hence, the result follows.

(d) Since $\hat{u}_{i,t} = x_{i,t} - \hat{\lambda}_i\hat{f}_t$ and $x_{i,t} = u_{i,t} + \lambda_i H^{-1}Hf_t$,

$$\begin{aligned} \hat{u}_{i,t} &= u_{i,t} + \lambda_i H^{-1}Hf_t - \hat{\lambda}_i\hat{f}_t, \\ &= u_{i,t} - \lambda_i H^{-1}(\hat{f}_t - Hf_t) - (\hat{\lambda}_i - \lambda_i H^{-1})\hat{f}_t. \quad (5) \end{aligned}$$

Squaring both sides, summing over t, and multiplying by T^{-1} yields

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\hat{u}_{i,t}^2 &= T^{-1}\sum_{t=1}^{T}u_{i,t}^2 + \lambda_i^2 H^{-2}T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)^2 + (\hat{\lambda}_i - \lambda_i H^{-1})^2 T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 \\ &\quad -2\lambda_i H^{-1}T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t} - 2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}\sum_{t=1}^{T}\hat{f}_t u_{i,t} \\ &\quad +2\lambda_i(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)\hat{f}_t, \\ &= T^{-1}\sum_{t=1}^{T}u_{i,t}^2 + I + II + III + IV + V. \end{aligned}$$

However, $I = O_p(\theta^{-2})$ by using Lemma A3 (b), $II = O_p(\frac{1}{\min\{T,N^2\}})$ by using Lemma A3 (f) and $T^{-1}\sum_{t=2}^{T}\hat{f}_t = 1$, and $III = O_p(\theta^{-2})$ by using Lemma A3 (c). We also have

$$\begin{aligned} IV &= -2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}\sum_{t=1}^{T}\hat{f}_t u_{i,t}, \\ &= -2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t} - 2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}H\sum_{t=1}^{T}f_t u_{i,t}, \\ &= O_p\left(\frac{1}{\min\{T,N^2\}}\right) \times O_p(\theta^{-2}) + O_p\left(\frac{1}{\min\{T,N^2\}}\right) \times O_p(T^{-1/2}), \end{aligned}$$

by using Lemma A3 (f) and Lemma A3 (c) for the first term and by using Lemma A3 (f) for the second term, and $V = O_p\left(\frac{1}{\min\{T,N^2\}}\right) \times O_p(\theta^{-2})$ by using Lemma A3 (f) and Lemma A3 (e). Hence, the result follows.

(e) We have

$$\begin{aligned} \hat{U}_{i,t} &= \sum_{s=1}^{t}\hat{u}_{i,s}, \\ &= \sum_{s=1}^{t}u_{i,s} - \lambda_i H^{-1}\sum_{s=1}^{t}(\hat{f}_s - Hf_s) - (\hat{\lambda}_i - \lambda_i H^{-1})\sum_{s=1}^{t}\hat{f}_s, \\ &= U_{i,t} - U_{i,0} - \lambda_i H^{-1}\sum_{s=1}^{t}(\hat{f}_s - Hf_s) - (\hat{\lambda}_i - \lambda_i H^{-1})\sum_{s=1}^{t}\hat{f}_s, \end{aligned}$$

from (5). Multiplying both sides by $T^{-1/2}$ would yield

$$\begin{aligned} T^{-1/2}\hat{U}_{i,t} &= T^{-1/2}U_{i,t} - T^{-1/2}U_{i,0} - \lambda_i H^{-1}\left[T^{-1/2}\sum_{s=1}^{t}(\hat{f}_s - Hf_s)\right] - (\hat{\lambda}_i - H^{-1}\lambda_i)T^{-1/2}\sum_{s=1}^{t}\hat{f}_s, \\ &= T^{-1/2}U_{i,t} + I + II + III. \end{aligned}$$

but $I = O_p(T^{-1/2})$ from Assumption 3 (a), $II = O_p(\theta^{-1})$ by using Lemma A3 (a), $III = O_p\left(\frac{1}{\min\{N,T^{1/2}\}}\right)$ by using Lemma A3 (f) and

$$\begin{aligned} T^{-1/2}\sum_{s=1}^{t}\hat{f}_s &= T^{-1/2}\hat{F}_t = T^{-1/2}HF_t + T^{-1/2}\sum_{s=1}^{t}(\hat{f}_s - Hf_s), \\ &= O_p(1) + O_p(\theta^{-1}), \end{aligned}$$

by using Lemma A1 (b) and Lemma A3 (a). This results in $T^{-1/2}\hat{U}_{i,t} = T^{-1/2}U_{i,t} + O_p(\theta^{-1})$ so that squaring both sides yields

$$\begin{aligned} T^{-1}\hat{U}_{i,t}^2 &= T^{-1}U_{i,t}^2 + O_p(\theta^{-2}) + O_p(\theta^{-1}) \times T^{-1/2}U_{i,t}, \\ &= T^{-1}U_{i,t}^2 + O_p(\theta^{-1}), \end{aligned} \tag{6}$$

by using Lemma A1 (e). Furthermore, summing over t yields

$$T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t}^{2}=T^{-1}\sum_{t=1}^{T}U_{i,t}^{2}+O_{p}(\theta^{-1})T^{-1/2}\sum_{t=1}^{T}U_{i,t}.$$

Multiplying both sides by T^{-1} yields

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t}^{2} &= T^{-2}\sum_{t=1}^{T}U_{i,t}^{2}+O_{p}(\theta^{-1})\underbrace{T^{-3/2}\sum_{t=1}^{T}U_{i,t}}_{=O_p(1)\text{ by Lemma A1(g)}}, \\ &= T^{-2}\sum_{t=1}^{T}U_{i,t}^{2}+O_{p}(\theta^{-1}). \end{aligned}$$

Hence, the result follows.

(f) We use an identity similar to (4) for $\hat{U}_{i,t}$

$$T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}\hat{u}_{i,t}=\frac{\hat{U}_{i,T}^{2}}{2T}-\frac{\hat{U}_{i,0}^{2}}{2T}-\frac{1}{2T}\sum_{t=1}^{T}\hat{u}_{i,t}^{2}, \tag{7}$$

and an identity similar to (3) for $U_{i,t}$

$$T^{-1}\sum_{t=1}^{T}U_{i,t-1}u_{i,t}=\frac{U_{i,T}^{2}}{2T}-\frac{U_{i,0}^{2}}{2T}-\frac{1}{2T}\sum_{t=1}^{T}u_{i,t}^{2}. \tag{8}$$

Subtracting (8) from (7) yields

$$\begin{aligned} &T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}\hat{u}_{i,t}-T^{-1}\sum_{t=1}^{T}U_{i,t-1}u_{i,t} \\ = &\frac{1}{2T}(\hat{U}_{i,T}^{2}-U_{i,T}^{2})-\frac{1}{2T}(\hat{U}_{i,0}^{2}-U_{i,0}^{2})-\frac{1}{2T}\left(\sum_{t=1}^{T}\hat{u}_{i,t}^{2}-\sum_{t=1}^{T}u_{i,t}^{2}\right), \\ = &I+II+III. \end{aligned}$$

However, I and II are $O_p(\theta^{-1})$ from (6) and III is $O_p(\theta^{-2})$ by using Lemma A4 (d). Hence, the result follows.

(g) Since $F_t^{c2}=(F_{t-1}+f_t-\bar{F})^2=(F_{t-1}^c+f_t)^2=F_{t-1}^{c2}+f_t^2+2F_{t-1}^cf_t$ by construction, we obtain

$$\begin{aligned} F_{t-1}^{c}f_{t} &= \frac{1}{2}(F_{t}^{c2}-F_{t-1}^{c2}-f_{t}^{2}), \\ &= \frac{1}{2}(F_{t}^{2}-F_{t-1}^{2}-2\bar{F}f_{t}-f_{t}^{2}). \end{aligned}$$

Summing over t and multiplying by T^{-1} yields

$$T^{-1}\sum_{t=1}^{T}F_{t-1}^{c}f_{t}=\frac{1}{2}(T^{-1}F_{T}^{2}-T^{-1}F_{0}^{2}-2\bar{F}\bar{f}-T^{-1}\sum_{t=1}^{T}f_{t}^{2}). \tag{9}$$

We also have by construction

$$\begin{aligned} \hat{F}_{t-1}^{c}\hat{f}_{t} &= \frac{1}{2}(\hat{F}_{t}^{c2}-\hat{F}_{t-1}^{c2}-\hat{f}_{t}^{2}), \\ &= \frac{1}{2}(\hat{F}_{t}^{2}-\hat{F}_{t-1}^{2}-2\overline{\hat{F}}\hat{f}_{t}-\hat{f}_{t}^{2}), \end{aligned}$$

so that

$$T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}^{c}\hat{f}_t = \frac{1}{2}(T^{-1}\hat{F}_T^2 - T^{-1}\hat{F}_0^2 - 2\overline{\hat{F}\hat{f}} - T^{-1}\sum_{t=1}^{T}\hat{f}_t^2). \tag{10}$$

Subtracting (9) multiplied by H^2 from (10) yields

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}^{c}\hat{f}_t &= T^{-1}H^2\sum_{t=1}^{T}F_{t-1}^{c}f_t + \frac{1}{2T}(\hat{F}_T^2 - H^2F_T^2) - \frac{1}{2T}(\hat{F}_0^2 - H^2F_0^2) \\ &\quad - \left(T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 - T^{-1}H^2\sum_{t=1}^{T}f_t^2\right) - (\overline{\hat{F}\hat{f}} - H^2\bar{F}\bar{f}), \\ &= T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t + I + II + III + IV. \end{aligned}$$

For the terms $I + II + III$, we follow the proof of part (c) to obtain $O_p(\theta^{-1})$. Term IV is

$$\begin{aligned} \overline{\hat{F}\hat{f}} - H^2\bar{F}\bar{f} &= (\bar{\hat{F}} - H\bar{F})H\bar{f} + H\bar{F}(\bar{\hat{f}} - H\bar{f}) + (\bar{\hat{F}} - H\bar{F})(\bar{\hat{f}} - H\bar{f}), \\ &= O_p(T^{1/2}\theta^{-1}) \times O_p(T^{-1/2}) \\ &\quad + O_p(T^{1/2}) \times O_p(T^{-1/2}\theta^{-1}) \\ &\quad + O_p(T^{1/2}\theta^{-1}) \times O_p(T^{-1/2}\theta^{-1}), \\ &= O_p(\theta^{-1}), \end{aligned}$$

because $H\bar{f} = O_p(T^{-1/2})$,

$$\begin{aligned} \bar{\hat{F}} &= T^{-1}H\sum_{t=1}^{T}F_{t-1} + T^{-1}\sum_{t=1}^{T}(\hat{F}_{t-1} - HF_{t-1}), \\ &= T^{-1}H\sum_{t=1}^{T}F_{t-1} + T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{t-1}(\hat{f}_s - Hf_s), \\ &= H\bar{F} + O_p(T^{1/2}\theta^{-1}), \end{aligned}$$

and

$$\begin{aligned} \bar{\hat{f}} &= T^{-1}H\sum_{t=1}^{T}f_t + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t), \\ &= H\bar{f} + O_p(T^{-1/2}\theta^{-1}). \end{aligned}$$

(h) We have

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{c2} &= T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2 - 2\bar{\hat{F}}T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1} + T^{-1}\bar{\hat{F}}^2, \\ &= T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2 - T^{-1}\bar{\hat{F}}^2 = I + II. \end{aligned}$$

However,

$$I = T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + O_p(\theta^{-1}),$$

by using Lemma A4 (b) and

$$II = T^{-1}H^2\bar{F}^2 + O_p(\theta^{-1}).$$

Hence,

$$\begin{aligned} I+II &= T^{-2}H^2\sum_{t=1}^T F_{t-1}^2 - T^{-1}H^2\bar{F}^2 + O_p(\theta^{-1}), \\ &= T^{-2}H^2\sum_{t=1}^T (F_{t-1}-\bar{F})^2 + O_p(\theta^{-1}). \end{aligned}$$

(i) The proof follows part (g) by using an identity similar to (10)

$$T^{-1}\sum_{t=1}^T \hat{U}_{i,t-1}^c \hat{u}_{i,t} = \frac{1}{2}(T^{-1}\hat{U}_{i,T}^2 - T^{-1}\hat{U}_{i,0}^2 - 2\overline{\hat{U}}_i\bar{\hat{u}}_i - T^{-1}\sum_{t=1}^T \hat{u}_{i,t}^2),$$

and an identity similar to (9) for $U_{i,t}$

$$T^{-1}\sum_{t=1}^T U_{i,t-1}^c u_{i,t} = \frac{1}{2}(T^{-1}U_{i,T}^2 - T^{-1}U_{i,0}^2 - 2\overline{U}_i\overline{u}_i - T^{-1}\sum_{t=1}^T u_{i,t}^2).$$

(j) We have

$$\begin{aligned} T^{-2}\sum_{t=1}^T (\hat{U}_{i,t-1}-\overline{\hat{U}}_i)^2 &= T^{-2}\sum_{t=1}^T \hat{U}_{i,t-1}^2 - 2T^{-2}\overline{\hat{U}}_i\sum_{t=1}^T \hat{U}_{i,t-1} + T^{-2}\sum_{t=1}^T \overline{\hat{U}}_i^2, \\ &= T^{-2}\sum_{t=1}^T \hat{U}_{i,t-1}^2 - T^{-1}\overline{\hat{U}}_i^2 = I+II, \end{aligned}$$

but

$$I = T^{-2}\sum_{t=1}^T U_{i,t-1}^2 + O_p(\theta^{-1}),$$

by using Lemma A4 (e) and

$$II = -T^{-1}\bar{U}_i^2 + O_p(\theta^{-1}).$$

Hence,

$$\begin{aligned} I+II &= T^{-2}\sum_{t=1}^T U_{i,t-1}^2 - T^{-1}\bar{U}_i^2 + O_p(\theta^{-1}), \\ &= T^{-2}\sum_{t=1}^T (U_{i,t-1}-\bar{U}_i)^2 + O_p(\theta^{-1}), \end{aligned}$$

and the result follows.■

Proof of Theorem SA-1. (i-a) The common test is

$$t_{\hat{F}} = \frac{T\hat{\delta}}{\hat{\sigma}\left(T^{-2}\sum_{t=2}^T \hat{F}_{t-1}^2\right)^{-1/2}}. \tag{11}$$

Under Assumptions 1–4 and A, we can use Lemma A4 (b) and (c) so that the numerator becomes

$$\begin{aligned} T\hat{\delta} &= \frac{T^{-1}\sum_{t=1}^T \hat{F}_{t-1}\hat{f}_t}{T^{-2}\sum_{t=1}^T \hat{F}_{t-1}^2}, \\ &= \frac{T^{-1}H^2\sum_{t=1}^T F_{t-1}f_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^T F_{t-1}^2 + O_p(\theta^{-1})}, \\ &= \frac{cT^{-2}H^2\sum_{t=1}^T F_{t-1}^2 + T^{-1}H^2\sum_{t=1}^T F_{t-1}e_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^T F_{t-1}^2 + O_p(\theta^{-1})}. \end{aligned} \tag{12}$$

The variance estimate

$$
\begin{aligned}
\hat{\sigma}^2 &= T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t-\hat{\delta}\hat{F}_{t-1}\right)^2, \\
&= T^{-1}\sum_{t=1}^{T}\left[Hf_t-\hat{\delta}^* HF_{t-1}+\left(\hat{f}_t-Hf_t\right)+\hat{\delta}^* HF_{t-1}-\hat{\delta}\hat{F}_{t-1}\right]^2, \\
&= T^{-1}\sum_{t=1}^{T}\left[He_t-(\hat{\delta}^*-\delta)HF_{t-1}+\left(\hat{f}_t-Hf_t\right)+\hat{\delta}^* HF_{t-1}-\hat{\delta}\hat{F}_{t-1}\right]^2, \\
&= T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{5} D_j\right)^2 \le T^{-1}\sum_{t=1}^{T} 5\left(\sum_{j=1}^{5} D_j^2\right),
\end{aligned}
$$

where $\hat{\delta}^* = \frac{T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t}{T^{-1}H^2\sum_{t=1}^{T}F_{t-1}^2}$. To ensure this is a consistent estimate, we compute the stochastic orders of the five terms. First, $T^{-1}\sum_{t=1}^{T} D_1^2 = T^{-1}\sum_{t=1}^{T} H^2 e_t^2$,

$$
T^{-1}\sum_{t=1}^{T} D_2^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^{T}F_{t-1}e_t\right]^2}{T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2} = O_p(T^{-1}),
$$

by using Lemma A1 (c) and (d).

$$
T^{-1}\sum_{t=1}^{T} D_3^2 = T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t-Hf_t\right)^2 = O_p(\theta^{-2}),
$$

by using Lemma A3 (b).

$$
T^{-1}\sum_{t=1}^{T} D_4^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t\right]^2}{T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2} = O_p(T^{-1}),
$$

by using Lemma A1 (c) and (d). Finally,

$$
T^{-1}\sum_{t=1}^{T} D_5^2 = T^{-1}\frac{\left[T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t\right]^2}{T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2} = O_p(T^{-1}),
$$

by using Lemma A4 (b) and (c). Therefore, the first term dominates and the variance estimate satisfies $\hat{\sigma}^2 = Q^{-2}\sigma^2 + O_p(\theta^{-2})$ with $Q^{-1} \equiv p\lim H$, for any fixed c. Hence, plugging (12) into (11) and applying Lemma A1 (c) and (d), we obtain

$$
t_{\hat{F}} \Rightarrow c\left[\int_0^1 W_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_c(r)dW(r)}{\left[\int_0^1 W_c(r)^2 dr\right]^{1/2}}.
$$

(i-b) We follow the same steps as above by replacing $\hat{f}_t$ and $\hat{F}_{t-1}$ with $\hat{u}_{i,t}$ and $\hat{U}_{i,t-1}$ and using the corresponding lemmas to show the results. Hence, the proof is condensed.

