

Two-Class Routing with Admission Control and Strict Priorities – Online Appendix

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A Proof of Lemma 5.4

A.1 Preliminaries

We make use of the following intermediate result, which states that in a finite-horizon setting, the benefit associated with being in a “better” system state increases with the length of the horizon.

Lemma A.1. *For all $n \geq 1$, $\alpha \in [0, 1)$, and applicable i and j , we have*

1. $v_{n,\alpha}(i, j) - v_{n,\alpha}(i + 1, j) \geq v_{n,\alpha}(i, j) - v_{n,\alpha}(i + 1, j)$
2. $v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) \geq v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1)$
3. $v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j) \geq v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j)$

Proof. We verify Statement 1 with a sample path argument; the proofs of Statements 2 and 3 are similar. Start two processes on the same probability space: one in state (i, j) and one in state $(i+1, j)$, each with n periods remaining in the horizon. Let Δ_n denote the difference in reward collected by the two processes until either coupling occurs or n time periods have elapsed. We show $\Delta_n \leq \Delta_{n+1}$ pathwise. On sample paths ω where coupling occurs in the first n periods, $\Delta_n(\omega) = \Delta_{n+1}(\omega)$. On all other paths, Process 1 has at least as many servers available as Process 2 after n periods, and collects a reward in the remaining period at least as large as that by Process 2. Thus, $\Delta_n(\omega) \leq \Delta_{n+1}(\omega)$. \square

To prove Lemma 5.4, it suffices to show that for every n that

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) \leq v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i, j + 2) \tag{1}$$

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) \leq v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \tag{2}$$

$$v_{n,\alpha}(i, N_B) - v_{n,\alpha}(i + 1, N_B) \leq v_{n,\alpha}(i + 1, N_B) - v_{n,\alpha}(i + 2, N_B) \tag{3}$$

$$v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \leq v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i, j + 2), \tag{4}$$

and we proceed via induction over the time periods. The base case ($n = 0$) is trivial, as we assume $v_{0,\alpha} = 0$. Now suppose that (1)–(4) hold over horizons of length up to n : our induction hypothesis. In the analysis that follows, we assume, for convenience, that $\alpha = 1$, allowing us to suppress α in our arguments; nearly identical reasoning can be used for the case where $\alpha < 1$.

A.2 Inductive Step, Inequality (1)

Fix $i \in \{0, 1, \dots, N_A\}$ and $j \in \{0, 1, \dots, N_B - 2\}$. We want to show that

$$v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i, j+1) \leq v_{n+1,\alpha}(i, j+1) - v_{n+1,\alpha}(i, j+2). \quad (5)$$

Using finite-horizon analogues of the optimality equations (3), we rewrite the left-hand side of (5):

$$\begin{aligned} v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i, j+1) &= \lambda_H \left[\mathbf{1}_{\{i < N_A\}} \alpha (v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1)) \right. \\ &\quad \left. + \mathbf{1}_{\{i = N_A\}} \alpha (v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2)) \right] \\ &\quad + \lambda_L \left[\max \{R_L + \alpha v_{n,\alpha}(i, j+1), \alpha v_{n,\alpha}(i, j)\} \right. \\ &\quad \left. - \max \{R_L + \alpha v_{n,\alpha}(i, j+2), \alpha v_{n,\alpha}(i, j+1)\} \right] \\ &\quad + i\mu\alpha [v_{n,\alpha}(i-1, j) - v_{n,\alpha}(i-1, j+1)] \\ &\quad + j\mu\alpha [v_{n,\alpha}(i, j-1) - v_{n,\alpha}(i, j)] \\ &\quad + \mu\alpha [v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j)] \\ &\quad + (N_A + N_B - i - j - 1)\mu\alpha [v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j+1)] \end{aligned} \quad (6)$$

It suffices to show that each term in brackets on the right-hand side of (6) is bounded above by $v_{n+1,\alpha}(i, j+1) - v_{n+1,\alpha}(i, j+2)$. Consider the first term. If $i < N_A$, we have that

$$\begin{aligned} v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1) &\leq v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2) \\ &\leq v_{n+1,\alpha}(i, j+1) - v_{n+1,\alpha}(i, j+2), \end{aligned}$$

where the first inequality follows by (4) and our induction hypothesis, and the second by Lemma A.1.

Similar reasoning applies if $i = N_A$. Now consider the second term. There are three possibilities:

1. $R_L + \alpha v_{n,\alpha}(i, j+1) \geq \alpha v_{n,\alpha}(i, j)$ and $R_L + \alpha v_{n,\alpha}(i, j+2) < \alpha v_{n,\alpha}(i, j+1)$,
2. $R_L + \alpha v_{n,\alpha}(i, j+1) \geq \alpha v_{n,\alpha}(i, j)$ and $R_L + \alpha v_{n,\alpha}(i, j+2) \geq \alpha v_{n,\alpha}(i, j+1)$, and
3. $R_L + \alpha v_{n,\alpha}(i, j+1) < \alpha v_{n,\alpha}(i, j)$ and $R_L + \alpha v_{n,\alpha}(i, j+2) < \alpha v_{n,\alpha}(i, j+1)$.

We cannot have $R_L + \alpha v_{n,\alpha}(i, j+1) < \alpha v_{n,\alpha}(i, j)$ and $R_L + \alpha v_{n,\alpha}(i, j+2) \geq \alpha v_{n,\alpha}(i, j+1)$, as by the induction hypothesis, $v_{n,\alpha}$ is convex in j . Consider the first case; the analysis for the latter two

cases is straightforward. We have

$$\begin{aligned}
& \max \{R_L + \alpha v_{n,\alpha}(i, j + 1), \alpha v_{n,\alpha}(i, j)\} - \max \{R_L + \alpha v_{n,\alpha}(i, j + 2), \alpha v_{n,\alpha}(i, j + 1)\} \\
&= R_L + \alpha v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i, j + 1) \\
&< v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i, j + 2) \\
&\leq v_{n+1,\alpha}(i, j + 1) - v_{n+1,\alpha}(i, j + 2),
\end{aligned}$$

where the first inequality follows by assumption, and the second by Lemma A.1. Now consider the third term. The induction hypothesis (specifically, inequalities (1) and (2)) and Lemma A.1 yield

$$\begin{aligned}
v_{n,\alpha}(i - 1, j) - v_{n,\alpha}(i - 1, j + 1) &\leq v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) \\
&\leq v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i, j + 2) \\
&\leq v_{n+1,\alpha}(i, j + 1) - v_{n+1,\alpha}(i, j + 2).
\end{aligned}$$

The remaining terms follow in a similar fashion.

A.3 Inductive Step, Inequality (2)

Fix $i \in \{0, \dots, N_A - 1\}$ and $j \in \{0, \dots, N_B - 1\}$. We want to show that

$$v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i, j + 1) \leq v_{n+1,\alpha}(i + 1, j) - v_{n+1,\alpha}(i + 1, j + 1), \quad (7)$$

and proceed using a sample path argument. Start four processes on the same probability space, each with $n + 1$ periods remaining in the horizon. Processes 1 and 4 begin in states (i, j) and $(i + 1, j + 1)$, respectively, and follow the optimal policy π^* . Processes 2 and 3 begin in states $(i, j + 1)$ and $(i + 1, j)$, respectively, and use potentially suboptimal policies π_2 and π_3 , respectively. These policies deviate from π^* only during the first time period; we describe them in more detail later.

Let Δ be the difference in reward collected by Processes 1 and 2 until coupling occurs; define Δ' analogously for Processes 3 and 4. Equation (1) implies that

$$\begin{aligned}
\mathbb{E}\Delta &= v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}^{\pi_2}(i, j + 1) \\
\mathbb{E}\Delta' &= v_{n+1,\alpha}^{\pi_3}(i + 1, j) - v_{n+1,\alpha}(i + 1, j + 1)
\end{aligned}$$

It suffices to show $\mathbb{E}\Delta \leq \mathbb{E}\Delta'$, as this implies

$$\begin{aligned} v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i, j + 1) &\leq v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}^{\pi_2}(i, j + 1) \\ &\leq v_{n+1,\alpha}^{\pi_3}(i + 1, j) - v_{n+1,\alpha}(i + 1, j + 1) \\ &\leq v_{n+1,\alpha}(i + 1, j) - v_{n+1,\alpha}(i + 1, j + 1). \end{aligned}$$

There is one Type B server that is busy in Processes 2 and 4, but idle in Processes 1 and 3; call it Server I. We construct our probability space so that Server I completes service in all four processes simultaneously (but triggers a dummy completion in Processes 1 and 3). Similarly, there is one Type A server that is busy in Processes 3 and 4, but idle in Processes 1 and 2; call it Server II. We probabilistically link this server across all four processes as we did with Server I. In the first time period, seven transitions are possible:

1. A Type H arrival
2. A Type L arrival
3. A Type A service completion.
4. A completion by Server I.
5. A completion by Server II.
6. A completion by any other Type B server.
7. A dummy transition due to uniformization.

For $k = 1, \dots, 7$, let A_k be the event in which the k^{th} transition occurs; these events partition the sample space. By the Tower Property, it suffices to show that $\mathbb{E}[\Delta | A_k] \leq \mathbb{E}[\Delta' | A_k]$ for each k .

Case 1 (Event A_1): If $i + 1 < N_A$, all four processes admit the Type H job with a Type A server, and transition to states $(i + 1, j)$, $(i + 1, j + 1)$, $(i + 2, j)$ and $(i + 2, j + 1)$, respectively. Since all four processes subsequently follow the optimal policy, we have that

$$\begin{aligned} \mathbb{E}[\Delta | A_1] &= v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \\ \mathbb{E}[\Delta' | A_1] &= v_{n,\alpha}(i + 2, j) - v_{n,\alpha}(i + 2, j + 1). \end{aligned}$$

By the induction hypothesis and (1), we have $\mathbb{E}[\Delta | A_1] \leq \mathbb{E}[\Delta' | A_1]$, as desired. If $i + 1 = N_A$, but $j + 1 < N_B$, then Processes 3 and 4 must admit the job with a Type B server, and we have that

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \leq v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 1, j + 2) = \mathbb{E}[\Delta' | A_1],$$

again, by the induction hypothesis and (1). Finally, if $i + 1 = N_A$ and $j + 1 = N_B$, then Process 3 admits the job with a Type B server, whereas Process 4 must turn the job away. We thus have

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1) \leq R_{HB} + v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+1, j+1) = \mathbb{E}[\Delta' | A_1],$$

where the inequality follows this time by Lemma 4.2.

