Fix Probabilities from LOP Geometry

Appendices A and B

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Appendix A: Expressions for the Distance of a Point to an LOP

This appendix provides expressions for the distance to LOP$_i$ from each point (x, y) on a coordinate grid. In eq. (2) of the main text, this distance is represented by $r_i(x, y)$. Which of the following expressions is most convenient depends on how the LOPs are represented mathematically, which in turn may depend on the type of observation they represent. Cartesian coordinates are assumed (flat-Earth approximation). See Fig. A1.

1. LOPs of the form $y = mx + b$, where m is the slope of the LOP, i.e., the tangent of its angle with the x axis, and b is the y-intercept. The distance of (x, y) to the closest point on LOP$_i$ is

$$r_i(x, y) = \frac{|m_i x - y + b_i|}{\sqrt{1 + m_i^2}} \tag{A1}$$

   where m_i and b_i are the values of m and b particular to LOP$_i$. If the magnitude of the slope m is large enough that the LOP is effectively parallel to the y-axis over the span of the grid, crossing the x-axis at c, we have simply that $r_i(x, y) = |x - c|$.

2. LOPs of the form $\mathbf{q} = \mathbf{g} + h\mathbf{d}$, where $\mathbf{q}$, $\mathbf{g}$, and $\mathbf{d}$ are vectors in the x-y plane: $\mathbf{q}$ represents the coordinates of any arbitrary point on the LOP, $\mathbf{g}$ represents the coordinates of a known point on the LOP, $\mathbf{d}$ represents the direction of the LOP (a unit vector), and h is a (signed) scalar whose absolute value is the distance of $\mathbf{q}$ from $\mathbf{g}$. The direction $\mathbf{d} = (\cos\theta, \sin\theta)$ or $(-\cos\theta, -\sin\theta)$, where θ is the angle the LOP makes with the x-axis, measured counter-clockwise. Then the minimum distance of a point (x, y) to LOP$_i$ is

$$r_i(x, y) = |d_{1i}(g_{2i} - y) - d_{2i}(g_{1i} - x)| \tag{A2}$$

   where d_{1i} and d_{2i} are the two components of the vector $\mathbf{d}$ for LOP$_i$, and g_{1i} and g_{2i} are the two components of the vector $\mathbf{g}$ for the same LOP.

3. LOPs of the form $a = x \sin Z + y \cos Z$, where a is the minimum distance between the LOP and the origin, and Z is the azimuth of that minimum, measured at the origin clockwise from the y-axis. Then the distance of (x, y) to LOP$_i$ is just

$$r_i(x, y) = a_i - x \sin Z_i - y \cos Z_i \tag{A3}$$

where a_i and Z_i are the values of a and Z for LOP_i.

The distance formulas above for forms 1 and 2 can be obtained from expressions in Hummel (1965) and other texts. Form 3 applied to celestial LOPs has been used by DeWit (1974) and HMNAO (2015) in developments for a least-squares solution for a fix, which is now Procedure 11 in the back of the Nautical Almanac.