(ii-a) The common test is

$$\bar{t}_{\hat{F}} = \frac{T\hat{\delta}}{\hat{\sigma}\left[T^{-2}\sum_{t=2}^{T}(\hat{F}_{t-1}-\overline{\hat{F}})^2\right]^{-1/2}}, \tag{13}$$

where

$$\begin{aligned} T\hat{\delta} &= \frac{T^{-1}\sum_{t=1}^{T}(\hat{F}_{t-1}-\overline{\hat{F}})\hat{f}_t}{T^{-2}\sum_{t=1}^{T}(\hat{F}_{t-1}-\overline{\hat{F}})^2}, \\ &= \frac{T^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})f_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})^2 + O_p(\theta^{-1})}, \\ &= \frac{cT^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})^2 + T^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})e_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})^2 + O_p(\theta^{-1})}, \end{aligned} \tag{14}$$

by using Lemma A4 (g) for the numerator and Lemma A4 (h) for the denominator. For the variance estimate, we can show that $\hat{\sigma}^2 = Q^{-2}\sigma^2 + O_p(\theta^{-2})$ as follows.

$$\begin{aligned} \hat{\sigma}^2 &= T^{-1}\sum_{t=1}^{T}\left[\hat{f}_t - \overline{\hat{f}} - \hat{\delta}\left(\hat{F}_{t-1}-\overline{\hat{F}}\right)\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left[Hf_t - H\bar{f} - \hat{\delta}\left(\hat{F}_{t-1}-\overline{\hat{F}}\right) + \left(\hat{f}_t - Hf_t\right) - \left(\overline{\hat{f}} - H\overline{f}\right)\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left[He_t - H\bar{e} + \delta\left(HF_{t-1}-H\bar{F}\right) - \hat{\delta}^*\left(HF_{t-1}-H\bar{F}\right) + \left(\hat{f}_t - Hf_t\right)\right. \\ &\quad \left. - \left(\overline{\hat{f}} - H\overline{f}\right) + \hat{\delta}^*\left(HF_{t-1}-H\bar{F}\right) - \hat{\delta}\left(\hat{F}_{t-1}-H\overline{\hat{F}}\right)\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left[He_t - H\bar{e} - \left(\hat{\delta}^* - \delta\right)\left(HF_{t-1}-H\bar{F}\right) + \left(\hat{f}_t - Hf_t\right) - \left(\overline{\hat{f}} - H\overline{f}\right)\right. \\ &\quad \left. + \hat{\delta}^*\left(HF_{t-1}-H\bar{F}\right) - \hat{\delta}\left(\hat{F}_{t-1}-\overline{\hat{F}}\right)\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{7}D_j\right)^2 \le T^{-1}\sum_{t=1}^{T}7\left(\sum_{j=1}^{7}D_j^2\right), \end{aligned}$$

where $\hat{\delta}^* = \frac{T^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})f_t}{T^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})^2}$. However, $T^{-1}\sum_{t=1}^{T}D_1^2 = T^{-1}H^2\sum_{t=1}^{T}e_t^2$, $T^{-1}\sum_{t=1}^{T}D_2^2 = H^2\bar{e}^2 = O_p(T^{-1})$,

$$T^{-1}\sum_{t=1}^{T}D_3^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})e_t\right]^2}{T^{-2}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})^2} = O_p(T^{-1}),$$

by using Lemma A1 (i) and (j),

$$T^{-1}\sum_{t=1}^{T}D_4^2 = T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t - Hf_t\right)^2 = O_p(\theta^{-2}),$$

by using Lemma A3 (b),

$$T^{-1}\sum_{t=1}^{T}D_5^2=\left(\overline{\hat{f}}-H\overline{f}\right)^2=\left[T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t-Hf_t\right)\right]^2=O_p(\theta^{-2}),$$

by using Lemma A3 (a),

$$T^{-1}\sum_{t=1}^{T}D_6^2=T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})f_t\right]^2}{T^{-2}H^2\sum_{t=1}^{T}(F_{t-1}-\bar{F})^2}=O_p(T^{-1}),$$

by using Lemma A1 (i) and (j),

$$T^{-1}\sum_{t=1}^{T}D_7^2=T^{-1}\frac{\left[T^{-1}\sum_{t=1}^{T}(\hat{F}_{t-1}-\overline{\hat{F}})\hat{f}_t\right]^2}{T^{-2}\sum_{t=1}^{T}(\hat{F}_{t-1}-\overline{\hat{F}})^2}=O_p(T^{-1}),$$

by using Lemma A4 (g) and (h). Therefore, by plugging (14) into (13) and applying Lemma A1 (i) and (j), we obtain

$$t_{\hat{F}}\Rightarrow c\left[\int_0^1\bar{W}_c(r)^2dr\right]^{1/2}+\frac{\int_0^1\bar{W}_c(r)dW(r)}{\left[\int_0^1\bar{W}_c(r)^2dr\right]^{1/2}},$$

as $N,T\to\infty$.

(ii-b) We follow the same steps as above by replacing $\hat{f}_t$ and $\hat{F}_{t-1}$ with $\hat{u}_{i,t}$ and $\hat{U}_{i,t-1}$ and using the corresponding lemmas to show the results. Hence, the proof is condensed.■

Lemma A5. Under Assumptions 1 and 5, the following hold.
(a) $k_T^{-1/2}\sum_{t=1}^{T}\alpha_T^{-t}e_t\Rightarrow N(0,\sigma^2/2c)$,
(b) $\sum_{t=1}^{T}F_t=O_p(k_TT^{1/2})+O_p(\alpha_T^Tk_T^{3/2})$,
(c) $\alpha_T^{-T}k_T^{-1}\sum_{t=1}^{T}F_{t-1}e_t=o_p(1)$,
(d) $\alpha_T^{-2T}k_T^{-2}\sum_{t=1}^{T}F_t^2=O_p(1)$,
(e) $T^{-1}\sum_{t=1}^{T}f_t^2=O_p(\alpha_T^{2T}T^{-1})+O_p(1)$.

Under Assumptions 3 (a) and 5, the following hold for all i:

(f) $k_T^{-1/2}\sum_{t=1}^{T}\rho_{i,T}^{-t}z_{i,t}\Rightarrow N(0,\sigma_i^2/2c_i)$,
(g) $\sum_{t=1}^{T}U_{i,t}=O_p(k_TT^{1/2})+O_p(\rho_{i,T}^Tk_T^{3/2})$,

(h) ${\rho_{i,T}}^{-T}k_T^{-1}\sum_{t=1}^{T}U_{i,t-1}z_{i,t}=o_p(1)$,
(i) ${\rho_{i,T}}^{-2T}k_T^{-2}\sum_{t=1}^{T}U_{i,t}^2=O_p(1)$,
(j) $T^{-1}\sum_{t=1}^{T}u_{i,t}^2=O_p(\rho_{i,T}^{2T}T^{-1})+O_p(1)$.

Proof of Lemma A5. Here, we present the proof of only parts (a) to (e). The proofs of parts (f) to (j) are shown in the same way but with $U_{i,t}$ instead of F_t and with Assumption 1 replaced by Assumption 3 (a). We suppress the proofs to conserve space.

(a) See Lemma 4.2 of Phillips and Magdalinos (2007).

(b) We start with the expression

$$\begin{aligned}
\textstyle\sum_{t=1}^{T} F_t &= \textstyle\sum_{t=0}^{T} \alpha_T^t e_0 + \sum_{t=0}^{T-1} \alpha_T^t e_1 + \sum_{t=0}^{T-2} \alpha_T^t e_2 + \cdots + e_T, \\
&= \frac{1}{1-\alpha_T}\left[(\alpha_T - \alpha_T^{T+1})F_0 + (1-\alpha_T^T)e_1 + (1-\alpha_T^{T-1})e_2 + \cdots + (1-\alpha_T)e_T\right], \\
&= \frac{k_T}{c}\left[\textstyle\sum_{t=1}^{T} e_t - \sum_{t=1}^{T} \alpha_T^{T+1-t} e_t + (\alpha_T - \alpha_T^{T+1})F_0\right], \\
&= \frac{k_T}{c}\textstyle\sum_{t=1}^{T} e_t - \dfrac{\alpha_T^{T+1}k_T}{c}\textstyle\sum_{t=1}^{T} \alpha_T^{-t} e_t + \dfrac{k_T}{c}(\alpha_T - \alpha_T^{T+1})F_0, \\
&= I + II + III.
\end{aligned}$$

However, $I = O_p(k_T T^{1/2})$ from Assumption 1, $II = O_p(\alpha_T^T k_T^{3/2})$ by using Lemma A5 (a), and $III = O_p(\alpha_T^T k_T)$ from Assumption 1. Hence, the result follows.

(c) We start with the expression for F_{t-1}

$$F_{t-1} = e_{t-1} + \alpha_T e_{t-2} + \ldots + \alpha_T^{t-2} e_1 + \alpha_T^{t-1} F_0 = \alpha_T^{t-1} \textstyle\sum_{s=1}^{t-1} \alpha_T^{-s} e_s + \alpha_T^{t-1} F_0.$$

Multiplying both sides by $\alpha_T^{-T} k_T^{-1} e_t$ and summing over t yield

$$\begin{aligned}
\alpha_T^{-T} k_T^{-1} \textstyle\sum_{t=1}^{T} F_{t-1} e_t &= \alpha_T^{-T} k_T^{-1} \textstyle\sum_{t=1}^{T} \left(\sum_{s=1}^{t-1} \alpha_T^{t-s-1} e_s\right) e_t + \alpha_T^{-T} k_T^{-1} F_0 \sum_{t=1}^{T} \alpha_T^{t-1} e_t, \\
&= I + II.
\end{aligned}$$

The expected value of this is zero because of Assumption 1. To show that this is $o_p(1)$, we confirm that the second moments of both terms are bounded as $T \to \infty$. For I, we can simplify the second moment as follows by using Assumption 1.

$$\begin{aligned}
&\mathbb{E}\left[\alpha_T^{-T} k_T^{-1} \textstyle\sum_{t=1}^{T} \left(\sum_{s=0}^{t-1} \alpha_T^{t-s-1} e_s\right) e_t\right]^2 \\
&= \alpha_T^{-2T} k_T^{-2} \sigma^4 \textstyle\sum_{t=1}^{T} \sum_{s=0}^{t-1} \alpha_T^{2(t-s-1)}, \\
&= \alpha_T^{-2T} \frac{\alpha_T^2 \sigma^4}{k_T(\alpha_T^2 - 1)} \left(\frac{\alpha_T^{2T} - 1}{k_T(\alpha_T^2 - 1)} - \frac{T}{k_T \alpha_T^2}\right), \\
&= \frac{\alpha_T^2 \sigma^4}{k_T(\alpha_T^2 - 1)} \left(\frac{1 - \alpha_T^{-2T}}{k_T(\alpha_T^2 - 1)} - \frac{\alpha_T^{-2T} T}{k_T \alpha_T^2}\right).
\end{aligned}$$

However, since $k_T(\alpha_T^2-1)\to 2c$, $\alpha_T^2\to 1$, and $\alpha_T^{-2T}T=o(1)$, this is $O(1)$. For II,

$$\begin{aligned} & \mathbb{E}\left[\alpha_T^{-T}k_T^{-1}F_0\sum_{t=1}^T\alpha_T^{t-1}e_t\right]^2, \\ = & \alpha_T^{-2T}k_T^{-1}\frac{\mathbb{E}(F_0^2)\sigma^2}{k_T(\alpha_T^2-1)}(\alpha_T^{2T}-1), \\ = & k_T^{-1}\frac{\mathbb{E}(F_0^2)\sigma^2}{k_T(\alpha_T^2-1)}(1-\alpha_T^{-2T})=O(k_T^{-1}), \end{aligned}$$

so that the second moment of II diminishes. Therefore, the result follows.

(d) We take squares of both sides of $F_t=\alpha_TF_{t-1}+e_t$ and take summations over $t=1,...,T$ to obtain

$$\begin{aligned} \sum_{t=1}^T F_{t-1}^2 &= \frac{1}{\alpha_T^2-1}\left\{F_T^2-F_0^2-\sum_{t=1}^T e_t^2-2\alpha_T\sum_{t=1}^T F_{t-1}e_t\right\}, \\ \alpha_T^{-2T}k_T^{-2}\sum_{t=1}^T F_{t-1}^2 &= \frac{1}{k_T(\alpha_T^2-1)}\left\{\frac{\alpha_T^{-2T}}{k_T}(F_T^2-F_0^2)-\frac{\alpha_h^{-2h}}{k_h}\sum_{t=1}^T e_t^2\right. \\ & \left.-\frac{2\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^T F_{t-1}e_t\right\}, \\ &= \frac{1}{k_T(\alpha_T^2-1)}\{I-II-III\}, \end{aligned}$$

where $k_T(\alpha_T^2-1)\to 2c$. Now we can show that $I=O_p(1)$, because

$$\begin{aligned} F_T &= \sum_{j=1}^T\alpha_T^{T-j}e_j=\alpha_T^Tk_T^{1/2}\left(\frac{1}{\sqrt{k_T}}\sum_{j=1}^T\alpha_T^{-j}e_j\right), \\ F_T^2 &= \alpha_T^{2T}k_T\left(\frac{1}{\sqrt{k_T}}\sum_{j=1}^T\alpha_T^{-j}e_j\right)^2=O_p(\alpha_T^{2T}k_T), \end{aligned}$$

so that

$$I=\frac{\alpha_T^{-2T}}{k_T}F_T^2-\frac{\alpha_T^{-2T}}{k_T}F_0^2=O_p(1)+O_p(\alpha_T^{-2T}k_T^{-1})=O_p(1).$$

For II,

$$II=\left(\frac{\alpha_T^{-2T}T}{k_T}\right)\frac{1}{T}\sum_{t=1}^T e_t^2=o\left(\frac{k_T}{T}\right)\times O_p(1)=o_p(1).$$

For III (divided by 2),

$$\begin{aligned} III &= \frac{\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^T F_{t-1}e_t \\ &= \frac{\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^T\left(\sum_{j=1}^{t-1}\alpha_T^{t-1-j}e_j\right)e_t+F_0\left(\frac{\alpha_T^{-T}}{\sqrt{k_T}}\right)\frac{1}{\sqrt{k_T}}\sum_{t=1}^T\alpha_T^{-(T-t)}e_t, \\ &= IIIa+IIIb. \end{aligned}$$

For $IIIa$, because we have

$$\begin{aligned}\mathbb{E}\left[\frac{\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^T\left(\sum_{j=1}^{t-1}\alpha_T^{t-1-j}e_j\right)e_t\right]^2 &= \frac{\sigma^4\alpha_T^{-4T}}{k_T^2}\sum_{t=1}^T\sum_{j=1}^{t-1}\alpha_T^{2(t-j-1)},\\ &= \frac{\sigma^4\alpha_T^{-4T}}{k_T^2(\alpha_T^2-1)}\left[\sum_{t=1}^T\alpha_T^{2(t-1)}-T\right]=O(\alpha_T^{-2T}),\end{aligned}$$

$IIIa=o_p(1)$. For $IIIb$, because $F_0=O_p(1)$, $\left(\frac{\alpha_T^{-T}}{\sqrt{k_T}}\right)=o\left(\frac{\sqrt{k_T}}{T}\right)=o_p(1)$, and $\left(\frac{1}{\sqrt{k_T}}\sum_{t=1}^T\alpha_T^{-(T-t)}e_t\right)=O_p(1)$, $IIIb=o_p(1)$. Hence, $III=o_p(1)$. The result follows.■

(e) We start with

$$f_t=\frac{c}{k_T}F_{t-1}+e_t.$$

Squaring both sides, summing over t, and multiplying by T^{-1} yields

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T f_t^2 &= \frac{c^2}{Tk_T^2}\sum_{t=1}^T F_{t-1}^2+\frac{2c}{Tk_T}\sum_{t=1}^T F_{t-1}e_t+\frac{1}{T}\sum_{t=1}^T e_t^2,\\ &= I+II+III.\end{aligned}$$

However, $I=O_p(\alpha_T^{2T}T^{-1})$ by using Lemma A5 (d), $II=o_p(\alpha_T^{T}T^{-1})$ by using Lemma A5 (c), and $III=O_p(1)$ by Assumption 1. Hence, the result follows.■

Proof of Theorem 1. We start with equation (A.1) of Bai and Ng (2004). Let $u_t=[u_{1,t}\ u_{2,t}\ \cdots\ u_{N,t}]$ be a $1\times N$ vector of the first differences of the idiosyncratic errors at time t.

$$\begin{aligned}\hat{f}_t &= Hf_t+V^{-1}N^{-1}T^{-1}\hat{f}'u\Lambda f_t+V^{-1}N^{-1}T^{-1}\hat{f}'f\Lambda' u_t'\\ &\quad +V^{-1}N^{-1}T^{-1}\hat{f}'uu_t',\end{aligned}$$

or

$$\begin{aligned}\hat{f}_t &= A_1f_t+A_2f_t+N^{-1}\sum_{i=1}^N a_{1,i}u_{i,t}+N^{-1}\sum_{i=1}^N a_{2,i}u_{i,t},\\ &= Af_t+N^{-1}\sum_{i=1}^N a_iu_{i,t},\end{aligned}$$

where $A=A_1+A_2$ and $a_i=a_{1,i}+a_{2,i}$. We also have $A_1=V^{-1}N^{-1}T^{-1}\hat{f}'f\Lambda'\Lambda$ from the definition of the H matrix, $A_2=V^{-1}N^{-1}T^{-1}\hat{f}'u\Lambda$, $a_{1,i}=V^{-1}T^{-1}\hat{f}'f\lambda_i'$, and $a_{2,i}=V^{-1}T^{-1}\hat{f}'u_i$. In the following, because V^{-1} appears in every component A_1, A_2, $a_{1,i}$ and $a_{2,i}$, we multiply them by V to ease computation and separately derive the bound of V.

