Appendix for the paper: 'The 3-step hedge-based valuation: fair valuation in the presence of systematic risks'

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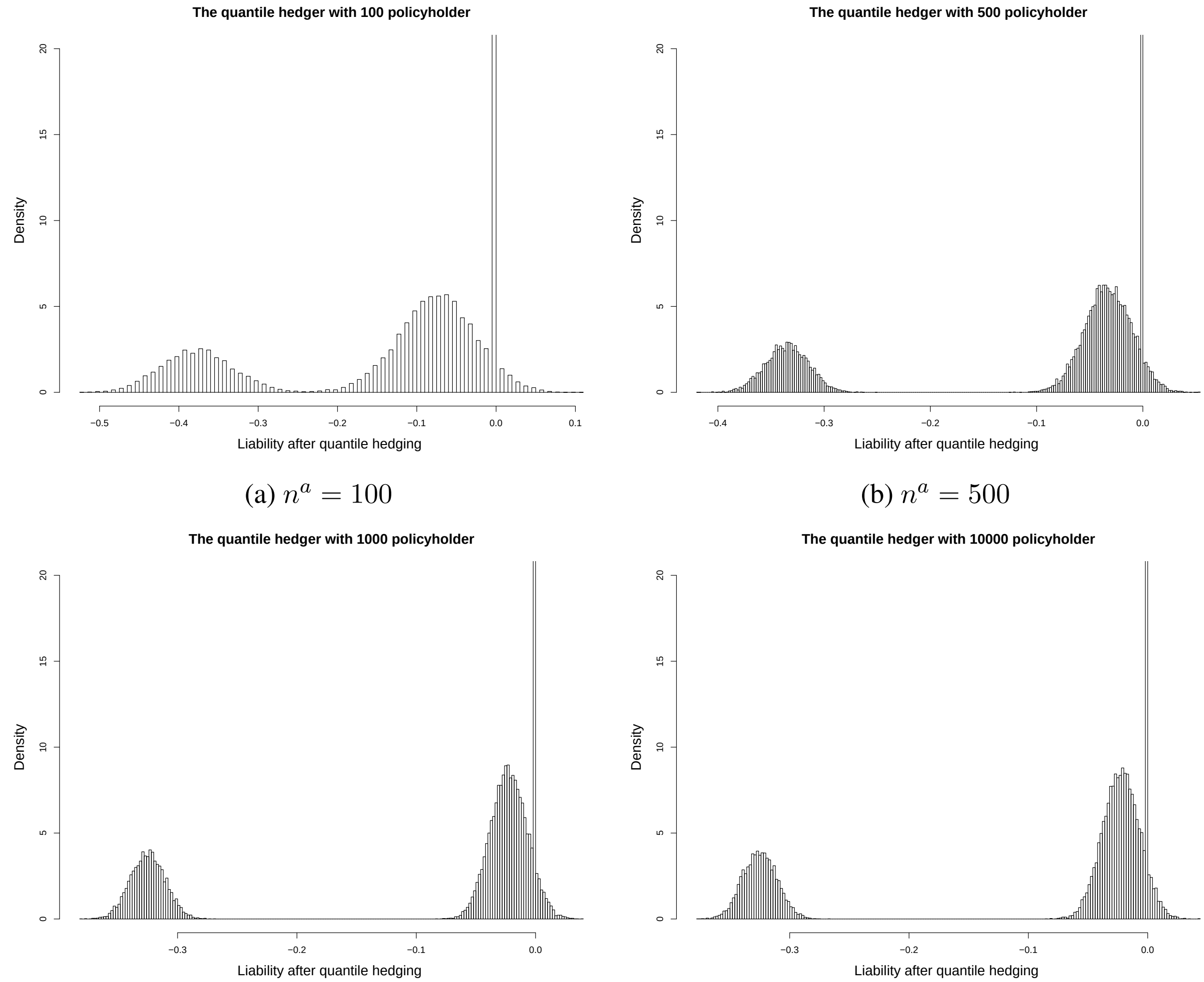
1 Additional material for Section 4.3

1.1 3-step valuation using quantile hedging valuation

The previous susbsections consider 3-step valuations where the standard deviation principle is used for the valuation of diversifiable claims. In this subsection, we directly use a quantile hedging approach for the residual claim, without using the conditional standard deviation principle; see Example 6. For simplicity, we put $i = 1$. This also makes the results comparable with the valuation based on a conditional standard deviation principle. The function l is given by $l(x) = \frac{p}{1-p}x_+ + x_-$, where $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$; see Koenker and Bassett (1978). The hedger $\boldsymbol{\eta}$ is defined in (4.2) and one can show that the hedger satisfies: $\text{VaR}_p\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} - \boldsymbol{\eta}_{S-\boldsymbol{\theta}_S \cdot \boldsymbol{Y}} \cdot \boldsymbol{Y}\right] = 0$. We use simulation to find the hedger $\boldsymbol{\eta}$. Figure 1 shows the liability after we take into account the quantile hedging strategy $\boldsymbol{\eta}$ is used to cover the residual part of the claim. Only in a few situations this liability is strictly positive, which implies that in the majority of the situations, the amount available to the insurer is sufficient to cover the realization of the claim.

