Supporting Material for the paper “Estimating the rate of defects under imperfect sampling inspection - a new approach”

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1 The problem

The paper considers the problem of estimating the defect rate λ per item in the population based on a sample of M items. Moreover, we assume that the number of defects in an item is Poisson-distributed with mean λ. Two independent imperfect inspectors exaimine each item, with inspector i ($i = 1, 2$) detecting each defect with probability p_i, independently of the second inspector.

The number of defects in item m ($1 \le m \le M$) is Poisson-distributed with mean λ,

$$N_m \sim Poisson(\lambda),$$

and by our assumption the random variables $N_1, N_2, \cdots, N_M$ are independent.

Let $R_{m,i}$ denote the number of defects detected by inspector i ($i = 1, 2$) in item m.

We assume that the defect rate λ, as well as the detection rates p_1, p_2, are not known to us.

In contrast with the well-known problem of Capture-Recapture here we *do not* have information regarding the number of defects found by *both* inspectors. So, the following notations will be useful in formulation of the problem.

Denote:

- $X_{m,i}$ is the number of defcts detected *only* by inspector i in item m.
- Y_m is the number of defects detected by *both* inspectors in item m.

Note that

$$R_{m,i} = X_{m,i} + Y_m, \quad i = 1, 2. \tag{1}$$

We claim that $\{X_{m,1}, X_{m,2}, Y_m\}_{m=1}^M$ are *independent* and Poisson distributed with

$$X_{m,1} \sim Poisson(\lambda p_1(1 - p_2)), \quad X_{m,2} \sim Poisson(\lambda p_2(1 - p_1)), \tag{2}$$

$$Y_m \sim Poisson(\lambda p_1 p_2). \tag{3}$$

These results follow from the decomposition property of Poisson process, where $p_1(1 - p_2)$ is the probability that each defect is detected only by inspector 1, $p_2(1 - p_1)$ is the probability that each defect is detected only by inspector 2, $p_1 p_2$ is the probability that each defect is detected by both inspectors.

Our problem is:

Given the values $\{r_{m,i}\}_{m=1}^M$ ($i = 1, 2$) of the random variables, $\{R_{m,i}\}_{m=1}^M$, generated by (1),(2),(3), with the independence assumption on the set $\{X_{m,1}, X_{m,2}, Y_m\}_{m=1}^M$, estimate λ, p_1, p_2.

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Note that $R_{m,1}, R_{m,2}$ are *dependent* but the sequence $\{(R_{m,1}, R_{m,2})\}_{m=1}^{M}$ are independent, and the distribution of $(R_{m,1}, R_{m,2})$ is the Bivariate-Poisson-Distribution (BPD). Our data $\{(r_{m,1}, r_{m,2})\}_{m=1}^{M}$ can thus be considered as M independent realizations of the BPD (R_1, R_2).

When convenient, we can remove the subscript m and refer to random variables R_1, R_2, which are given by

$$R_1 = X_1 + Y, \quad R_2 = X_2 + Y,$$

where (X_1, X_2, Y) are independent, with

$$X_1 \sim Poisson(\lambda p_1(1 - p_2)), \quad X_2 \sim Poisson(\lambda p_2(1 - p_1)), \tag{4}$$

$$Y \sim Poisson(\lambda p_1 p_2). \tag{5}$$

By composition of independent Poisson random variables,

$$R_i \sim Poisson(\lambda p_i), \quad i = 1, 2.$$

so that

$$E(R_i) = VAR(R_i) = \lambda p_i, \quad i = 1, 2.$$

Also, using independence, and by (5),

$$Cov(R_1, R_2) = Cov\,(X_1 + Y, X_2 + Y) = VAR(Y) = \lambda p_1 p_2.$$

The study of properties of the moment-type estimators below will be based on approximations for the mean and variance of a ratio of two random variables. The Taylor expansions for the ratio of two random variables R and S are given by [1]:

$$E\left(\frac{R}{S}\right) \approx \frac{E(R)}{E(S)} - \frac{COV(R,S)}{E^2(S)} + \frac{VAR(S) \cdot E(R)}{E^3(S)}, \tag{6}$$

$$VAR\left(\frac{R}{S}\right) \approx \frac{E^2(R)}{E^2(S)}\left[\frac{VAR(R)}{E^2(R)} - 2\frac{COV(R,S)}{E(R) \cdot E(S)} + \frac{VAR(S)}{E^2(S)}\right]. \tag{7}$$

2 Moment-type estimators

Define the **sample mean** for $E(R_i)$ and the **sample covariance** for $COV(R_1, R_2)$ as follows:

$$\bar{r}_i = \frac{1}{M}\sum_{m=1}^{M} r_{m,i}, \quad i = 1, 2$$

$$\hat{S}_{1,2} = \frac{1}{M-1}\sum_{m=1}^{M}(r_{m,1} - \bar{r}_1)(r_{m,2} - \bar{r}_2).$$

The moment-type estimators for p_1, p_2 and λ are given by

$$\hat{p}_1 = \frac{\hat{S}_{1,2}}{\bar{r}_2} = \frac{\frac{1}{M-1}\sum_{m=1}^{M}(r_{m,1} - \bar{r}_1)(r_{m,2} - \bar{r}_2)}{\frac{1}{M}\sum_{m=1}^{M} r_{m,2}}, \tag{8}$$

$$\hat{p}_2 = \frac{\hat{S}_{1,2}}{\bar{r}_1} = \frac{\frac{1}{M-1}\sum_{m=1}^{M}(r_{m,1} - \bar{r}_1)(r_{m,2} - \bar{r}_2)}{\frac{1}{M}\sum_{m=1}^{M} r_{m,1}}, \tag{9}$$

$$\hat{\lambda} = \frac{\bar{r}_1 \cdot \bar{r}_2}{\hat{S}_{1,2}} = \frac{\left(\frac{1}{M}\sum_{m=1}^{M} r_{m,1}\right) \cdot \left(\frac{1}{M}\sum_{m=1}^{M} r_{m,2}\right)}{\frac{1}{M-1}\sum_{m=1}^{M}(r_{m,1} - \bar{r}_1)(r_{m,2} - \bar{r}_2)}. \tag{10}$$

2.1 Preliminary calculations for $(r_{m,1}, r_{m,2})$

The proof of Theorem 1 requires some preliminary calculations concerning our data $\{(r_{m,1}, r_{m,2})\}_{m=1}^{M}$ that are, as explained above, M independent realizations of the random variable (R_1, R_2).

$$\begin{aligned}
E(r_{m,i}) &= VAR(r_{m,i}) = \lambda p_i \qquad i = 1, 2,\\
E(r_{m,i}^2) &= VAR(r_{m,i}) + (E(r_{m,i}))^2 = \lambda p_i(\lambda p_i + 1) \quad i = 1, 2,\\
Cov(r_{m,1}, r_{m,2}) &= \lambda p_1 p_2,\\
E(r_{m,1} \cdot r_{m,2}) &= Cov(r_{m,1}, r_{m,2}) + E(r_{m,1})E(r_{m,2})\\
&= \lambda p_1 p_2 + \lambda^2 p_1 p_2 = \lambda(\lambda + 1)p_1 p_2.
\end{aligned}$$

If $m \neq n$ then

$$\begin{aligned}
E(r_{m,1} \cdot r_{n,2}) &= Cov(r_{m,1}, r_{n,2}) + E(r_{m,1})E(r_{n,2}) = \lambda^2 p_1 p_2\\
E(r_{m,1} \cdot r_{m,2} \cdot r_{n,2}) &= E(r_{m,1} \cdot r_{m,2})E(r_{n,2}) = \lambda^2(\lambda + 1)p_1 p_2^2.
\end{aligned}$$

Also,

$$\begin{aligned}
& E(r_{m,1} \cdot r_{m,2}^2) = E((X_{m,1} + Y_m)(X_{m,2} + Y_m)^2)\\
= \ & E(X_{m,1}X_{m,2}^2) + E(X_{m,1}Y_m^2) + 2E(X_{m,1}X_{m,2}Y_m) + E(Y_m X_{m,2}^2) + E(Y_m^3) + 2E(X_{m,2}Y_m^2)\\
= \ & E(X_{m,1})\left(E(X_{m,2}^2) + E(Y_m^2) + 2E(X_{m,2})E(Y_m)\right) + E(Y_m)E(X_{m,2}^2) + E(Y_m^3) + 2E(X_{m,2})E(Y_m^2)\\
= \ & \lambda p_1 p_2(\lambda^2 p_2 + (2p_2 + 1)\lambda + 1),
\end{aligned}$$

$$E(r_{m,1}^2 \cdot r_{m,2}) = \lambda p_1 p_2(\lambda^2 p_1 + (2p_1 + 1)\lambda + 1),$$

and

$$\begin{aligned}
& E\left(r_{m,1}^2 \cdot r_{m,2}^2\right) =\\
= \ & E\left(X_{m,1}^2 \cdot X_{m,2}^2\right) + 2E\left(X_{m,1}^2 \cdot X_{m,2} \cdot Y_m\right) + E\left(X_{m,1}^2 \cdot Y_m^2\right) + 2E\left(X_{m,1} \cdot X_{m,2}^2 \cdot Y_m\right)\\
+ \ & 4E\left(X_{m,1} \cdot X_{m,2} \cdot Y_m^2\right) + 2E\left(X_{m,1} \cdot Y_m^3\right) + E\left(X_{m,2}^2 \cdot Y_m^2\right) + 2E\left(X_{m,2} \cdot Y_m^3\right) + E\left(Y_m^4\right)\\
= \ & \lambda p_1 p_2\left[1 + \lambda^3 p_1 p_2 + \lambda^2(p_1 + p_2 + 4p_1 p_2) + \lambda(2p_1 + 2p_2 + 2p_1 p_2 + 1)\right].
\end{aligned}$$

2.2 Results concerning sample mean and sample covariance

Since $\{(r_{m,1}, r_{m,2})\}_{m=1}^{M}$ are M independent realizations of the random variable (R_1, R_2)

$$E(\bar{r}_i) = \lambda p_i, \ \ VAR(\bar{r}_i) = \frac{\lambda p_i}{M} \qquad i = 1, 2,$$

$$E(r_{m,1} \cdot \bar{r}_2) = E(r_{m,2} \cdot \bar{r}_1) = E(\bar{r}_1 \cdot \bar{r}_2) = \lambda p_1 p_2\left(\lambda + \frac{1}{M}\right).$$

Since

$$E((r_{m,1} - \bar{r}_1)(r_{m,2} - \bar{r}_2)) = \lambda p_1 p_2\left(1 - \frac{1}{M}\right),$$

the mean of sample covariance is

$$E(\hat{S}_{1,2}) = \lambda p_1 p_2.$$

To calculate the covariance between $\hat{S}_{1,2}$ and $\bar{r}_2$, we perform the following calculations:

For $m \neq n$,

$$\begin{aligned}
& E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_2) \\
= & \frac{1}{M} \sum_{j \neq m,n} E(r_{m,1} \cdot r_{n,2} \cdot r_{j,2}) + \frac{1}{M} E(r_{m,1} \cdot r_{n,2} \cdot r_{m,2}) + \frac{1}{M} E(r_{m,1} \cdot r_{n,2}^2) \\
= & \frac{M-2}{M} \lambda^3 p_1 p_2^2 + \frac{1}{M} \lambda^2 (\lambda + 1) p_1 p_2^2 + \frac{1}{M} \lambda^2 p_1 p_2 (\lambda p_2 + 1) \\
= & \lambda^3 p_1 p_2^2 + \frac{1}{M} \lambda^2 p_1 p_2 \left(p_2 + 1\right),
\end{aligned}$$

and similarly

$$E(r_{m,2} \cdot r_{n,2} \cdot \bar{r}_1) = \lambda^3 p_1 p_2^2 + \frac{2}{M} \lambda^2 p_1 p_2^2.$$

We now calculate $E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_2)$

$$E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_2) = \lambda^2 (\lambda + 1) p_1 p_2^2 + \frac{1}{M} \lambda^2 p_1 p_2^2 + \frac{1}{M} \lambda p_1 p_2 (\lambda + 1),$$

and

$$\begin{aligned}
& E(\bar{r}_1 \cdot \bar{r}_2 \cdot r_{n,2}) = \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1}^{M} E(r_{i,1} \cdot r_{j,2} \cdot r_{n,2}) \\
= & \frac{1}{M^2} \sum_{i \neq n} \sum_{j \neq n} E(r_{i,1} \cdot r_{j,2} \cdot r_{n,2}) + \frac{1}{M^2} \sum_{j \neq n}^{M} E(r_{n,1} \cdot r_{j,2} \cdot r_{n,2}) + \frac{1}{M^2} \sum_{i \neq n} E(r_{i,1} \cdot r_{n,2}^2) + \frac{1}{M^2} E(r_{n,1} \cdot r_{n,2}^2) \\
= & \frac{1}{M^2} E(r_{n,2}) \sum_{i \neq n} \sum_{j \neq n} E(r_{i,1} \cdot r_{j,2}) + \frac{1}{M^2} E(r_{n,1} \cdot r_{n,2}) \sum_{j \neq n}^{M} E(r_{j,2}) + \frac{1}{M^2} E(r_{n,2}^2) \sum_{i \neq n} E(r_{i,1}) + \frac{1}{M^2} E(r_{n,1} \cdot r_{n,2}^2) \\
= & \frac{1}{M^2} E(r_{n,2}) \cdot \sum_{i \neq n} \sum_{j \neq n,i} E(r_{i,1}) E(r_{j,2}) + \frac{1}{M^2} E(r_{n,2}) \cdot \sum_{i \neq n} E(r_{i,1} \cdot r_{i,2}) \\
+ & \frac{1}{M^2} E(r_{n,1} \cdot r_{n,2}) \sum_{j \neq n}^{M} E(r_{j,2}) + \frac{1}{M^2} E(r_{n,2}^2) \sum_{i \neq n} E(r_{i,1}) + \frac{1}{M^2} E(r_{n,1} \cdot r_{n,2}^2) \\
= & \frac{(M-1)(M-2)}{M^2} E(r_{m,1}) E^2(r_{m,2}) + \frac{M-1}{M^2} E(r_{m,2}) E(r_{m,1} \cdot r_{m,2}) \\
+ & \frac{M-1}{M^2} E(r_{m,1} \cdot r_{m,2}) E(r_{m,2}) + \frac{M-1}{M^2} E(r_{m,1}) E(r_{m,1}^2) + \frac{1}{m^2} E(r_{m,1} \cdot r_{m,2}^2) \\
= & \lambda^3 p_1 p_2^2 + \frac{1}{M} \lambda^2 p_1 p_2 \left(2p_2 + 1\right) + \frac{1}{M^2} \lambda p_1 p_2.
\end{aligned}$$

To make the derivations somewhat simpler, in some of the following calculations the interim result will be given in terms of the moments, without substituting the values of these moments in terms of the parameters p_1, p_2 and λ until the final stage.