(i) If $c > 0$ and $c_i = 0$ for all i, then

$$
\begin{aligned}
VA_1 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(\alpha_T^T T^{-1/2})\text{ by Lemma A5(e)}} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)\text{ from Assumption 2}} = O_p(\alpha^T T^{-1/2}), \\
VA_2 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}T^{-1}\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(1)} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)} = O_p(1), \\
Va_{1,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(\alpha_T^T T^{-1/2})} |\lambda_i| = O_p(\alpha^T T^{-1/2}), \\
Va_{2,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(1)} = O_p(1).
\end{aligned}
$$

(ii) If $c = 0$ and $c_i > 0$ for all i, then

$$
\begin{aligned}
VA_1 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(1)} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)} = O_p(1), \\
VA_2 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}T^{-1}\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(\rho_{i,T}^T T^{-1/2})\text{ by Lemma A5 (j)}} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)} = O_p(\rho_{i,T}^T T^{-1/2}), \\
Va_{1,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(1)} |\lambda_i| = O_p(1), \\
Va_{2,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(\rho_{i,T}^T T^{-1/2})} = O_p(\rho_{i,T}^T T^{-1/2}).
\end{aligned}
$$

The largest eigenvalue V of $N^{-1}T^{-1}xx'$ satisfies $V^{1/2} = \left\|N^{-1/2}T^{-1/2}x\right\|$, where $\|\cdot\|$ denotes the Euclidean norm, so that

$$
\begin{aligned}
V &= N^{-1}T^{-1}\|x\|^2, \\
&= N^{-1}T^{-1}\textstyle\sum_{i=1}^{N}\sum_{t=1}^{T}x_{i,t}^2, \\
&= \begin{cases} O_p(\alpha_T^T T^{-1/2}), & \text{for case (i)} \\ O_p(\rho_{i,T}^T T^{-1/2}), & \text{for case (ii)} \end{cases} .
\end{aligned}
\tag{15}
$$

Hence, the results follow if the stochastic bound for V is sharp and the expression is asymptotically dominated by the mildly explosive components.■

Appendix III. Proof of Factor Estimation Errors in Theorem 1(i)

In the following, we provide two additional facts pertaining to consistency of factor estimation in Theorem 1 (i).

First, we show that the differenced factors can be consistently estimated. From Proof of Theorem 1 (i) in Appendix II, we have

$$\begin{aligned} \hat{f}_t - (A_1 + A_2) f_t &= N^{-1} \textstyle\sum_{i=1}^N a_{1,i} u_{i,t} + N^{-1} \sum_{i=1}^N a_{2,i} u_{i,t}, \\ &= I + II, \end{aligned} \tag{16}$$

where $A_1 = V^{-1} N^{-1} T^{-1} \hat{f}' f \Lambda' \Lambda$, $A_2 = V^{-1} N^{-1} T^{-1} \hat{f}' u \Lambda$, $a_{1,i} = V^{-1} T^{-1} \hat{f}' f \lambda_i'$, and $a_{2,i} = V^{-1} T^{-1} \hat{f}' u_i$ as we previously defined. Then,

$$I = V^{-1} (T^{-1} \hat{f}' f) N^{-1} \textstyle\sum_{i=1}^N \lambda_i u_{i,t},$$

with $V = O_p(\alpha_T^T T^{-1/2})$ from (15) and

$$(T^{-1} \hat{f}' f) \leq \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T \hat{f}_s^2 \right|^{1/2}}_{=1} \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T f_s^2 \right|^{1/2}}_{=O_p(\alpha_T^T T^{-1/2}) \text{ by Lemma A5(e)}} = O_p(\alpha_T^T T^{-1/2}).$$

Hence,

$$I = O_p(\alpha_T^{-T} T^{1/2}) \times O_p(\alpha_T^T T^{-1/2}) \times o_p(1) = o_p(1).$$

We also have, by using the definition of $a_{2,i}$ in Proof of Theorem 1,

$$\begin{aligned} II &= V^{-1} T^{-1} N^{-1} \textstyle\sum_{s=1}^T \hat{f}_s \sum_{i=1}^N u_{i,s} u_{i,t}, \\ &\leq V^{-1} T^{-1} \textstyle\sum_{s=1}^T \hat{f}_s \left| N^{-1} \sum_{i=1}^N u_{i,s}^2 \right|^{1/2} \left| N^{-1} \sum_{i=1}^N u_{i,t}^2 \right|^{1/2}, \\ &\leq V^{-1} \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T \hat{f}_s^2 \right|}_{=1} \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T \left| N^{-1} \sum_{i=1}^N u_{i,s}^2 \right| \right|}_{=O_p(1)}^{1/2} \left| N^{-1} \textstyle\sum_{i=1}^N u_{i,t}^2 \right|^{1/2}, \end{aligned}$$

but $V = O_p(\alpha_T^T T^{-1/2})$ from (15), $T^{-1} \sum_{s=1}^T \left| N^{-1} \sum_{i=1}^N u_{i,s}^2 \right| = O_p(1)$, and $N^{-1} \sum_{i=1}^N u_{i,t}^2 = O_p(1)$ under $c_i = 0$. Hence,

$$II = O_p(\alpha_T^{-T} T^{1/2}) \times O_p(1) = o_p(1).$$

We also show that the level factors involve factor estimation errors of order $O_p(T^{1/2} N^{-1/2})$. Using (16), the level factor estimate is

$$\hat{F}_t = \textstyle\sum_{s=1}^t \hat{f}_s = (A_1 + A_2) \sum_{s=1}^t f_s + \sum_{s=1}^t N^{-1} \sum_{i=1}^N a_{1,i} u_{i,s} + \sum_{s=1}^t N^{-1} \sum_{i=1}^N a_{2,i} u_{i,s},$$

or

$$\begin{aligned} \hat{F}_t - (A_1 + A_2) F_t &= \textstyle\sum_{s=1}^t N^{-1} \sum_{i=1}^N a_{1,i} u_{i,s} + \sum_{s=1}^t N^{-1} \sum_{i=1}^N a_{2,i} u_{i,s}, \\ &= I + II = O_p(T^{1/2} N^{-1/2}), \end{aligned} \tag{17}$$

because

$$
\begin{aligned}
I &= V^{-1}(T^{-1}\hat{f}'f)\sum_{s=1}^{t} N^{-1}\sum_{i=1}^{N}\lambda_i u_{i,t}, \\
&= V^{-1}(T^{-1}\hat{f}'f)T^{1/2}N^{-1/2}(T^{-1/2}N^{-1/2}\sum_{s=1}^{t}\sum_{i=1}^{N}\lambda_i u_{i,t}),
\end{aligned}
$$

where $V^{-1}(T^{-1}\hat{f}'f) = O_p(1)$ because $V^{-1} = O_p(\alpha_T^{-T}T^{1/2})$ and $T^{-1}\hat{f}'f = O_p(\alpha_T^{T}T^{-1/2})$ as we showed in term I of the differenced factor. We also have

$$T^{-1/2}N^{-1/2}\sum_{s=1}^{t}\sum_{i=1}^{N}\lambda_i u_{i,t} = O_p(1).$$

Hence,

$$I = O_p(1) \times T^{1/2}N^{-1/2} \times O_p(1) = O_p(T^{1/2}N^{-1/2}).$$

and

$$
\begin{aligned}
II &= V^{-1}T^{-1}N^{-1}\sum_{l=1}^{T}\hat{f}_l\sum_{i=1}^{N}u_{i,l}\sum_{s=1}^{t}u_{i,s}, \\
&\leq V^{-1}\left|T^{-1}\sum_{l=1}^{T}\hat{f}_l^2\right| \quad \left|T^{-1}\sum_{l=1}^{T}\left|N^{-1}\sum_{i=1}^{N}u_{i,l}^2\right|\right|^{1/2}\left|N^{-1}\sum_{i=1}^{N}\sum_{s=1}^{t}u_{i,s}^2\right|^{1/2},
\end{aligned}
$$

where $V = O_p(\alpha_T^T T^{-1/2})$ from (15), $T^{-1}\sum_{l=1}^{T}\hat{f}_l^2 = 1$, $T^{-1}\sum_{l=1}^{T}\left|N^{-1}\sum_{i=1}^{N}u_{i,l}^2\right| = O_p(1)$, and $\left|N^{-1}\sum_{i=1}^{N}\sum_{s=1}^{t}u_{i,s}^2\right|^{1/2} = O_p(T^{1/2})$. Hence, II is dominated by I. Therefore, (16) implies that the factor estimation errors in the differenced factor are $o_p(1)$ and (17) implies that the factor estimation errors in the level factor are $O_p(T^{1/2}N^{-1/2})$.

Appendix IV. Finite Sample Properties of the CS Tests when the Training Sample is Selected by a Statistical Method

In Appendix IV, we investigate the size and power of the CS tests when we split the sample at the estimated origination date of explosive behavior in the cross-sectional average (or an index) $\bar{X}_t = N^{-1}\sum_{i=1}^{N} X_{i,t}$ using the time-stamping method proposed by Phillips et al. (2011). This method sequentially computes the DF test statistics $t_{\bar{X}}(\tau)$ from a subsample of $\{\bar{X}_t\}_{t=1}^{[\tau(T+h)]}$ where $[\cdot]$ denotes the integer part. τ can take values over $\tau_0 \leq \tau \leq 1$, where τ_0 is a small positive trimming value before which no explosive behavior is assumed. In particular, we follow Phillips et al. (2011) and use $\tau_0 = 0.01 + 1.8/\sqrt{T+h}$. Then, the break date (as a fraction of the total sample size $T+h$) is estimated by

$$\hat{\tau} = \inf_{\tau \in [\tau_0, 1]} \{\tau : t_{\bar{X}}(\tau) > cv(\tau)\}$$

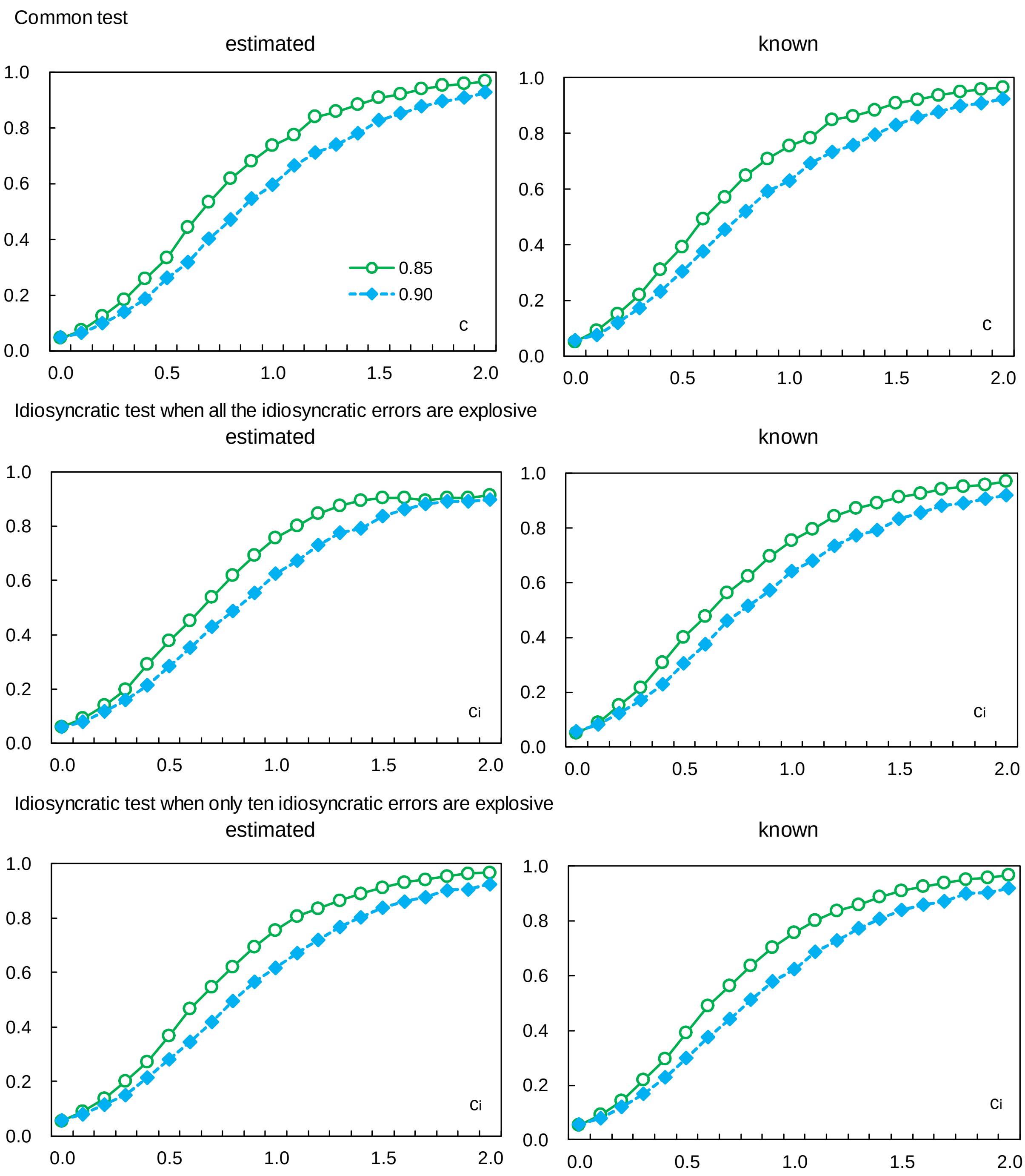
where $cv(\tau)$ is the 5% critical value. Under our setting of Section 4, all the cross-sectional units have the same origination dates, thus $\{\bar{X}_t\}_{t=1}^{T+h}$ is considered to be a single time series which follows a random walk for $1 \leq t \leq T$ and an explosive process for $T+1 \leq t \leq T$ regardless of N. Hence, the time-stamping method of Phillips et al. (2011) is straightforwardly applied. It is natural to use the critical value such as $cv(\tau) = \log(\log(\tau(T+h))$, which diverges as $T+h \to \infty$ as the t statistic diverges as $T+h \to \infty$, although developing theoretical properties of the break date estimator is beyond the scope of this paper. We obtain the 5% critical value via 5,000 Monte Carlo simulation. We also adopt a strategy that $\hat{\tau}$ is identified only if $t_{\bar{X}}(\tau)$ exceeds $cv(\tau)$ for $2\log(T+h)$ consecutive periods.[2] Once $\hat{\tau}$ is obtained, the CS tests are computed by setting the training sample $1 \leq t \leq [\hat{\tau}(T+h)]$ and the explosive window $[\hat{\tau}(T+h)] + 1 \leq t \leq T+h$.