Case 2 (Event A_2): Processes 2 and 3 take (potentially suboptimal) actions based upon those taken by Processes 1 and 4. Specifically:

- If Processes 1 and 4 admit the arriving Type L job, so do Processes 2 and 3.
- If Processes 1 and 4 reject the arriving Type L job, so do Processes 2 and 3.
- If Process 1 admits the job, and Process 4 rejects, then Process 2 rejects and Process 3 admits.

We need not consider the case when Process 1 rejects the job in state (i, j) , and Process 4 admits the job in state $(i + 1, j + 1)$, as by the induction hypothesis, we can assume that π^* is a monotone switching curve policy. Suppose $j + 1 < N_B$. If Processes 1 and 4 both admit the job, then

$$\mathbb{E}[\Delta | A_2] = v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2) \leq v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+1, j+2) = \mathbb{E}[\Delta' | A_2],$$

by (2) and the induction hypothesis. If Processes 1 and 4 both reject, all four processes remain in the same state, and we can again leverage (2) and the induction hypothesis. Finally, if Process 1 admits and Process 4 rejects, we have that

$$\mathbb{E}[\Delta | A_2] = R_L + v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+1) = R_L + v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+1, j+1) = \mathbb{E}[\Delta' | A_2].$$

The case where $j + 1 = N_B$ follows via similar arguments, which we omit for brevity.

Case 3 (Event A_3): The four processes transition to states $(i - 1, j)$, $(i - 1, j + 1)$, (i, j) , and $(i, j + 1)$, respectively, and

$$\mathbb{E}[\Delta | A_3] = v_{n,\alpha}(i - 1, j) - v_{n,\alpha}(i - 1, j + 1) \leq v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) = \mathbb{E}[\Delta' | A_3],$$

by (2) and the induction hypothesis.

Case 4 (Event A_4): Processes 1 and 2 both transition to state (i, j) , and coupling occurs. Processes 3 and 4 transition to state $(i + 1, j)$, and couple as well. Thus $\mathbb{E}[\Delta | A_4] = \mathbb{E}[\Delta' | A_4] = 0$.

Case 5 (Event A_5): Processes 1 and 3 both transition to state (i, j) , while Processes 2 and 4 both transition to state $(i + 1, j)$. We thus have $\mathbb{E}[\Delta | A_5] = \mathbb{E}[\Delta' | A_5] = v_{n,\alpha}(i, j) - v_{n,\alpha}(i + 1, j)$.

Case 6 (Event A_6): The four processes transition to states $(i, j - 1)$, (i, j) , $(i + 1, j - 1)$, and $(i + 1, j)$, respectively, and

$$\mathbb{E}[\Delta | A_6] = v_{n,\alpha}(i, j - 1) - v_{n,\alpha}(i, j) \leq v_{n,\alpha}(i + 1, j - 1) - v_{n,\alpha}(i + 1, j) = \mathbb{E}[\Delta' | A_6],$$

by (2) and the induction hypothesis.

Case 7 (Event A_7): The four processes do not change state, and $\mathbb{E}[\Delta | A_7] \leq \mathbb{E}[\Delta' | A_7]$ by (2) and the induction hypothesis.

Thus, $\mathbb{E}\Delta \leq \mathbb{E}\Delta'$, as desired. \square

A.4 Inductive Step, Inequality (3)

Fix $i \in \{0, 1, \dots, N_A - 2\}$. We want to show that

$$v_{n+1,\alpha}(i, N_B) - v_{n+1,\alpha}(i + 1, N_B) \leq v_{n+1,\alpha}(i + 1, N_B) - v_{n+1,\alpha}(i + 2, N_B). \quad (8)$$

As in the inductive proof of (1), we can rewrite the right-hand side of (8) as

$$\begin{aligned} v_{n+1,\alpha}(i, N_B) - v_{n+1,\alpha}(i + 1, N_B) &= \lambda_H (v_{n,\alpha}(i + 1, N_B) - v_{n,\alpha}(i + 2, N_B)) \quad (9) \\ &\quad + \lambda_L \left[\max \{ R_L + \alpha v_{n,\alpha}(i + 1, N_B), \alpha v_{n,\alpha}(i, N_B) \} \right. \\ &\quad \quad \left. - \max \{ R_L + \alpha v_{n,\alpha}(i + 2, N_B), \alpha v_{n,\alpha}(i + 1, N_B) \} \right] \\ &\quad + i\mu\alpha [v_{n,\alpha}(i - 1, N_B) - v_{n,\alpha}(i, N_B)] \\ &\quad + \mu\alpha [v_{n,\alpha}(i, N_B) - v_{n,\alpha}(i, N_B)] \\ &\quad + j\mu\alpha [v_{n,\alpha}(i, N_B - 1) - v_{n,\alpha}(i + 1, N_B - 1)] \\ &\quad + (N_A + N_B - i - j)\mu\alpha [v_{n,\alpha}(i, N_B) - v_{n,\alpha}(i + 1, N_B)] \end{aligned}$$

It again suffices to show that each term in brackets on the right-hand side of (9) is bounded above by $v_{n+1,\alpha}(i + 1, N_B) - v_{n+1,\alpha}(i + 2, N_B)$. We consider only the second term, as the analysis of the remaining terms is straightforward. There are again three possibilities:

1. $R_L + \alpha v_{n,\alpha}(i + 1, N_B) \geq \alpha v_{n,\alpha}(i, N_B)$ and $R_L + \alpha v_{n,\alpha}(i + 2, N_B) < \alpha v_{n,\alpha}(i + 1, N_B)$,
2. $R_L + \alpha v_{n,\alpha}(i + 1, N_B) \geq \alpha v_{n,\alpha}(i, N_B)$ and $R_L + \alpha v_{n,\alpha}(i + 2, N_B) \geq \alpha v_{n,\alpha}(i + 1, N_B)$, and
3. $R_L + \alpha v_{n,\alpha}(i + 1, N_B) < \alpha v_{n,\alpha}(i, N_B)$ and $R_L + \alpha v_{n,\alpha}(i + 2, N_B) < \alpha v_{n,\alpha}(i + 1, N_B)$.

We cannot have $R_L + \alpha v_{n,\alpha}(i + 1, N_B) < \alpha v_{n,\alpha}(i, N_B)$ and $R_L + \alpha v_{n,\alpha}(i + 1, N_B) \geq \alpha v_{n,\alpha}(i, N_B)$, as by the induction hypothesis, $v_{n,\alpha}$ is convex in i when $j = N_B$. Consider the first case; the analysis

for the latter two cases is straightforward. We have

$$\begin{aligned}
& \max \{R_L + \alpha v_{n,\alpha}(i + 1, N_B), \alpha v_{n,\alpha}(i, N_B)\} - \max \{R_L + \alpha v_{n,\alpha}(i + 2, N_B), \alpha v_{n,\alpha}(i + 1, N_B)\} \\
&= R_L + v_{n,\alpha}(i + 1, N_B) - v_{n,\alpha}(i + 1, N_B) \\
&< v_{n,\alpha}(i + 1, N_B) - v_{n,\alpha}(i + 2, N_B) \\
&\leq v_{n+1,\alpha}(i + 1, N_B) - v_{n+1,\alpha}(i + 2, N_B),
\end{aligned}$$

as desired.

A.5 Inductive Step, Inequality (4)

Fix $i \in \{0, \dots, N_A - 1\}$ and $j \in \{0, \dots, N_B - 2\}$. We want to show that

$$v_{n+1,\alpha}(i + 1, j) - v_{n+1,\alpha}(i + 1, j + 1) \leq v_{n+1,\alpha}(i, j + 1) - v_{n+1,\alpha}(i, j + 2), \quad (10)$$

and do so using another sample path argument. Start four processes on the same probability space, each with $n + 1$ periods remaining in the horizon. Processes 1 and 4 begin in states $(i + 1, j)$ and $(i, j + 2)$, respectively, and follow the optimal policy π^* . Processes 2 and 3 begin in states $(i + 1, j + 1)$ and $(i, j + 1)$, respectively, and use potentially suboptimal policies π_2 and π_3 , respectively. These policies deviate from π^* only during the first time period, in a way that we specify later.

Let the random variable Θ denote the difference in reward collected by Processes 1 and 2 until coupling occurs; define Θ' analogously for Processes 3 and 4. It suffices to show that $E\Theta \leq E\Theta'$. There are i Type A and j Type B servers that are busy in all four processes, and $N_A - i - 1$ Type A and $N_B - j - 2$ Type B servers that are idle in all four processes. We probabilistically the remaining three servers (one Type A, two Type B) as in Table 1.

	Server I	Server II	Server III
Process 1, State $(i + 1, j)$	Idle Type B	Busy Type A	Idle Type B
Process 2, State $(i + 1, j + 1)$	Busy Type B	Busy Type A	Idle Type B
Process 3, State $(i, j + 1)$	Idle Type B	Busy Type B	Idle Type A
Process 4, State $(i, j + 2)$	Busy Type B	Busy Type B	Idle Type A

Table 1: Marking scheme for servers in the sample path argument for Equation (4).

Note that our marking scheme does not require units to be of the same type in every process, but are linked so that completions of all servers marked as Server I (similarly, Servers II and III) occur simultaneously in all four processes. In the first time period, eight transitions are possible:

1. A Type H arrival
2. A Type L arrival
3. A completion by Server I.
4. A completion by Server II.
5. A completion by Server III.
6. A completion by an unmarked Type A server.
7. A completion by an unmarked Type B server.
8. A dummy transition due to uniformization.

For $k = 1, \dots, 8$, let B_k be the event in which transition k occurs. Again, it suffices to show that $\mathbb{E}[\Theta | B_k] \leq \mathbb{E}[\Theta' | B_k]$ for each k .