We have

$$
\begin{aligned}
& E\left(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2\right) \\
= & \frac{1}{M^2} \sum_{m'=1}^{M} \sum_{n'=1}^{M} E(r_{m,1} \cdot r_{m,2} \cdot r_{m',1} \cdot r_{n',2}) \\
= & \frac{1}{M^2} \sum_{m' \neq m} \sum_{n' \neq m,m'} E(r_{m,1} \cdot r_{m,2} \cdot r_{m',1} \cdot r_{n',2}) + \frac{1}{M^2} \sum_{m' \neq m} E(r_{m,1} \cdot r_{m,2} \cdot r_{m',1} \cdot r_{m',2}) \\
+ & \frac{1}{M^2} \sum_{n' \neq m} E(r_{m,1}^2 \cdot r_{m,2} \cdot r_{n',2}) + \frac{1}{M^2} \sum_{m' \neq m} E(r_{m,1} \cdot r_{m,2}^2 \cdot r_{m',1}) + \frac{1}{M^2} E(r_{m,1}^2 \cdot r_{m,2}^2) \\
= & \frac{(M-1)(M-2)}{M^2} \cdot E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + \frac{M-1}{M^2} \cdot E^2(r_{m,1} \cdot r_{m,2}) \\
+ & \frac{M-1}{M^2} \cdot E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) + \frac{M-1}{M^2} \cdot E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) + \frac{1}{M^2} E(r_{m,1}^2 \cdot r_{m,2}^2) \\
= & \frac{M-3}{M} \cdot E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + \frac{1}{M} \cdot E^2(r_{m,1} \cdot r_{m,2}) \\
+ & \frac{1}{M} \cdot E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) + \frac{1}{M} \cdot E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) + O\left(\frac{1}{M^2}\right)
\end{aligned}
$$

$$
E\left(r_{m,1}^2 \cdot r_{m,2} \cdot \bar{r}_2\right) = \frac{1}{M} \sum_{j=1}^{M} E(r_{m,1}^2 \cdot r_{m,2} \cdot r_{j,2}) = \frac{M-1}{M} \cdot E(r_{m,2}) \cdot E\left(r_{m,1}^2 \cdot r_{m,2}\right) + \frac{1}{M} E\left(r_{m,1}^2 \cdot r_{m,2}^2\right),
$$

$$
E\left(r_{m,1} \cdot r_{m,2}^2 \cdot \bar{r}_1\right) = \frac{1}{M} \sum_{j=1}^{M} E(r_{m,1} \cdot r_{j,1} \cdot r_{m,2}^2) = \frac{M-1}{M} \cdot E(r_{m,1}) \cdot E(r_{m,1} \cdot r_{m,2}^2) + \frac{1}{M} \cdot E(r_{m,1}^2 \cdot r_{m,2}^2),
$$

$$
E(r_{m,2}^2 \cdot \bar{r}_1) = \frac{1}{M} \sum_{j \neq m} E(r_{j,1}) E(r_{m,2}^2) + \frac{1}{M} E(r_{m,1} \cdot r_{m,2}^2) = \lambda^2 p_1 p_2 (\lambda p_2 + 1) + \frac{1}{M} \lambda p_1 p_2 (2\lambda p_2 + 1),
$$

$$
\begin{aligned}
& E\left(r_{m,1}^2 \cdot \bar{r}_2^2\right) = \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1}^{M} E\left(r_{m,1}^2 \cdot r_{i,2} \cdot r_{j,2}\right) \\
= & \frac{1}{M^2} \sum_{i \neq m} \sum_{j \neq m,i} E\left(r_{m,1}^2\right) \cdot E\left(r_{i,2} \cdot r_{j,2}\right) + \frac{1}{M^2} \sum_{i \neq m} E\left(r_{m,1}^2\right) \cdot E\left(r_{i,2}^2\right) + \frac{1}{M^2} \sum_{j \neq m} E\left(r_{m,1}^2 \cdot r_{m,2}\right) \cdot E\left(r_{j,2}\right) \\
+ & \frac{1}{M^2} \sum_{i \neq m} E\left(r_{m,1}^2 \cdot r_{m,2}\right) \cdot E\left(r_{i,2}\right) + \frac{1}{M^2} E\left(r_{m,1}^2 \cdot r_{m,2}^2\right) \\
= & \frac{(M-1)(M-2)}{M^2} E\left(r_{m,1}^2\right) \cdot E^2(r_{m,2}) + \frac{M-1}{M^2} E\left(r_{m,1}^2\right) \cdot E\left(r_{m,2}^2\right) \\
+ & 2 \cdot \frac{M-1}{M^2} E\left(r_{m,1}^2 \cdot r_{m,2}\right) \cdot E\left(r_{m,2}\right) + \frac{1}{M^2} E\left(r_{m,1}^2 \cdot r_{m,2}^2\right) \\
= & \frac{M-3}{M} E\left(r_{m,1}^2\right) \cdot E^2(r_{m,2}) + \frac{1}{M} E\left(r_{m,1}^2\right) \cdot E\left(r_{m,2}^2\right) + 2 \cdot \frac{1}{M} E\left(r_{m,1}^2 \cdot r_{m,2}\right) \cdot E\left(r_{m,2}\right) + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

By symmetry

$$
\begin{aligned}
E\left(r_{m,2}^2 \cdot \bar{r}_1^2\right) &= \frac{M-3}{M} E\left(r_{m,2}^2\right) \cdot E^2(r_{m,1}) + \frac{1}{M} E\left(r_{m,2}^2\right) \cdot E\left(r_{m,1}^2\right) \\
&+ 2 \cdot \frac{1}{M} E\left(r_{m,2}^2 \cdot r_{m,1}\right) \cdot E\left(r_{m,1}\right) + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

To calculate the expectation of $(r_{m,1}\cdot\bar{r}_1\cdot\bar{r}_2^2)$ we first simplify some terms:

$$
\begin{aligned}
& E\left(r_{m,1}\cdot\bar{r}_1\cdot\bar{r}_2^2\right) = M^{-3}\sum_{n=1}^{M}\sum_{n'=1}^{M}\sum_{m'=1}^{M} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}\cdot r_{m',2}) \\
= \;& M^{-3}\sum_{n\neq m}\sum_{n'=1}^{M}\sum_{n'=1}^{M} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}\cdot r_{m',2}) + M^{-3}\sum_{n'=1}^{M}\sum_{n'=1}^{M} E(r_{m,1}^2\cdot r_{n',2}\cdot r_{m',2}) \\
= \;& M^{-3}\sum_{n\neq m}\sum_{n'\neq n,m}\sum_{n'=1}^{M} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}\cdot r_{m',2}) + M^{-3}\sum_{n\neq m}\sum_{m'=1}^{M} E(r_{m,1}\cdot r_{n,1}\cdot r_{n,2}\cdot r_{m',2}) \\
+ \;& M^{-3}\sum_{n\neq m}\sum_{m'=1}^{M} E(r_{m,1}\cdot r_{n,1}\cdot r_{m,2}\cdot r_{m',2}) + M^{-3}\sum_{n'\neq m}\sum_{m'\neq m} E(r_{m,1}^2\cdot r_{n',2}\cdot r_{m',2}) \\
+ \;& M^{-3}\sum_{m'\neq m} E(r_{m,1}^2\cdot r_{m,2}\cdot r_{m',2}) + M^{-3}\sum_{n'\neq m} E(r_{m,1}^2\cdot r_{m,2}\cdot r_{n',2}) + M^{-3}E(r_{m,1}^2\cdot r_{m,2}^2) \\
= \;& M^{-3}\sum_{n\neq m}\sum_{n'\neq n,m}\sum_{m'\neq n',n,m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}\cdot r_{m',2}) + M^{-3}\sum_{n\neq m}\sum_{n'\neq n,m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}^2) + \\
+ \;& M^{-3}\sum_{n\neq m}\sum_{n'\neq n,m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}\cdot r_{n,2}) + M^{-3}\sum_{n\neq m}\sum_{n'\neq n,m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n',2}\cdot r_{m,2}) \\
+ \;& M^{-3}\sum_{n\neq m}\sum_{m'\neq n,m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n,2}\cdot r_{m',2}) + M^{-3}\sum_{n\neq m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n,2}^2) \\
+ \;& M^{-3}\sum_{n\neq m} E(r_{m,1}\cdot r_{n,1}\cdot r_{n,2}\cdot r_{m,2}) + M^{-3}\sum_{n\neq m}\sum_{m'\neq n,m} E(r_{m,1}\cdot r_{n,1}\cdot r_{m,2}\cdot r_{m',2}) \\
+ \;& M^{-3}\sum_{n\neq m} E(r_{m,1}\cdot r_{n,1}\cdot r_{m,2}\cdot r_{n,2}) + M^{-3}\sum_{n\neq m} E(r_{m,1}\cdot r_{n,1}\cdot r_{m,2}^2) \\
+ \;& M^{-3}\sum_{n'\neq m}\sum_{m'\neq m,n'} E(r_{m,1}^2\cdot r_{n',2}\cdot r_{m',2}) + M^{-3}\sum_{n'\neq m} E(r_{m,1}^2\cdot r_{n',2}\cdot r_{m,2}) \\
+ \;& M^{-3}\sum_{n'\neq m} E(r_{m,1}^2\cdot r_{n',2}^2) + \sum_{m'\neq m} E(r_{m,1}^2\cdot r_{m,2}\cdot r_{m',2}) + M^{-3}\sum_{n'\neq m} E(r_{m,1}^2\cdot r_{m,2}\cdot r_{n',2}) + M^{-3}E(r_{m,1}^2\cdot r_{m,2}^2),
\end{aligned}
$$

So,

$$
\begin{aligned}
& E\left(r_{m,1}\cdot\bar{r}_1\cdot\bar{r}_2^2\right) = \frac{(M-1)(M-2)(M-3)}{M^3}\cdot E^2(r_{m,1})\cdot E^2(r_{m,2}) \\
+ \;& \frac{(M-1)(M-2)}{M^3}\cdot E^2(r_{m,1})\cdot E(r_{m,2}^2) + \frac{(M-1)(M-2)}{M^3}\cdot E(r_{m,1}^2)\cdot E^2(r_{m,2}) \\
+ \;& 4\cdot\frac{(M-1)(M-2)}{M^3}\cdot E(r_{m,1})\cdot E(r_{m,2})\cdot E(r_{m,1}\cdot r_{m,2}) \\
+ \;& 2\cdot\frac{M-1}{M^3}\cdot E(r_{m,1})\cdot E(r_{m,1}\cdot r_{m,2}^2) + 2\cdot\frac{M-1}{M^3}\cdot E^2(r_{m,1}\cdot r_{m,2}) \\
+ \;& 3\cdot\frac{M-1}{M^3}\cdot E(r_{m,1}^2\cdot r_{m,2})\cdot E(r_{m,2}) + \frac{M-1}{M^3}\cdot E(r_{m,1}^2)\cdot E(r_{m,2}^2) + \frac{1}{M^3}\cdot E(r_{m,1}^2\cdot r_{m,2}^2) \\
= \;& \frac{M-6}{M}\cdot E^2(r_{m,1})\cdot E^2(r_{m,2}) + \frac{1}{M}\cdot E^2(r_{m,1})\cdot E(r_{m,2}^2) + \frac{1}{M}\cdot E(r_{m,1}^2)\cdot E^2(r_{m,2}) \\
+ \;& 4\cdot\frac{1}{M}\cdot E(r_{m,1})\cdot E(r_{m,2})\cdot E(r_{m,1}\cdot r_{m,2}) + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

By symmetry:

$$
\begin{aligned}
& E\left(r_{m,2} \cdot \bar{r}_1^2 \cdot \bar{r}_2\right) = \frac{(M-1)(M-2)(M-3)}{M^3} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) \\
+ \quad & \frac{(M-1)(M-2)}{M^3} \cdot E^2(r_{m,1}) \cdot E(r_{m,2}^2) + \frac{(M-1)(M-2)}{M^3} \cdot E(r_{m,1}^2) \cdot E^2(r_{m,2}) \\
+ \quad & 4 \cdot \frac{(M-1)(M-2)}{M^3} \cdot E(r_{m,1}) \cdot E(r_{m,2}) \cdot E(r_{m,1} \cdot r_{m,2}) \\
+ \quad & 2 \cdot \frac{M-1}{M^3} \cdot E(r_{m,2}) \cdot E(r_{m,2} \cdot r_{m,1}^2) + 2 \cdot \frac{M-1}{M^3} \cdot E^2(r_{m,1} \cdot r_{m,2}) \\
+ \quad & 3 \cdot \frac{M-1}{M^3} \cdot E(r_{m,2}^2 \cdot r_{m,1}) \cdot E(r_{m,1}) + \frac{M-1}{M^3} \cdot E(r_{m,1}^2) \cdot E(r_{m,2}^2) + \frac{1}{M^3} \cdot E(r_{m,1}^2 \cdot r_{m,2}^2) \\
= \quad & \frac{M-6}{M} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) + \frac{1}{M} \cdot E^2(r_{m,1}) \cdot E(r_{m,2}^2) + \frac{1}{M} \cdot E(r_{m,1}^2) \cdot E^2(r_{m,2}) \\
+ \quad & 4 \cdot \frac{1}{M} \cdot E(r_{m,1}) \cdot E(r_{m,2}) \cdot E(r_{m,1} \cdot r_{m,2}) + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