To see how this procedure affects the size and power of the CS tests, we use the same DGP as Section 4.3 with $N = 100$, $T = 50$, $h = 100$, and $\kappa = 0.85$, 0.90. The power functions of the common and the idiosyncratic tests are reported in the left panels of Figure IV, while the power functions under the known breakpoint assumption are presented in the right panels for comparison. The top panels show power functions of the common test and the middle panels present those of the idiosyncratic test when all the cross-sections have explosive behaviors. These figures show that the size and power are qualitatively unchanged even if we use the estimated breakpoint to select the training sample. In addition, it is concerning that the break date estimator becomes less precise when a smaller number of idiosyncratic components have explosive behaviors. To see this, the bottom panels of Figure IV show power functions of the idiosyncratic tests when only 10 cross-sections have explosive components. We observe that the power functions are almost equivalent to the previous case, because there is less contamination in the estimation window in such a case. Finally, the

[2] Applying the time-stamping method to $\{\bar{X}_t\}_{t=1}^{T+h}$ is not the only method to estimate the demarcation date. It is stressed that the goal of this study is not to consistently estimate the break point but to provide tests for explosive behaviors in the common and the idiosyncratic components. Indeed, if the break points are heterogenous, there is no method of consistently estimate the "true break point".

power functions of the CS tests are all monotonic and standard even when the demarcation point is estimated by a statistical method. This verifies the advantage of the CS method over the PANIC approach in detecting mild or large explosive behaviors as we show in the lower right panel of Figure 1 of the main paper.

Figure IV. Power functions of the CS tests
when the demarcation point is estimated



Appendix V. Proof of Theorem SA-2 and Theorem 2

Throughout Appendix V, we let $F_{t-1}^c = F_{t-1} - \bar{F}$, where $\bar{F} = h^{-1}\sum_{t=T+1}^{T+h} F_{t-1}$ and $\widetilde{F}_{t-1}^c = \widetilde{F}_{t-1} - \overline{\overline{F}}$, where $\overline{\overline{F}} = h^{-1}\sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}$. Let also $f_t^c = f_t - \bar{f}$, where $\bar{f} = h^{-1}\sum_{t=T+1}^{T+h} f_t$ and $\widetilde{f}_t^c = \widetilde{f}_t - \overline{\overline{f}}$, where $\overline{\overline{f}} = h^{-1}\sum_{t=T+1}^{T+h} \widetilde{f}_t$. In addition, we let $U_{i,t-1}^c = U_{i,t-1} - \bar{U}_i$, where $\bar{U}_i = h^{-1}\sum_{t=T+1}^{T+h} U_{i,t-1}$ and $\widetilde{U}_{i,t-1}^c = \widetilde{U}_{i,t-1} - \overline{\overline{U}}_i$, where $\overline{\overline{U}}_i = h^{-1}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}$. Let also $u_{i,t}^c = u_{i,t} - \bar{u}_i$, where $\bar{u}_i = h^{-1}\sum_{t=T+1}^{T+h} u_{i,t-1}$ and $\widetilde{u}_{i,t-1}^c = \widetilde{u}_{i,t-1} - \overline{\overline{u}}_i$, where $\overline{\overline{u}}_i = h^{-1}\sum_{t=T+1}^{T+h} \widetilde{u}_{i,t-1}$. We also let $\rho_h = \max_i \rho_{i,h}$.

Lemma B1. Under Assumptions 1–5, the following hold:

(a) For $t = T+1, ..., T+h$ uniformly in t,

$$F_t = O_p(\alpha_h^h k_h^{1/2}),$$

(b) For $t = T+1, ..., T+h$ uniformly in t and all i,

$$U_{i,t} = O_p(\rho_h^h k_h^{1/2}),$$

(c) For $t = T+1, ..., T+h$ uniformly in t,

$$\widetilde{F}_t - HF_t = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(d)

$$h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1} - HF_{t-1})^2 = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),$$

(e)

$$h^{-1}\textstyle\sum_{t=T+1}^{T+h} F_{t-1}(\widetilde{F}_{t-1} - HF_{t-1}) = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(f)

$$\begin{aligned} h^{-1}\textstyle\sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}(\widetilde{F}_{t-1} - HF_{t-1}) \;\; &= \;\; O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right), \end{aligned}$$

(g)

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}-HF_{t-1})e_t = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(h) For $t = T+1, ..., T+h$ uniformly in t,

$$f_t = O_p(\alpha_h^h k_h^{-1/2}),$$

(i) For $t = T+1, ..., T+h$ uniformly in t and all i,

$$u_{i,t} = O_p(\rho_h^h k_h^{-1/2}),$$

(j) For $t = T+1, ..., T+h$ uniformly in t,

$$\widetilde{f}_t - Hf_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(k)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_t - Hf_t)^2 = O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right),$$

(l)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}-HF_{t-1})(\widetilde{f}_t - Hf_t) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right),$$

(m)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}-HF_{t-1})f_t = O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(n)

$$h^{-1}\sum_{t=T+1}^{T+h} F_{t-1}(\widetilde{f}_t - Hf_t) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(o)

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t) &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right), \end{aligned}$$

(p)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_t - Hf_t)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(q)

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2 &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 + O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N, T\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\{N^{1/2}, T^{1/2}\}}\right), \end{aligned}$$

(r)

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}e_t &= h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}e_t \\ &\quad + O_p\left(\frac{\alpha_h^h h^{-1/2}k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right). \end{aligned}$$

Proof of Lemma B1.
(a) For $t = T + j$ with $j = 1, ..., h$,

$$\begin{aligned} F_t &= e_t + \alpha_h e_{t-1} + ... + \alpha_h^{j-1}e_{t+j-1} + \alpha_h^j F_T, \\ &= \alpha_h^j \sum_{s=1}^{j}\alpha_h^{-s}e_{t+j-s} + \alpha_h^j F_T = O_p(\alpha_h^h k_h^{1/2}), \end{aligned}$$

by using Lemma A5 (a). (b) The proof is the same as part (a).

(c) We start with

$$\widetilde{F}_t = \frac{\sum_{i=1}^N \hat{\lambda}_i^* X_{it}}{\sum_{i=1}^N \hat{\lambda}_i^{*2}} = \frac{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^*\lambda_i}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}}F_t + \frac{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^* U_{i,t}}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} = I + II.$$

For I,

$$\begin{aligned} I &= \frac{N^{-1}\sum_{i=1}^N (H\hat{\lambda}_i^{*2} + \hat{\lambda}_i^*\lambda_i - H\hat{\lambda}_i^{*2})}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}}F_t, \\ &= \left[H - \frac{HN^{-1}\sum_{i=1}^N \hat{\lambda}_i^*(\hat{\lambda}_i^* - H^{-1}\lambda_i)}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}}\right]F_t, \end{aligned}$$

but

$$N^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^*(\hat{\lambda}_i^*-H^{-1}\lambda_i) = N^{-1}\sum_{i=1}^{N}H^{-1}\lambda_i(\hat{\lambda}_i^*-H^{-1}\lambda_i)+N^{-1}\sum_{i=1}^{N}(\hat{\lambda}_i^*-H^{-1}\lambda_i)^2,$$
$$= O_p\left(\frac{1}{\min\{N,T^{1/2}\}}\right),$$

by using Lemma A3 (f). For II,

$$II=\frac{N^{-1}\sum_{i=1}^{N}H^{-1}\lambda_iU_{i,t}+N^{-1}\sum_{i=1}^{N}(\hat{\lambda}_i^*-H^{-1}\lambda_i)U_{i,t}}{N^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^{*2}}.$$

Therefore,

$$\begin{aligned}\widetilde{F}_t-HF_t &= -\frac{HN^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^*(\hat{\lambda}_i^*-H^{-1}\lambda_i)}{N^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^{*2}}F_t \\ &\quad +\frac{N^{-1}\sum_{i=1}^{N}H^{-1}\lambda_iU_{i,t}}{N^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^{*2}}+\frac{N^{-1}\sum_{i=1}^{N}(\hat{\lambda}_i^*-H^{-1}\lambda_i)U_{i,t}}{N^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^{*2}}, \\ &= O_p\left(\frac{\alpha_h^hk_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^hk_h^{1/2}}{N^{1/2}}\right)+O_p\left(\frac{\rho_h^hk_h^{1/2}}{\min\{N,T^{1/2}\}}\right), \\ &= O_p\left(\frac{\alpha_h^hk_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^hk_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

uniformly in t by using (a) and (b). (d) is straightforwardly shown from (c).

(e) Since $F_t=O_p(\alpha_h^hk_h^{1/2})$,

$$\begin{aligned}F_t(\widetilde{F}_t-HF_t) &= O_p(\alpha_h^hk_h^{1/2})\times\left[O_p\left(\frac{\alpha_h^hk_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^hk_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^hk_h}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

uniformly in t for $t=T+1,\cdots,T+h$ by using (a) and (c). Hence,

$$h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}(\widetilde{F}_{t-1}-HF_{t-1})=O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^hk_h}{\min\{N^{1/2},T^{1/2}\}}\right).$$

(f) By using (e) and (d),

$$\begin{aligned}
\widetilde{F}_t(\widetilde{F}_t - HF_t) &= HF_t(\widetilde{F}_t - HF_t) + (\widetilde{F}_t - HF_t)^2, \\
&= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right), \\
&= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),
\end{aligned}$$

uniformly in t for $t = T+1, \cdots, T+h$. Hence,

$$\begin{aligned}
h^{-1} \textstyle\sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}(\widetilde{F}_{t-1} - HF_{t-1}) &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right).
\end{aligned}$$

(g) Since

$$(\widetilde{F}_{t-1} - HF_{t-1})e_t = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

uniformly in t and $(\widetilde{F}_{t-1} - HF_{t-1})$ and e_t are independent with e_t being i.i.d. with $\mathbb{E}(e_t) = 0$, we have

$$h^{-1/2} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})e_t = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right).$$

(h) From (a), we obtain

$$\begin{aligned}
f_t &= F_t - F_{t-1} = (1 + \frac{c}{k_h})F_{t-1} + e_t - F_{t-1}, \\
&= \frac{c}{k_h} F_{t-1} + e_t = O_p(\alpha_h^h k_h^{-1/2}),
\end{aligned}$$

uniformly in t. (i) The proof is the same as (h).

(j) We start with

$$\widetilde{f}_t = \frac{\sum_{i=1}^N \hat{\lambda}_i^* x_{it}}{\sum_{i=1}^N \hat{\lambda}_i^{*2}} = \frac{N^{-1} \sum_{i=1}^N \hat{\lambda}_i^* \lambda_i}{N^{-1} \sum_{i=1}^N \hat{\lambda}_i^{*2}} f_t + \frac{N^{-1} \sum_{i=1}^N \hat{\lambda}_i^* u_{i,t}}{N^{-1} \sum_{i=1}^N \hat{\lambda}_i^{*2}} = I + II,$$

where

$$
\begin{aligned}
I &= \left[H - \frac{HN^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^*(\hat{\lambda}_i^* - \lambda_i H^{-1})}{N^{-1}\sum_{i=1}^{N}\hat{\lambda}_i^{*2}}\right] f_t, \\
&= Hf_t + O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right),
\end{aligned}
$$

and $II = O_p\left(\frac{\rho_h^h k_h^{-1/2}}{N^{1/2}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right) = O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)$. Hence,

$$
\widetilde{f}_t - Hf_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),
$$

uniformly in t. (k) is straightforwardly shown from (j).

(l) (c) and (j) imply that

$$
\begin{aligned}
(\widetilde{F}_{t-1} - HF_{t-1})(\widetilde{f}_t - Hf_t) &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right] \\
&\quad \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right),
\end{aligned}
$$

uniformly in t, which yields the result.

(m) Since

$$
\begin{aligned}
(\widetilde{F}_{t-1} - HF_{t-1})f_t &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right] \times O_p(\alpha_h^h k_h^{-1/2}), \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}
$$

uniformly in t, which yields the result.

(n) (a) and (j) imply that

$$
\begin{aligned}
F_{t-1}(\widetilde{f}_t - Hf_t) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}
$$

uniformly in t, which yields the result.

(o) By using (n) and (l), we get

$$\begin{aligned}\widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t) &= HF_{t-1}(\widetilde{f}_t - Hf_t) + (\widetilde{F}_t - HF_t)(\widetilde{f}_t - Hf_t), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{3/2}, T\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right),\end{aligned}$$

uniformly in t, which yields the result.

(p) From part (j), we obtain

$$h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{f}_t - Hf_t)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right).$$

(q) We have

$$\begin{aligned}\sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})^2 &= \sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}^2 + H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 - 2H \sum_{t=T+1}^{T+h} \widetilde{F}_{t-1} F_{t-1}, \\ &= \sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}^2 + H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 \\ &\quad - 2H \sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})F_{t-1} - 2H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2, \\ &= \sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}^2 - H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 \\ &\quad - 2H \sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})F_{t-1}.\end{aligned}$$

This yields

$$\begin{aligned}\sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}^2 &= H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 + \sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})^2 \\ &\quad + 2H \sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})F_{t-1},\end{aligned}$$

or

$$\begin{aligned}h^{-2} \sum_{t=T+1}^{T+h} \widetilde{F}_{t-1}^2 &= h^{-2} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 + h^{-2} \sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})^2 \\ &\quad + 2h^{-2} H \sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})F_{t-1}, \\ &= h^{-2} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 \\ &\quad + O_p\left(\frac{\alpha_h^{2h} h^{-1} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} h^{-1} k_h}{\min\{N, T\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h} h^{-1} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h h^{-1} k_h}{\min\{N^{1/2}, T^{1/2}\}}\right),\end{aligned}$$

by using (d) and (e).

(r) We have

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}e_t &= h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}e_t + h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}-HF_{t-1})e_t, \\ &= h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}e_t \\ &\quad +O_p\left(\frac{\alpha_h^h k_h^{1/2}h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

by using (g).■

Lemma B2. Suppose Assumptions 1–4 and A hold or Assumptions 1–5 and the following condition hold:

$$\frac{\alpha_h^h\rho_h^h h^{1/2}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\to 0.$$

Then, we have

$$\hat{\sigma}^2\xrightarrow{p}Q^{-2}\sigma^2,$$

as $N,T,h\to\infty$, where

$$\hat{\sigma}^2 = h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_t-\hat{\delta}^*\widetilde{F}_{t-1})^2,$$

with

$$\hat{\delta}^* = \frac{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}\widetilde{f}_t}{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2}.$$

Proof of Lemma B2.

We start with the AR coefficient estimator,

$$\hat{\delta}^* = \frac{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}\widetilde{f}_t}{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2}.$$

Since

$$\begin{aligned} \widetilde{f}_t &= \widetilde{F}_t-\widetilde{F}_{t-1}, \\ &= HF_t-HF_{t-1}+(\widetilde{F}_t-HF_t)-(\widetilde{F}_{t-1}-HF_{t-1}), \\ &= H\frac{c}{k_h}F_{t-1}+He_t+(\widetilde{F}_t-HF_t)-(\widetilde{F}_{t-1}-HF_{t-1}), \\ &= \frac{c}{k_h}\widetilde{F}_{t-1}+He_t+(\widetilde{f}_t-Hf_t)-\frac{c}{k_h}(\widetilde{F}_{t-1}-HF_{t-1}), \end{aligned}$$

we obtain

$$\begin{aligned}\hat{\delta}^* &= \frac{c}{k_h} + \frac{H\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}e_t}{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} + \frac{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t)}{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} \\ &\quad + \frac{c}{k_h}\frac{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{F}_{t-1} - HF_{t-1})}{\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2}, \\ &= \frac{c}{k_h} + \frac{\alpha_h^{-2h}k_h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} + \frac{\alpha_h^{-2h}k_h^{-2}H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1} - HF_{t-1})e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} \\ &\quad + \frac{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t)}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} + \frac{c}{k_h}\frac{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{F}_{t-1} - HF_{t-1})}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2}.\end{aligned}$$

This yields

$$\begin{aligned}\alpha_h^{-h}k_h^{-1}\left(\hat{\delta}^* - \frac{c}{k_h}\right) &= \frac{\alpha_h^{-h}k_h^{-1}H^2\sum_{t=T+1}^{T+h}F_{t-1}e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} + \frac{\alpha_h^{-h}k_h^{-1}H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1} - HF_{t-1})e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} \\ &\quad + \frac{\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t)}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2} + \frac{c}{k_h}\frac{\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{F}_{t-1} - HF_{t-1})}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2}, \\ &= I + II + III + IV.\end{aligned}$$

For the denominator,

$$\begin{aligned}\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2 &= \alpha_h^{-2h}k_h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 + O_p\left(\frac{hk_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^{-2h}\rho_h^{2h}hk_h^{-1}}{\min\{N, T\}}\right) \\ &\quad + O_p\left(\frac{hk_h^{-1}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^{-h}\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right),\end{aligned}$$

by using Lemma B1 (q) and the four terms of the factor estimation errors are $o_p(1)$ under the stated conditions.

The numerator of I is $o_p(1)$ by using Lemma A5 (c). For the numerator of II,

$$\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1} - HF_{t-1})e_t = O_p\left(\frac{h^{1/2}k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^{-h}\rho_h^{h}h^{1/2}k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

by using Lemma B1 (g) and it is $o_p(1)$ under the stated conditions.

For the numerator of III,

$$\begin{aligned}\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t) &= O_p\left(\frac{\alpha_h^{h}hk_h^{-1}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{-h}\rho_h^{2h}hk_h^{-1}}{\min\{N, T\}}\right) = o_p(1),\end{aligned}$$

by using Lemma B1 (o) and it is $o_p(1)$ under the stated conditions.