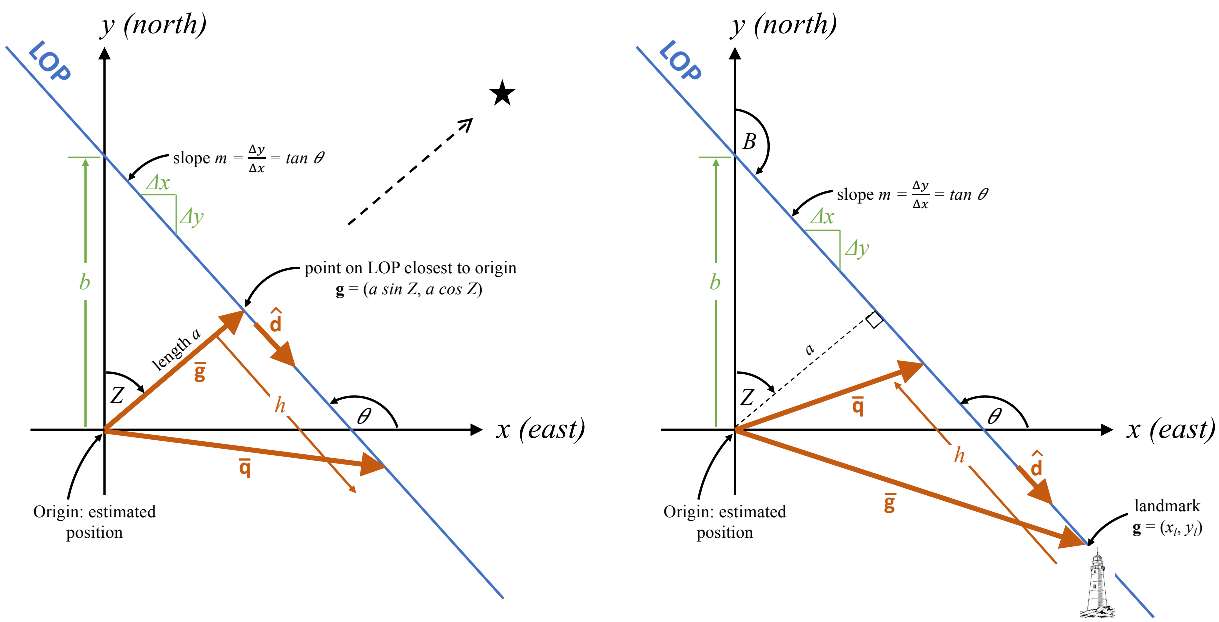

Figure A1 Variables used to describe an LOP defined by an observation of (a) a celestial object's angular altitude, or (b) the bearing of a landmark. The azimuthal angles are related as follows: $\theta = 180° - Z = 270° - B; Z = B - 90°$.

The remaining piece involves translating measured parameters ("observables") to one of the above LOP representations. For a reduced celestial observation yielding an intercept a (observed minus computed altitude) at star azimuth Z, form 3 can be used directly. For form 1, $m = -\tan Z$ and $b = a \sec Z$. For form 2, $\mathbf{d} = (\cos Z, -\sin Z)$ and $\mathbf{g} = (a \sin Z, a \cos Z)$.

For a bearing-line observation of a landmark at (x_l, y_l) measured at true bearing B, for form 1, $m = \cot B$ and $b = y_l - m x_l$. For form 2, $\mathbf{d} = (\sin B, \cos B)$ and $\mathbf{g} = (x_l, y_l)$. For form 3, $Z = B - 90°$ and $a = y_l \sin B - x_l \cos B$. The landmark's coordinates for this purpose are $x_l = 60(\lambda_l - \lambda_o)\cos\phi_o$ and $y_l = 60(\phi_l - \phi_o)$, where λ_l and λ_o are the longitudes of the landmark and the origin (at the estimated position of the observer), respectively, ϕ_l and ϕ_o are the latitudes of the two points, and the factor 60 converts from degrees to nautical miles.[1] The landmark does not have to lie within the grid used for evaluating the probability values, although if it is far outside the grid, spherical trigonometry formulas may have to be used.

[1]See footnote 2 of the main text. The use of 60 nautical miles per degree is an approximation that can result in $\leq 0.5\%$ errors (up to 100 m over 10 nmi) in the grid coordinates of the landmark. This is greater than the errors due to the "flat Earth" approximation for 10 nmi square grids at $|\phi_0| < 70°$.

Appendix B: LOP Triangle Probabilities for Observations with Different Uncertainties

This appendix considers the total probability that the observer's true position is within the triangle formed by three LOPs defined by observations of different uncertainties. The main text considers the simpler problem where the observations have equal uncertainty.

According to standard statistical theory, if the observations, labeled $i = 1, 2,$ and 3, have different uncertainties σ_i, then in determining a fix by least squares (or equivalent means), each observation should be assigned a weight w_i proportional to $1/\sigma_i^2$. No change is required in the scheme for computing the probability density function described in this paper, because the $1/\sigma_i^2$ factor is already included in eqs. (1) and (2).

The general computational strategy was described in the main text; a large number of test cases were generated in a Monte-Carlo scheme, each with three LOPs defined by synthetic observations. The errors in the positions of the LOPs were randomly taken from three normal distributions, characterized by standard deviations of $\sigma_{\mathrm{LOP(1)}}$, $\sigma_{\mathrm{LOP(2)}}$, and $\sigma_{\mathrm{LOP(3)}}$. For each test case, a 101×101 grid of probabilities was generated using eq. (2) of the main text, representing the area surrounding the observer's estimated position, and the total probability within the LOP triangle computed. A test run consisting of 100 of these randomly generated test cases therefore generated 100 computed probabilities that the observer's true position is within the LOP triangle. An independent weighted least-squares solution for the fix location was also computed for each test case.

In the first set of runs it was assumed that the σ_i values to be used in eq. (2) and for the observational weights in the least-squares fits were the same as the known standard deviations of the error distributions of the LOP positions, $\sigma_{\mathrm{LOP(i)}}$, which were input values for each run. The results of these runs are given in Table B1, col. 3, and the numbers listed there are similar to those shown in the same column in Table 1 in the main text.