We now calculate

$$
\begin{aligned}
& E\left(\bar{r}_1^2 \cdot \bar{r}_2^2\right) = M^{-4} E\left(\left(\sum_{m=1}^{M} r_{m,1}\right)^2 \left(\sum_{n=1}^{M} r_{n,2}\right)^2\right) \\
= \quad & M^{-4} E\left(\left(\sum_{m=1}^{M} \sum_{m'=1}^{M} r_{m,1} r_{m',1}\right) \left(\sum_{n=1}^{M} \sum_{n'=1}^{M} r_{n,2} r_{n',2}\right)\right) = M^{-4} \sum_{m=1}^{M} \sum_{m'=1}^{M} \sum_{n=1}^{M} \sum_{n'=1}^{M} E\left(r_{m,1} r_{m',1} r_{n,2} r_{n',2}\right) \\
= \quad & M^{-4} \sum_{m=1}^{M} \sum_{n' \neq n} \sum_{n=1}^{M} \sum_{n'=1}^{M} E\left(r_{m,1} r_{m',1} r_{n,2} r_{n',2}\right) + M^{-4} \sum_{m=1}^{M} \sum_{n=1}^{M} \sum_{n'=1}^{M} E\left(r_{m,1}^2 r_{n,2} r_{n',2}\right) \\
= \quad & M^{-4} \sum_{m=1}^{M} \sum_{m' \neq m} \sum_{n \neq m,m'} \sum_{n'=1}^{M} E\left(r_{m,1} r_{m',1} r_{n,2} r_{n',2}\right) + M^{-4} \sum_{m=1}^{M} \sum_{n' \neq m} \sum_{n'=1}^{M} E\left(r_{m,1} r_{m',1} r_{m,2} r_{n',2}\right) \\
+ \quad & M^{-4} \sum_{m=1}^{M} \sum_{m' \neq m} \sum_{n'=1}^{M} E\left(r_{m,1} r_{m',1} r_{n',2} r_{n',2}\right) \\
+ \quad & M^{-4} \sum_{m=1}^{M} \sum_{n \neq m} \sum_{n'=1}^{M} E\left(r_{m,1}^2 r_{n,2} r_{n',2}\right) + M^{-4} \sum_{m=1}^{M} \sum_{n'=1}^{M} E\left(r_{m,1}^2 r_{m,2} r_{n',2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'}\sum_{n'\neq m,m',n} E\left(r_{m,1}r_{m',1}r_{n,2}r_{n',2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'} E\left(r_{m,1}r_{m',1}r_{n,2}r_{m,2}\right) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{n'\neq m}\sum_{n\neq m,m'} E\left(r_{m,1}r_{m',1}r_{m',2}r_{n',2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'} E\left(r_{m,1}r_{m',1}r_{n,2}^{2}\right) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n'\neq m,m'} E\left(r_{m,1}r_{m',1}r_{m,2}r_{n',2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E\left(r_{m,1}r_{m',1}r_{m,2}^{2}\right) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E\left(r_{m,1}r_{m',1}r_{m,2}r_{m',2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n'\neq m,m'} E\left(r_{m,1}r_{m',1}r_{m',2}r_{n',2}\right) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E\left(r_{m,1}r_{m',1}r_{m',2}r_{m,2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E\left(r_{m,1}r_{m',1}r_{m',2}^{2}\right) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{n\neq m}\sum_{n'\neq m,n} E\left(r_{m,1}^{2}r_{n,2}r_{n',2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{n\neq m} E\left(r_{m,1}^{2}r_{n,2}r_{m,2}\right) + M^{-4}\sum_{m=1}^{M}\sum_{n\neq m}^{M} E\left(r_{m,1}^{2}r_{n,2}^{2}\right) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{n'\neq m} E\left(r_{m,1}^{2}r_{m,2}r_{n',2}\right) + M^{-4}\sum_{m=1}^{M} E\left(r_{m,1}^{2}r_{m,2}^{2}\right), \\
= \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'}\sum_{n'\neq m,m',n} E(r_{m,1})E(r_{m',1})E(r_{n,2})E(r_{n',2})
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'} E(r_{m,1}r_{m,2})E(r_{m',1})E(r_{n,2}) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'} E(r_{m,1})E(r_{m',1}r_{m',2})E(r_{n',2}) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n\neq m,m'} E(r_{m,1})E(r_{m',1})E(r_{n,2}^{2}) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n'\neq m,m'} E(r_{m,1}r_{m,2})E(r_{m',1})E(r_{n',2}) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E(r_{m,1}r_{m,2}^{2})E(r_{m',1}) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E(r_{m,1}r_{m,2})E(r_{m',1}r_{m',2}) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m}\sum_{n'\neq m,m'} E(r_{m,1})E(r_{m',1}r_{m',2})E(r_{n',2}) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E(r_{m,1}r_{m,2})E(r_{m',1}r_{m',2}) + M^{-4}\sum_{m=1}^{M}\sum_{m'\neq m} E(r_{m,1})E(r_{m',1}r_{m',2}^{2}) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{n\neq m}\sum_{n'\neq m,n} E(r_{m,1}^{2})E(r_{n,2})E(r_{n',2}) + M^{-4}\sum_{m=1}^{M}\sum_{n\neq m} E(r_{m,1}^{2}r_{m,2})E(r_{n,2}) \\
+ \quad & M^{-4}\sum_{m=1}^{M}\sum_{n\neq m}^{M} E(r_{m,1}^{2})E(r_{n,2}^{2}) + M^{-4}\sum_{m=1}^{M}\sum_{n'\neq m} E(r_{m,1}^{2}r_{m,2})E(r_{n',2}) + M^{-4}\sum_{m=1}^{M} E(r_{m,1}^{2}r_{m,2}^{2}),
\end{aligned}
$$

So

$$
\begin{aligned}
& E\left(\bar{r}_1^2 \cdot \bar{r}_2^2\right) = \\
= \; & \frac{(M-1)(M-2)(M-3)}{M^3} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) + 4 \cdot \frac{(M-1)(M-2)}{M^3} \cdot E(r_{m,1} r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) \\
+ \; & \frac{(M-1)(M-2)}{M^3} E^2(r_{m,1}) E(r_{m,2}^2) + 2 \cdot \frac{M-1}{M^3} E(r_{m,1} r_{m,2}^2) E(r_{m,1}) + 2 \cdot \frac{M-1}{M^3} E^2(r_{m,1} r_{m,2}) \\
+ \; & \frac{(M-1)(M-2)}{M^3} E(r_{m,1}^2) E^2(r_{m,2}) + 2 \cdot \frac{M-1}{M^3} E(r_{m,1}^2 r_{m,2}) E(r_{m,2}) \\
+ \; & \frac{M-1}{M^3} E(r_{m,1}^2) E(r_{m,2}^2) + \frac{1}{M^3} E(r_{m,1}^2 r_{m,2}^2) \\
= \; & \frac{M-6}{M} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) + 4 \cdot \frac{1}{M} \cdot E(r_{m,1} r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + \frac{1}{M} E^2(r_{m,1}) E(r_{m,2}^2) \\
+ \; & \frac{1}{M} E(r_{m,1}^2) E^2(r_{m,2}) + O\left(\frac{1}{M^2}\right) \\
= \; & \lambda^4 p_1^2 p_2^2 + \frac{\lambda^3 p_1 p_2}{M} \left[4 p_1 p_2 + p_1 + p_2\right] + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

Therefore, the variance of $(\bar{r}_1 \cdot \bar{r}_2)$ is

$$
\begin{aligned}
& VAR(\bar{r}_1 \cdot \bar{r}_2) = E((\bar{r}_1)^2 \cdot (\bar{r}_2)^2) - E^2(\bar{r}_1 \cdot \bar{r}_2) \\
= \; & \lambda^4 p_1^2 p_2^2 + \frac{\lambda^3 p_1 p_2}{M}[4 p_1 p_2 + p_1 + p_2] - [\lambda^2 p_1 p_2 + \frac{1}{M} \lambda p_1 p_2]^2 + O\left(\frac{1}{M^2}\right) \\
= \; & \frac{\lambda^3 p_1 p_2}{M}[p_1 + p_2 + 2 p_1 p_2] + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

For $n \neq m$,

$$
\begin{aligned}
& E(r_{n,1} \cdot r_{m,1} \cdot r_{n,2} \cdot \bar{r}_2) = E(r_{n,1} \cdot r_{m,1} \cdot r_{m,2} \cdot \bar{r}_2) = \frac{1}{M} \sum_{m'=1}^{M} E(r_{n,1} \cdot r_{m,1} \cdot r_{m,2} \cdot r_{m',2}) \\
= \; & \frac{1}{M} \sum_{m' \neq n,m} E(r_{n,1}) E(r_{m,1} \cdot r_{m,2}) E(r_{m',2}) + \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) + \frac{1}{M} E(r_{n,1}) E(r_{m,1} \cdot r_{m,2}^2) \\
= \; & \frac{M-2}{M} E(r_{m,1}) E(r_{m,2}) E(r_{m,1} \cdot r_{m,2}) + \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) + \frac{1}{M} E(r_{m,1}) E(r_{m,1} \cdot r_{m,2}^2)
\end{aligned}
$$

and

$$
\begin{aligned}
& E(r_{m,2} \cdot r_{n,1} \cdot r_{n,2} \cdot \bar{r}_1) = E(r_{n,2} \cdot r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1) = \frac{1}{M} \sum_{m'=1}^{M} E(r_{n,2} \cdot r_{m,1} \cdot r_{m,2} \cdot r_{m',1}) \\
= \; & \frac{1}{M} \sum_{m' \neq n,m} E(r_{n,2}) E(r_{m,1} \cdot r_{m,2}) E(r_{m',1}) + \frac{1}{M} E(r_{m,2}) E(r_{m,1}^2 \cdot r_{m,2}) + \frac{1}{M} E^2(r_{n,2} \cdot r_{n,1}) \\
= \; & \frac{M-2}{M} E(r_{m,1}) E(r_{m,2}) E(r_{m,1} \cdot r_{m,2}) + \frac{1}{M} E(r_{m,2}) E(r_{m,1}^2 \cdot r_{m,2}) + \frac{1}{M} E^2(r_{m,2} \cdot r_{m,1}).
\end{aligned}
$$

Also, for $n \neq m$

$$
\begin{aligned}
& E(r_{m,1} \cdot r_{n,1} \cdot (\bar{r}_2)^2) = \frac{1}{M^2} \sum_{m'=1}^{M} \sum_{n'=1}^{M} E(r_{m,1} \cdot r_{n,1} \cdot r_{m',2} \cdot r_{n',2}) \\
= \quad & \frac{1}{M^2} \cdot \sum_{m' \neq m,n} \sum_{n' \neq m,n,m'} E(r_{m,1} \cdot r_{n,1} \cdot r_{m',2} \cdot r_{n',2}) + \frac{1}{M^2} \cdot \sum_{m' \neq m,n} E(r_{m,1} \cdot r_{n,1} \cdot r_{m',2} \cdot r_{m',2}) \\
+ \quad & \frac{1}{M^2} \cdot \sum_{m' \neq m,n} E(r_{m,1} \cdot r_{n,1} \cdot r_{m',2} \cdot r_{m,2}) + \frac{1}{M^2} \cdot \sum_{m' \neq m,n} E(r_{m,1} \cdot r_{n,1} \cdot r_{m',2} \cdot r_{n,2}) \\
+ \quad & \frac{1}{M^2} \cdot \sum_{n' \neq m,n} E(r_{m,1} \cdot r_{n,1} \cdot r_{m,2} \cdot r_{n',2}) + \frac{1}{M^2} \cdot \sum_{n' \neq m,n} E(r_{m,1} \cdot r_{n,1} \cdot r_{n,2} \cdot r_{n',2}) \\
+ \quad & \frac{1}{M^2} \cdot E(r_{m,1} \cdot r_{n,1} \cdot r_{m,2} \cdot r_{m,2}) + \frac{1}{M^2} \cdot E(r_{m,1} \cdot r_{n,1} \cdot r_{m,2} \cdot r_{n,2}) \\
+ \quad & \frac{1}{M^2} \cdot E(r_{m,1} \cdot r_{n,1} \cdot r_{n,2} \cdot r_{m,2}) + \frac{1}{M^2} \cdot E(r_{m,1} \cdot r_{n,1} \cdot r_{n,2}^2) \\
= \quad & \frac{(M-2)(M-3)}{M^2} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) + \frac{M-2}{M^2} \cdot E^2(r_{m,1}) \cdot E(r_{m,2}^2) \\
+ \quad & 4 \cdot \frac{M-2}{M^2} \cdot E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) \\
+ \quad & \frac{2}{M^2} \cdot E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) + \frac{2}{M^2} \cdot E^2(r_{m,1} \cdot r_{m,2}) \\
= \quad & \frac{M-5}{M} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) + \frac{1}{M} \cdot E^2(r_{m,1}) \cdot E(r_{m,2}^2) \\
+ \quad & 4 \cdot \frac{1}{M} \cdot E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + O\left(\frac{1}{M^2}\right),
\end{aligned}
$$

and by symmetry,

$$
\begin{aligned}
& E(r_{m,2} \cdot r_{n,2} \cdot (\bar{r}_1)^2) = \frac{(M-2)(M-3)}{M^2} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) \\
+ \quad & \frac{M-2}{M^2} \cdot E(r_{m,1}^2) \cdot E^2(r_{m,2}) + 4 \cdot \frac{M-2}{M^2} \cdot E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{M,2}) \\
+ \quad & \frac{2}{M^2} \cdot E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) + \frac{2}{M^2} \cdot E^2(r_{m,1} \cdot r_{m,2}) \\
= \quad & \frac{M-5}{M} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) + \frac{1}{M} \cdot E(r_{m,1}^2) \cdot E^2(r_{m,2}) \\
+ \quad & 4 \cdot \frac{1}{M} \cdot E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