Case 1 (Event B_1): If $i + 1 < N_A$, the analysis is straightforward. If $i + 1 = N_A$, then Processes 1 and 2 assign the Type H job to a Type B server, whereas Processes 3 and 4 route the job to a Type A server. We have that

$$\mathbb{E}[\Theta | B_1] = v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 1, j + 2) = v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 1, j + 2) = \mathbb{E}[\Theta' | B_1].$$

Case 2 (Event B_2): Processes 2 and 3 take (potentially suboptimal) actions based upon those taken by Processes 1 and 4. In particular:

- If Processes 1 and 4 admit the arriving Type L job, so do Processes 2 and 3.
- If Processes 1 and 4 reject the arriving Type L job, so do Processes 2 and 3.
- If Process 1 admits the job, and Process 4 rejects, then Process 2 rejects and Process 3 admits.

We need not consider the case when Process 1 rejects the job in state $(i + 1, j)$, and Process 4 admits the job in state $(i, j + 2)$, as by the induction hypothesis, we can assume that π^* is a monotone switching curve policy with a slope of at least -1 . The analysis for the case where both Processes 1 and 4 reject the job is straightforward, as a dummy transition occurs.

Now consider the case where Processes 1 and 4 both admit the job. If $j + 2 < N_B$, the analysis is again straightforward. If $j + 2 = N_B$, then Process 4 assigns a Type A server to the job, whereas all other processes assign Type B servers, and we have that

$$\mathbb{E}[\Theta | B_2] = v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 1, j + 2) \leq v_{n,\alpha}(i, j + 2) - v_{n,\alpha}(i + 1, j + 2) = \mathbb{E}[\Theta' | B_2],$$

where the inequality follows by Lemma 4.1. Finally, consider the case where Process 1 admits the job, and Process 4 rejects. Transitions to states $(i+1, j+1)$, $(i+1, j)$, $(i, j+2)$, and $(i, j+1)$ occur. Processes 1 and 2 couple, as do Processes 3 and 4, and we have that $\mathbb{E}[\Theta | B_2] = \mathbb{E}[\Theta' | B_2] = R_L$.

Case 3 (Event B_3): The four processes transition to states $(i+1, j)$, $(i+1, j)$, $(i, j+1)$, and $(i, j+1)$, respectively. Processes 1 and 2 couple, as do Processes 3 and 4, and $\mathbb{E}[\Theta | B_3] = \mathbb{E}[\Theta' | B_3] = 0$.

Case 4 (Event B_4): The four processes transition to states (i, j) , $(i, j+1)$, (i, j) , and $(i, j+1)$. Processes 1 and 3 couple, as do Processes 2 and 4, and $\mathbb{E}[\Theta | B_4] = \mathbb{E}[\Theta' | B_4]$.

Case 5 (Event B_5): A dummy transition occurs, and by the induction hypothesis, we have that $\mathbb{E}[\Theta | B_5] = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1) \leq v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2) = \mathbb{E}[\Theta' | B_5]$.

Case 6 (Event B_6): The four processes transition to states (i, j) , $(i, j+1)$, $(i-1, j+1)$, and $(i-1, j+2)$, respectively, and we leverage the induction hypothesis to show that $\mathbb{E}[\Theta | B_6] \leq \mathbb{E}[\Theta' | B_6]$.

Case 7 (Event B_7): The analysis is identical to that for Case 5.

Thus, $\mathbb{E}\Theta \leq \mathbb{E}\Theta'$, as desired. \square

B Proof of Proposition 5.6

Fix $\alpha \in (0, 1)$, and suppose

$$R_{HB} \leq R_L \leq R_{HB} + \frac{\mu}{\lambda_L} R_{HB} + \frac{\mu}{\lambda_H} \left(1 + \frac{\mu}{\lambda_L} + \frac{\lambda_H}{\lambda_L} \right) R_{HA}. \quad (11)$$

We claim that condition (11) is sufficient for the value functions v_α and h to be convex in i :

Lemma B.1. *If $R_L > R_{HB}$, then for every $n \geq 0$ and applicable i and j , we have that*

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i+1, j) \leq v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+2, j) \quad (12)$$

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j+1) \leq v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2) \quad (13)$$

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i+1, j) \leq v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1) + R_{HA} - R_{HB} \quad (14)$$

We defer the proof until Appendix E, as it is lengthy, and involves arguments very similar to those used in the proof of Proposition 5.4 in Appendix A. In proving that inequalities (12)–(14) hold, we make use of an intermediate result:

Lemma B.2. *If condition (11) holds, then for each $n \geq 0$, we have*

$$v_{n,\alpha}(N_A - 1, N_B) - v_{n,\alpha}(N_A, N_B) \leq R_{HA}. \quad (15)$$

B.1 Proof of Lemma B.1

By a fairly unconventional sample path argument. Start two processes in the same probability space. Process 1 begins in state $(N_A - 1, N_B)$ and follows the optimal policy π^* , whereas Process 2 begins in state (N_A, N_B) and uses a potentially suboptimal policy π that rejects Type L jobs arriving in any states (i, j) where $j = N_B$. Let Δ denote the difference in reward collected by the two processes until coupling occurs. We have $\mathbb{E}\Delta = v_\alpha(N_A - 1, N_B) - v_\alpha^\pi(N_A, N_B)$, and it suffices to show $\mathbb{E}\Delta \leq R_{HA}$. Both processes move in parallel until one of the following occurs:

1. A service completion from the Type A server that is idle in Process 1, but busy in Process 2.
2. A Type L arrival when $j = N_B$ that Process 1 admits (but Process 2, by assumption, redirects).
3. A Type H arrival when Process 1 is in state $(N_A - 1, N_B)$, and Process 2 is in state (N_A, N_B) .
4. A Type H arrival when the two processes are in states $(N_A - 1, j)$ and (N_A, j) , for some $j < N_B$.

Let $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 be the set of paths on which Events 1, 2, 3, and 4 occur first, respectively; these sets partition Ω . We further partition the set Ω_4 . After Event 4 occurs, Processes 1 and 2 are in states (N_A, j) and $(N_A, j + 1)$, respectively, and Process 1 has collected $R_{HA} - R_{HB}$ more reward than Process 2. From here, Process 2 switches to a policy that imitates the decisions made by Process 1 (and that no longer rejects jobs in states (i, j) where $j = N_B$, unless Process 1 does so). Both processes continue to move in parallel until one of the following occurs.

- 4.1 A Type L service completion seen by Process 2, but not by Process 1.
- 4.2 A Type H arrival when Process 1 is in state $(N_A, N_B - 1)$, and Process 2 is in state (N_A, N_B) .
- 4.3 A Type L arrival when Process 1 is in state $(N_A, N_B - 1)$, and Process 2 is in state (N_A, N_B) .

Let Ω_4^1, Ω_4^2 and Ω_4^3 be the set of paths on which events 4.1, 4.2, and 4.3 occur first, respectively. Then

$$\Delta(\omega) = \begin{cases} 0 & \omega \in \Omega_1 \\ R_L & \omega \in \Omega_2 \\ R_{HA} & \omega \in \Omega_3 \\ R_{HA} - R_{HB} & \omega \in \Omega_4^1 \\ R_{HA} & \omega \in \Omega_4^2 \\ R_{HA} + R_L - R_{HB} & \omega \in \Omega_4^3 \end{cases}, \quad (16)$$

from which it follows that

$$\mathbb{E}\Delta = R_L\mathbb{P}(\Omega_2) + R_{HA}\mathbb{P}(\Omega_3) + (R_{HA} - R_{HB})\mathbb{P}(\Omega_4^1) + R_{HA}\mathbb{P}(\Omega_4^2) + (R_{HA} + R_L - R_{HB})\mathbb{P}(\Omega_4^3). \quad (17)$$

Note that $\Delta(\omega) > R_{HA}$ only when $\omega \in \Omega_4^3$, and so we proceed by showing that $\mathbb{P}(\Omega_4^3)$ is relatively small. Suppose, for the time being, that the following inequalities hold (which we later prove):

$$\mathbb{P}(\Omega_1) \geq \frac{\mu}{\lambda_H} \mathbb{P}(\Omega_4) \quad (18)$$

$$\mathbb{P}(\Omega_4^1) \geq \frac{\mu}{\lambda_L} \mathbb{P}(\Omega_4^3) \quad (19)$$

$$\mathbb{P}(\Omega_4^2) = \frac{\lambda_H}{\lambda_L} \mathbb{P}(\Omega_4^3) \quad (20)$$

We demonstrate that (11), when combined with inequalities (18)–(20), imply $\mathbb{E}\Delta \leq R_{HA}$. Indeed:

$$\begin{aligned} R_L &\leq R_{HB} + \frac{\mu}{\lambda_L} R_{HB} + \frac{\mu}{\lambda_H} \left(1 + \frac{\mu}{\lambda_L} + \frac{\lambda_H}{\lambda_L} \right) R_{HA} \\ &\iff (R_L - R_{HB})P(\Omega_4^3) \leq \left[\frac{\mu}{\lambda_L} R_{HB} + \frac{\mu}{\lambda_H} \left(1 + \frac{\mu}{\lambda_L} + \frac{\lambda_H}{\lambda_L} \right) R_{HA} \right] P(\Omega_4^3) \\ &\implies (R_L - R_{HB})P(\Omega_4^3) \leq R_{HB}\mathbb{P}(\Omega_4^1) + R_{HA}\frac{\mu}{\lambda_H} [\mathbb{P}(\Omega_4^1) + \mathbb{P}(\Omega_4^2) + \mathbb{P}(\Omega_4^3)] \\ &\implies (R_L - R_{HB})P(\Omega_4^3) \leq R_{HB}\mathbb{P}(\Omega_4^1) + R_{HA}\mathbb{P}(\Omega_1) \\ &\implies -R_{HA}P(\Omega_1) - R_{HB}\mathbb{P}(\Omega_4^1) + (R_L - R_{HB})P(\Omega_4^3) \leq 0 \\ &\implies -R_{HA}P(\Omega_1) + (R_L - R_{HA})\mathbb{P}(\Omega_2) - R_{HB}\mathbb{P}(\Omega_4^1) + (R_L - R_{HB})P(\Omega_4^3) \leq 0 \\ &\iff \mathbb{E}\Delta - R_{HA} \leq 0, \end{aligned}$$

where the second line follows by inequalities (19) and (20), the third by (18) and the fact that $\mathbb{P}(\Omega_4) = \mathbb{P}(\Omega_4^1) + \mathbb{P}(\Omega_4^2) + \mathbb{P}(\Omega_4^3)$, and the final line by Equation (17).