Table B1. Probabilities of Observer Inside LOP Triangle for 3 LOPs of Different Uncertainties

Run	$\sigma_{\mathrm{LOP(i)}}$	Avg. probability (%) with $\sigma_i = \sigma_{\mathrm{LOP(i)}}$	Avg. probability (%) with $\sigma_i = f \times \sigma_{\mathrm{LOP(i)}}$	No. actually inside out of 100
1	0.8, 1.0, 1.2	25.1 ± 20.3	33.8 ± 0.8	30
2	1.3, 0.7, 0.9	27.1 ± 22.4	34.4 ± 1.0	24
3	1.1, 1.1, 0.8	26.3 ± 25.7	33.9 ± 0.7	23
4	0.8, 0.6, 0.9	25.5 ± 20.4	33.7 ± 0.7	21
5	0.7, 1.2, 0.6	28.1 ± 19.7	34.7 ± 1.1	26
6	0.9, 0.8, 0.6	23.0 ± 24.4	33.8 ± 0.7	20
7	1.1, 1.6, 1.4	23.9 ± 20.9	33.9 ± 0.9	29
8	1.3, 0.8, 1.0	23.2 ± 21.9	34.0 ± 0.8	24
9	1.4, 1.6, 0.9	26.4 ± 23.7	34.2 ± 0.8	25
10	1.1, 1.4, 0.8	24.8 ± 21.6	34.4 ± 0.9	23
Avg. of all runs		25.3	34.1	24.5

Each run is 100 random test cases

In the second set of runs, it was assumed that the σ_i values to be used in eq. (2) and for the observational weights were not well known in an absolute sense but that their ratios should be preserved, which also preserves the relative weights. In these runs, the σ_i values were normalized based on the variance of the residuals (variance of the fit), that is, the average value of the square of the distances between the fix (= the point of maximum probability) and the LOPs, each distance appropriately weighted. The variance of the residuals, v, is computed according to the formula

$$v = \frac{\dfrac{1}{N-p} \displaystyle\sum_{i=1}^{N} \left\{ \frac{1}{\sigma_i^2} \Big[r_i(x_f, y_f) \Big]^2 \right\}}{\dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \frac{1}{\sigma_i^2}} \tag{B1}$$

from Bevington & Robinson (2003), where N is the number of observations (here, $N = 3$) p is the number of parameters to be determined (here, $p = 2$, for the coordinates of the fix); and $N - p = \nu$ is the number of degrees of freedom in the problem. The quantity $r_i(x_f, y_f)$ is a residual, the distance of LOP$_i$ from the fix at (x_f, y_f). Note that this expression is insensitive to changes in the σ_i values as long as their ratios are preserved and the same set of LOPs is used. If $w_{\rm avg} = (1/N) \sum(1/\sigma_i^2)$ is the average weight, then in each test case in this second set of runs, the normalization condition was

$$v \cdot w_{\rm avg} = 1 \tag{B2}$$

and, if not satisfied, then each uncertainty σ_i was multiplied by $f = \sqrt{v \cdot w_{\rm avg}}$ to make it so. The software then re-evaluated eq. (2) for the entire grid using the normalized σ_i values. Multiplication by a constant maintains the ratios of the σ_i and w_i values and, as previously noted, does not affect the value of v. The above condition can also be expressed as

$$\frac{1}{N-p} \sum_{i=1}^{N} \left\{ \frac{1}{\sigma_i^2} \Big[r_i(x_f, y_f) \Big]^2 \right\} = \chi_\nu^2 = 1 \tag{B3}$$

where χ_ν^2 is the chi-squared per degree of freedom. The condition $\chi_\nu^2 = 1$ should hold if the problem has been properly modeled and the σ_i values are consistent with the distribution of residuals.

The results for the set of runs with the σ_i values normalized, in each case, by the factor f are given in Table B1, col. 4. Again, these results are similar to those in the same column of Table 1 in the main text, and are higher than the probabilities reported in col. 3. Cautions expressed in the main text regarding the use of RMSE values for computing probabilities apply equally to the normalized σ_i values used to produce the probabilities listed in col. 4 here. That is, the normalization, based on the variance of the post-fit residuals, tends to inflate the probabilities unrealistically for problems where N is small.

The least-squares fits were unaffected by the change in the σ_i. The least-squares software uses weights relative to the average weight, $w_{\rm avg}$, as per Bevington & Robinson (2003), so that the computed formal errors of x_f and y_f are not affected by changes in the magnitudes of the σ_i as long as the ratios remain the same.

As noted in the main text, for each test case, the software used for these tests can determine whether the observer's hypothetical position is actually within the LOP triangle. For each run of 100 cases, the number of observer positions inside the triangles is listed in Table B1, col. 5. These numbers, considered to be a percentage, are comparable to those listed in col. 3.

For observations that all have the same unknown uncertainty σ, the formula for variance reduces to $v = \sum[r_i(x_f, y_f)]^2/(N-p)$, the square root of which is the root mean squared error (RMSE) referred to in the main text. For these cases, $w_{\rm avg} = 1/\sigma^2$, so the above normalization condition becomes $\mathrm{RMSE}^2/\sigma^2 = 1$, that is, $\sigma = \mathrm{RMSE}$. This was the value used in eq. (2) for the second set of computational test runs described in Sec. 4.1 of the main text. Therefore, $\chi^2_\nu = 1$ describes these cases also.

References

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