Finally,

$$E(r_{n,1}\cdot r_{m,2}\cdot \bar r_1\cdot \bar r_2) = E(r_{m,1}\cdot r_{n,2}\cdot \bar r_1\cdot \bar r_2) = \frac{1}{M^2}\sum_{m'=1}^{M}\sum_{n'=1}^{M}E(r_{m,1}\cdot r_{n,2}\cdot r_{m',1}\cdot r_{n',2})$$

$$= \frac{1}{M^2}\sum_{m'\neq m,n}\sum_{m'\neq m,n,m'}E(r_{m,1}\cdot r_{n,2}\cdot r_{m',1}\cdot r_{n',2}) + \frac{1}{M^2}\sum_{m'\neq m,n}E(r_{m,1}\cdot r_{n,2}\cdot r_{m',1}\cdot r_{m',2})$$

$$+ \frac{1}{M^2}\sum_{n'\neq m,n}E(r_{m,1}\cdot r_{n,2}\cdot r_{m,1}\cdot r_{n',2}) + \frac{1}{M^2}\sum_{n'\neq m,n}E(r_{m,1}\cdot r_{n,2}\cdot r_{n,1}\cdot r_{n',2})$$

$$+ \frac{1}{M^2}\sum_{m'\neq m,n}E(r_{m,1}\cdot r_{n,2}\cdot r_{m',1}\cdot r_{m,2}) + \frac{1}{M^2}\sum_{m'\neq m,n}E(r_{m,1}\cdot r_{n,2}\cdot r_{m',1}\cdot r_{n,2})$$

$$+ \frac{1}{M^2}E(r_{m,1}\cdot r_{n,2}\cdot r_{m,1}\cdot r_{m,2}) + \frac{1}{M^2}E(r_{m,1}\cdot r_{n,2}\cdot r_{n,1}\cdot r_{n,2})$$

$$+ \frac{1}{M^2}E(r_{m,1}\cdot r_{n,2}\cdot r_{m,1}\cdot r_{n,2}) + \frac{1}{M^2}E(r_{m,1}\cdot r_{m,2}\cdot r_{n,1}\cdot r_{m,2})$$

$$= \frac{(M-2)(M-3)}{M^2}\cdot E^2(r_{m,1})\cdot E^2(r_{n,2}) + 2\cdot\frac{M-2}{M^2}\cdot E(r_{m,1})\cdot E(r_{m,2})\cdot E(r_{m,1}\cdot r_{m,2})$$

$$+ \frac{M-2}{M^2}\cdot E(r_{m,1}^2)\cdot E^2(r_{m,2}) + \frac{M-2}{M^2}\cdot E^2(r_{m,1})\cdot E(r_{m,2}^2)$$

$$+ \frac{M-2}{M^2}\cdot E(r_{m,1}\cdot r_{m,2})\cdot E(r_{m,1})\cdot E(r_{m,2})$$

$$+ \frac{1}{M^2}E(r_{m,1}\cdot r_{n,2}\cdot r_{m,1}\cdot r_{m,2}) + \frac{1}{M^2}E(r_{m,1}\cdot r_{n,2}\cdot r_{n,1}\cdot r_{n,2})$$

$$+ \frac{1}{M^2}E(r_{m,1}\cdot r_{n,2}\cdot r_{m,1}\cdot r_{n,2}) + \frac{1}{M^2}E(r_{m,1}\cdot r_{m,2}\cdot r_{n,1}\cdot r_{m,2})$$

$$= \frac{M-5}{M}\cdot E^2(r_{m,1})\cdot E^2(r_{n,2}) + 2\cdot\frac{1}{M}\cdot E(r_{m,1})\cdot E(r_{m,2})\cdot E(r_{m,1}\cdot r_{m,2})$$

$$+ \frac{1}{M}\cdot E(r_{m,1}^2)\cdot E^2(r_{m,2}) + \frac{1}{M}\cdot E^2(r_{m,1})\cdot E(r_{m,2}^2)$$

$$+ \frac{1}{M}\cdot E(r_{m,1}\cdot r_{m,2})\cdot E(r_{m,1})\cdot E(r_{m,2}) + O\left(\frac{1}{M^2}\right).$$

The expectation of the product between $\hat S_{1,2}$ and $\bar r_2$ equals to:

$$E(\hat S_{1,2}\cdot \bar r_2) = \frac{1}{M(M-1)}\sum_{m=1}^{M}\sum_{n=1}^{M}E\left((r_{m,1}-\bar r_1)(r_{m,2}-\bar r_2)\cdot r_{n,2}\right)$$

$$= \frac{1}{M(M-1)}\sum_{m=1}^{M}\sum_{n\neq m}E\left((r_{m,1}-\bar r_1)(r_{m,2}-\bar r_2)\cdot r_{n,2}\right) + \frac{1}{M(M-1)}\sum_{m=1}^{M}E\left((r_{m,1}-\bar r_1)(r_{m,2}-\bar r_2)\cdot r_{m,2}\right)$$

$$= \frac{1}{M(M-1)}\sum_{m=1}^{M}\sum_{n\neq m}\left[E(r_{m,1}\cdot r_{m,2}\cdot r_{n,2}) - E(r_{m,1}\cdot r_{n,2}\cdot \bar r_2) - E(r_{m,2}\cdot r_{n,2}\cdot \bar r_1) + E(\bar r_1\cdot \bar r_2\cdot r_{n,2})\right]$$

$$+ \frac{1}{M(M-1)}\sum_{m=1}^{M}\left[E(r_{m,1}\cdot r_{m,2}^2) - E(r_{m,1}\cdot r_{m,2}\cdot \bar r_2) - E(\bar r_1\cdot r_{m,2}^2) + E(r_{m,2}\cdot \bar r_1\cdot \bar r_2)\right]$$

$$= \lambda^2 p_1 p_2^2 + \frac{1}{M^2}\lambda p_1 p_2 - \frac{1}{M}\lambda^2 p_1 p_2^2$$

$$+ \frac{1}{M-1}\left[\lambda p_1 p_2(p_2\lambda+1) - \frac{1}{M}\lambda p_1 p_2\left(\lambda p_2+2\right) + \frac{1}{M^2}\lambda p_1 p_2\right],$$

and finally the covariance between $\hat{S}_{1,2}$ and $\bar{r}_2$ is given by:

$$\begin{aligned} & Cov(\hat{S}_{1,2}, \bar{r}_2) = E(\hat{S}_{1,2} \cdot \bar{r}_2) - E(\hat{S}_{1,2})E(\bar{r}_2) \\ = \ & \lambda^2 p_1 p_2^2 + \frac{1}{M^2}\lambda p_1 p_2 - \frac{1}{M}\lambda^2 p_1 p_2^2 \\ + \ & \frac{1}{M-1}\left[\lambda p_1 p_2 (p_2 \lambda + 1) - \frac{1}{M}\lambda p_1 p_2 (\lambda p_2 + 2) + \frac{1}{M^2}\lambda p_1 p_2\right] - \lambda p_1 p_2 \cdot \lambda p_2 \\ = \ & \frac{1}{M}\lambda p_1 p_2. \end{aligned}$$

In order to calculate the expectation of $(\bar{r}_1 \cdot \bar{r}_2 \cdot \hat{S}_{1,2})$ we make some preliminaries calculations: For $n \neq m, m' \neq m, n$

$$\begin{aligned} & E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot \bar{r}_2) = \frac{1}{M}\sum_{i=1}^{M} E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot r_{i,2}) \\ = \ & \frac{M-3}{M}E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot r_{i,2}) + \frac{1}{M}E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot r_{m,2}) \\ + \ & \frac{1}{M}E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot r_{n,2}) + \frac{1}{M}E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot r_{m',2}) \\ = \ & \frac{M-3}{M}E^2(r_{m,1}) \cdot E^2(r_{m,2}) + \frac{2}{M}E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) \\ + \ & \frac{1}{M}E(r_{m,2}^2) \cdot E^2(r_{m,1}), \end{aligned}$$

and by symmetry,

$$\begin{aligned} & E(r_{m,1} \cdot r_{n,2} \cdot r_{m',2} \cdot \bar{r}_1) = \\ = \ & \frac{M-3}{M}E^2(r_{m,1}) \cdot E^2(r_{m,2}) + \frac{2}{M}E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + \frac{1}{M}E(r_{m,1}^2) \cdot E^2(r_{m,2}). \end{aligned}$$

Therefore,

$$\begin{aligned} & E(r_{m,1} \cdot r_{n,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) = E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot r_{m',2}) \\ - \ & E(r_{m,1} \cdot r_{n,2} \cdot r_{m',1} \cdot \bar{r}_2) - E(r_{m,1} \cdot r_{n,2} \cdot r_{m',2} \cdot \bar{r}_1) + E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\ = \ & E(r_{m,1}) \cdot E(r_{m,2}) \cdot E(r_{m,1} \cdot r_{m,2}) - 2\frac{M-3}{M}E^2(r_{m,1}) \cdot E^2(r_{m,2}) \\ - \ & \frac{4}{M}E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) - \frac{1}{M}E^2(r_{m,1}) \cdot E(r_{m,2}^2) - \frac{1}{M}E(r_{m,1}^2) \cdot E^2(r_{m,2}) \\ + \ & E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2). \end{aligned}$$

For $n \neq m$

$$\begin{aligned} & E(r_{m,1} \cdot r_{n,2}^2 \cdot \bar{r}_1) = \frac{1}{M}\sum_{i=1}^{M} E(r_{m,1} \cdot r_{n,2}^2 \cdot r_{i,1}) \\ = \ & \frac{M-2}{M}E^2(r_{m,1}) \cdot E(r_{m,2}^2) + \frac{1}{M}E(r_{m,1}) \cdot E(r_{m,2}^2 \cdot r_{m,1}) + \frac{1}{M}E(r_{m,1}^2) \cdot E(r_{m,2}^2), \end{aligned}$$

and by symmetry

$$\begin{aligned} & E(r_{m,1}^2 \cdot r_{n,2} \cdot \bar{r}_2) \\ = \ & \frac{M-2}{M}E^2(r_{m,2}) \cdot E(r_{m,1}^2) + \frac{1}{M}E(r_{m,2}) \cdot E(r_{m,1}^2 \cdot r_{m,2}) + \frac{1}{M}E(r_{m,1}^2) \cdot E(r_{m,2}^2). \end{aligned}$$

Therefore,

$$
\begin{aligned}
& E(r_{m,1} \cdot r_{n,2} \cdot (r_{n,1} - \bar{r}_1) \cdot (r_{n,2} - \bar{r}_2)) = E(r_{m,1} \cdot r_{n,2}^2 \cdot r_{n,1}) \\
- \quad & E(r_{m,1} \cdot r_{n,2} \cdot r_{n,1} \cdot \bar{r}_2) - E(r_{m,1} \cdot r_{n,2}^2 \cdot \bar{r}_1) + E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & E(r_{m,1}) \cdot E(r_{m,1} \cdot r_{m,2}^2) \\
- \quad & \frac{M-2}{M} E(r_{M,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) - \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) - \frac{1}{M} E(r_{m,1}) \cdot E(r_{m,2}^2 \cdot r_{m,1}) \\
- \quad & \frac{M-2}{M} E^2(r_{m,1}) \cdot E(r_{m,2}^2) - \frac{1}{M} E(r_{m,1}^2) \cdot E(r_{m,2}^2) - \frac{1}{M} E(r_{m,1}) \cdot E(r_{m,2}^2 \cdot r_{m,1}) + \\
+ \quad & E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & \frac{M-2}{M} E(r_{m,1}) \cdot E(r_{m,1} \cdot r_{m,2}^2) \\
- \quad & \frac{M-2}{M} E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) - \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) \\
- \quad & \frac{M-2}{M} E^2(r_{m,1}) \cdot E(r_{m,2}^2) - \frac{1}{M} E(r_{m,1}^2) \cdot E(r_{m,2}^2) \\
+ \quad & E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2).
\end{aligned}
$$

For $n \neq m$,

$$
\begin{aligned}
& E\left(r_{m,1} \cdot r_{n,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)\right) = E(r_{m,1}^2 \cdot r_{m,2} \cdot r_{n,2}) \\
- \quad & E(r_{m,1}^2 \cdot r_{n,2} \cdot \bar{r}_2) - E(r_{m,1} \cdot r_{m,2} \cdot r_{n,2} \cdot \bar{r}_1) + E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) \\
- \quad & \frac{M-2}{M} E(r_{m,1}^2) \cdot E^2(r_{m,2}) - \frac{1}{M} E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) - \frac{1}{M} E(r_{m,1}^2) \cdot E(r_{m,2}^2) \\
- \quad & \frac{M-2}{M} E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) - \frac{1}{M} E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) - \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) \\
+ \quad & E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & \frac{M-2}{M} E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) - \frac{M-2}{M} E(r_{m,1}^2) \cdot E^2(r_{m,2}) - \frac{1}{M} E(r_{m,1}^2) \cdot E(r_{m,2}^2) \\
- \quad & \frac{M-2}{M} E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) - \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) \\
+ \quad & E(r_{m,1} \cdot r_{n,2} \cdot \bar{r}_1 \cdot \bar{r}_2).
\end{aligned}
$$