For the numerator of IV,

$$\alpha_h^{-h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{F}_{t-1}-HF_{t-1}) = O_p\left(\frac{\alpha_h^h h k_h^{-1}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right) +O_p\left(\frac{\alpha_h^h\rho_h^{2h}hk_h^{-1}}{\min\{N,T\}}\right),$$

by using Lemma B1 (f) and it is $o_p(1)$ under the stated conditions so that we proceed with $\hat{\delta}^* - \frac{c}{k_h} = O_p(\alpha_h^{-h}k_h^{-1})$ and $\hat{\delta}^* = O_p(k_h^{-1})$. Then,

$$\begin{aligned}
\hat{\sigma}^2 &= h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_t - \hat{\delta}^*\widetilde{F}_{t-1})^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[Hf_t - \hat{\delta}^*HF_{t-1} + (\widetilde{f}_t - Hf_t) - \hat{\delta}^*(\widetilde{F}_{t-1}-HF_{t-1})]^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[Hf_t - H\frac{c}{k_h}F_{t-1} - (\hat{\delta}^* - \frac{c}{k_h})HF_{t-1} + (\widetilde{f}_t - Hf_t) - \hat{\delta}^*(\widetilde{F}_{t-1}-HF_{t-1})]^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[He_t - (\hat{\delta}^* - \frac{c}{k_h})HF_{t-1} + (\widetilde{f}_t - Hf_t) - \hat{\delta}^*(\widetilde{F}_{t-1}-HF_{t-1})]^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[H^2e_t^2 + (\hat{\delta}^* - \frac{c}{k_h})^2H^2F_{t-1}^2 + (\widetilde{f}_t - Hf_t)^2 + \hat{\delta}^{*2}(\widetilde{F}_{t-1}-HF_{t-1})^2\\
&\quad +2H^2(\hat{\delta}^* - \frac{c}{k_h})F_{t-1}e_t + 2H(\widetilde{f}_t - Hf_t)e_t - 2\hat{\delta}^*(\widetilde{F}_{t-1}-HF_{t-1})He_t\\
&\quad +2(\hat{\delta}^* - \frac{c}{k_h})HF_{t-1}(\widetilde{f}_t - Hf_t) - 2(\hat{\delta}^* - \frac{c}{k_h})\hat{\delta}^*HF_{t-1}(\widetilde{F}_{t-1}-HF_{t-1})\\
&\quad +2\hat{\delta}^*(\widetilde{f}_t - Hf_t)(\widetilde{F}_{t-1}-HF_{t-1})],\\
&= h^{-1}\sum_{t=T+1}^{T+h}H^2e_t^2 + \sum_{k=1}^{9}D_k,
\end{aligned}$$

has nine terms of the factor estimation errors. We now show that they are all $o_p(1)$ under the stated conditions. For D_1,

$$\begin{aligned}
D_1 &= (\hat{\delta}^* - \frac{c}{k_h})^2H^2h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^2,\\
&= O_p(\alpha_h^{-2h}k_h^{-2})\times O_p\left(\alpha_h^{2h}h^{-1}k_h^2\right),\\
&= O_p(h^{-1}) = o_p(1).
\end{aligned}$$

For D_2,

$$\begin{aligned}
D_2 &= h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_t - Hf_t)^2,\\
&= O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N,T\}}\right),
\end{aligned}$$

by using Lemma B1 (k) and it is $o_p(1)$ under the stated conditions. For D_3,

$$\begin{aligned} D_3 &= \hat{\delta}^{*2} h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1})^2, \\ &= O_p(k_h^{-2}) \times \left[O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}} \right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}} \right) \right], \\ &= O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}} \right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}} \right), \end{aligned}$$

by using Lemma B1 (d) and it is $o_p(1)$ under the stated conditions. For D_4,

$$\begin{aligned} D_4 &= 2H^2(\hat{\delta}^* - \frac{c}{k_h}) h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} e_t, \\ &= O_p(\alpha_h^{-h} k_h^{-1}) \times o_p\left(\alpha_h^h h^{-1} k_h \right), \\ &= o_p(h^{-1}) = o_p(1), \end{aligned}$$

by using Lemma A5 (c). For D_5,

$$\begin{aligned} D_5 &= 2Hh^{-1} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{f}_t - Hf_t) e_t, \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}} \right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B1 (p) and it is $o_p(1)$ under the stated conditions. For D_6,

$$\begin{aligned} D_6 &= 2\hat{\delta}^* H h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{F}_{t-1} - HF_{t-1}) e_t, \\ &= O_p(k_h^{-1}) \times \left[O_p\left(\frac{\alpha_h^h h^{1/2} k_h^{1/2}}{\min\{N, T^{1/2}\}} \right) + O_p\left(\frac{\rho_h^h h^{1/2} k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}} \right) \right], \\ &= O_p\left(\frac{\alpha_h^h h^{1/2} k_h^{-1/2}}{\min\{N, T^{1/2}\}} \right) + O_p\left(\frac{\rho_h^h h^{1/2} k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B1 (g) and it is $o_p(1)$ under the stated conditions. For D_7,

$$\begin{aligned} D_7 &= 2(\hat{\delta}^* - \frac{c}{k_h}) H h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} (\widetilde{f}_t - Hf_t), \\ &= O_p(\alpha_h^{-h} k_h^{-1}) \times \left[O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}} \right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}} \right) \right], \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1}}{\min\{N, T^{1/2}\}} \right) + O_p\left(\frac{\rho_h^h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B1 (o) and it is $o_p(1)$ under the stated conditions. For D_8,

$$
\begin{aligned}
D_8 &= 2(\hat{\delta}^* - \frac{c}{k_h})\hat{\delta}^* H h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1}(\widetilde{F}_{t-1} - HF_{t-1}), \\
&= O_p(\alpha_h^{-h} k_h^{-1}) \times O_p(k_h^{-1}) \times \left[O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^h k_h^{-1}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B1 (e) and it is $o_p(1)$ under the stated conditions. For D_9,

$$
\begin{aligned}
D_9 &= 2\hat{\delta}^* h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{f}_t - Hf_t)(\widetilde{F}_{t-1} - HF_{t-1}), \\
&= O_p\left(k_h^{-1}\right) \times \left[O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{3/2}, T\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h^{-1}}{\min\{N^{3/2}, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right),
\end{aligned}
$$

by using Lemma B1 (l) and it is $o_p(1)$ under the stated conditions. Therefore,

$$
\hat{\sigma}^2 = h^{-1} \textstyle\sum_{t=T+1}^{T+h} H^2 e_t^2 + o_p(1),
$$

under the stated conditions, which yields the result. Note that carefully investigating all 17 terms of the factor estimation errors gives the dominating terms $O_p\left(\frac{\alpha_h^h h k_h^{-1}}{\min\{N, T^{1/2}\}}\right)$, and $O_p\left(\frac{\rho_h^h h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right)$ that appear in the numerator of IV.■

Lemma B3. Under Assumptions 1–5, the following hold:

(a) For $t = T+1, ..., T+h$ uniformly in t,

$$
\widetilde{U}_{i,t} - U_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),
$$

(b)

$$
h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1} - U_{i,t-1})^2 = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),
$$

(c)

$$
h^{-1} \textstyle\sum_{t=T+1}^{T+h} U_{i,t-1}(\widetilde{U}_{i,t-1} - U_{i,t-1}) = O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N^{1/2}, T^{1/2}\}}\right),
$$

(d)

$$h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}(\widetilde{U}_{i,t-1}-U_{i,t-1}) = O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\{N,T^{1/2}\}}\right) +O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(e)

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(f) For $t=T+1,...,T+h$ uniformly in t,

$$\widetilde{u}_{i,t}-u_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(g)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t}-u_{i,t})^2 = O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N,T\}}\right),$$

(h)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})(\widetilde{u}_{i,t}-u_{i,t}) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N^{3/2},T\}}\right) +O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),$$

(i)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})u_{i,t} = O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(j)

$$h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}(\widetilde{u}_{i,t}-u_{i,t}) = O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(k)

$$h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}(\widetilde{u}_{i,t}-u_{i,t}) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right) +O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right),$$

(l)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t}-u_{i,t})z_{i,t}=O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(m)

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2 &= h^{-2}H^2\sum_{t=T+1}^{T+h}U_{i,t-1}^2+O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) \\ &+O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

(n)

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}z_{i,t} &= h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t} \\ &+O_p\left(\frac{\alpha_h^h h^{-1/2}k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right). \end{aligned}$$

Proof of Lemma B3.
(a) Since

$$\begin{aligned} \widetilde{U}_{i,t}-U_{i,t} &= (\hat{\lambda}_i^*-H^{-1}\lambda_i)HF_t+H^{-1}\lambda_i^*(\widetilde{F}_t-HF_t) \\ &+(\hat{\lambda}_i^*-H^{-1}\lambda_i)(\widetilde{F}_t-HF_t), \\ &= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right) \\ &+O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right) \\ &+O_p\left(\frac{1}{\min\{N,T^{1/2}\}}\right) \\ &\times\left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B1 (c) for $t=T+1,\cdots,T+h$ uniformly in t. (b) The result is straightforwardly shown from (a).

(c) Since

$$U_{i,t}(\widetilde{U}_{i,t}-U_{i,t}) = O_p\left(\rho_h^h k_h^{1/2}\right)\times\left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right],$$
$$= O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),$$

uniformly in t, the result follows.

(d) By using (b) and (c),

$$\widetilde{U}_{i,t}(\widetilde{U}_{i,t}-U_{i,t}) = U_{i,t}(\widetilde{U}_{i,t}-U_{i,t})+(\widetilde{U}_{i,t}-U_{i,t})^2,$$
$$= O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N^{1/2},T^{1/2}\}}\right)$$
$$+O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N,T\}}\right),$$
$$= O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),$$

uniformly in t for $t=T+1,\cdots,T+h$. Hence,

$$h^{-1}\textstyle\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}(\widetilde{U}_{i,t-1}-U_{i,t-1}) = O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\{N,T^{1/2}\}}\right)$$
$$+O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N^{1/2},T^{1/2}\}}\right).$$

(e) Since

$$(\widetilde{U}_{i,t-1}-U_{i,t-1})z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

and $(\widetilde{U}_{i,t-1}-U_{i,t-1})$ and $z_{i,t}$ are independent with $z_{i,t}$ being i.i.d. with $\mathbb{E}(z_{i,t})=0$, we have

$$h^{-1/2}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right).$$

(f) This is because

$$\widetilde{u}_{i,t}-u_{i,t} = (\hat{\lambda}_i^*-H^{-1}\lambda_i)Hf_t+H^{-1}\lambda_i(\widetilde{f}_t-Hf_t)$$
$$+(\hat{\lambda}_i^*-H^{-1}\lambda_i)(\widetilde{f}_t-Hf_t),$$
$$= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

uniformly in t by using Lemma B1 (j).

(g) is straightforwardly shown from (f).

(h) (a) and (f) imply that

$$\begin{aligned}(\widetilde{U}_{i,t-1}-U_{i,t-1})(\widetilde{u}_{i,t}-u_{i,t}) &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right] \\ &\quad \times\left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),\end{aligned}$$

uniformly in t, which yields the result.

(i) Since

$$\begin{aligned}(\widetilde{U}_{i,t-1}-U_{i,t-1})u_{i,t} &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right]\times O_p(\rho_h^h k_h^{-1/2}), \\ &= O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

uniformly in t, which yields the result.

(j) Lemma B1 (b) and Lemma B3 (f) imply that

$$\begin{aligned}U_{i,t-1}(\widetilde{u}_{i,t}-u_{i,t}) &= O_p(\rho_h^h k_h^{1/2})\times\left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

uniformly in t, which yields the result.

(k) By using (j) and (h), we get

$$\begin{aligned}\widetilde{U}_{i,t-1}(\widetilde{u}_{i,t}-u_{i,t}) &= U_{i,t-1}(\widetilde{u}_{i,t}-u_{i,t})+(\widetilde{U}_{i,t-1}-U_{i,t-1})(\widetilde{u}_{i,t}-u_{i,t}), \\ &= O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right) \\ &\quad +O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N^{3/2},T\}}\right) \\ &\quad +O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right) \\ &\quad +O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

uniformly in t, which yields the result.

(l) From (f), we obtain

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t}-u_{i,t})z_{i,t}=O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right).$$

(m) We have

$$\begin{aligned}
\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})^2 &= \sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2+\sum_{t=T+1}^{T+h}U_{i,t-1}^2-2\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}U_{i,t-1},\\
&= \sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2+\sum_{t=T+1}^{T+h}U_{i,t-1}^2\\
&\quad -2\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1}-2\sum_{t=T+1}^{T+h}U_{i,t-1}^2,\\
&= \sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2-\sum_{t=T+1}^{T+h}U_{i,t-1}^2-2\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1}.
\end{aligned}$$

Solving the first term on the right-hand side gives

$$\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2=\sum_{t=T+1}^{T+h}U_{i,t-1}^2+\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})^2+2\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1},$$

so that

$$\begin{aligned}
h^{-2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2 &= h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^2+h^{-2}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})^2\\
&\quad +2h^{-2}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1},\\
&= h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^2\\
&\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right)\\
&\quad +O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}$$

by using (b) and (c).

(n) We have

$$\begin{aligned}
h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}z_{i,t} &= h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t}+h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})z_{i,t},\\
&= h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t}\\
&\quad +O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}$$

by using (e).■

Lemma B4. Suppose Assumptions 1–4 and A hold or Assumptions 1–5 and the following conditions hold:

$$\frac{\alpha_h^h h k_h^{-1}}{\min\{N, T^{1/2}\}} \to 0 \text{ and } \frac{\rho_h^h h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}} \to 0.$$

Then, we have

$$\hat{\sigma}_i^2 \xrightarrow{p} \sigma_i^2,$$

for any i as $N, T, h \to \infty$, where

$$\hat{\sigma}_i^2 = h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\widetilde{u}_{i,t} - \hat{\delta}_i^* \widetilde{U}_{i,t-1})^2,$$

with

$$\hat{\delta}_i^* = \frac{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} \widetilde{u}_{i,t}}{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2}.$$

Proof of Lemma B4.

We start with the AR coefficient estimator,

$$\hat{\delta}_i^* = \frac{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} \widetilde{u}_{i,t}}{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2}.$$

Since

$$\begin{aligned}
\widetilde{u}_{i,t} &= \widetilde{U}_{i,t} - \widetilde{U}_{i,t-1}, \\
&= U_{i,t} - U_{i,t-1} + (\widetilde{U}_{i,t} - U_{i,t}) - (\widetilde{U}_{i,t-1} - U_{i,t-1}), \\
&= \frac{c_i}{k_h} U_{i,t-1} + z_{i,t} + (\widetilde{U}_{i,t} - U_{i,t}) - (\widetilde{U}_{i,t-1} - U_{i,t-1}), \\
&= \frac{c_i}{k_h} \widetilde{U}_{i,t-1} + z_{i,t} + (\widetilde{u}_{i,t} - u_{i,t}) - \frac{c_i}{k_h} (\widetilde{U}_{i,t-1} - U_{i,t-1}),
\end{aligned}$$

we obtain

$$\begin{aligned}
\hat{\delta}_i^* &= \frac{c_i}{k_h} + \frac{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} z_{i,t}}{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} + \frac{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} (\widetilde{u}_{i,t} - u_{i,t})}{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} \\
&\quad + \frac{c_i}{k_h} \frac{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} (\widetilde{U}_{i,t-1} - U_{i,t-1})}{\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2}, \\
&= \frac{c_i}{k_h} + \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1} z_{i,t}}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} + \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t}}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} \\
&\quad + \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} (\widetilde{u}_{i,t} - u_{i,t})}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} + \frac{c_i}{k_h} \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1} (\widetilde{U}_{i,t-1} - U_{i,t-1})}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2}.
\end{aligned}$$

This yields

$$
\begin{aligned}
\rho_h^{-h} k_h^{-1}\left(\hat{\delta}_i^* - \frac{c_i}{k_h}\right) &= \frac{\rho_h^{-h} k_h^{-1}\sum_{t=T+1}^{T+h} U_{i,t-1} z_{i,t}}{\rho_h^{-2h} k_h^{-2}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} + \frac{\rho_h^{-h} k_h^{-1}\sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t}}{\rho_h^{-2h} k_h^{-2}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} \\
&\quad + \frac{\rho_h^{-h} k_h^{-1}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}(\widetilde{u}_{i,t} - u_{i,t})}{\rho_h^{-2h} k_h^{-2}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2} + \frac{c_i}{k_h}\frac{\rho_h^{-h} k_h^{-1}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}(\widetilde{U}_{i,t-1} - U_{i,t-1})}{\rho_h^{-2h} k_h^{-2}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2}, \\
&= I + II + III + IV.
\end{aligned}
$$

For the denominator,

$$
\begin{aligned}
\rho_h^{-2h} k_h^{-2}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^2 &= \rho_h^{-2h} k_h^{-2}\sum_{t=T+1}^{T+h} U_{i,t-1}^2 + O_p\left(\frac{\alpha_h^{2h}\rho_h^{-2h} h k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{h k_h^{-1}}{\min\{N, T\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^{h}\rho_h^{-h} h k_h^{-1}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B3 (m) and the four terms of the factor estimation errors are $o_p(1)$ under the stated conditions.