It remains to show that (18) – (20) hold. We prove (18) here; inequalities (19) and (20) follow in a similar fashion. Because our MDP is uniformizable, and we have specified the policies by which Processes 1 and 2 operate, we can model them jointly as a discrete-time Markov chain. Randomness in this Markov chain is fully characterized by a sequence of i.i.d. uniform random variables $\{U_n : n \geq 1\}$, where U_i governs the state transition occurring immediately before the i^{th} decision epoch.

However, each sample path ω can be described more succinctly. In formulating our MDP, we assumed that $\Lambda = \lambda_H + \lambda_L + N_A\mu + N_B\mu = 1$. This allows us to partition the interval $[0, 1]$ into one subinterval of width λ_H , one subinterval of width λ_L , and $(N_A + N_B)$ subintervals of width μ . We can associate each subinterval with a possible state transition in our MDP model. For instance, we can use the subinterval $[0, \lambda_H)$ for transitions resulting from a Type H arrival, and $[\lambda_H + \lambda_L, \lambda_H + \lambda_L + \mu)$ for service completions resulting from a specific Type A server in the system (or for a dummy transitions if the server is idle). Thus, a sample path ω can be summarized by the type of transition that occurs in each decision epoch; we describe this using a sequence of random variables $\{X_n : n \geq 1\}$. We

allow the X_n to take on a value in the set $\{H, L, A_1, \dots, A_{N_A}, B_1, \dots, B_{N_B}\}$.

Fix $\omega \in \Omega_4$, and let $T(\omega)$ be the time at which Event 4 occurs on this path; note $X_{T(\omega)}(\omega) = H$. Consider a transformed sample path ω' that is identical to ω , except that its $T(\omega)^{\text{th}}$ element is A_1 instead of H . (Assume, without loss of generality, that the first Type A server is the one that is initially idle in Process 1, but busy in Process 2.) Let Ω'_1 be the set of all paths in Ω_4 that are transformed in this way. Observe that $\Omega'_1 \subseteq \Omega_4$, as Event 1 occurs on ω' instead of Event 4. It suffices to show $\mathbb{P}(\Omega'_1) = \frac{\mu}{\lambda_H} \mathbb{P}(\Omega_4)$. Indeed:

$$\begin{aligned}
\mathbb{P}(\Omega'_1) &= \sum_{n=1}^{\infty} \mathbb{P}(\Omega'_1, T = n) \\
&= \sum_{n=1}^{\infty} \sum_{(x_1, \dots, x_{n-1}): T=n} \mathbb{P}(X_1 = x_1, \dots, X_{n-1} = x_{n-1}, X_n = H) \\
&= \sum_{n=1}^{\infty} \sum_{(x_1, \dots, x_{n-1}): T=n} \mathbb{P}(X_1 = x_1, \dots, X_{n-1} = x_{n-1}) \mathbb{P}(X_n = H) \\
&= \sum_{n=1}^{\infty} \sum_{(x_1, \dots, x_{n-1}): T=n} \mathbb{P}(X_1 = x_1, \dots, X_{n-1} = x_{n-1}) \left[\frac{\mu}{\lambda_H} \mathbb{P}(X_n = A_1) \right] \\
&= \sum_{n=1}^{\infty} \frac{\mu}{\lambda_H} \mathbb{P}(\Omega_4, T = n) = \frac{\mu}{\lambda_H} \mathbb{P}(\Omega_4)
\end{aligned}$$

where the second line follows because the X_n are i.i.d. Thus, inequality (18) holds. \square

C Proof of Proposition 5.7

C.1 Proposition 5.7, Undiscounted Rewards

For convenience, we refer to the system with rewards (R_{HA}, R_{HB}, R_L) as the *original system*, and the system with rewards (R'_{HA}, R'_{HB}, R'_L) as the *modified system*. Because the MDP is unichain, the long-run average reward under any policy is constant and independent of the starting state. Let π_i be the threshold policy in which Type L jobs are admitted in all states (i', N_B) where $i' \leq i$. We use π_{-1} to denote the policy that never admits Type L jobs when all Type B servers are busy. Finally, let J_i and J'_i denote the long-run average reward obtained attained by policy π_i under the original and modified systems, respectively.

For the original system, let π_{i^*} denote the best policy among $\{\pi_{-1}, \pi_0, \pi_1, \dots, \pi_{N_A}\}$; if multiple policies are optimal, let π_{i^*} denote the one with the largest threshold. To avoid trivialities, assume $i^* < N_A$. By construction, $J_{i^*} > J_i$ for all $i > i^*$. It suffices to show that $J'_{i^*} > J'_i$ for all $i > i^*$ as well. Fix $i > i^*$, and construct two stochastic processes on the same probability space, both under

the original system. The process $\{X(t) : t \geq 0\}$ follows policy π_{i^*} , whereas $\{\tilde{X}(t) : t \geq 0\}$ follows π_i . Assume $X(0) = \tilde{X}(0) = (i^* + 1, N_B)$. Let $\tau_0 = 0$, and for $n \geq 0$, define the stopping times

$$\tau_{n+1} = \inf \left\{ t > \tau_n : (X(t), \tilde{X}(t)) = ((i^* + 1, N_B), (i^* + 1, N_B)) \right\},$$

the times at which both processes return to state $(i^* + 1, N_B)$. By memorylessness, $\{\tau_n : n \geq 1\}$ constitute a renewal process. Let $R(t)$ and $\tilde{R}(t)$ denote the reward collected by processes X and \tilde{X} by time t . Similarly, let R_i and \tilde{R}_i denote the reward collected by X and \tilde{X} during the i^{th} renewal epoch. By the Renewal Reward Theorem,

$$J_{i^*} = \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{R(t)}{t} \right] = \frac{\mathbb{E}R_1}{\mathbb{E}\tau_1} \quad J_i = \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{\tilde{R}(t)}{t} \right] = \frac{\mathbb{E}\tilde{R}_1}{\mathbb{E}\tau_1}. \quad (21)$$

The above limits are well-defined, as $\{(X(t), \tilde{X}(t)) : t \geq 0\}$ is an irreducible continuous-time Markov chain with finite state space, implying $\mathbb{E}\tau_1 < \infty$. Thus, we have

$$J_{i^*} - J_i = \frac{\mathbb{E}R_1 - \mathbb{E}\tilde{R}_1}{\mathbb{E}\tau_1} > 0. \quad (22)$$

Consider $\mathbb{E}R_1 - \mathbb{E}\tilde{R}_1$, the expected difference in reward collected by X and \tilde{X} over a single renewal epoch. Discrepancies between the reward collected by X and \tilde{X} can occur in four ways. Let

1. N_1 be the number of Type L jobs that X rejects due to policy considerations, but \tilde{X} admits, and assigns to Type A servers.
2. N_2 be the number of Type H jobs that X admits, but \tilde{X} rejects because all servers are busy.
3. N_3 be the number of Type H jobs that X assigns to Type A servers, but \tilde{X} is forced to assign to Type B servers.
4. N_4 be the number of Type L jobs that X admits, but \tilde{X} rejects, either because all servers are busy, or due to policy considerations.

We make two claims, that we later prove: that the above list of discrepancies is comprehensive, and that $N_1 \geq N_4$ pathwise. Supposing these to be true for the time being, it follows that

$$\mathbb{E}[R_1 - \tilde{R}_1] = -R_L \mathbb{E}[N_1] + R_{HA} \mathbb{E}[N_2] + (R_{HA} - R_{HB}) \mathbb{E}[N_3] + R_L \mathbb{E}[N_4] > 0. \quad (23)$$

We can similarly construct two processes $\{X'(t) : t \geq 0\}$ and $\{\tilde{X}'(t) : t \geq 0\}$, under the modified system. Let R'_1 , and \tilde{R}'_1 be defined as in (21). Since changing rewards does not affect the evolution

of the processes X' and \tilde{X}' , we have

$$\begin{aligned}
\mathbb{E}[R'_1 - \tilde{R}'_1] &= -R'_L \mathbb{E}[N_1] + R'_{HA} \mathbb{E}[N_2] + (R'_{HA} - R'_{HB}) \mathbb{E}[N_3] + R'_L \mathbb{E}[N_4] \\
&= R'_{HA} \mathbb{E}[N_2] + (R'_{HA} - R'_{HB}) \mathbb{E}[N_3] - R'_L \mathbb{E}[N_1 - N_4] \\
&\geq R_{HA} \mathbb{E}[N_2] + (R_{HA} - R_{HB}) \mathbb{E}[N_3] - R_L \mathbb{E}[N_1 - N_4] \\
&> 0.
\end{aligned}$$

By an equation analogous to (22), $J'_{i^*} > J'_i$, and it is preferable under the modified system to set the threshold to i^* than to any $i > i^*$. It remains to prove the two aforementioned claims: that the events described by the random variables N_1, \dots, N_4 are the only transitions in which processes X and \tilde{X} do not collect the same reward, and that $N_1 \geq N_4$ pathwise.

To show the first claim, it suffices to show that if \tilde{X} admits a Type H (Type L) job with a Type A (Type B) server, X does as well. Let $X_A(t)$ and $X_B(t)$ be the number of free Type A and Type B servers in process X at time t , respectively. Define $\tilde{X}_A(t)$ and $\tilde{X}_B(t)$ analogously. Since X and \tilde{X} are defined on the same probability space, every service completion seen by \tilde{X} is also observed by X . It follows that $X_A(t) \geq \tilde{X}_A(t)$ and $X_B(t) \geq \tilde{X}_B(t)$ for all t on every sample path.