For $m' \neq m$,

$$
\begin{aligned}
& E(r_{m,1} \cdot r_{m,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) = E(r_{m,1} \cdot r_{m,2} \cdot r_{m',1} \cdot r_{m',2}) \\
- \quad & E(r_{m,1} \cdot r_{m,2} \cdot r_{m',1} \cdot \bar{r}_2) - E(r_{m,1} \cdot r_{m,2} \cdot r_{m',2} \cdot \bar{r}_1) + E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & \frac{M-2}{M} E^2(r_{m,1} \cdot r_{m,2}) - 2\frac{M-2}{M} E(r_{m,1} \cdot r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) \\
- \quad & \frac{1}{M} E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) - \frac{1}{M} E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) \\
+ \quad & E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2),
\end{aligned}
$$

and

$$
\begin{aligned}
& E(r_{m,1} \cdot r_{m,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) = E(r_{m,1}^2 \cdot r_{m,2}^2) \\
- \quad & E(r_{m,1}^2 \cdot r_{m,2} \cdot \bar{r}_2) - E(r_{m,1} \cdot r_{m,2}^2 \cdot \bar{r}_1) + E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & E(r_{m,1}^2 \cdot r_{m,2}^2) - \frac{M-1}{M} E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) \\
- \quad & 2\frac{1}{M} E(r_{m,1}^2 \cdot r_{m,2}^2) - \frac{M-1}{M} E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) + E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2) \\
= \quad & \frac{M-2}{M} E(r_{m,1}^2 \cdot r_{m,2}^2) \\
- \quad & \frac{M-1}{M} E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) - \frac{M-1}{M} E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) \\
+ \quad & E(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2).
\end{aligned}
$$

Thus the expectation of $(\bar{r}_1 \cdot \bar{r}_2 \cdot \hat{S}_{1,2})$ is

$$E(\bar{r}_1 \cdot \bar{r}_2 \cdot \hat{S}_{1,2}) = \frac{1}{M^2(M-1)} \sum_{m=1}^{M} \sum_{n=1}^{M} \sum_{m'=1}^{M} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2))$$

$$\begin{aligned}
= \quad & \frac{1}{M^2(M-1)} \sum_{m=1}^{M} \sum_{n \neq m} \sum_{m' \neq m} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) + \\
+ \quad & \frac{1}{M^2(M-1)} \sum_{m=1}^{M} \sum_{n \neq m} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) \\
+ \quad & \frac{1}{M^2(M-1)} \sum_{m=1}^{M} \sum_{m' \neq m} E(r_{m,1} \cdot r_{m,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) + \\
+ \quad & \frac{1}{M^2(M-1)} \sum_{m=1}^{M} E(r_{m,1} \cdot r_{m,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) = \\
= \quad & \frac{1}{M(M-1)} \sum_{n \neq m} \sum_{m' \neq m,n} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) + \\
+ \quad & \frac{1}{M(M-1)} \sum_{n \neq m} E(r_{m,1} \cdot r_{n,2} \cdot (r_{n,1} - \bar{r}_1) \cdot (r_{n,2} - \bar{r}_2)) + \\
+ \quad & \frac{1}{M(M-1)} \sum_{n \neq m} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) + \\
+ \quad & \frac{1}{M(M-1)} \sum_{m' \neq m} E(r_{m,1} \cdot r_{m,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) + \\
+ \quad & \frac{1}{M(M-1)} E(r_{m,1} \cdot r_{m,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) \\
= \quad & \frac{M-2}{M} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) + \frac{1}{M} E(r_{m,1} \cdot r_{n,2} \cdot (r_{n,1} - \bar{r}_1) \cdot (r_{n,2} - \bar{r}_2)) \\
+ \quad & \frac{1}{M} E(r_{m,1} \cdot r_{n,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) + \frac{1}{M} E(r_{m,1} \cdot r_{m,2} \cdot (r_{m',1} - \bar{r}_1) \cdot (r_{m',2} - \bar{r}_2)) \\
+ \quad & \frac{1}{M(M-1)} E(r_{m,1} \cdot r_{m,2} \cdot (r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)) \\
= \quad & \frac{M-6}{M} E(r_{m,1}) \cdot E(r_{m,2}) \cdot E(r_{m,1} \cdot r_{m,2}) - \frac{M-5}{M} E^2(r_{m,1}) \cdot E^2(r_{m,2}) \\
- \quad & \frac{1}{M} E^2(r_{m,1}) \cdot E(r_{m,2}^2) - \frac{1}{M} E(r_{m,1}^2) \cdot E^2(r_{m,2}) \\
+ \quad & \frac{1}{M} E(r_{m,1}) \cdot E(r_{m,1} \cdot r_{m,2}^2) + \frac{1}{M} E(r_{m,2}) \cdot E(r_{m,1}^2 \cdot r_{m,2}) + \frac{1}{M} E^2(r_{m,1} \cdot r_{m,2}) + O\left(\frac{1}{M^2}\right) \\
= \quad & \frac{M-6}{M} \lambda^2 p_1 p_2 \cdot \lambda(\lambda+1) p_1 p_2 - \frac{M-5}{M} \lambda^4 p_1^2 p_2^2 - \frac{1}{M} \lambda^2 p_1^2 \cdot \lambda p_2 (\lambda p_2 + 1) - \frac{1}{M} \lambda^2 p_2^2 \cdot \lambda p_1 (\lambda p_1 + 1) \\
+ \quad & \frac{1}{M} \lambda p_1 \cdot \lambda p_1 p_2 (p_2 \lambda^2 + (2p_2+1)\lambda + 1) \\
+ \quad & \frac{1}{M} \lambda p_2 \cdot \lambda p_1 p_2 (p_1 \lambda^2 + (2p_1+1)\lambda + 1) + \frac{1}{M} (\lambda(\lambda+1) p_1 p_2)^2 + O\left(\frac{1}{M^2}\right) \\
= \quad & p_1^2 p_2^2 \cdot \lambda^3 + \frac{1}{M} p_1 p_2 \lambda^2 (p_1 + p_2 + p_1 p_2) + O\left(\frac{1}{M^2}\right).
\end{aligned}$$

Hence, the covariance between $(\bar{r}_1 \cdot \bar{r}_2)$ and $\hat{S}_{1,2}$ is given by:

$$\begin{aligned}
& COV(\bar{r}_1 \cdot \bar{r}_2, \hat{S}_{1,2}) = E(\bar{r}_1 \cdot \bar{r}_2 \cdot \hat{S}_{1,2}) - E(\bar{r}_1 \cdot \bar{r}_2) \cdot E(\hat{S}_{1,2}) \\
= \quad & p_1^2 p_2^2 \cdot \lambda^3 + \frac{1}{M} p_1 p_2 \lambda^2 \left(p_1 + p_2 + p_1 p_2\right) \\
- \quad & \left(\lambda^2 p_1 p_2 + \frac{1}{M} \lambda p_1 p_2\right) \lambda p_1 p_2 + O\left(\frac{1}{M^2}\right) \\
= \quad & \frac{1}{M} p_1 p_2 \lambda^2 \left(p_1 + p_2\right) + O\left(\frac{1}{M^2}\right).
\end{aligned}$$

We now calculate the variance of $\hat{S}_{1,2}$.

$$\begin{aligned}
& VAR\left(\hat{S}_{1,2}\right) = \frac{1}{(M-1)^2} \sum_{m=1}^{M} \sum_{n=1}^{M} COV\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right), \left(r_{n,1} - \bar{r}_1\right) \cdot \left(r_{n,2} - \bar{r}_2\right)\right) \\
= \quad & \frac{1}{(M-1)^2} \sum_{m=1}^{M} \sum_{n \neq m} COV\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right), \left(r_{n,1} - \bar{r}_1\right) \cdot \left(r_{n,2} - \bar{r}_2\right)\right) \\
+ \quad & \frac{1}{(M-1)^2} \sum_{m=1}^{M} VAR\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right)\right) \\
= \quad & \frac{1}{(M-1)^2} \sum_{m=1}^{M} \sum_{n \neq m} COV\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right), \left(r_{n,1} - \bar{r}_1\right) \cdot \left(r_{n,2} - \bar{r}_2\right)\right) \\
+ \quad & \frac{M}{(M-1)^2} \cdot VAR\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right)\right).
\end{aligned}$$

We begin with the second element above. The variance of $(r_{m,1} - \bar{r}_1) \cdot (r_{m,2} - \bar{r}_2)$ is

$$\begin{aligned}
& VAR\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right)\right) = E\left(\left(r_{m,1} - \bar{r}_1\right)^2 \cdot \left(r_{m,2} - \bar{r}_2\right)^2\right) - E^2\left(\left(r_{m,1} - \bar{r}_1\right) \cdot \left(r_{m,2} - \bar{r}_2\right)\right) \\
= \quad & E\left(r_{m,1}^2 \cdot r_{m,2}^2\right) + E\left((\bar{r}_1)^2 \cdot (\bar{r}_2)^2\right) + 4E\left(r_{m,1} \cdot r_{m,2} \cdot \bar{r}_1 \cdot \bar{r}_2\right) \\
- \quad & 2E\left(r_{m,1}^2 \cdot r_{m,2} \cdot \bar{r}_2\right) - 2E\left(r_{m,1} \cdot r_{m,2}^2 \cdot \bar{r}_1\right) \\
+ \quad & E\left(r_{m,1}^2 \cdot (\bar{r}_2)^2\right) + E\left(r_{m,2}^2 \cdot (\bar{r}_1)^2\right) \\
- \quad & 2E\left(r_{m,1} \cdot \bar{r}_1 \cdot (\bar{r}_2)^2\right) - 2E\left(r_{m,2} \cdot (\bar{r}_1)^2 \cdot \bar{r}_2\right) \\
- \quad & \lambda^2 p_1^2 p_2^2 \left(1 - \frac{1}{M}\right)^2 \\
= \quad & \frac{M-4}{M} \cdot E\left(r_{m,1}^2 \cdot r_{m,2}^2\right) - 3 \cdot \frac{M-6}{M} \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) \\
+ \quad & \frac{4M-24}{M} \cdot E(r_{m,1} r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) + \frac{M-6}{M} E^2(r_{m,1}) E(r_{m,2}^2) - \frac{M-6}{M} E(r_{m,1}^2) E^2(r_{m,2}) \\
+ \quad & 4 \cdot \frac{1}{M} \cdot E^2(r_{m,1} \cdot r_{m,2}) + \frac{8-2M}{M} \cdot E(r_{m,1}^2 \cdot r_{m,2}) \cdot E(r_{m,2}) + \frac{8-2M}{M} \cdot E(r_{m,1} \cdot r_{m,2}^2) \cdot E(r_{m,1}) \\
+ \quad & 2 \frac{1}{M} E\left(r_{m,1}^2\right) \cdot E\left(r_{m,2}^2\right) - \lambda^2 p_1^2 p_2^2 \left(1 - \frac{2}{M}\right) + O\left(\frac{1}{M^2}\right) \\
= \quad & E\left(r_{m,1}^2 \cdot r_{m,2}^2\right) - 3 \cdot E^2(r_{m,1}) \cdot E^2(r_{m,2}) \\
+ \quad & 4 \cdot E(r_{m,1} r_{m,2}) \cdot E(r_{m,1}) \cdot E(r_{m,2}) - 2 \cdot E(r_{m,2}) \cdot E\left(r_{m,1}^2 \cdot r_{m,2}\right) - 2 \cdot E(r_{m,1}) \cdot E\left(r_{m,2}^2 \cdot r_{m,1}\right) \\
+ \quad & E\left(r_{m,1}^2\right) \cdot E^2(r_{m,2}) + E\left(r_{m,2}^2\right) \cdot E^2(r_{m,1}) - \lambda^2 p_1^2 p_2^2 + O\left(\frac{1}{M}\right).
\end{aligned}$$

The covariance between $(r_{m,1}-\bar{r}_1)\cdot(r_{m,2}-\bar{r}_2)$ and $(r_{n,1}-\bar{r}_1)\cdot(r_{n,2}-\bar{r}_2)$ for $n \neq m$ is