The numerator of I is $o_p(1)$ by using Lemma A5 (h). For the numerator of II,

$$
\rho_h^{-h} k_h^{-1}\sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t} = O_p\left(\frac{\alpha_h^{h}\rho_h^{-h} h^{1/2} k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{h^{1/2} k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),
$$

by using Lemma B3 (e) and it is $o_p(1)$ under the stated conditions. For the numerator of III,

$$
\begin{aligned}
\rho_h^{-h} k_h^{-1}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}(\widetilde{u}_{i,t} - u_{i,t}) &= O_p\left(\frac{\alpha_h^{2h}\rho_h^{-h} h k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{h} h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^{h} h k_h^{-1}}{\min\{N, T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B3 (k) and it is $o_p(1)$ under the stated conditions. For the numerator of IV,

$$
\begin{aligned}
\rho_h^{-h} k_h^{-2}\sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}(\widetilde{U}_{i,t-1} - U_{i,t-1}) &= O_p\left(\frac{\alpha_h^{2h}\rho_h^{-h} h k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^{h} h k_h^{-1}}{\min\{N, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{h} h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B3 (d) and it is $o_p(1)$ under the stated conditions. Therefore, we proceed with $\hat{\delta}_i^* - \frac{c_i}{k_h} = O_p(\rho_h^{-h} k_h^{-1})$ and $\hat{\delta}_i^* = O_p(k_h^{-1})$. We now consider

$$
\begin{aligned}
\hat{\sigma}_i^2 &= h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t} - \hat{\delta}_i^*\widetilde{U}_{i,t-1})^2, \\
&= h^{-1}\textstyle\sum_{t=T+1}^{T+h}[u_{i,t} - \hat{\delta}_i^* U_{i,t-1} + (\widetilde{u}_{i,t} - u_{i,t}) - \hat{\delta}_i^*(\widetilde{U}_{i,t-1} - U_{i,t-1})]^2, \\
&= h^{-1}\textstyle\sum_{t=T+1}^{T+h}[u_{i,t} - \dfrac{c_i}{k_h}U_{i,t-1} - (\hat{\delta}_i^* - \dfrac{c_i}{k_h})U_{i,t-1} + (\widetilde{u}_{i,t} - u_{i,t}) - \hat{\delta}_i^*(\widetilde{U}_{i,t-1} - U_{i,t-1})]^2, \\
&= h^{-1}\textstyle\sum_{t=T+1}^{T+h}[z_{i,t} - (\hat{\delta}_i^* - \dfrac{c_i}{k_h})U_{i,t-1} + (\widetilde{u}_{i,t} - u_{i,t}) - \hat{\delta}_i^*(\widetilde{U}_{i,t-1} - U_{i,t-1})]^2, \\
&= h^{-1}\textstyle\sum_{t=T+1}^{T+h}[z_{i,t}^2 + (\hat{\delta}_i^* - \dfrac{c_i}{k_h})^2 U_{i,t-1}^2 + (\widetilde{u}_{i,t} - u_{i,t})^2 + \hat{\delta}_i^{*2}(\widetilde{U}_{i,t-1} - U_{i,t-1})^2 \\
&\quad +2(\hat{\delta}_i^* - \dfrac{c_i}{k_h})U_{i,t-1}z_{i,t} + 2(\widetilde{u}_{i,t} - u_{i,t})z_{i,t} - 2\hat{\delta}_i^*(\widetilde{U}_{i,t-1} - U_{i,t-1})z_{i,t} \\
&\quad +2(\hat{\delta}_i^* - \dfrac{c_i}{k_h})U_{i,t-1}(\widetilde{u}_{i,t} - u_{i,t}) - 2(\hat{\delta}_i^* - \dfrac{c_i}{k_h})\hat{\delta}_i^* U_{i,t-1}(\widetilde{U}_{i,t-1} - U_{i,t-1}) \\
&\quad +2\hat{\delta}_i^*(\widetilde{u}_{i,t} - u_{i,t})(\widetilde{U}_{i,t-1} - U_{i,t-1})], \\
&= h^{-1}\textstyle\sum_{t=T+1}^{T+h} z_{i,t}^2 + \sum_{k=1}^{9} D_k,
\end{aligned}
$$

has nine terms of the factor estimation errors. We now show that they are all $o_p(1)$ under the stated conditions. For D_1,

$$
\begin{aligned}
D_1 &= (\hat{\delta}_i^* - \frac{c_i}{k_h})^2 h^{-1}\textstyle\sum_{t=T+1}^{T+h} U_{i,t-1}^2, \\
&= O_p(\rho_h^{-2h}k_h^{-2}) \times O_p\left(\rho_h^{2h}h^{-1}k_h^2\right), \\
&= O_p(h^{-1}) = o_p(1).
\end{aligned}
$$

For D_2,

$$
\begin{aligned}
D_2 &= h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t} - u_{i,t})^2, \\
&= O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N, T\}}\right),
\end{aligned}
$$

by using Lemma B3 (g) and it is $o_p(1)$ under the stated conditions. For D_3,

$$
\begin{aligned}
D_3 &= \hat{\delta}^{*2}h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1} - U_{i,t-1})^2, \\
&= O_p(k_h^{-2}) \times \left[O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N, T\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N, T\}}\right),
\end{aligned}
$$

by using Lemma B3 (b) and it is $o_p(1)$ under the stated conditions. For D_4,

$$
\begin{aligned}
D_4 &= 2(\hat{\delta}_i^* - \frac{c_i}{k_h})h^{-1}\textstyle\sum_{t=T+1}^{T+h} U_{i,t-1}z_{i,t}, \\
&= O_p(\rho_h^{-h}k_h^{-1}) \times o_p\left(\rho_h^h k_h h^{-1}\right) = o_p(1).
\end{aligned}
$$

For D_5,

$$\begin{aligned} D_5 &= 2h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t}-u_{i,t})z_{i,t}, \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B3 (l) and it is $o_p(1)$ under the stated conditions. For D_6,

$$\begin{aligned} D_6 &= 2\hat{\delta}_i^* h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}-U_{i,t-1})z_{i,t}, \\ &= O_p(k_h^{-1})\times\left[O_p\left(\frac{\alpha_h^h h^{1/2}k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h h^{1/2}k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^h h^{1/2}k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h h^{1/2}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B3 (e) and it is $o_p(1)$ under the stated conditions. For D_7,

$$\begin{aligned} D_7 &= 2(\hat{\delta}_i^*-\frac{c_i}{k_h})h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}(\widetilde{u}_{i,t}-u_{i,t}), \\ &= O_p(\rho_h^{-h}k_h^{-1})\times\left[O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B3 (j) and it is $o_p(1)$ under the stated conditions. For D_8,

$$\begin{aligned} D_8 &= 2(\hat{\delta}_i^*-\frac{c_i}{k_h})\hat{\delta}_i^* h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}(\widetilde{U}_{i,t-1}-U_{i,t-1}), \\ &= O_p(\rho_h^{-h}k_h^{-1})\times\left[O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h}{\min\{N^{1/2},T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B3 (c) and it is $o_p(1)$ under the stated conditions. For D_9,

$$\begin{aligned} D_9 &= 2\hat{\delta}_i^* h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t}-u_{i,t})(\widetilde{U}_{i,t-1}-U_{i,t-1}), \\ &= O_p\left(k_h^{-1}\right)\times\left[O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N,T\}}\right), \end{aligned}$$

by using Lemma B3 (h) and it is $o_p(1)$ under the stated conditions. Therefore,

$$\hat{\sigma}_i^2 = h^{-1}\sum_{t=T+1}^{T+h}z_{i,t}^2+o_p(1),$$

under the stated conditions. Note that the two conditions

$$\frac{\alpha_h^h h k_h^{-1}}{\min\{N, T^{1/2}\}} \to 0 \text{ and } \frac{\rho_h^h h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}} \to 0,$$

are obtained by carefully investigating all 17 terms of the factor estimation errors.■

Lemma B5. Under Assumptions 1–5, the following hold:

(a)

$$\overline{\widetilde{F}} - H\bar{F} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(b)

$$\overline{\widetilde{f}} - H\bar{f} = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right).$$

Proof of Lemma B5. (a) We have

$$\overline{\widetilde{F}} - H\bar{F} = h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1} - HF_{t-1}) = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

by using Lemma B1 (c). (b) We have

$$\overline{\widetilde{f}} - H\bar{f} = h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{f}_t - Hf_t) = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

by using Lemma B1 (j).■

Lemma B6. Under Assumptions 1–5, the following hold:

(a) For $t = T+1, ..., T+h$ uniformly in t,

$$F_t^c = O_p(\alpha_h^h k_h^{1/2}),$$

(b) For $t = T+1, ..., T+h$ uniformly in t and all i,

$$U_{i,t}^c = O_p(\rho_h^h k_h^{1/2}),$$

(c) For $t = T+1, ..., T+h$ uniformly in t,

$$\widetilde{F}_t^c - HF_t^c = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(d)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})^{2}=O_{p}\left(\frac{\alpha_{h}^{2h}k_{h}}{\min\{N^{2},T\}}\right)+O_{p}\left(\frac{\rho_{h}^{2h}k_{h}}{\min\{N,T\}}\right),$$

(e)

$$h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^{c}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})=O_{p}\left(\frac{\alpha_{h}^{2h}k_{h}}{\min\{N,T^{1/2}\}}\right)+O_{p}\left(\frac{\alpha_{h}^{h}\rho_{h}^{h}k_{h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(f)

$$\begin{aligned}h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c}) \quad &= \quad O_{p}\left(\frac{\alpha_{h}^{2h}k_{h}}{\min\{N,T^{1/2}\}}\right)+O_{p}\left(\frac{\alpha_{h}^{h}\rho_{h}^{h}k_{h}}{\min\{N^{1/2},T^{1/2}\}}\right)\\ &\quad +O_{p}\left(\frac{\rho_{h}^{2h}k_{h}}{\min\{N,T\}}\right),\end{aligned}$$

(g)

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})e_{t}=O_{p}\left(\frac{\alpha_{h}^{h}k_{h}^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_{p}\left(\frac{\rho_{h}^{h}k_{h}^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(h) For $t=T+1,...,T+h$ uniformly in t,

$$f_{t}^{c}=O_{p}(\alpha_{h}^{h}k_{h}^{-1/2}),$$

(i) For $t=T+1,...,T+h$ uniformly in t and all i,

$$u_{i,t}^{c}=O_{p}(\rho_{h}^{h}k_{h}^{-1/2}),$$

(j) For $t=T+1,...,T+h$ uniformly in t,

$$\widetilde{f}_{t}^{c}-Hf_{t}^{c}=O_{p}\left(\frac{\alpha_{h}^{h}k_{h}^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_{p}\left(\frac{\rho_{h}^{h}k_{h}^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(k)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_{t}^{c}-Hf_{t}^{c})^{2}=O_{p}\left(\frac{\alpha_{h}^{2h}k_{h}^{-1}}{\min\{N^{2},T\}}\right)+O_{p}\left(\frac{\rho_{h}^{2h}k_{h}^{-1}}{\min\{N,T\}}\right),$$

(l)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})(\widetilde{f}_{t}^{c}-Hf_{t}^{c})=O_{p}\left(\frac{\alpha_{h}^{2h}}{\min\{N^{2},T\}}\right)+O_{p}\left(\frac{\rho_{h}^{2h}}{\min\{N,T\}}\right),$$

(m)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})f_t^c=O_p\left(\frac{\alpha_h^{2h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(n)

$$h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^{c}(\widetilde{f}_t^c-Hf_t^c)=O_p\left(\frac{\alpha_h^{2h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(o)

$$\begin{aligned}h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}(\widetilde{f}_t^c-Hf_t^c) &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N^{1/2},T^{1/2}\}}\right)\\ &\quad +O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),\end{aligned}$$

(p)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{f}_t^c-Hf_t^c)e_t=O_p\left(\frac{\alpha_h^{h}k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(q)

$$\begin{aligned}h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2} &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right)\\ &\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

(r)

$$\begin{aligned}h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}e_t &= h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t\\ &\quad +O_p\left(\frac{\alpha_h^{h}h^{-1/2}k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}h^{-1/2}k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right).\end{aligned}$$

Proof of Lemma B6.
(a) We have

$$\begin{aligned}F_{t-1}^{c} &= F_{t-1}-h^{-1}\sum_{t=T+1}^{T+h}F_{t-1},\\ &= O_p(\alpha_h^{h}k_h^{1/2}),\end{aligned}$$

by using Lemma B1 (a). (b) The proof is the same as (a).

(c) We have

$$\begin{aligned}
\widetilde{F}^c_{t-1} - HF^c_{t-1} &= (\widetilde{F}_{t-1} - HF_{t-1}) - (\overline{\widetilde{F}} - H\overline{F}), \\
&= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}$$

uniformly in t, for $t = T+1, \cdots T+h$, by using Lemmas B1 (c) and B5 (a). (d) is straightforwardly shown from (c).

(e) We have

$$\begin{aligned}
F^c_{t-1}(\widetilde{F}^c_{t-1} - HF^c_{t-1}) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}$$

uniformly in t, for $t = T+1, \cdots T+h$, by using (a) and (c). The result follows.

(f) We have

$$\begin{aligned}
\widetilde{F}^c_{t-1}(\widetilde{F}^c_{t-1} - HF^c_{t-1}) &= HF^c_{t-1}(\widetilde{F}^c_{t-1} - HF^c_{t-1}) + (\widetilde{F}^c_{t-1} - HF^c_{t-1})^2, \\
&= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right), \\
&= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),
\end{aligned}$$

uniformly in t, for $t = T+1, \cdots T+h$, by using results derived in (e) and (d). The result follows.

(g) We have

$$(\widetilde{F}^c_{t-1} - HF^c_{t-1})e_t = (\widetilde{F}_{t-1} - HF_{t-1})e_t - (\overline{\widetilde{F}} - H\overline{F})e_t,$$

so that

$$
\begin{aligned}
h^{-1/2}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})e_t &= h^{-1/2}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}-HF_{t-1})e_t+(\overline{\widetilde{F}}-H\overline{F})h^{-1/2}\textstyle\sum_{t=T+1}^{T+h}e_t, \\
&= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right), \\
&= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B1 (g) and Lemma B5 (a).

(h) We have

$$
f_t^c = f_t - h^{-1}\textstyle\sum_{t=T+1}^{T+h} f_t = O_p(\alpha_h^h k_h^{-1/2}).
$$

(i) can be shown same as (h).

(j) We have

$$
\begin{aligned}
\widetilde{f}_t^c - Hf_t^c &= (\widetilde{f}_t - Hf_t)-(\overline{\widetilde{f}}-H\bar{f}), \\
&= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

uniformly in t, for $t=T+1,\cdots T+h$, by using Lemma B1 (j) and Lemma B5 (b). (k) is straightforwardly shown from part (j).

(l) We have

$$
\begin{aligned}
(\widetilde{F}_{t-1}^c - HF_{t-1}^c)(\widetilde{f}_t^c - Hf_t^c) &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right] \\
&\quad \times\left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),
\end{aligned}
$$

uniformly in t, for $t=T+1,\cdots T+h$, by using the results obtained in (c) and (j). This yields the result.

(m) We have

$$
\begin{aligned}
(\widetilde{F}_{t-1}^c - HF_{t-1}^c)f_t^c &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right]\times O_p(\alpha_h^h k_h^{-1/2}), \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

uniformly in t, for $t = T+1, \cdots T+h$, by using the result obtained in (c). This yields the result.

(n) (a) and (j) imply that

$$\begin{aligned} F_{t-1}^c(\widetilde{f}_t^c - Hf_t^c) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right), \end{aligned}$$

uniformly in t, for $t = T+1, \cdots T+h$, which yields the result.

(o) We have

$$\begin{aligned} \widetilde{F}_{t-1}^c(\widetilde{f}_t^c - Hf_t^c) &= HF_{t-1}^c(\widetilde{f}_t^c - Hf_t^c) + (\widetilde{F}_t^c - HF_t^c)(\widetilde{f}_t^c - Hf_t^c), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{3/2}, T\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right), \end{aligned}$$

uniformly in t, for $t = T+1, \cdots T+h$, by using the results obtained in (n) and (l). This yields the result.