To show the second claim, consider $X_A(t) + X_B(t) - [\tilde{X}_A(t) - \tilde{X}_B(t)]$. This difference increases by one whenever \tilde{X} admits a Type L job with a Type A server that X is forced to reject (that is, when N_1 increases by one). Similarly, this difference decreases by one when X admits a job that \tilde{X} rejects (that is, when N_2 or N_4 increases by one), or when the aforementioned Type A server completes service in \tilde{X} , but not in X . At the start of any renewal epoch, this difference equals zero, and so we must have $N_1 \geq N_2 + N_4$. \square

D Proof of Proposition 5.5

If Type H calls are subject to admission control, the optimality equations must be modified slightly. For brevity, we include only those for the long-run discounted reward criterion.

$$\begin{aligned}
v_\alpha(i, j) &= \lambda_H \left[\mathbf{1}_{\{i < N_A\}} \max \{ R_{HA} + \alpha v_\alpha(i+1, j), \alpha v_\alpha(i, j) \} \right. \\
&\quad \left. + \mathbf{1}_{\{i=N_A, j < N_B\}} \max \{ R_{HB} + \alpha v_\alpha(i, j+1), \alpha v_\alpha(i, j) \} + \mathbf{1}_{\{i=N_A, j=N_B\}} \alpha v_\alpha(i, j) \right] \\
&+ \lambda_L \left[\mathbf{1}_{\{j < N_B\}} \max \{ R_L + \alpha v_\alpha(i, j+1), \alpha v_\alpha(i, j) \} \right. \\
&\quad \left. + \mathbf{1}_{\{i < N_A, j=N_B\}} \max \{ R_L + \alpha v_\alpha(i+1, j), \alpha v_\alpha(i, j) \} + \mathbf{1}_{\{i=N_A, j=N_B\}} \alpha v_\alpha(i, j) \right] \\
&+ i\mu \alpha v_\alpha(i-1, j) + j\mu \alpha v_\alpha(i, j-1) + (N_A + N_B - i - j)\mu \alpha v_\alpha(i, j)
\end{aligned} \tag{24}$$

In this modified setting, the structural properties stated in Section 4 of the main paper all still hold; they follow via identical sample path arguments as the ones presented therein. It suffices to show that $v_{n,\alpha}$ (and consequently, v_α and h) satisfies the following structural properties:

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i + 1, j) \leq v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 2, j) \quad (25)$$

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) \leq v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \quad (26)$$

$$v_{n,\alpha}(N_A, j) - v_{n,\alpha}(N_A, j + 1) \leq v_{n,\alpha}(N_A, j + 1) - v_{n,\alpha}(N_A, j + 2) \quad (27)$$

We proceed via induction on n . The base case ($n = 0$) is trivial, as we assume $v_{\alpha,0}(i, j) = 0$ for all i and j . Now suppose that (25)–(27) hold over horizons of length up to n : our induction hypothesis. We prove in separate subsections that these inequalities hold over horizons of length $n + 1$.

D.1 Inductive Step, Inequality (25)

Fix $i \in \{0, \dots, N_A - 2\}$ and $j \in \{0, \dots, N_B\}$. We want to show that

$$v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i + 1, j) \leq v_{n+1,\alpha}(i + 1, j) - v_{n+1,\alpha}(i + 2, j), \quad (28)$$

and proceed using a sample path argument. Start four processes on the same probability space, each with $n + 1$ periods remaining in the horizon. Processes 1 and 4 begin in states (i, j) and $(i + 2, j)$, respectively, and follow the optimal policy π^* . Processes 2 and 3 both begin in state $(i + 1, j)$, and use potentially suboptimal policies π_2 and π_3 , respectively, that deviate from π^* only during the first time period. Specifically, Processes 2 and 3 take actions that depend on those made by Processes 1 and 4 when a job arrives:

- If Processes 1 and 4 admit the arriving job, so do Processes 2 and 3.
- If Processes 1 and 4 reject the arriving job, so do Processes 2 and 3.
- If Process 1 admits the job, and Process 4 rejects, then Process 2 rejects and Process 3 admits.

We need not consider the case where Process 1 rejects in state (i, j) and Process 4 admits in state $(i + 2, j)$, as by the induction hypothesis, we can assume that π^* is a monotone switching curve policy. Let Δ be the difference in reward collected by Processes 1 and 2 until coupling occurs; define Δ' analogously for Processes 3 and 4. We show that $\mathbb{E}\Delta \leq \mathbb{E}\Delta'$. There are i Type A and j Type B servers that are busy in all four processes, and $N_A - i - 2$ Type A and $N_B - j$ Type B servers that are idle in all four processes. We probabilistically link the remaining two Type A servers according to the scheme in Table 2.

	Server I	Server II
Process 1, State (i, j)	Idle	Idle
Process 2, State $(i + 1, j)$	Busy	Idle
Process 3, State $(i + 1, j)$	Idle	Busy
Process 4, State $(i + 2, j)$	Busy	Busy

Table 2: Marking scheme for units in the sample path argument for Equation (25).

Note that our marking scheme probabilistically links servers linked so that completions of all servers marked as Server I (similarly, Server II) occur simultaneously in all four processes. In the first time period, eight transitions are possible:

1. A Type H arrival
2. A Type L arrival
3. A completion by Server I.
4. A completion by Server II.
5. A completion by any other Type A server.
6. A Type A service completion.
7. A dummy transition due to uniformization.

For $k = 1, \dots, 7$, let A_k be the event in which transition k occurs; it suffices to show that $\mathbb{E}[\Delta | A_k] \leq \mathbb{E}[\Delta' | A_k]$ for each k . We proceed case-by-case.

Case 1 (Event A_1): If Processes 1 and 4 both admit the job, then by construction of policies π_2 and π_3 , Processes 2 and 3 do so as well, and

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 2, j) \leq v_{n,\alpha}(i + 2, j) - v_{n,\alpha}(i + 3, j) = \mathbb{E}[\Delta' | A_1].$$

by the induction hypothesis and (25). If Processes 1 and 4 both reject, a dummy transition occurs, and the analysis is straightforward. Finally, if Process 1 admits, and Process 4 rejects, then

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j) \leq v_{n,\alpha}(i + 2, j) - v_{n,\alpha}(i + 2, j) = \mathbb{E}[\Delta' | A_1].$$

Now suppose $i + 2 = N_A$. Process 4 must admit Type H jobs with a Type B server (provided

$j < N_B$). If Process 1 and 4 both admit the job, then

$$\begin{aligned}\mathbb{E}[\Delta | A_1] &= v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+2, j) \\ &\leq v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+2, j+1) \\ &\leq v_{n,\alpha}(i+2, j) + R_{HA} - v_{n,\alpha}(i+2, j+1) - R_{HB} = \mathbb{E}[\Delta' | A_1]\end{aligned}$$

where the inequalities follows by the induction hypothesis and (26), respectively.

Case 2 (Event A_2): If $j < N_B$, then all four processes admit the Type L job, and the analysis is straightforward. When $j = N_B$, then admitting in all four processes entails using a Type A server. If Processes 1 and 4 both admit the job, then

$$\mathbb{E}[\Delta | A_2] = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+2, j) \leq v_{n,\alpha}(i+2, j) - v_{n,\alpha}(i+3, j) = \mathbb{E}[\Delta' | A_2],$$

by (25) and the induction hypothesis. If Processes 1 and 4 both reject, a dummy transition occurs, and the analysis is again straightforward. Finally, if Process 1 admits and Process 4 rejects, we have

$$\mathbb{E}[\Delta | A_2] = R_L + v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j) = R_L + v_{n,\alpha}(i+2, j) - v_{n,\alpha}(i+2, j) = \mathbb{E}[\Delta' | A_2].$$

Case 3 (Event A_3): Processes 1 and 2 both transition to state (i, j) , and coupling occurs. Processes 3 and 4 transition to state $(i+1, j)$, and couple as well. Thus $\mathbb{E}[\Delta | A_3] = \mathbb{E}[\Delta' | A_3] = 0$.

Case 4 (Event A_4): Processes 1 and 3 both transition to state (i, j) , while Processes 2 and 4 both transition to state $(i+1, j)$. We thus have $\mathbb{E}[\Delta | A_4] = \mathbb{E}[\Delta' | A_4] = v_{n,\alpha}(i, j) - v_{n,\alpha}(i+1, j)$.

Case 5 (Event A_5): The four processes transition to states $(i-1, j)$, (i, j) , (i, j) , and $(i+1, j)$, respectively, and

$$\mathbb{E}[\Delta | A_5] = v_{n,\alpha}(i-1, j) - v_{n,\alpha}(i, j) \leq v_{n,\alpha}(i, j) - v_{n,\alpha}(i+1, j) = \mathbb{E}[\Delta' | A_5],$$

by (25) and the induction hypothesis.

Case 6 (Event A_6): The four processes transition to states $(i, j-1)$, $(i+1, j-1)$, $(i+1, j-1)$, and $(i+2, j-1)$, respectively, and

$$\mathbb{E}[\Delta | A_6] = v_{n,\alpha}(i, j-1) - v_{n,\alpha}(i+1, j-1) \leq v_{n,\alpha}(i+1, j-1) - v_{n,\alpha}(i+2, j-1) = \mathbb{E}[\Delta' | A_6],$$

by (25) and the induction hypothesis.

Case 7 (Event A_7): A dummy transition occurs, and the analysis is straightforward.