$$
\begin{aligned}
& COV\left((r_{m,1}-\bar{r}1)\cdot(r_{m,2}-\bar{r}_2),(r_{n,1}-\bar{r}_1)\cdot(r_{n,2}-\bar{r}_2)\right) \\
= \; & E\left((r_{m,1}-\bar{r}_1)\cdot(r_{m,2}-\bar{r}_2)\cdot(r_{n,1}-\bar{r}_1)\cdot(r_{n,2}-\bar{r}_2)\right)-\lambda^2 p_1^2 p_2^2\left(1-\frac{1}{M}\right)^2 \\
= \; & E^2(r_{m,1}\cdot r_{m,2})-E(r_{n,1}\cdot r_{m,1}\cdot r_{m,2}\cdot\bar{r}_2)-E(r_{n,2}\cdot r_{m,1}\cdot r_{m,2}\cdot\bar{r}_1)+ \\
+ \; & E(r_{m,1}\cdot r_{m,2}\cdot\bar{r}_1\cdot\bar{r}_2)-E(r_{n,1}\cdot r_{m,1}\cdot r_{n,2}\cdot\bar{r}_2)+E(r_{m,1}\cdot r_{n,1}\cdot(\bar{r}_2)^2) \\
+ \; & E(r_{m,1}\cdot r_{n,2}\cdot\bar{r}_1\cdot\bar{r}_2)-E(r_{m,1}\cdot\bar{r}_1\cdot(\bar{r}_2)^2)-E(r_{m,2}\cdot r_{n,1}\cdot r_{n,2}\cdot\bar{r}_1) \\
+ \; & E(r_{n,1}\cdot r_{m,2}\cdot\bar{r}_1\cdot\bar{r}_2)+E(r_{m,2}\cdot r_{n,2}\cdot(\bar{r}_1)^2)-E(r_{m,2}\cdot(\bar{r}_1)^2\cdot\bar{r}_2) \\
+ \; & E(r_{n,1}\cdot r_{n,2}\cdot\bar{r}_1\cdot\bar{r}_2)-E(r_{n,1}\cdot\bar{r}_1\cdot(\bar{r}_2)^2)-E(r_{n,2}\cdot(\bar{r}_1)^2\cdot\bar{r}_2) \\
+ \; & E\left((\bar{r}_1)^2\cdot(\bar{r}_2)^2\right)-\lambda^2 p_1^2 p_2^2\left(1-\frac{1}{M}\right)^2 \\
= \; & E^2(r_{m,1}\cdot r_{m,2})-2E(r_{n,1}\cdot r_{m,1}\cdot r_{m,2}\cdot\bar{r}_2)-2E(r_{n,2}\cdot r_{m,1}\cdot r_{m,2}\cdot\bar{r}_1) \\
+ \; & 2E(r_{m,1}\cdot r_{m,2}\cdot\bar{r}_1\cdot\bar{r}_2)+E(r_{m,1}\cdot r_{n,1}\cdot(\bar{r}_2)^2) \\
+ \; & 2E(r_{m,1}\cdot r_{n,2}\cdot\bar{r}_1\cdot\bar{r}_2)-2E(r_{m,1}\cdot\bar{r}_1\cdot(\bar{r}_2)^2) \\
+ \; & E(r_{m,2}\cdot r_{n,2}\cdot(\bar{r}_1)^2)-2E(r_{m,2}\cdot(\bar{r}_1)^2\cdot\bar{r}_2) \\
+ \; & E\left((\bar{r}_1)^2\cdot(\bar{r}_2)^2\right)-\lambda^2 p_1^2 p_2^2\left(1-\frac{1}{M}\right)^2 \\
= \; & -2\cdot\frac{M-2}{M}E(r_{m,1})E(r_{m,2})E(r_{m,1}\cdot r_{m,2})+\frac{M-2}{M}E^2(r_{m,1}\cdot r_{m,2}) \\
+ \; & \frac{M-2}{M}\cdot E^2(r_{m,1})\cdot E^2(r_{m,2}) \\
- \; & \lambda^2 p_1^2 p_2^2\left(1-\frac{2}{M}\right)+O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& VAR\left(\hat{S}_{1,2}\right) \\
= \; & \frac{M}{M-1}\cdot COV\left((r_{m,1}-\bar{r}_1)\cdot(r_{m,2}-\bar{r}_2),(r_{n,1}-\bar{r}_1)\cdot(r_{n,2}-\bar{r}_2)\right) \\
+ \; & \frac{M}{(M-1)^2}\cdot VAR\left((r_{m,1}-\bar{r}_1)\cdot(r_{m,2}-\bar{r}_2)\right) \\
= \; & -2\cdot\frac{M-2}{M-1}E(r_{m,1})E(r_{m,2})E(r_{m,1}\cdot r_{m,2})+\frac{M-2}{M-1}E^2(r_{m,1}\cdot r_{m,2}) \\
+ \; & \frac{M-2}{M-1}\cdot E^2(r_{m,1})\cdot E^2(r_{m,2})-\lambda^2 p_1^2 p_2^2\cdot\frac{M-2}{M-1}+O\left(\frac{1}{M^2}\right) \\
+ \; & \frac{1}{M}E\left(r_{m,1}^2\cdot r_{m,2}^2\right)-3\frac{1}{M}\cdot E^2(r_{m,1})\cdot E^2(r_{m,2})+4\frac{1}{M}\cdot E(r_{m,1}r_{m,2})\cdot E(r_{m,1})\cdot E(r_{m,2}) \\
- \; & 2\frac{1}{M}\cdot E(r_{m,2})\cdot E\left(r_{m,1}^2\cdot r_{m,2}\right)-2\frac{1}{M}\cdot E(r_{m,1})\cdot E\left(r_{m,2}^2\cdot r_{m,1}\right)+\frac{1}{M}E\left(r_{m,1}^2\right)\cdot E^2(r_{m,2}) \\
+ \; & \frac{1}{M}E\left(r_{m,2}^2\right)\cdot E^2(r_{m,1})-\frac{1}{M}\lambda^2 p_1^2 p_2^2+O\left(\frac{1}{M^2}\right) \\
= \; & \frac{1}{M}\lambda p_1 p_2\Big[\lambda\left(p_1 p_2+1\right)+1\Big]+O\left(\frac{1}{M^2}\right).
\end{aligned}
$$

2.3 Asymptotic bias of the moment estimators

Theorem 1 gives the expectation of moment estimators defined by (8),(9),(10).

Theorem 1. *Expectations of the moment estimators are given, as $M \to \infty$, by*

$$E(\hat{p}_1) = p_1 + O\left(\frac{1}{M^2}\right), \quad E(\hat{p}_2) = p_2 + O\left(\frac{1}{M^2}\right),$$

$$E(\hat{\lambda}) = \lambda + \frac{(\lambda+1)(p_1p_2+1) - (p_1+p_2)}{p_1p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right).$$

Proof. Based on the moment-type estimators defined by (8),(9),(10) and the Taylor approximation for the mean of the ratio between two random variables (6), we can compute the expectation of our moment estimators as follows:

$$E(\hat{p}_1) = \frac{E(\hat{S}_{1,2})}{E(\bar{r}_2)} - \frac{COV(\hat{S}_{1,2}, \bar{r}_2)}{E^2(\bar{r}_2)} + \frac{VAR(\bar{r}_2) \cdot E(\hat{S}_{1,2})}{E^3(\bar{r}_2)} + O\left(\frac{1}{M^2}\right), \tag{11}$$

$$E(\hat{p}_2) = \frac{E(\hat{S}_{1,2})}{E(\bar{r}_1)} - \frac{COV(\hat{S}_{1,2}, \bar{r}_1)}{E^2(\bar{r}_1)} + \frac{VAR(\bar{r}_1) \cdot E(\hat{S}_{1,2})}{E^3(\bar{r}_1)} + O\left(\frac{1}{M^2}\right), \tag{12}$$

$$E(\hat{\lambda}) = \frac{E(\bar{r}_1 \cdot \bar{r}_2)}{E(\hat{S}_{1,2})} - \frac{COV(\bar{r}_1 \cdot \bar{r}_2, \hat{S}_{1,2})}{E^2(\hat{S}_{1,2})} + \frac{VAR(\hat{S}_{1,2}) \cdot E(\bar{r}_1 \cdot \bar{r}_2)}{E^3(\hat{S}_{1,2})} + O\left(\frac{1}{M^2}\right). \tag{13}$$

Substituting the above results in (11),(12),(13) completes the proof of Theorem 1. □

2.4 Asymptotic variance and of the moment estimators

Theorem 2 gives the variance of the moment estimators defined by (8),(9),(10).

Theorem 2. *The variance of the moment estimators is given, as $M \to \infty$, by*

$$VAR(\hat{p}_1) = \frac{p_1}{\lambda p_2}\left[\lambda(p_1p_2+1) + 1 - p_1\right] \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right),$$

$$VAR(\hat{p}_2) = \frac{p_2}{\lambda p_1}\left[\lambda(p_1p_2+1) + 1 - p_2\right] \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right),$$

$$VAR(\hat{\lambda}) = \frac{\lambda^2(p_1p_2+1) + \lambda(1 + 2p_1p_2 - p_1 - p_2)}{p_1p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right).$$

Proof. Based on the moment-type estimators defined by (8),(9),(10) and the Taylor approximation for the variance of the ratio between two random variables (7), we can compute the expectation of our moment estimators as follows:

$$VAR(\hat{p}_1) = \frac{E^2(\hat{S}_{1,2})}{E^2(\bar{r}_2)}\left[\frac{VAR(\hat{S}_{1,2})}{E^2(\hat{S}_{1,2})} - 2\frac{COV(\hat{S}_{1,2}, \bar{r}_2)}{E(\hat{S}_{1,2}) \cdot E(\bar{r}_2)} + \frac{VAR(\bar{r}_2)}{E^2(\bar{r}_2)}\right] + O\left(\frac{1}{M^2}\right) \tag{14}$$

$$VAR(\hat{p}_2) = \frac{E^2(\hat{S}_{1,2})}{E^2(\bar{r}_1)}\left[\frac{VAR(\hat{S}_{1,2})}{E^2(\hat{S}_{1,2})} - 2\frac{COV(\hat{S}_{1,2}, \bar{r}_1)}{E(\hat{S}_{1,2}) \cdot E(\bar{r}_1)} + \frac{VAR(\bar{r}_1)}{E^2(\bar{r}_1)}\right] + O\left(\frac{1}{M^2}\right), \tag{15}$$

$$VAR(\hat{\lambda}) = \frac{E^2(\bar{r}_1 \cdot \bar{r}_2)}{E^2(\hat{S}_{1,2})}\left[\frac{VAR(\bar{r}_1 \cdot \bar{r}_2)}{E^2(\bar{r}_1 \cdot \bar{r}_2)} - 2 \cdot \frac{COV(\bar{r}_1 \cdot \bar{r}_2, \hat{S}_{1,2})}{E(\bar{r}_1 \cdot \bar{r}_2) \cdot E(\hat{S}_{1,2})} + \frac{VAR(\hat{S}_{1,2})}{E^2(\hat{S}_{1,2})}\right] + O\left(\frac{1}{M^2}\right). \tag{16}$$

Substituting the above results in (14),(15),(16) complete the proof of Theorem 2.

□

3 Maximum Likelihood estimators: the Fisher information matrix

Recall that

$$P(r_1,r_2) = P(R_1=r_1,R_2=r_2) = P(X_1+Y=r_1,X_2+Y=r_2)$$

$$= \sum_{l=0}^{\min(r_1,r_2)} P(Y=l)P(X_1=r_1-l)P(X_2=r_2-l).$$

$$= e^{-\lambda[1-(1-p_1)(1-p_2)]} \frac{(\lambda p_1(1-p_2))^{r_1}}{r_1!} \frac{(\lambda p_2(1-p_1))^{r_2}}{r_2!} \cdot \sum_{l=0}^{min(r_1,r_2)} \binom{r_1}{l}\binom{r_2}{l} l!\, (\lambda(1-p_1)(1-p_2))^{-l}\,.$$

We compute

$$\begin{aligned} \frac{\partial P(r_1,r_2)}{\partial \lambda} &= -[1-(1-p_1)(1-p_2)]P(r_1,r_2) \\ &+ \frac{r_1+r_2}{\lambda}P(r_1,r_2) - p_1p_2P(r_1-1,r_2-1) \end{aligned} \tag{17}$$

$$\frac{\partial P(r_1,r_2)}{\partial p_1} = -\lambda(1-p_2)P(r_1,r_2) + \frac{r_1}{p_1}P(r_1,r_2) - \lambda p_2 P(r_1,r_2-1) \tag{18}$$

$$\frac{\partial P(r_1,r_2)}{\partial p_2} = -\lambda(1-p_1)P(r_1,r_2) + \frac{r_2}{p_2}P(r_1,r_2) - \lambda p_1 P(r_1-1,r_2) \tag{19}$$

Using (17),

$$\begin{aligned} \frac{\partial \log P(r_1,r_2)}{\partial \lambda} &= \frac{\frac{\partial P(r_1,r_2)}{\partial \lambda}}{P(r_1,r_2)} \\ &= = -[1-(1-p_1)(1-p_2)] + \frac{r_1+r_2}{\lambda} - p_1p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)} \end{aligned}$$

Using (18),

$$\frac{\partial \log P(r_1,r_2)}{\partial p_1} = \frac{\frac{\partial P(r_1,r_2)}{\partial p_1}}{P(r_1,r_2)} = -\lambda(1-p_2) + \frac{r_1}{p_1} - \lambda p_2\frac{P(r_1,r_2-1)}{P(r_1,r_2)}$$

Using (19),

$$\frac{\partial \log P(r_1,r_2)}{\partial p_2} = \frac{\frac{\partial P(r_1,r_2)}{\partial p_2}}{P(r_1,r_2)} = -\lambda(1-p_1) + \frac{r_2}{p_2} - \lambda p_1\frac{P(r_1-1,r_2)}{P(r_1,r_2)}.$$

Using the above we compute the second derivatives

$$
\begin{aligned}
\frac{\partial^2 \log P(r_1,r_2)}{\partial\lambda^2} &= -\frac{r_1+r_2}{\lambda^2} - p_1p_2\frac{\frac{\partial P(r_1-1,r_2-1)}{\partial\lambda}}{P(r_1,r_2)} + p_1p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{\frac{\partial P(r_1,r_2)}{\partial\lambda}}{P(r_1,r_2)} \\
&= -\frac{r_1+r_2}{\lambda^2} + p_1^2p_2^2\frac{P(r_1-2,r_2-2)}{P(r_1,r_2)} \\
&- p_1^2p_2^2\left[\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\right]^2 + \frac{2p_1p_2}{\lambda}\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)} \\
\frac{\partial^2 \log P(r_1,r_2)}{\partial\lambda\partial p_1} &= -(1-p_2) - p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)} \\
&- p_1p_2\frac{\frac{\partial P(r_1-1,r_2-1)}{\partial p_1}}{P(r_1,r_2)} + p_1p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{\frac{\partial P(r_1,r_2)}{\partial p_1}}{P(r_1,r_2)} \\
&= -(1-p_2) + \lambda p_1p_2^2\frac{P(r_1-1,r_2-2)}{P(r_1,r_2)} \\
&- \lambda p_1p_2^2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{P(r_1,r_2-1)}{P(r_1,r_2)} \\
\frac{\partial^2 \log P(r_1,r_2)}{\partial\lambda\partial p_2} &= -(1-p_1) - p_1\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)} \\
&- p_1p_2\frac{\frac{\partial P(r_1-1,r_2-1)}{\partial p_2}}{P(r_1,r_2)} + p_1p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{\frac{\partial P(r_1,r_2)}{\partial p_2}}{P(r_1,r_2)} \\
&= -(1-p_1) + \lambda p_1^2p_2\frac{P(r_1-2,r_2-1)}{P(r_1,r_2)} \\
&- \lambda p_1^2p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{P(r_1-1,r_2)}{P(r_1,r_2)} \\
\frac{\partial^2 \log P(r_1,r_2)}{\partial p_1^2} &= -\frac{r_1}{p_1^2} - \lambda p_2\frac{\frac{\partial P(r_1,r_2-1)}{\partial p_1}}{P(r_1,r_2)} + \lambda p_2\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\frac{\frac{\partial P(r_1,r_2)}{\partial p_1}}{P(r_1,r_2)} \\
&= -\frac{r_1}{p_1^2} + \lambda^2p_2^2\frac{P(r_1,r_2-2)}{P(r_1,r_2)} - \lambda^2p_2^2\left[\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\right]^2 \\
\frac{\partial^2 \log P(r_1,r_2)}{\partial p_1\partial p_2} &= \lambda - \lambda\frac{P(r_1,r_2-1)}{P(r_1,r_2)} - \lambda p_2\frac{\frac{\partial P(r_1,r_2-1)}{\partial p_2}}{P(r_1,r_2)} \\
&+ \lambda p_2\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\frac{\frac{\partial P(r_1,r_2)}{\partial p_2}}{P(r_1,r_2)} \\
&= \lambda + \lambda^2p_1p_2\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)} \\
&- \lambda^2p_1p_2\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\frac{P(r_1-1,r_2)}{P(r_1,r_2)} \\
\frac{\partial^2 \log P(r_1,r_2)}{\partial p_2^2} &= -\frac{r_2}{p_2^2} - \lambda p_1\frac{\frac{\partial P(r_1-1,r_2)}{\partial p_2}}{P(r_1,r_2)} + \lambda p_1\frac{P(r_1-1,r_2)}{P(r_1,r_2)}\frac{\frac{\partial P(r_1,r_2)}{\partial p_2}}{P(r_1,r_2)} \\
&= -\frac{r_2}{p_2^2} + \lambda^2p_1^2\frac{P(r_1-2,r_2)}{P(r_1,r_2)} - \lambda^2p_1^2\left[\frac{P(r_1-1,r_2)}{P(r_1,r_2)}\right]^2
\end{aligned}
$$