(p) We have

$$\begin{aligned} h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{f}_t^c - Hf_t^c)e_t &= h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{f}_t - Hf_t)e_t + (\overline{\widetilde{f}} - H\bar{f})h^{-1}\textstyle\sum_{t=T+1}^{T+h} e_t, \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right), \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B1 (p) and Lemma B5 (b). Hence,

$$h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{f}_t^c - Hf_t^c)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right).$$

(q) We have

$$
\begin{aligned}
\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})^{2} &= \textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}+H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}-2H\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}F_{t-1}^{c},\\
&= \textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}+H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}\\
&\quad \textstyle -2H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})F_{t-1}-2H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2},\\
&= \textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}-H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}\\
&\quad \textstyle -2H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})F_{t-1}^{c}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2} &= \textstyle H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})^{2}\\
&\quad \textstyle +2H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})F_{t-1}^{c},
\end{aligned}
$$

or

$$
\begin{aligned}
h^{-2}\textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2} &= \textstyle h^{-2}H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+h^{-2}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})^{2}\\
&\quad \textstyle +2h^{-2}H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})F_{t-1}^{c},\\
&= \textstyle h^{-2}H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}\\
&\quad +O_{p}\left(\frac{\alpha_{h}^{2h}h^{-1}k_{h}}{\min\{N^{2},T\}}\right)+O_{p}\left(\frac{\rho_{h}^{2h}h^{-1}k_{h}}{\min\{N,T\}}\right)\\
&\quad +O_{p}\left(\frac{\alpha_{h}^{2h}h^{-1}k_{h}}{\min\{N,T^{1/2}\}}\right)+O_{p}\left(\frac{\alpha_{h}^{h}\rho_{h}^{h}h^{-1}k_{h}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using (d) and (e).

(r) We have

$$
\begin{aligned}
h^{-1}\textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}e_{t} &= \textstyle h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_{t}+h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})e_{t},\\
&= \textstyle h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_{t}\\
&\quad +O_{p}\left(\frac{\alpha_{h}^{h}k_{h}^{1/2}h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_{p}\left(\frac{\rho_{h}^{h}k_{h}^{1/2}h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using (g).■

Lemma B7. Suppose Assumptions 1–4 and A hold or Assumptions 1–5 and the following condition hold:

$$
\frac{\alpha_{h}^{h}\rho_{h}^{h}h^{1/2}k_{h}^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\to 0.
$$

Then, we have

$$\hat{\sigma}^2 \xrightarrow{p} Q^{-2}\sigma^2,$$

as $N, T, h \to \infty$, where

$$\hat{\sigma}^2 = h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{f}_t^c - \hat{\delta}^* \widetilde{F}_{t-1}^c)^2.$$

Proof of Lemma B7. The proof follows that of Lemma B2 by replacing Lemma B1 with Lemma B6. Thus, it is not repeated.

Lemma B8. Under Assumptions 1–5, the following hold:

(a) For $t = T+1, ..., T+h$ uniformly in t,

$$\widetilde{U}_{i,t}^c - U_{i,t}^c = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(b)

$$h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1}^c - U_{i,t-1}^c)^2 = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),$$

(c)

$$h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c (\widetilde{U}_{i,t-1}^c - U_{i,t-1}^c) = O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(d)

$$\begin{aligned} h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^c (\widetilde{U}_{i,t-1}^c - U_{i,t-1}^c) &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N^{1/2}, T^{1/2}\}}\right), \end{aligned}$$

(e)

$$h^{-1/2} \sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1}^c - U_{i,t-1}^c) z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(f) For $t = T+1, ..., T+h$ uniformly in t,

$$\widetilde{u}_{i,t}^c - u_{i,t}^c = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

(g)

$$h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{u}_{i,t}^c - u_{i,t}^c)^2 = O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right),$$

(h)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}^{c}-U_{i,t-1}^{c})(\widetilde{u}_{i,t}^{c}-u_{i,t}^{c}) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N^{3/2},T\}}\right) +O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),$$

(i)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}^{c}-U_{i,t-1}^{c})u_{i,t}^{c}=O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(j)

$$h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}^{c}(\widetilde{u}_{i,t}^{c}-u_{i,t}^{c})=O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(k)

$$h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^{c}(\widetilde{u}_{i,t}^{c}-u_{i,t}^{c}) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right) +O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N,T^{1/2}\}}\right),$$

(l)

$$h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{u}_{i,t}^{c}-u_{i,t}^{c})z_{i,t}=O_p\left(\frac{\alpha_h^{h}k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(m)

$$h^{-2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^{c2} = h^{-2}H^2\sum_{t=T+1}^{T+h}U_{i,t-1}^{c2}+O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) +O_p\left(\frac{\alpha_h^{h}\rho_h^{h}h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),$$

(n)

$$h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^{c}z_{i,t} = h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}^{c}z_{i,t} +O_p\left(\frac{\alpha_h^{h}h^{-1/2}k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}h^{-1/2}k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right).$$

Proof of Lemma B8. They can be shown straightforwardly from Lemma B3 as we did in Lemma B6 from Lemma B1. Thus, the proof is not repeated.■

Lemma B9. Suppose Assumptions 1–4 and A hold or Assumptions 1–5 and the following conditions hold:

$$\frac{\alpha_h^h h k_h^{-1}}{\min\{N, T^{1/2}\}} \to 0 \text{ and } \frac{\rho_h^h h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}} \to 0.$$

Then, we have

$$\hat{\sigma}_i^2 \xrightarrow{p} \sigma_i^2,$$

for any i as $N, T, h \to \infty$, where

$$\hat{\sigma}_i^2 = h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{u}_{i,t}^c - \hat{\delta}_i^* \widetilde{U}_{i,t-1}^c)^2.$$

Proof of Lemma B9. The proof follows that of Lemma B4 by replacing Lemma B3 with Lemma B8. Thus, it is not repeated.■

Lemma B10. Let $\Theta \sim N(0, \sigma^2/2c)$ and $\Theta_i \sim N(0, \sigma_i^2/2c_i)$. Under Assumptions 1-5, the following hold as T and $h \to \infty$.

(a) Suppose $c > 0$.

If $T/k_h \to 0$, then

$$\alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \frac{1}{2c}\Theta^2.$$

If $T/k_h \to \pi$ $(0 < \pi < \infty)$, then

$$\alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \frac{1}{2c}\left(\frac{F_T}{\sqrt{T}}\sqrt{\pi} + \Theta\right)^2.$$

If $T/k_h \to \infty$, then

$$\alpha_h^{-2h} k_h^{-1} T^{-1} \sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \frac{1}{2c}\left(\frac{F_T}{\sqrt{T}}\right)^2.$$

(b) Suppose $c_i > 0$.

If $T/k_h \to 0$, then

$$\rho_{i,h}^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 \approx \frac{1}{2c_i}\Theta_i^2.$$

If $T/k_h \to \pi$ $(0 < \pi < \infty)$, then

$$\rho_{i,h}^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 \approx \frac{1}{2c_i}\left(\frac{U_{i,T}}{\sqrt{T}}\sqrt{\pi} + \Theta_i\right)^2.$$

If $T/k_h \to \infty$, then

$$\rho_{i,h}^{-2h} k_h^{-1} T^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 \approx \frac{1}{2c_i}\left(\frac{U_{i,T}}{\sqrt{T}}\right)^2.$$

Proof of Lemma B10.

(a) We take squares of both sides of $F_t = \alpha_T F_{t-1} + e_t$ to obtain

$$\begin{aligned} F_t^2 &= \alpha_h^2 F_{t-1}^2 + 2\alpha_h F_{t-1} e_t + e_t^2, \\ (\alpha_h^2 - 1) F_{t-1}^2 &= F_t^2 - F_{t-1}^2 - 2\alpha_h F_{t-1} e_t - e_t^2. \end{aligned}$$

We then take summations over $t = T+1, ..., T+h$ to obtain

$$\begin{aligned} (\alpha_h^2 - 1)\sum_{t=T+1}^{T+h} F_{t-1}^2 &= F_{T+h}^2 - F_T^2 - \sum_{t=T+1}^{T+h} e_t^2 - 2\alpha_h \sum_{t=T+1}^{T+h} F_{t-1} e_t, \\ \sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{\alpha_h^2 - 1}\left\{F_{T+h}^2 - F_T^2 - \sum_{t=T+1}^{T+h} e_t^2 - 2\alpha_h \sum_{t=T+1}^{T+h} F_{t-1} e_t\right\}, \\ \alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{k_h^2(\alpha_h^2 - 1)}\left\{\alpha_h^{-2h}(F_{T+h}^2 - F_T^2) - \alpha_h^{-2h}\sum_{t=T+1}^{T+h} e_t^2 \right. \\ &\quad \left. -2\alpha_h^{-2h+1}\sum_{t=T+1}^{T+h} F_{t-1} e_t\right\}, \\ &= \frac{1}{k_h(\alpha_h^2 - 1)}\left\{\frac{\alpha_h^{-2h}}{k_h}(F_{T+h}^2 - F_T^2) - \frac{\alpha_h^{-2h}}{k_h}\sum_{t=T+1}^{T+h} e_t^2 \right. \\ &\quad \left. -\frac{2\alpha_h^{-2h+1}}{k_h}\sum_{t=T+1}^{T+h} F_{t-1} e_t\right\}, \\ &= \frac{1}{k_h(\alpha_h^2 - 1)}\left\{I - II - III\right\}. \end{aligned}$$

We now consider terms II, III, and I in order. For II,

$$\frac{\alpha_h^{-2h}}{k_h}\sum_{t=T+1}^{T+h} e_t^2 = \left(\alpha_h^{-2h}\frac{h}{k_h}\right)\left(\frac{1}{h}\sum_{t=T+1}^{T+h} e_t^2\right) = o(1) \times O_p(1) = o_p(1),$$

by using Proposition A.1 of Phillips and Magdalinos (2007) $\alpha_h^{-2h} = o(k_h^2 h^{-2})$ for the first component and the weak law of large numbers for the second component.

For III, by plugging

$$F_{t-1} = \alpha_h^{t-T-1} F_T + \sum_{j=1}^{t-T-j-1} \alpha_h^{t-T-j-1} e_{T+j},$$

in III (divided by 2) yields

$$\begin{aligned}
\frac{\alpha_h^{-2h+1}}{k_h} \sum_{t=T+1}^{T+h} F_{t-1} e_t &= \frac{\alpha_h^{-2h+1}}{k_h} \sum_{t=T+1}^{T+h} \left(\sum_{j=1}^{t-T-1} \alpha_h^{t-T-j-1} e_{T+j} \right) e_t \\
&\quad + \frac{1}{k_h} F_T \alpha_h^{-2h+1} \sum_{t=T+1}^{T+h} \alpha_h^{t-T-1} e_t, \\
&= \frac{\alpha_h^{-2h}}{k_h} \sum_{t=T+1}^{T+h} \left(\sum_{j=1}^{t-T-1} \alpha_h^{t-T-j} e_{T+j} \right) e_t + \frac{1}{k_h} F_T \alpha_h^{-h} \sum_{t=T+1}^{T+h} \alpha_h^{t-T-h} e_t, \\
&= \frac{\alpha_h^{-2h}}{k_h} \sum_{t=T+1}^{T+h} \left(\sum_{j=1}^{t-T-1} \alpha_h^{t-T-j} e_{T+j} \right) e_t \\
&\quad + \underbrace{\left(\frac{F_T}{\sqrt{T}} \right)}_{=O_p(1)} \underbrace{\left(\sqrt{\frac{T}{k_h}} \alpha_h^{-h} \right)}_{=o(T^{1/2} h^{-1/2})} \underbrace{\left(\frac{1}{\sqrt{k_h}} \sum_{t=T+1}^{T+h} \alpha_h^{t-T-h} e_t \right)}_{=O_p(1)}, \\
&= \frac{\alpha_h^{-2h}}{k_h} \sum_{t=T+1}^{T+h} \left(\sum_{j=1}^{t-T-1} \alpha_T^{t-T-j} e_{T+j} \right) e_t + o_p\left(T^{1/2} h^{-1/2} \right),
\end{aligned}$$

because $\sqrt{\frac{T}{k_h}} \alpha_h^{-h} = \sqrt{\frac{T}{k_h}} \times o\left(k_h h^{-1}\right) = o\left(T^{1/2} k_h^{1/2} h^{-1}\right) = o\left(T^{1/2} h^{-1/2}\right)$. For $\frac{1}{\sqrt{k_h}} \sum_{t=T+1}^{T+h} \alpha_h^{t-T-h} e_t = O_p(1)$, we used Lemma 4.2 of Phillips and Magdalinos (2007). In addition, we can show

$$\frac{\alpha_h^{-2h}}{k_h} \sum_{t=T+1}^{T+h} \left(\sum_{j=1}^{t-T-h} \alpha_h^{t-T-j} e_{T+j} \right) e_t = O_p(\alpha_h^{-h}) = o_p(1),$$

by following Phillips and Magdalinos (2007).

Finally, we consider term I as follows.

$$\begin{aligned}
\alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{k_h(\alpha_h^2 - 1)} \left\{ \frac{\alpha_h^{-2h}}{k_h} (F_{T+h}^2 - F_T^2) + o_p\left(T^{1/2} h^{-1/2} \right) \right\}, \\
&= \frac{1}{k_h(\alpha_h^2 - 1)} \left(\frac{\alpha_h^{-2h}}{k_h} \right) F_{T+h}^2 \\
&\quad - \frac{1}{k_h(\alpha_h^2 - 1)} \left(\alpha_h^{-2h} \frac{T}{k_h} \right) \frac{F_T^2}{T} + o_p\left(T^{1/2} h^{-1/2} \right),
\end{aligned}$$

because $k_h(\alpha_h^2 - 1) \to 2c$. But the second term is $o_p(Th^{-1})$ because $\alpha_h^{-2h} = o(k_h h^{-1})$ so that

$\alpha_h^{-2h}(T/k_h) = o(Th^{-1})$ and $F_T^2/T = O_p(1)$. Furthermore,

$$
\begin{aligned}
\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{\alpha_h^{-2h}}{k_h}\right)F_{T+h}^2 + o_p\left(Th^{-1}\right), \\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{\alpha_h^{-2h}}{k_h}\right)\left(\alpha_h^h F_T + \textstyle\sum_{j=1}^{h}\alpha_h^{h-j}e_{T+j}\right)^2 + o_p\left(Th^{-1}\right), \\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{1}{k_h}\right)\left(F_T + \textstyle\sum_{j=1}^{h}\alpha_h^{-j}e_{T+j}\right)^2 + o_p\left(Th^{-1}\right), \\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{F_T}{\sqrt{T}}\sqrt{\frac{T}{k_h}} + \frac{1}{\sqrt{k_h}}\textstyle\sum_{j=1}^{h}\alpha_h^{-j}e_{T+j}\right)^2 + o_p\left(Th^{-1}\right).
\end{aligned}
$$

Therefore, if $T/k_h \to 0$, then $T/h \to 0$ by Assumption 5 and

$$
\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \frac{1}{2c}\left(\frac{1}{\sqrt{k_n}}\textstyle\sum_{j=1}^{h}\alpha_h^{-j}e_{T+j}\right)^2 \Rightarrow \frac{1}{2c}\Theta^2,
$$

by Lemma A5 (a). If $T/k_h \to \pi$ $(0 < \pi < \infty)$, then $T/h \to 0$ and

$$
\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \frac{1}{2c}\left(\frac{F_T}{\sqrt{T}}\sqrt{\pi} + \Theta\right)^2.
$$

If $T/k_h \to \infty$, then

$$
\begin{aligned}
\alpha_h^{-2h}k_h^{-1}T^{-1}\textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{k_h(\alpha_h^2-1)}\frac{k_h}{T}\left(\frac{F_T}{\sqrt{T}}\sqrt{\frac{T}{k_h}} + \frac{1}{\sqrt{k_h}}\textstyle\sum_{j=1}^{h}\alpha_h^{-j}e_{T+j}\right)^2 + o_p\left(k_h h^{-1}\right), \\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{F_T}{\sqrt{T}} + \underbrace{\sqrt{\frac{k_h}{T}}}_{=o(1)}\underbrace{\frac{1}{\sqrt{k_h}}\textstyle\sum_{j=1}^{h}\alpha_h^{-j}e_{T+j}}_{=O_p(1)\text{ by Lemma A5(a)}}\right)^2 + o_p\left(k_h h^{-1}\right), \\
&\approx \frac{1}{2c}\left(\frac{F_T}{\sqrt{T}}\right)^2.
\end{aligned}
$$

(b) We follow the same steps as above by replacing F_t^c and F_t with $U_{i,t}^c$ and $U_{i,t}$ to show the results. Hence, the proof is condensed.■

We now provide a proof for the asymptotic properties of the CS tests under the LTU framework (Theorem SA-2) provided in Appendix I and under the MLTU framework (Theorem 2) presented in Section 4.

Proof of Theorem SA-2.