Thus, Equation (28) holds, as desired. \square

D.2 Inductive Step, Inequality (26)

Fix $i \in \{0, \dots, N_A - 1\}$ and $j \in \{0, \dots, N_B - 1\}$. We want to show that

$$v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i, j + 1) \leq v_{n+1,\alpha}(i + 1, j) - v_{n+1,\alpha}(i + 1, j + 1). \quad (29)$$

The proof follows by a sample path argument nearly identical to that used in the proof of Proposition 5.4, to show that the value function in that setting is supermodular. Thus, we restrict attention to points in the proof that deviate from the original proof. We construct four stochastic processes and define random variables Δ and Δ' as before; it again suffices to show that $\mathbb{E}\Delta \leq \mathbb{E}\Delta'$. We mark servers as before, and operate Processes 2 and 3 according to the same suboptimal policies described in Section D.1. Of the seven transitions that can occur in the next time period, the analysis for the latter five cases (all of the possible transitions in which arrivals do not occur) is identical. We consider only two possibilities below:

Case 1: Suppose a Type H arrival occurs in the next time period. This time, there is a decision to make in all four processes with respect to the Type H arrival. Consider first the case where $i + 1 < N_A$. If Processes 1 and 4 both admit the job, then the four processes transition to states $(i + 1, j)$, $(i + 1, j + 1)$, $(i + 2, j)$ and $(i + 2, j + 1)$, respectively, and

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \leq v_{n,\alpha}(i + 2, j) - v_{n,\alpha}(i + 2, j + 1) = \mathbb{E}[\Delta' | A_1].$$

If Processes 1 and 4 both reject the job, dummy transitions occur and

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j + 1) \leq v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) = \mathbb{E}[\Delta' | A_1]$$

Finally, if Process 1 admits and Process 4 rejects, then

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) = \mathbb{E}[\Delta' | A_1]$$

Now suppose $i + 1 = N_A$. If Process 1 and 4 admit both admit the job (the latter with a Type B server), so do Processes 2 and 3 (the latter, again, with a Type B server), and by (26), we have that

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) \leq v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 1, j + 2) = \mathbb{E}[\Delta' | A_1].$$

If Processes 1 and 4 both reject, a dummy transition occurs, and the analysis is straightforward. Finally, if Process 1 admits and Process 4 rejects,

$$\mathbb{E}[\Delta | A_1] = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1) = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+1, j+1) = \mathbb{E}[\Delta' | A_1].$$

Case 2: Suppose a Type L arrival occurs. Since it is optimal for Type L jobs to be admitted when Type B servers are available, the analysis when $j+1 < N_B$ is straightforward. Suppose $j+1 = N_B$. If Processes 1 and 4 both admit the job, then

$$\mathbb{E}[\Delta | A_2] = v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+1, j+1) \leq v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+2, j+1) = \mathbb{E}[\Delta' | A_2],$$

by Lemma 4.1 and the induction hypothesis. If Processes 1 and 4 both reject, a dummy transition occurs. Finally, if Process 1 admits and Process 4 rejects, we have that

$$\mathbb{E}[\Delta | A_2] = R_L + v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+1) = R_L + v_{n,\alpha}(i+1, j+1) - v_{n,\alpha}(i+1, j+1) = \mathbb{E}[\Delta' | A_2].$$

Thus, Equation (26) holds, as desired. \square

D.3 Inductive Step, Inequality (27)

Fix $j \in \{0, 1, \dots, N_B - 2\}$. We want to show that

$$v_{n+1,\alpha}(N_A, j) - v_{n+1,\alpha}(N_A, j+1) \leq v_{n+1,\alpha}(N_A, j+1) - v_{n+1,\alpha}(N_A, j+2). \quad (30)$$

We can rewrite the left-hand side of (30) as

$$\begin{aligned} v_{n+1,\alpha}(N_A, j) - v_{n+1,\alpha}(N_A, j+1) &= \lambda_H \left[\max \{ R_{HB} + \alpha v_{n,\alpha}(N_A, j+1), \alpha v_{n,\alpha}(N_A, j) \} \right. \\ &\quad \left. - \max \{ R_{HB} + \alpha v_{n,\alpha}(N_A, j+2), \alpha v_{n,\alpha}(N_A, j+1) \} \right] \\ &\quad + \lambda_L \left[R_L + \alpha v_{n,\alpha}(N_A, j+1) - R_L - \alpha v_{n,\alpha}(N_A, j+2) \right] \\ &\quad + N_A \mu \alpha \left[v_{n,\alpha}(N_A - 1, j) - v_{n,\alpha}(N_A - 1, j+1) \right] \\ &\quad + j \mu \alpha \left[v_{n,\alpha}(N_A, j-1) - v_{n,\alpha}(N_A, j) \right] \\ &\quad + \mu \alpha \left[v_{n,\alpha}(N_A, j) - v_{n,\alpha}(N_A, j) \right] \\ &\quad + (N_A + N_B - i - j) \mu \alpha \left[v_{n,\alpha}(N_A, j) - v_{n,\alpha}(N_A, j+1) \right] \end{aligned} \quad (31)$$

Note that in the second term, we leveraged the fact that when $R_L > R_{HB}$ it is optimal to admit Type L jobs whenever Type B servers are available. It again suffices to show that each term in brackets

on the right-hand side of (31) is bounded above by $v_{n+1,\alpha}(N_A, j+1) - v_{n+1,\alpha}(N_A, j+2)$. Consider the first term. There are three possibilities:

1. $R_{HA} + \alpha v_{n,\alpha}(N_A, j+1) \geq \alpha v_{n,\alpha}(N_A, j)$ and $R_{HA} + \alpha v_{n,\alpha}(N_A, j+2) \geq \alpha v_{n,\alpha}(N_A, j+1)$
2. $R_{HA} + \alpha v_{n,\alpha}(N_A, j+1) < \alpha v_{n,\alpha}(N_A, j)$ and $R_{HA} + \alpha v_{n,\alpha}(N_A, j+2) < \alpha v_{n,\alpha}(N_A, j+1)$
3. $R_{HA} + \alpha v_{n,\alpha}(N_A, j+1) \geq \alpha v_{n,\alpha}(N_A, j)$ and $R_{HA} + \alpha v_{n,\alpha}(N_A, j+2) < \alpha v_{n,\alpha}(N_A, j+1)$

We need not consider the remaining possibility, as by the induction hypothesis, $v_{n,\alpha}$ is convex in j when $i = N_A$. Consider the third case; the analysis for the remaining two cases is straightforward.

We have that

$$\begin{aligned} & \max \{R_{HA} + \alpha v_{n,\alpha}(N_A, j+1), \alpha v_{n,\alpha}(N_A, j)\} - \max \{R_{HA} + \alpha v_{n,\alpha}(N_A, j+2), \alpha v_{n,\alpha}(N_A, j+1)\} \\ &= R_{HA} \\ &< v_{n,\alpha}(N_A, j+1) - v_{n,\alpha}(N_A, j+2) \\ &\leq v_{n+1,\alpha}(N_A, j+1) - v_{n+1,\alpha}(N_A, j+2) \end{aligned}$$

where the first inequality follows by assumption, and the second by Lemma A.1. Now consider the third term. The induction hypothesis (specifically, inequalities (25) and (26)) yields

$$\begin{aligned} v_{n,\alpha}(N_A - 1, j) - v_{n,\alpha}(N_A - 1, j+1) &\leq v_{n,\alpha}(N_A, j) - v_{n,\alpha}(N_A, j+1) \\ &\leq v_{n,\alpha}(N_A, j+1) - v_{n,\alpha}(N_A, j+2) \\ &\leq v_{n+1,\alpha}(N_A, j+1) - v_{n+1,\alpha}(N_A, j+2). \end{aligned}$$

The remaining terms follow in a similar fashion.

E Proof of Lemma B.1

We begin with an intermediate result:

Lemma E.1. *If $R_L > R_{HB}$, then for every $n \geq 0$ and applicable i and j , we have that*

$$v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j+1) \leq \frac{\lambda_H}{\lambda_H + \lambda_L} R_{HB} + \frac{\lambda_L}{\lambda_H + \lambda_L} R_L. \quad (32)$$

Inequality (32) is slightly stronger than that in Statement 2 of Lemma 4.2 (which states that $v_{n,\alpha}(i, j) - v_{n,\alpha}(i, j+1) \leq R_L$), by leveraging the fact that optimal policies always admit Type L calls when Type B servers are available.

Proof. We use a sample path argument. Fix $\alpha \in [0, 1]$, $i \in \{0, \dots, N_A\}$, and $j \in \{0, \dots, N_B - 1\}$, and construct two stochastic processes on the same probability space. Process 2 begins in state $(i, j + 1)$ and follows the optimal policy π^* , while Process 1 begins in state (i, j) and follows the suboptimal policy that imitates the decisions made by Process 2, with one exception: Process 1 admits any job that arrives when Process 2 is in state (N_A, N_B) . Let Δ denote the difference in reward collected by the two processes until coupling occurs; it suffices to show that

$$\mathbb{E}\Delta \leq \frac{\lambda_H}{\lambda_H + \lambda_L} R_{HB} + \frac{\lambda_L}{\lambda_H + \lambda_L} R_L.$$

Both processes move in parallel until they couple, which can occur in two ways:

1. Process 2 sees a Type B service completion not observed by Process 1.
2. Process 1 is in state $(N_A, N_B - 1)$, Process 2 is in state (N_A, N_B) , and an arrival occurs.

Let A_1 and A_2 be the events in which coupling occurs via the first and second possibilities, respectively. Conditional on A_1 occurring, we have $\Delta = 0$. Conditional on A_2 occurring, with probability $\lambda_H/(\lambda_H + \lambda_L)$, a Type H arrival occurs, and $\Delta = R_{HB}$. Similarly, with probability $\lambda_L/(\lambda_H + \lambda_L)$, a Type L arrival occurs, and $\Delta = R_L$. It follows that

$$E[\Delta | A_2] = \frac{\lambda_H}{\lambda_H + \lambda_L} R_{HB} + \frac{\lambda_L}{\lambda_H + \lambda_L} R_L,$$

and we are done. □

To prove Lemma B.1, we again use induction over the time periods. The base case ($n = 0$) is trivial, as we assume $v_{\alpha,0}(i, j) = 0$ for all i and j . Now suppose that (12)–(14) hold over horizons of length $k \leq n$: our induction hypothesis. In the analysis that follows, we assume once again (for notational convenience) that $\alpha = 1$; nearly identical reasoning can be used when $\alpha < 1$.