The elements of Fisher Information matrix are thus given by

$$
\begin{aligned}
-E\left(\frac{\partial^2}{\partial\lambda^2}\ln P(r_1,r_2)\right) &= \frac{p_1+p_2}{\lambda} - p_1^2p_2^2E\left[\frac{P(r_1-2,r_2-2)}{P(r_1,r_2)}\right] \\
&+ p_1^2p_2^2E\left[\left\{\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\right\}^2\right] - \frac{2p_1p_2}{\lambda}E\left[\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\right] \\
-E\left(\frac{\partial^2}{\partial\lambda\partial p_1}\ln P(r_1,r_2)\right) &= (1-p_2) - \lambda p_1p_2^2E\left[\frac{P(r_1-1,r_2-2)}{P(r_1,r_2)}\right] \\
&+ \lambda p_1p_2^2E\left[\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\right] \\
-E\left(\frac{\partial^2}{\partial\lambda\partial p_2}\ln P(r_1,r_2)\right) &= (1-p_1) - \lambda p_1^2p_2E\left[\frac{P(r_1-2,r_2-1)}{P(r_1,r_2)}\right] \\
&+ \lambda p_1^2p_2E\left[\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\frac{P(r_1-1,r_2)}{P(r_1,r_2)}\right] \\
-E\left(\frac{\partial^2}{\partial p_1^2}\ln P(r_1,r_2)\right) &= \frac{\lambda}{p_1} - \lambda^2p_2^2E\left[\frac{P(r_1,r_2-2)}{P(r_1,r_2)}\right] \\
&+ \lambda^2p_2^2E\left[\left\{\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\right\}^2\right] \\
-E\left(\frac{\partial^2}{\partial p_1\partial p_2}\ln P(r_1,r_2)\right) &= -\lambda - \lambda^2p_1p_2E\left[\frac{P(r_1-1,r_2-1)}{P(r_1,r_2)}\right] \\
&+ \lambda^2p_1p_2E\left[\frac{P(r_1,r_2-1)}{P(r_1,r_2)}\frac{P(r_1-1,r_2)}{P(r_1,r_2)}\right] \\
-E\left(\frac{\partial^2}{\partial p_2^2}\ln P(r_1,r_2)\right) &= \frac{\lambda}{p_2} - \lambda^2p_1^2E\left[\frac{P(r_1-2,r_2)}{P(r_1,r_2)}\right] \\
&+ \lambda^2p_1^2E\left[\left\{\frac{P(r_1-1,r_2)}{P(r_1,r_2)}\right\}^2\right]
\end{aligned}
$$

Let us note that, for any $l,m\geq 0$, we have

$$
\begin{aligned}
E\left[\frac{P(r_1-l,r_2-m)}{P(r_1,r_2)}\right] &= \sum_{r_1=l}^{\infty}\sum_{r_2=m}^{\infty}P(r_1,r_2)\cdot\frac{P(r_1-l,r_2-m)}{P(r_1,r_2)} \\
&= \sum_{r_1=l}^{\infty}\sum_{r_2=m}^{\infty}P(r_1-l,r_2-m) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}P(r_1,r_2) = 1,
\end{aligned}
$$

so that the above expressions simplify to

$$
\begin{aligned}
-E\left(\frac{\partial^2}{\partial\lambda^2}\ln P(r_1,r_2)\right) &= \frac{p_1+p_2-2p_1p_2}{\lambda} - p_1^2p_2^2 + p_1^2p_2^2\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{(P(r_1-1,r_2-1))^2}{P(r_1,r_2)} \\
-E\left(\frac{\partial^2}{\partial\lambda\partial p_1}\ln P(r_1,r_2)\right) &= (1-p_2) - \lambda p_1p_2^2 \\
&+ \lambda p_1p_2^2\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{P(r_1-1,r_2-1)P(r_1,r_2-1)}{P(r_1,r_2)} \\
-E\left(\frac{\partial^2}{\partial\lambda\partial p_2}\ln P(r_1,r_2)\right) &= (1-p_1) - \lambda p_1^2p_2 \\
&+ \lambda p_1^2p_2\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{P(r_1-1,r_2-1)P(r_1-1,r_2)}{P(r_1,r_2)} \\
-E\left(\frac{\partial^2}{\partial p_1^2}\ln P(r_1,r_2)\right) &= \frac{\lambda}{p_1} - \lambda^2p_2^2 + \lambda^2p_2^2\sum_{r_1=0}^{\infty}\sum_{r_2=1}^{\infty}\frac{(P(r_1,r_2-1))^2}{P(r_1,r_2)} \\
-E\left(\frac{\partial^2}{\partial p_1\partial p_2}\ln P(r_1,r_2)\right) &= -\lambda - \lambda^2p_1p_2 \\
&+ \lambda^2p_1p_2\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{P(r_1,r_2-1)P(r_1-1,r_2)}{P(r_1,r_2)} \\
-E\left(\frac{\partial^2}{\partial p_2^2}\ln P(r_1,r_2)\right) &= \frac{\lambda}{p_2} - \lambda^2p_1^2 + \lambda^2p_1^2\sum_{r_1=1}^{\infty}\sum_{r_2=0}^{\infty}\frac{(P(r_1-1,r_2))^2}{P(r_1,r_2)}
\end{aligned}
$$

4 Comparison with Capture-Recapture (CR) Estimation

4.1 Derivation of Capture-Recapture Estimators

As explained in the paper in Capture-Recapture method the information regarding the number of events which were recorded by both observers *is* available. Thus the precondition applying the capture-recapture method is that we have the values of the variables $X_{m,1}, X_{m,2}, Y_m$: the number of events observed only by observer 1, only by observer 2, and by both observers, respectively. These are assumed to be independent and distributed according to (2),(3).

Lemma 1. *The maximum likelihood estimators* $(\hat{p}_{1,CR}, \hat{p}_{2,CR}, \hat{\lambda}_{CR})$ *of the parameters* (p_1, p_2, λ) *in the capture-recapture method are given by:*

$$\hat{p}_{1,CR} = \frac{\bar{y}}{\bar{x}_2+\bar{y}}, \tag{20}$$

$$\hat{p}_{2,CR} = \frac{\bar{y}}{\bar{x}_1+\bar{y}}, \tag{21}$$

$$\hat{\lambda}_{CR} = \frac{(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})}{\bar{y}}, \tag{22}$$

where

$$\bar{x}_i = \frac{1}{M}\sum_{m=1}^{M} x_{m,i}, \quad i=1,2, \quad \bar{y} = \frac{1}{M}\sum_{m=1}^{M} y_m.$$

The values $\{(x_{m,1}, x_{m,2}, y_m)\}_{m=1}^{M}$ of the random variables, $\{(X_{m,1}, X_{m,2}, Y_m)\}_{m=1}^{M}$, generated by (2) and (3), with the independence assumption on the set, both for the same m and for different values of m $(m = 1, 2, \cdots, M)$.

We note that these estimators can be obtained both using a moment-type approach and using a maximum-likelihood approach - in contrast with the problem which is the main focus of our paper, here the two approaches yield identical estimators.

Proof. Following the independence of the random variables $(X_{m,1}, X_{m,2}, Y_m)$ and the Poisson-distributed (2)-(3), the likelihood function is given by:

$$\begin{aligned}
& P(X_{m,1} = x_{m,1}, X_{m,2} = x_{m,2}, Y_m = y_m) \\
= \;& e^{-\lambda p_1(1-p_2)} \frac{(\lambda p_1(1-p_2))^{x_{m,1}}}{x_{m,1}!} \cdot e^{-\lambda p_2(1-p_1)} \frac{(\lambda p_2(1-p_1))^{x_{m,2}}}{x_{m,2}!} \cdot e^{-\lambda p_1 p_2} \frac{(\lambda p_1 p_2)^{y_m}}{y_m!} \\
= \;& e^{-\lambda[1-(1-p_1)(1-p_2)]} \cdot \lambda^{x_{m,1}+x_{m,2}+y_m} \frac{(p_1(1-p_2))^{x_{m,1}}}{x_{m,1}!} \cdot \frac{(p_2(1-p_1))^{x_{m,2}}}{x_{m,2}!} \cdot \frac{(p_1 p_2)^{y_m}}{y_m!}. \qquad (23)
\end{aligned}$$

Therefore, the log-likelihood, neglecting terms not depending on the parameters, is:

$$\begin{aligned}
LL(\lambda, p_1, p_2) = -M\lambda[1-(1-p_1)(1-p_2)] + \left(\sum_{m=1}^{M} (x_{m,1} + x_{m,2} + y_m)\right) \ln(\lambda) \\
+ \left(\sum_{m=1}^{M} x_{m,1}\right) \ln(p_1(1-p_2)) + \left(\sum_{m=1}^{M} x_{m,2}\right) \ln(p_2(1-p_1)) + \left(\sum_{m=1}^{M} y_m\right) \ln(p_1 p_2). \qquad (24)
\end{aligned}$$

To maximize the log-likelihood we calculate:

$$LL_\lambda(\lambda, p_1, p_2) = -M[1-(1-p_1)(1-p_2)] + \left(\sum_{m=1}^{M} (x_{m,1} + x_{m,2} + y_m)\right) \frac{1}{\lambda}, \qquad (25)$$

and

$$LL_{p_1}(\lambda, p_1, p_2) = -M\lambda(1-p_2) + \left(\sum_{m=1}^{M} (x_{m,1} + y_m)\right) \frac{1}{p_1} - \left(\sum_{m=1}^{M} x_{m,2}\right) \frac{1}{1-p_1}, \qquad (26)$$

and finally,

$$LL_{p_2}(\lambda, p_1, p_2) = -M\lambda(1-p_1) + \left(\sum_{m=1}^{M} (x_{m,2} + y_m)\right) \frac{1}{p_2} - \left(\sum_{m=1}^{M} x_{m,1}\right) \frac{1}{1-p_2}. \qquad (27)$$

Setting the partial derivatives to 0, we have

$$-[1-(1-p_1)(1-p_2)] + (\bar{x}_1 + \bar{x}_2 + \bar{y}) \frac{1}{\lambda} = 0,$$

$$-\lambda(1-p_2) + (\bar{x}_1 + \bar{y}) \frac{1}{p_1} - \bar{x}_2 \cdot \frac{1}{1-p_1} = 0,$$

$$-\lambda(1-p_1) + (\bar{x}_2 + \bar{y}) \frac{1}{p_2} - \bar{x}_1 \cdot \frac{1}{1-p_2} = 0.$$

Solving these equations we obtain

$$\hat{\lambda}_{CR} = \frac{(\bar{x}_1 + \bar{y})(\bar{x}_2 + \bar{y})}{\bar{y}}, \qquad (28)$$

$$\hat{p}_{1,CR} = \frac{\bar{y}}{\bar{x}_2 + \bar{y}}, \qquad (29)$$

$$\hat{p}_{2,CR} = \frac{\bar{y}}{\bar{x}_2 + \bar{y}}. \qquad (30)$$

□

4.2 Properties of estimators in the Capture-Recpature method

In this section we will make use of Taylor expansions defined in (6)-(7) for the following cases

$$E\left(\frac{1}{X}\right) \approx \frac{1}{E(X)} + \frac{VAR(X)}{E^3(X)}, \tag{31}$$

$$VAR\left(\frac{1}{X}\right) \approx \frac{VAR(X)}{E^4(X)}, \tag{32}$$

$$E\left(\frac{1}{X^2}\right) \approx \frac{1}{E^2(X)} + \frac{3VAR(X)}{E^4(X)}. \tag{33}$$

Theorem 3 states the expectation of the maximum-likelihood estimators defined by (28),(29),(30).

Theorem 3.

$$E(\hat{p}_{1,CR}) = p_1, \quad E(\hat{p}_{2,CR}) = p_2,$$

that is the estimators $\hat{p}_{1,CR}, \hat{p}_{2,CR}$ are unbiased.

As $M \to \infty$,

$$E(\hat{\lambda}_{CR}) = \lambda + \frac{(1-p_1)(1-p_2)}{p_1 p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right).$$

Proof. The maximum-likelihood estimators for p_1 defined by (29) can be written as:

$$\hat{p}_{1,CR} = \frac{M\bar{y}}{M\bar{x}_2 + M\bar{y}} \tag{34}$$

It is well-known that the conditional distribution of $(M\bar{y})$ given $(M\bar{x}_2 + M\bar{y})$ is Binomial with the parameters $(M\bar{x}_2 + M\bar{y}\ ,\ p_1)$. So, using the total law of expectation,

$$\begin{aligned} E(\hat{p}_{1,CR}) &= E\left(E\left(\frac{M\bar{y}}{M\bar{x}_2 + M\bar{y}}\middle| M\bar{x}_2 + M\bar{y}\right)\right) = E\left(\frac{1}{M\bar{x}_2 + M\bar{y}} E\left(M\bar{y} | M\bar{x}_2 + M\bar{y}\right)\right) \\ &= E\left(\frac{1}{M\bar{x}_2 + M\bar{y}}\left(M\bar{x}_2 + M\bar{y}\right) p_1\right) = p_1. \end{aligned} \tag{35}$$

The same for the p_2. We therefore conclude that these estimators are unbiased.