(i-a) The t test statistic is

$$t^*_{\widetilde{F}} = \frac{h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}\widetilde{f}_t}{\hat{\sigma}\left(h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2\right)^{1/2}}.$$

The numerator is

$$\begin{aligned}
h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}\widetilde{f}_t &= \frac{c}{h^2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2 + h^{-1}H^2\sum_{t=T+1}^{T+h}F_{t-1}e_t \\
&\quad -\frac{c}{h^2}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^2 - H^2F_{t-1}^2) \\
&\quad +h^{-1}H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1} - HF_{t-1})f_t \\
&\quad +h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}(\widetilde{f}_t - Hf_t), \\
&= \frac{c}{h^2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2 + I + II + III + IV,
\end{aligned}$$

but $I = o_p(1)$. Further, II, III, and IV are shown to be $o_p(1)$ by using Lemma B1 (q), (m), and (o) because $\alpha_h^h, \rho_h^h = O(1)$ when $k_h = h$. For the denominator,

$$\begin{aligned}
h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^2 &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 + O_p\left(\frac{1}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right) \\
&\quad +O_p\left(\frac{1}{\min\{N,T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h}{\min\{N^{1/2},T^{1/2}\}}\right) = o_p(1),
\end{aligned}$$

by using Lemma B1 (q). The consistency of $\hat{\sigma}$ is shown in Lemma B2 because under $k_h = h$, $\frac{\alpha_h^h\rho_h^h h^{1/2}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}} = o(1)$. Therefore,

$$t^*_{\widetilde{F}} = c\left(h^{-2}\sum_{t=T+1}^{T+h}F_{t-1}^2\right)^{1/2} + \frac{h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}e_t}{\sigma\left(h^{-2}\sum_{t=T+1}^{T+h}F_{t-1}^2\right)^{1/2}} + o_p(1),$$

which leads to the result.

(i-b) The t test statistic is

$$t^*_{\widetilde{U}}(i) = \frac{h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}\widetilde{u}_{i,t}}{\hat{\sigma}_i\left(h^{-2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2\right)^{1/2}}.$$

The numerator is

$$\begin{aligned}
h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}\widetilde{u}_{i,t} &= \frac{c_i}{h^2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2 + h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t} - \frac{c_i}{h^2}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1}^2 - U_{i,t-1}^2) \\
&\quad +h^{-1}\sum_{t=T+1}^{T+h}(\widetilde{U}_{i,t-1} - U_{i,t-1})u_{i,t} + h^{-1}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}(\widetilde{u}_{i,t} - u_{i,t}), \\
&= \frac{c}{h^2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^2 + I + II + III + IV,
\end{aligned}$$

but $I = o_p(1)$. Further, II, III, and IV are shown to be $o_p(1)$ by using Lemma B3 (m), (i), and (k). For the denominator, under $k_h = h$

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\widetilde{U}_{i,t-1}^{2} &= h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^{2}+O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2,T\right\}}\right) \\ &+O_p\left(\frac{1}{\min\left\{N,T\right\}}\right)+O_p\left(\frac{\alpha_h^{h}}{\min\left\{N,T^{1/2}\right\}}\right) \\ &+O_p\left(\frac{1}{\min\left\{N^{1/2},T^{1/2}\right\}}\right), \end{aligned}$$

by using Lemma B3 (m) and the four terms of the factor estimation errors are $o_p(1)$. The consistency of $\hat{\sigma}_i$ is shown in Lemma B4 because under $k_h = h$, $\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N,T^{1/2}\right\}} = o_p(1)$ and $\frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2},T^{1/2}\right\}} = o_p(1)$. Therefore,

$$t_{\widetilde{U}}^{*}(i) = c_i\left(h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^{2}\right)^{1/2}+\frac{h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t}}{\sigma_i\left(h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^{2}\right)^{1/2}}+o_p(1),$$

which leads to the result.

(ii-a) The result is directly shown from (i-a) by using Lemmas B6 and B7 instead of Lemmas B1 and B2.

(ii-b) The result is directly shown from (i-b) by using Lemmas B8 and B9 instead of Lemmas B3 and B4.■

Proof of Theorem 2.

(a) When $c = 0$, the t test statistic is

$$\bar{t}_{\widetilde{F}}^{*} = \frac{h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}\widetilde{f}_{t}^{c}}{\hat{\sigma}\left(h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}\right)^{1/2}}.$$

The numerator is

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}\widetilde{f}_{t}^{c} &= h^{-1}H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t+h^{-1}H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})e_t \\ &+h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}(\widetilde{f}_{t}^{c}-Hf_{t}^{c}), \\ &= h^{-1}H^{2}\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t+O_p\left(\frac{h^{-1/2}k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right) \\ &+O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right)+O_p\left(\frac{1}{\min\left\{N,T^{1/2}\right\}}\right) \\ &+O_p\left(\frac{\rho_h^{h}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\left\{N,T\right\}}\right), \end{aligned}$$

by using Lemma B6 (g) and (o) and the five terms of the factor estimation errors are $o_p(1)$ under the stated condition. For the denominator,

$$
\begin{aligned}
h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2} &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+O_p\left(\frac{h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right)\\
&+O_p\left(\frac{h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B6 (q) and the four terms of the factor estimation errors are $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}$ is shown in Lemma B7 under the same condition. Therefore,

$$
\bar{t}_{\widetilde{F}}^{*}=\frac{h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t}{\sigma\left(h^{-2}\sum_{t=T+1}^{T+h}F_{t-1}^{c2}\right)^{1/2}}+o_p(1),
$$

which leads to the result.

When $c>0$, the t test statistic is

$$
t_{\widetilde{F}}^{*}=\frac{k_h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}\widetilde{f}_t^{c}}{\hat{\sigma}\left(k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}\right)^{1/2}}.
$$

The numerator is

$$
\begin{aligned}
k_h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}\widetilde{f}_t^{c} &= \frac{c}{k_h^2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}+k_h^{-1}H^2\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t\\
&-\frac{c}{k_h^2}\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c2}-H^2F_{t-1}^{c2})\\
&+k_h^{-1}H\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})f_t\\
&+k_h^{-1}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}(\widetilde{f}_t^{c}-Hf_t^{c}),\\
&= \frac{c}{k_h^2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}+I+II+III+IV.
\end{aligned}
$$

Therefore, if we scale the t test by α_h^{-h}

$$
\begin{aligned}
\alpha_h^{-h}t_{\widetilde{F}}^{*} &= \frac{c}{\hat{\sigma}}\left(\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}\right)^{1/2}\\
&+\frac{\alpha_h^{-2h}}{\hat{\sigma}\left(\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2}\right)^{1/2}}(I+II+III+IV).
\end{aligned}
$$

We now show that the first term is asymptotically equal to a positive value or diverges to positive infinity and the second term disappears. The first term is

$$\begin{aligned}\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c2} &= \alpha_h^{-2h}k_h^{-2}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c2}-H^2F_{t-1}^{c2}),\\ &= \alpha_h^{-2h}k_h^{-2}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+O_p\left(\dfrac{hk_h^{-1}}{\min\{N^2,T\}}\right)\\ &\quad +O_p\left(\dfrac{\alpha_h^{-2h}\rho_h^{2h}hk_h^{-1}}{\min\{N,T\}}\right)+O_p\left(\dfrac{hk_h^{-1}}{\min\{N,T^{1/2}\}}\right)\\ &\quad +O_p\left(\dfrac{\alpha_h^{-h}\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

by using Lemma B6 (q) and the four terms of the factor estimation errors are $o_p(1)$ under the stated condition. We next consider the second term.

$$\alpha_h^{-2h}\times I=\alpha_h^{-2h}k_h^{-1}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t=o_p(1),$$

by using Lemma A5 (c),

$$\alpha_h^{-2h}\times II=\frac{c\alpha_h^{-2h}}{k_h^2}\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c2}-H^2F_{t-1}^{c2})=o_p(1),$$

as shown in the first term,

$$\begin{aligned}\alpha_h^{-2h}\times III &= \alpha_h^{-2h}k_h^{-1}H\textstyle\sum_{t=T+1}^{T+h}(\widetilde{F}_{t-1}^{c}-HF_{t-1}^{c})f_t^c,\\ &= O_p\left(\dfrac{hk_h^{-1}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\dfrac{\alpha_h^{-h}\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right),\end{aligned}$$

by using Lemma B6 (m) and it is $o_p(1)$ under the stated condition, and

$$\begin{aligned}\alpha_h^{-2h}\times IV &= \alpha_h^{-2h}k_h^{-1}\textstyle\sum_{t=T+1}^{T+h}\widetilde{F}_{t-1}^{c}(\widetilde{f}_t^c-Hf_t^c),\\ &= O_p\left(\dfrac{hk_h^{-1}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\dfrac{\alpha_h^{-h}\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right)\\ &\quad +O_p\left(\dfrac{\alpha_h^{-2h}\rho_h^{2h}hk_h^{-1}}{\min\{N,T\}}\right),\end{aligned}$$

by using Lemma B6 (o) and it is $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}$ is shown in Lemma B7 under the stated condition. Therefore,

$$\alpha_h^{-h}\bar{t}_{\widetilde{F}}^{*}=\frac{c}{\sigma}\left(\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}\right)^{1/2}+o_p(1),$$

under the stated conditions.

Finally,

$$\begin{aligned}\alpha_h^{-h}\bar{t}_{\widetilde{F}}^{*} &= \frac{c}{\sigma}\left(\alpha_h^{-2h}k_h^{-2}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}\right)^{1/2}+o_p(1),\\ &= \frac{c}{\sigma}\left(\alpha_h^{-2h}k_h^{-2}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{2}-\alpha_h^{-2h}k_h^{-2}hH^2\bar{F}^2\right)^{1/2}+o_p(1), \qquad (18)\end{aligned}$$

but because $\bar{F}^2 = O_p(k_h^2 h^{-1}) + O_p(\alpha_h^{2h} k_h^3 h^{-2})$ from Lemma A5 (b) and $k_h h^{-1} = o_p(1)$,

$$\alpha_h^{-2h} k_h^{-2} h H^2 \bar{F}^2 = O_p(\alpha_h^{-2h}) + O_p(k_h h^{-1}) = o_p(1).$$

By using Lemma B10 (a), $\alpha_h^{-2h} k_h^{-2} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2$ or $\alpha_h^{-2h} k_h^{-1} T^{-1} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2$ is asymptotically equal to the stated values. Plugging these into (18) yields the final results.

(b) When $c_i = 0$, the t test statistic is

$$\bar{t}_{\widetilde{U}}^*(i) = \frac{h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^c \widetilde{u}_{i,t}^c}{\hat{\sigma}_i \left(h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^{c2} \right)^{1/2}}.$$

The numerator is

$$\begin{aligned}
h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^c \widetilde{u}_{i,t}^c &= h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c z_{i,t} + h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{U}_{i,t-1}^c - U_{i,t-1}^c) z_{i,t} \\
&\quad + h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^c (\widetilde{u}_{i,t}^c - u_{i,t}^c), \\
&= h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c z_{i,t} + O_p\left(\frac{\alpha_h^h h^{-1/2} k_h^{1/2}}{\min\{N, T^{1/2}\}} \right) \\
&\quad + O_p\left(\frac{\rho_h^h h^{-1/2} k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}} \right) + O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}} \right) \\
&\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2}, T^{1/2}\}} \right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\{N, T^{1/2}\}} \right),
\end{aligned}$$

by using Lemma B8 (e) and (k) and the five terms of the factor estimation errors are $o_p(1)$ under the stated condition. For the denominator,

$$\begin{aligned}
h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^{c2} &= h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2} + O_p\left(\frac{\alpha_h^{2h} h^{-1} k_h}{\min\{N^2, T\}} \right) \\
&\quad + O_p\left(\frac{\rho_h^{2h} h^{-1} k_h}{\min\{N, T\}} \right) + O_p\left(\frac{\alpha_h^h \rho_h^h h^{-1} k_h}{\min\{N, T^{1/2}\}} \right) \\
&\quad + O_p\left(\frac{\rho_h^{2h} h^{-1} k_h}{\min\{N^{1/2}, T^{1/2}\}} \right),
\end{aligned}$$

by using Lemma B8 (m) and the four terms of the factor estimation errors are $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}_i$ is shown in Lemma B9 under the same condition. Therefore,

$$\bar{t}_{\widetilde{U}}^*(i) = \frac{h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c z_{i,t}}{\sigma_i \left(h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2} \right)^{1/2}} + o_p(1),$$

which leads to the result.

When $c_i > 0$, the t test statistic is

$$\bar{t}^*_{\widetilde{U}}(i) = \frac{k_h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}^c_{i,t-1} \widetilde{u}^c_{i,t}}{\hat{\sigma}_i \left(k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}^{c2}_{i,t-1} \right)^{1/2}}.$$

The numerator is

$$\begin{aligned} k_h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}^c_{i,t-1} \widetilde{u}^c_{i,t} &= \frac{c_i}{k_h^2} \sum_{t=T+1}^{T+h} \widetilde{U}^{c2}_{i,t-1} + k_h^{-1} \sum_{t=T+1}^{T+h} U^c_{i,t-1} z_{i,t} - \frac{c_i}{k_h^2} \sum_{t=T+1}^{T+h} (\widetilde{U}^{c2}_{i,t-1} - U^{c2}_{i,t-1}) \\ &\quad + k_h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{U}^c_{i,t-1} - U^c_{i,t-1}) u_{i,t} + k_h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}^c_{i,t-1} (\widetilde{u}^c_{i,t} - u^c_{i,t}), \\ &= \frac{c_i}{k_h^2} \sum_{t=T+1}^{T+h} \widetilde{U}^{c2}_{i,t-1} + I + II + III + IV. \end{aligned}$$

Therefore, if we scale the t test by ρ_h^{-h}, then

$$\begin{aligned} \rho_h^{-h} \bar{t}^*_{\widetilde{U}}(i) &= \frac{c_i}{\hat{\sigma}_i} \left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}^{c2}_{i,t-1} \right)^{1/2} \\ &\quad + \frac{\rho_h^{-2h}}{\hat{\sigma}_i \left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}^{c2}_{i,t-1} \right)^{1/2}} (I + II + III + IV). \end{aligned}$$

We now show that the first term is equal to a positive value or diverges to positive infinity and the second term disappears. First,

$$\begin{aligned} \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \widetilde{U}^2_{i,t-1} &= \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U^2_{i,t-1} + \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} (\widetilde{U}^2_{i,t-1} - U^2_{i,t-1}), \\ &= \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U^2_{i,t-1} + O_p \left(\frac{\alpha_h^{2h} \rho_h^{-2h} h k_h^{-1}}{\min\{N^2, T\}} \right) + O_p \left(\frac{h k_h^{-1}}{\min\{N, T\}} \right) \\ &\quad + O_p \left(\frac{\alpha_h^h \rho_h^{-h} h k_h^{-1}}{\min\{N, T^{1/2}\}} \right) + O_p \left(\frac{h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B8 (m) and the last four terms of the factor estimation errors are $o_p(1)$ under $\frac{\alpha_h^h h k_h^{-1}}{\min\{N, T^{1/2}\}} \to 0$. We next consider the second term.

$$\rho_h^{-2h} \times I = \rho_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} U^c_{i,t-1} z_{i,t} = o_p(1),$$

by using Lemma A5 (h),

$$\rho_h^{-2h} \times II = \frac{c_i \rho_h^{-2h}}{k_h^2} \sum_{t=T+1}^{T+h} (\widetilde{U}^{c2}_{i,t-1} - U^{c2}_{i,t-1}) = o_p(1),$$

as shown in the first term,

$$\begin{aligned} \rho_h^{-2h} \times III &= \rho_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} (\widetilde{U}^c_{i,t-1} - U^c_{i,t-1}) u_{i,t}, \\ &= O_p \left(\frac{\alpha_h^h \rho_h^{-h} h k_h^{-1}}{\min\{N, T^{1/2}\}} \right) + O_p \left(\frac{h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B8 (i) and it is $o_p(1)$ under the stated condition, and

$$\begin{aligned}\rho_h^{-2h} \times IV &= \rho_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} \widetilde{U}_{i,t-1}^{c} (\widetilde{u}_{i,t} - u_{i,t}), \\ &= O_p\left(\frac{\alpha_h^{2h}\rho_h^{-2h} h k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{h}\rho_h^{-h} h k_h^{-1}}{\min\{N, T^{1/2}\}}\right),\end{aligned}$$

by using Lemma B8 (k) and it is $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}_i$ is shown in Lemma B9 under the stated condition. Therefore,

$$\rho_h^{-h} \bar{t}_{\widetilde{U}}^{*}(i) = \frac{c_i}{\sigma_i}\left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2}\right)^{1/2} + o_p(1).$$

Finally,

$$\begin{aligned}\rho_h^{-h} \bar{t}_{\widetilde{U}}^{*}(i) &= \frac{c_i}{\sigma_i}\left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2}\right)^{1/2} + o_p(1), \\ &= \frac{c_i}{\sigma_i}\left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{t-1}^{2} - \rho_h^{-2h} k_h^{-2} h \bar{U}^2\right)^{1/2} + o_p(1), \qquad (19)\end{aligned}$$

but because $\bar{U}^2 = O_p(k_h^2 h^{-1}) + O_p(\alpha_h^{2h} k_h^3 h^{-2})$ from Lemma A5 (g) and $k_h h^{-1} = o_p(1)$,

$$\rho_h^{-2h} k_h^{-2} h \bar{U}^2 = O_p(\alpha_h^{-2h}) + O_p(k_h h^{-1}) = o_p(1).$$

By using Lemma B10 (b), $\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{t-1}^2$ or $\rho_h^{-2h} k_h^{-1} T^{-1} \sum_{t=T+1}^{T+h} U_{t-1}^2$ is asymptotically equal to the stated values. Plugging these results into (19) yields the final results.■

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