E.1 Inductive Proof, Inequality (12)

Fix $i \in \{0, \dots, N_A - 2\}$ and $j \in \{0, \dots, N_B\}$. We show that

$$v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i + 1, j) \leq v_{n+1,\alpha}(i + 1, j) - v_{n+1,\alpha}(i + 2, j), \quad (33)$$

using a sample path argument. Start four processes on the same probability space, each with $n + 1$ periods remaining in the horizon. Processes 1 and 4 begin in states (i, j) and $(i + 2, j)$, respectively, and follow the optimal policy π^* . Processes 2 and 3 begin in state $(i + 1, j)$, and use potentially

suboptimal policies π_2 and π_3 , respectively, that deviate from π^* during the first time period. The actions Processes 2 and 3 take when a job arrives depend on those taken by Processes 1 and 4:

- If Processes 1 and 4 admit the arriving job, so do Processes 2 and 3.
- If Processes 1 and 4 reject the arriving job, so do Processes 2 and 3.
- If Process 1 admits the job, and Process 4 rejects, then Process 2 rejects and Process 3 admits.

We can ignore the case where Process 1 rejects in state (i, j) and Process 4 admits in state $(i+2, j)$, as by the induction hypothesis, we can assume that π^* is a monotone switching curve policy.

Let Δ be the difference in reward collected by Processes 1 and 2 until coupling occurs; define Δ' analogously for Processes 3 and 4; it suffices to show $\mathbb{E}\Delta \leq \mathbb{E}\Delta'$. There are i Type A and j Type B servers that are busy in all four processes, and $N_A - i - 2$ Type A and $N_B - j$ Type B servers that are idle in all four processes. We probabilistically link the remaining two Type A servers according to the scheme in Table 3.

	Server I	Server II
Process 1, State (i, j)	Idle Type A	Idle Type A
Process 2, State $(i + 1, j)$	Busy Type A	Idle Type A
Process 3, State $(i + 1, j)$	Idle Type A	Busy Type A
Process 4, State $(i + 2, j)$	Busy Type A	Busy Type A

Table 3: Marking scheme for units in the sample path argument for Equation (12).

In the first time period, eight transitions are possible:

1. A Type H arrival
2. A Type L arrival
3. A completion by Server I.
4. A completion by Server II.
5. A completion by any other Type A server.
6. A completion by a Type B server.
7. A dummy transition due to uniformization.

For $k = 1, \dots, 8$, let A_k be the event in which transition k occurs. It suffices to show that $\mathbb{E}[\Theta | A_k] \leq \mathbb{E}[\Theta' | A_k]$ for each k , and we proceed case-by-case.

Case 1 (Event A_1): If $i + 2 < N_A$, the analysis is straightforward. If $i + 2 = N_A$, Process 4 routes the job to a Type A server, and we have

$$\mathbb{E}[\Theta | A_1] = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+2, j) \leq v_{n,\alpha}(i+2, j) - v_{n,\alpha}(i+2, j+1) + R_{HA} - R_{HB} = \mathbb{E}[\Theta' | A_1],$$

by the induction hypothesis and (14). Finally, if $i + 2 = N_A$ and $j = N_B$, then Processes 3 and 4 find themselves in state $(i + 2, j)$, and by inequality (12), we have

$$\mathbb{E}[\Theta | A_1] = v_{n,\alpha}(i+1, j) - v_{n,\alpha}(i+2, j) \leq R_{HA} = \mathbb{E}[\Theta' | A_1].$$

Case 2 (Event A_2): By Proposition 4.3, we need only consider the case where $j = N_B$, as all four processes would otherwise admit the arriving Type L job, and the analysis is straightforward. Processes 2 and 3 take (potentially suboptimal) actions based upon those taken by Processes 1 and 4. In particular:

- If Processes 1 and 4 admit the arriving Type L job, so do Processes 2 and 3.
- If Processes 1 and 4 reject the arriving Type L job, so do Processes 2 and 3.
- If Process 1 admits the job, and Process 4 rejects, then Process 2 rejects and Process 3 admits.

We need not consider the case when Process 1 rejects the job in state $(i + 1, j)$, and Process 4 admits the job in state $(i, j + 2)$, as by the induction hypothesis, we can assume that π^* is a threshold-type policy. The analysis for the case where both Processes 1 and 4 reject the job is straightforward; all four processes either reject the job or admit it with a Type A server, and we leverage the induction hypothesis. For the latter possibility, the four processes transition to state $(i + 1, j)$, $(i + 1, j)$, $(i + 2, j)$, and $(i + 2, j)$, respectively, and Processes 1 and 3 alone collect a reward R_L . Thus, we have that $\mathbb{E}[\Theta | A_2] = R_L = \mathbb{E}[\Theta' | A_2]$.

Case 3 (Event A_3): The four processes transition to states (i, j) , (i, j) , $(i + 1, j)$, and $(i + 1, j)$. Processes 1 and 2 couple, as do Processes 3 and 4, and we have $\mathbb{E}[\Theta | A_3] = \mathbb{E}[\Theta' | A_3] = 0$.

Case 4 (Event A_4): The four processes transition to states (i, j) , $(i + 1, j)$, (i, j) , and $(i + 1, j)$. Processes 1 and 3 couple, as do Processes 2 and 4, and we have $\mathbb{E}[\Delta | A_4] = \mathbb{E}[\Delta' | A_4]$.

Case 5 (Event A_5): The four processes transition to states $(i - 1, j)$, (i, j) , (i, j) , and $(i + 1, j)$, respectively, and we can leverage the induction hypothesis to show that $\mathbb{E}[\Delta | A_5] = \mathbb{E}[\Delta' | A_5]$.

Case 6 (Event A_6): The four processes transition to states $(i, j - 1)$, $(i + 1, j - 1)$, $(i + 1, j - 1)$, and $(i + 2, j - 1)$, respectively, and the induction hypothesis applies.

Case 7 (Event A_7): No process changes state, and the induction hypothesis applies.

Thus, Inequality (12) holds for horizons of length $n + 1$.

E.2 Inductive Proof, Inequality (13)

Fix $i \in \{0, \dots, N_A - 2\}$ and $j \in \{0, \dots, N_B\}$. We want to show that

$$v_{n+1, \alpha}(i, j) - v_{n+1, \alpha}(i, j + 1) \leq v_{n+1, \alpha}(i, j + 1) - v_{n+1, \alpha}(i, j + 2), \quad (34)$$

and proceed using a sample path argument. Start four processes on the same probability space, each with $n + 1$ periods remaining in the horizon. Processes 1 and 4 begin in states (i, j) and $(i, j + 2)$, respectively, and follow the optimal policy π^* . Processes 2 and 3 both begin in state $(i, j + 1)$, and use potentially suboptimal policies π_2 and π_3 that mimic those described in Section E.1.

Let Θ be the difference in reward collected by Processes 1 and 2 until coupling occurs; define Θ' analogously for Processes 3 and 4. It suffices to show $\mathbb{E}\Theta \leq \mathbb{E}\Theta'$. There are i Type A and j Type B servers that are busy in all four processes, and $N_A - i$ Type A and $N_B - j - 2$ Type B servers that are idle in all four processes. We probabilistically link the remaining two Type B servers according to the scheme in Table 4.

	Server I	Server II
Process 1, State (i, j)	Idle Type B	Idle Type B
Process 2, State $(i, j + 1)$	Busy Type B	Idle Type B
Process 3, State $(i, j + 1)$	Idle Type B	Busy Type B
Process 4, State $(i, j + 2)$	Busy Type B	Busy Type B

Table 4: Marking scheme for units in the sample path argument for Equation (13).

In the first time period, eight transitions are possible:

1. A Type H arrival
2. A Type L arrival
3. A completion by a Type A server.
4. A completion by Server I.
5. A completion by Server II.
6. A completion by any other Type B server.
7. A dummy transition due to uniformization.

For $k = 1, \dots, 8$, let B_k be the event in which transition k occurs. It suffices to show that $\mathbb{E}[\Theta | B_k] \leq \mathbb{E}[\Theta' | B_k]$ for each k , and we proceed case-by-case.

Case 1 (Event B_1): For the time being, assume that either $i < N_A$ or $j + 2 < N_B$; we handle later the special case in which both inequalities do not hold. If $i < N_A$, then the four processes transition to states $(i + 1, j)$, $(i + 1, j + 1)$, $(i + 1, j + 1)$, and $(i + 1, j + 2)$, respectively, and the induction hypothesis applies. If $i = N_A$ but $j + 2 < N_B$, transitions to states $(i, j + 1)$, $(i, j + 2)$, $(i, j + 2)$, and $(i, j + 3)$ occur instead, and we again leverage the induction hypothesis.

Case 2 (Event B_2): As with Case 1, assume that either $i < N_A$ or $j + 2 < N_B$ for the time being. If $j + 2 < N_B$, the analysis is straightforward. Suppose $j + 2 = N_B$ but $i < N_A$. If Processes 1 and 4 both reject, the analysis is straightforward. If Processes 1 and 4 both admit, Processes 2 and 3 use a Type A server, and we have that

$$\begin{aligned}
\mathbb{E}[\Theta \mid B_2] &= v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j + 1) \\
&\leq v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j + 1) - R_{HA} + R_{HB} \\
&\leq v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 1, j + 2) \\
&= \mathbb{E}[\Theta' \mid B_2],
\end{aligned}$$

where the first inequality follows by Lemma 4.2, and the second by the induction hypothesis on inequality (14). Finally, if Process 1 accepts and Process 4 rejects, then $\mathbb{E}[\Theta \mid B_2] = R_L = \mathbb{E}[\Theta' \mid B_2]$.

Case 3 (Event B_3): The four processes transition to states $(i - 1, j)$, $(i - 1, j + 1)$, $(i - 1, j + 1)$, and $(i - 1, j + 2)$, respectively, and the induction hypothesis applies.

Case 4 (Event B_4): The four processes transition to states (i, j) , (i, j) , $(i, j + 1)$, and $(i, j + 1)$. It follows that $\mathbb{E}[\Theta \mid B_4] = \mathbb{E}[\Theta' \mid B_4] = 0$.

Case 5 (Event B_5): The four processes transition to states (i, j) , $(i, j + 1)$, (i, j) , and $(i, j + 1)$, respectively, and we have $\mathbb{E}[\Theta \mid B_5] = \mathbb{E}[\Theta' \mid B_5]$.