The estimator for detection rate defined in (22) can be re-written as

$$\hat{\lambda}_{CR} = \frac{(\bar{x}_1 + \bar{y})(\bar{x}_2 + \bar{y})}{\bar{y}} = \frac{\bar{x}_1 \bar{x}_2}{\bar{y}} + \bar{x}_1 + \bar{x}_2 + \bar{y}. \tag{36}$$

This estimator is asymptotically unbiased, since:

$$\begin{aligned} & E(\hat{\lambda}_{CR}) = E\left(E\left(\frac{\bar{x}_1\bar{x}_2}{\bar{y}} + \bar{x}_1 + \bar{x}_2 + \bar{y} \,\middle|\, \bar{y}\right)\right) \\ = \quad & E\left(\frac{1}{\bar{y}} E(\bar{x}_1)E(\bar{x}_2) + E(\bar{x}_1) + E(\bar{x}_2) + \bar{y}\right) \\ = \quad & E\left(\frac{1}{\bar{y}}\right) \lambda p_1(1-p_2)\lambda p_2(1-p_1) + \lambda p_1(1-p_2) + \lambda p_2(1-p_1) + \lambda p_1 p_2 \\ = \quad & \left[\frac{1}{\lambda p_1 p_2} + \frac{1}{M(\lambda p_1 p_2)^2}\right] \lambda^2 p_1 p_2 (1-p_1)(1-p_2) \\ + \quad & \lambda p_1(1-p_2) + \lambda p_2(1-p_1) + \lambda p_1 p_2 + +O\left(\frac{1}{M^2}\right) \\ = \quad & \lambda + \frac{(1-p_1)(1-p_2)}{p_1 p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right). \end{aligned} \tag{37}$$

□

Theorem 4 gives the variance of the maximum-likelihood estimators defined by (28),(29),(30).

Theorem 4. *As* $M \to \infty$*,*

$$VAR(\hat{p}_{1,CR}) = \frac{p_1(1-p_1)}{\lambda p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right), \tag{38}$$

$$VAR(\hat{p}_{2,CR}) = \frac{p_2(1-p_2)}{\lambda p_1} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right), \tag{39}$$

$$VAR(\hat{\lambda}_{CR}) = \frac{\lambda(1+2p_1p_2-p_1-p_2)}{p_1p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right). \tag{40}$$

Proof. We use the law of total variance and the fact that the conditional distribution of $(M\bar{y})$ given $(M\bar{x}_2 + M\bar{y})$ is Binomial with the parameters $(M\bar{x}_2 + M\bar{y}\ ,\ p_1)$, to find the variance of $\hat{p}_{1,CR}$.

$$\begin{aligned} VAR(\hat{p}_{1,CR}) &= VAR\left(E\left(\frac{M\bar{y}}{M\bar{x}_2+M\bar{y}}\middle| M\bar{x}_2+M\bar{y}\right)\right) + E\left(VAR\left(\frac{M\bar{y}}{M\bar{x}_2+M\bar{y}}\middle| M\bar{x}_2+M\bar{y}\right)\right) \\ &= VAR\left(\frac{1}{M\bar{x}_2+M\bar{y}}\,(M\bar{x}_2+M\bar{y})\,p_1\right) + E\left(\frac{1}{(M\bar{x}_2+M\bar{y})^2}(M\bar{x}_2+M\bar{y})p_1(1-p_1)\right) \\ &= p_1(1-p_1)E\left(\frac{1}{M\bar{x}_2+M\bar{y}}\right). \end{aligned} \tag{41}$$

Using (31) and the fact that $(M\bar{x}_2 + M\bar{y})$ is Poisson-distributed with $(M\lambda p_2)$ gives:

$$VAR(\hat{p}_{1,CR}) = \frac{p_1(1-p_1)}{\lambda p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2}\right). \tag{42}$$

The proof for $\hat{p}_{2,CR}$ is similar. We conclude that these estimators are consistent.

We proceed to calculate the variance of $\hat{\lambda}_{CR}$. Again, we use the law of total variance by conditioning on $\bar{y}$.

$$\begin{aligned} VAR(\hat{\lambda}_{CR}) &= VAR\left(E\left(\frac{(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})}{\bar{y}}\ \middle|\ \bar{y}\right)\right) \\ &+ E\left(VAR\left(\frac{(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})}{\bar{y}}\ \middle|\ \bar{y}\right)\right). \end{aligned} \tag{43}$$

We first calculate the interior elements in (43) by conditioning on $\bar{y}$ and using the independence between $\bar{x}_1, \bar{x}_2$ and $\bar{y}$. So,

$$\begin{aligned} & E\left(\frac{(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})}{\bar{y}}\ \middle|\ \bar{y}\right) = E\left(\frac{\bar{x}_1\cdot\bar{x}_2}{\bar{y}} + \bar{x}_1 + \bar{x}_2 + \bar{y}\middle|\ \bar{y}\right) \\ = & \frac{1}{\bar{y}}E(\bar{x}_1)\cdot E(\bar{x}_2) + E(\bar{x}_1) + E(\bar{x}_2) + \bar{y} \\ = & \frac{\lambda^2 p_1p_2(1-p_1)(1-p_2)}{\bar{y}} + \lambda p_1(1-p_2) + \lambda p_2(1-p_1) + \bar{y} \end{aligned} \tag{44}$$

and

$$
\begin{aligned}
& VAR\left(\frac{(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})}{\bar{y}}\;\middle|\;\bar{y}\right) = VAR\left(\frac{(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})}{\bar{y}}\;\middle|\;\bar{y}\right) = \frac{1}{\bar{y}^2}VAR\left((\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})\right) \\
= \;& \frac{1}{\bar{y}^2}\left\{E\left[(\bar{x}_1+\bar{y})^2(\bar{x}_2+\bar{y})^2\right] - E^2[(\bar{x}_1+\bar{y})(\bar{x}_2+\bar{y})]\right\} \\
= \;& \frac{1}{\bar{y}^2}\left\{E((\bar{x}_1+\bar{y})^2)E((\bar{x}_2+\bar{y})^2) - E^2(\bar{x}_1+\bar{y})E^2(\bar{x}_2+\bar{y})\right\} \\
= \;& \frac{1}{\bar{y}^2}\left\{[VAR(\bar{x}_1)+E^2(\bar{x}_1+\bar{y})][VAR(\bar{x}_2)+E^2(\bar{x}_2+\bar{y})] - E^2(\bar{x}_1+\bar{y})E^2(\bar{x}_2+\bar{y})\right\} \\
= \;& \frac{1}{\bar{y}^2}\left\{VAR(\bar{x}_1)VAR(\bar{x}_2) + VAR(\bar{x}_1)E^2(\bar{x}_2+\bar{y}) + VAR(\bar{x}_2)E^2(\bar{x}_1+\bar{y})\right\} \\
= \;& \frac{1}{\bar{y}^2}\left\{VAR(\bar{x}_1)VAR(\bar{x}_2) + VAR(\bar{x}_1)(E^2(\bar{x}_2)+2\bar{y}E(\bar{x}_2)+\bar{y}^2) + VAR(\bar{x}_2)(E^2(\bar{x}_1)+2\bar{y}E(\bar{x}_1)+\bar{y}^2)\right\} \\
= \;& \frac{1}{\bar{y}^2}VAR(\bar{x}_1)VAR(\bar{x}_2) + \frac{1}{\bar{y}^2}VAR(\bar{x}_1)E^2(\bar{x}_2) + 2\frac{1}{\bar{y}}VAR(\bar{x}_1)E(\bar{x}_2) + VAR(\bar{x}_1) \\
+ \;& \frac{1}{\bar{y}^2}VAR(\bar{x}_2)E^2(\bar{x}_1) + 2\frac{1}{\bar{y}}VAR(\bar{x}_2)E(\bar{x}_1) + VAR(\bar{x}_2) \\
= \;& \frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M^2\bar{y}^2} + \frac{\lambda^3 p_1 p_2^2(1-p_2)(1-p_1)^2}{M\bar{y}^2} + \frac{2\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M\bar{y}} + \frac{\lambda p_1(1-p_2)}{M} \\
+ \;& \frac{\lambda^3 p_1^2 p_2(1-p_1)(1-p_2)^2}{M\bar{y}^2} + \frac{2\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M\bar{y}} + \frac{\lambda p_2(1-p_1)}{M} \\
= \;& \frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}\left\{\frac{1}{M} + \lambda p_2(1-p_1) + \lambda p_1(1-p_2)\right\}\frac{1}{\bar{y}^2} + \frac{4\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}\frac{1}{\bar{y}} + \\
+ \;& \frac{\lambda p_1(1-p_2)}{M} + \frac{\lambda p_2(1-p_1)}{M}. \qquad (45)
\end{aligned}
$$

Substituting (44) and (45) in (43) gives:

$$
\begin{aligned}
& VAR(\hat{\lambda}_{CR}) = VAR\left(\bar{y} + \lambda p_1(1-p_2) + \lambda p_2(1-p_1) + \frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{\bar{y}}\right) + \\
+ \;& E\left(\frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}\left\{\frac{1}{M} + \lambda p_2(1-p_1) + \lambda p_1(1-p_2)\right\}\frac{1}{\bar{y}^2}\right) \\
+ \;& E\left(\frac{4\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}\frac{1}{\bar{y}} + \frac{\lambda p_1(1-p_2)}{M} + \frac{\lambda p_2(1-p_1)}{M}\right) \\
= \;& VAR\left(\bar{y} + \frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{\bar{y}}\right) + \\
+ \;& \frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}\left\{\frac{1}{M} + \lambda p_2(1-p_1) + \lambda p_1(1-p_2)\right\}E\left(\frac{1}{\bar{y}^2}\right) + \\
+ \;& \frac{4\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}E\left(\frac{1}{\bar{y}}\right) + \frac{\lambda p_1(1-p_2)}{M} + \frac{\lambda p_2(1-p_1)}{M} \\
= \;& \frac{\lambda p_1 p_2}{M} + \lambda^4 p_1^2 p_2^2(1-p_1)^2(1-p_2)^2 VAR\left(\frac{1}{\bar{y}}\right) + 2\lambda^2 p_1 p_2(1-p_1)(1-p_2)COV\left(\bar{y},\frac{1}{\bar{y}}\right) \\
+ \;& \frac{\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}\left\{\frac{1}{M} + \lambda p_2(1-p_1) + \lambda p_1(1-p_2)\right\}E\left(\frac{1}{\bar{y}^2}\right) \\
+ \;& \frac{4\lambda^2 p_1 p_2(1-p_1)(1-p_2)}{M}E\left(\frac{1}{\bar{y}}\right) + \frac{\lambda p_1(1-p_2)}{M} + \frac{\lambda p_2(1-p_1)}{M}. \qquad (46)
\end{aligned}
$$

Using (31),(32) and (33), we get:

$$
\begin{aligned}
& VAR(\hat{\lambda}_{CR}) = \frac{\lambda p_1 p_2}{M} + \lambda^4 p_1^2 p_2^2 (1-p_1)^2 (1-p_2)^2 \left(\frac{1}{(\lambda p_1 p_2)^3 \; M} \right) \\
+ \quad & 2\lambda^2 p_1 p_2 (1-p_1)(1-p_2) \left(1 - \lambda p_1 p_2 \left(\frac{1}{\lambda p_1 p_2} + \frac{1}{(\lambda p_1 p_2)^2 \; M} \right) \right) \\
+ \quad & \frac{\lambda^2 p_1 p_2 (1-p_1)(1-p_2)}{M} \left\{ \frac{1}{M} + \lambda p_2 (1-p_1) + \lambda p_1 (1-p_2) \right\} \left(\frac{1}{(\lambda p_1 p_2)^2} + \frac{3}{(\lambda p_1 p_2)^3 \; M} \right) \\
+ \quad & \frac{4\lambda^2 p_1 p_2 (1-p_1)(1-p_2)}{M} \left(\frac{1}{\lambda p_1 p_2} + \frac{1}{(\lambda p_1 p_2)^2 \; M} \right) \\
+ \quad & \frac{\lambda p_1 (1-p_2)}{M} + \frac{\lambda p_2 (1-p_1)}{M} + O\left(\frac{1}{M^2} \right) \\
= \quad & \frac{\lambda p_1 p_2}{M} + \frac{\lambda p_1 (1-p_2)}{M} + \frac{\lambda p_2 (1-p_1)}{M} + \frac{2\lambda (1-p_1)(1-p_2)}{M} + \frac{\lambda (1-p_1)^2 (1-p_2)^2}{M p_1 p_2} \\
+ \quad & \frac{\lambda (1-p_1)^2 (1-p_2)}{M p_1} + \frac{\lambda (1-p_1)(1-p_2)^2}{M p_2} + \frac{5(1-p_1)(1-p_2)}{M^2 p_1 p_2} \\
+ \quad & \frac{3(1-p_1)(1-p_2)^2}{M^2 p_1 p_2^2} + \frac{3(1-p_1)^2(1-p_2)}{M^2 p_1^2 p_2} + \frac{3(1-p_1)(1-p_2)}{M^3 \lambda p_1^2 p_2^2} + O\left(\frac{1}{M^2} \right) \\
= \quad & \frac{\lambda (1 + 2p_1 p_2 - p_1 - p_2)}{p_1 p_2} \cdot \frac{1}{M} + O\left(\frac{1}{M^2} \right). \qquad (47)
\end{aligned}
$$

□

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