Case 6 (Event B_6): The four processes transition to states $(i, j - 1)$, (i, j) , (i, j) , and $(i, j + 1)$, respectively, and the induction hypothesis applies.

Case 7 (Event B_7): No process changes state, and the induction hypothesis applies.

It remains to show that $\mathbb{E}[\Theta \mid B_1] \leq \mathbb{E}[\Theta' \mid B_1]$ and $\mathbb{E}[\Theta \mid B_2] \leq \mathbb{E}[\Theta' \mid B_2]$ when $i = N_A$ and $j + 2 = N_B$. However, we have that $\mathbb{E}[\Theta \mid B_1] = v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i, j + 2)$ and $\mathbb{E}[\Theta' \mid B_1] = R_{HB}$, and so the former inequality may not hold in general. We instead prove a slightly weaker claim, that $\mathbb{E}[\Theta \mid B_1 \cup B_2] \leq \mathbb{E}[\Theta' \mid B_1 \cup B_2]$, which still suffices to show that $E[\Theta] \leq E[\Theta']$. Conditional on $B_1 \cup B_2$ occurring, a Type H job arrives with probability $\lambda_H/(\lambda_H + \lambda_L)$, and a Type L job arrives

with probability $\lambda_L/(\lambda_H + \lambda_L)$. Noting that $\mathbb{E}[\Theta | B_2] = v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2)$ as well, we have

$$\begin{aligned} \mathbb{E}[\Theta | B_1 \cup B_2] &= \frac{\lambda_H}{\lambda_H + \lambda_L} \mathbb{E}[\Theta | B_1] + \frac{\lambda_L}{\lambda_H + \lambda_L} \mathbb{E}[\Theta | B_2] \\ &= v_{n,\alpha}(i, j+1) - v_{n,\alpha}(i, j+2) \\ &\leq \frac{\lambda_H}{\lambda_H + \lambda_L} R_{HB} + \frac{\lambda_L}{\lambda_H + \lambda_L} R_L \\ &\leq \frac{\lambda_H}{\lambda_H + \lambda_L} \mathbb{E}[\Theta' | B_1] + \frac{\lambda_L}{\lambda_H + \lambda_L} \mathbb{E}[\Theta' | B_2] \\ &= \mathbb{E}[\Theta' | B_1 \cup B_2], \end{aligned}$$

as desired. Thus, Inequality (13) holds for horizons of length $n + 1$.

E.2.1 Inductive Proof, Inequality (14)

Fix $i \in \{0, \dots, N_A - 2\}$ and $j \in \{0, \dots, N_B\}$. Once again, we show that

$$v_{n+1,\alpha}(i, j) - v_{n+1,\alpha}(i+1, j) \leq v_{n+1,\alpha}(i+1, j) - v_{n+1,\alpha}(i+1, j+1) + R_{HA} - R_{HB} \quad (35)$$

using a sample path argument. Start four processes on the same probability space, each with $n + 1$ periods remaining in the horizon. Processes 1 and 4 begin in states (i, j) and $(i+1, j+1)$, respectively, and follow the optimal policy π^* . Processes 2 and 3 begin in state $(i+1, j)$, and use potentially suboptimal policies π_2 and π_3 that mimic those described in Section E.1.

Let Ψ be the difference in reward collected by Processes 1 and 2 until coupling occurs; define Ψ' analogously for Processes 3 and 4. It suffices to show $\mathbb{E}\Psi \leq \mathbb{E}\Psi' + R_{HA} - R_{HB}$. There are i Type A and j Type B servers that are busy in all four processes, and $N_A - i - 1$ Type A and $N_B - j - 1$ Type B servers that are idle in all four processes. We probabilistically link the remaining Type A server and Type B server according to the scheme in Table 5.

	Server I	Server II
Process 1, State (i, j)	Idle Type A	Idle Type B
Process 2, State $(i+1, j)$	Busy Type A	Idle Type B
Process 3, State $(i+1, j)$	Idle Type B	Busy Type A
Process 4, State $(i+1, j+1)$	Busy Type B	Busy Type A

Table 5: Marking scheme for units in the sample path argument for Equation (14).

In the first time period, eight transitions are possible:

1. A Type H arrival

2. A Type L arrival
3. A completion by Server I.
4. A completion by Server II.
5. A completion by an unmarked Type A server.
6. A completion by an unmarked Type B server.
7. A dummy transition due to uniformization.

For $k = 1, \dots, 8$, let C_k be the event in which transition k occurs. It suffices to show for each k that $\mathbb{E}[\Psi | C_k] \leq \mathbb{E}[\Psi' | C_k] + R_{HA} - R_{HB}$ for each k , and we proceed case-by-case.

Case 1 (Event C_1): Assume, for the time being, that either $i + 1 < N_A$ or $j + 1 < N_B$; we handle later the special case in which both inequalities are violated. If $i + 1 < N_A$, the analysis is straightforward. If $i + 1 = N_A$ but $j + 1 < N_B$, then Process 1 admits the call with a Type A server, while all other processes use Type B servers instead, and

$$\begin{aligned}
\mathbb{E}[\Psi | C_1] &= v_{n,\alpha}(i + 1, j) + R_{HA} - v_{n,\alpha}(i + 1, j + 1) - R_{HB} \\
&\leq v_{n,\alpha}(i + 1, j + 1) + R_{HA} - v_{n,\alpha}(i + 1, j + 2) - R_{HB} \\
&= \mathbb{E}[\Psi' | C_1] + R_{HA} - R_{HB},
\end{aligned}$$

where the inequality follows by the induction hypothesis on inequality (13).

Case 2 (Event C_2): As in Case 1, assume temporarily that either $i + 1 < N_A$ or $j + 1 < N_B$. If $j + 1 < N_B$, the analysis is straightforward. If $j + 2 = N_B$ but $i + 1 < N_A$, suppose that Processes 2 and 3 always admit the incoming Type L job (with Type B servers), regardless of the actions taken by the other processes. If Process 4 admits (with a Type A server), then

$$\mathbb{E}[\Psi | C_2] = v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j + 1) \leq v_{n,\alpha}(i + 1, j + 1) - v_{n,\alpha}(i + 2, j + 1) = \mathbb{E}[\Psi' | C_2]$$

where we have used the induction hypothesis on inequality (12). If Process 4 rejects instead, then

$$\begin{aligned}
\mathbb{E}[\Psi | C_2] &= v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j + 1) \\
&\leq v_{n,\alpha}(i + 1, j + 1) + R_L - v_{n,\alpha}(i + 1, j + 1) + R_{HA} - R_{HB} \\
&= \mathbb{E}[\Psi' | C_2] + R_{HA} - R_{HB},
\end{aligned}$$

where the inequality follows by combining both statements of Lemma 4.2.

Case (Event C_3): The four processes transition to states (i, j) , (i, j) , $(i + 1, j)$, and $(i + 1, j)$.

Coupling occurs, and it follows that $\mathbb{E}[\Theta | C_3] = \mathbb{E}[\Theta' | C_3] = 0$.

Case 4 (Event C_4): The four processes transition to states (i, j) , $(i + 1, j)$, (i, j) , and $(i + 1, j)$. Coupling occurs, and we have $\mathbb{E}[\Delta | C_4] = \mathbb{E}[\Delta' | C_4]$.

Case 5 (Event C_5): The four processes transition to states $(i - 1, j)$, (i, j) , (i, j) , and $(i, j + 1)$, respectively, and the induction hypothesis applies.

Case 6 (Event C_6): The four processes transition to states $(i, j - 1)$, $(i + 1, j - 1)$, $(i + 1, j - 1)$, and $(i + 1, j)$, respectively, and the induction hypothesis applies.

Case 7 (Event C_7): No process changes state, and the induction hypothesis applies.

It remains to show that $\mathbb{E}[\Psi | B_1] \leq \mathbb{E}[\Psi' | B_1]$ and $\mathbb{E}[\Psi | B_2] \leq \mathbb{E}[\Psi' | B_2]$ when $i = N_A$ and $j + 2 = N_B$. Once again, these inequalities may not hold in general, and so we again show that $\mathbb{E}[\Psi | B_1 \cup B_2] \leq \mathbb{E}[\Psi' | B_1 \cup B_2]$. Reasoning similar to that used in the inductive proof of inequality (13) yields

$$\begin{aligned}
\mathbb{E}[\Psi | B_1 \cup B_2] &= \frac{\lambda_H}{\lambda_H + \lambda_L} \left[v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) + R_{HA} - R_{HB} \right] \\
&\quad + \frac{\lambda_L}{\lambda_H + \lambda_L} \left[v_{n,\alpha}(i, j + 1) - v_{n,\alpha}(i + 1, j + 1) \right] \\
&\leq \frac{\lambda_H}{\lambda_H + \lambda_L} \left[v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) + R_{HA} - R_{HB} \right] \\
&\quad + \frac{\lambda_L}{\lambda_H + \lambda_L} \left[v_{n,\alpha}(i + 1, j) - v_{n,\alpha}(i + 1, j + 1) + R_{HA} - R_{HB} \right] \\
&\leq \frac{\lambda_H}{\lambda_H + \lambda_L} \left[R_{HB} + R_{HA} - R_{HB} \right] + \frac{\lambda_L}{\lambda_H + \lambda_L} \left[R_L + R_{HA} - R_{HB} \right] \\
&= \frac{\lambda_H}{\lambda_H + \lambda_L} \left[v_{n,\alpha}(i + 1, j + 1) + R_{HB} - v_{n,\alpha}(i + 1, j + 1) \right] \\
&\quad + \frac{\lambda_L}{\lambda_H + \lambda_L} \left[v_{n,\alpha}(i + 1, j + 1) + R_L - v_{n,\alpha}(i + 1, j + 1) \right] + R_{HA} - R_{HB} \\
&= \mathbb{E}[\Psi' | B_1 \cup B_2] + R_{HA} - R_{HB},
\end{aligned}$$

where the first inequality follows by Lemma 4.2, and the second by Lemma E.1.