1.2 Comparison between the valuations

Table 1 shows the values of the different 3-step hedge-based valuations considered in this example, for different portfolio sizes. If the portfolio size increases, the value of the liability decreases because the claim is more diversified. The valuation described in Section 4.3.3 uses a risk measure (the VaR in this particular case) for the valuation of the systematic part, whereas the valuation described in Section 4.3.4 uses a pricing measure (the Esscher transform). Therefore, the values obtained using the Esscher transform are smaller than the values determined using the VaR. The last column of Table 1 contains the values of the claim when a quantile hedging approach is used for the residual part of the claim.



(a) $n^a = 100$

(b) $n^a = 500$

(c) $n^a = 1000$

(d) $n^a = 10000$

Figure 1: Histogram of the liability S given by (4.13) when a quantile hedger is used to determine the value of the residual part.

Table 1: The valuation of the hybrid claim in (4.14) using different approaches of the systematic valuation.

Number of policyholders n^a	Conditional standard deviation		Quantile hedging
	VaR	Esscher transform	
100	0.387	0.319	0.33
500	0.360	0.305	0.310
1000	0.354	0.302	0.305
10000	0.343	0.296	0.298

2 Numerical results for Section 5.2

We determine numerical values for the claim S given by (5.13), which represents the per-policy liability of a unit-linked portfolio. The parameter values used to obtain the numerical values are summarized in Table 2. Table 3 shows the numerical values for the claim S for different portfolio sizes and different valuations. Columns 2, 3 and 4 are valuations based on the modified standard deviation principle introduced in Example 5, for different choices of ρ^s. More precisely, in columns 2 and 3, we use the valuation $\rho^{Esscher}$ which was derived in (5.19). By changing the parameter θ^s, we consider different specifications for the Esscher transform. The fourth column assumes the valuation ρ^s is based on the VaR; see also Section 5.1.2. The last column is the hedge-based valuation ρ^{HB} which was derived in Section 5.2.3. The hedge-based valuation leads to the lowest values, whereas the valuation using the VaR leads to the largest values. Increasing the portfolio size leads to lower values for the claim.

The hedge-based valuation determines the hedgeable part by using the mean-variance hedger. The residual part of the claim is valuated using the standard deviation principle. This valuation does not distinguish between the systematic and the diversifiable risks. Indeed, only the distribution function of $S - H_S^h$ is used to determine the value of the residual part. The 3-step hedge-based valuation considers the conditional distribution of $S - H_S^h$, given the systematic risk Z. By putting more weight on the 'bad' scenarios, we increase the value of the claim. This effect is illustrated in Figure 2, where we determine the histogram for the residual part $S - H_S^h$, given a value for the random variance Z. In the left panel, we have that $Z = -0.1$, which corresponds with a bad scenario where more policyholders than expected survive, leading to an underhedging of the claim S using the mean-variance hedge. The right panel considers the case where $Z = -0.9$, which corresponds with a good longevity scenario. Note, for example, that $\mathbb{E}\left[S - H_S^h\right] = 0$, whereas $\mathbb{E}\left[S - H_S^h \middle| Z = -0.1\right] = 0.4$ and $\mathbb{E}\left[S - H_S^h \middle| Z = -0.1\right] = -0.3$.

Table 2: Parameter values for the unit linked contract. The distribution for the stock returns are given by (5.15) and the distribution for the systematic longevity risk is given by (5.18).

Contract		
Premium of the policyholder	P	0.663743
Maturity	T	15
Internal rate of return	v	0.009
Bonus share	α	0.4019
Distributions		
Mean of the stock return	μ^f	0.06
Vol of the stock return	σ^f	0.2
Mean of the longevity risk	μ^s	-0.5
Vol of the longevity risk	σ^s	0.2
Valuation		
Systematic valuation	θ^s	-0.95
Actuarial valuation	β	0.3
Risk free rate	r	0

Table 3: The valuation of a portfolio with unit-linked contracts using different approaches of the systematic valuation. The second and third column contain the values when an Esscher transform is used a systematic valuation, with different values for the parameter θ^s. The third column uses the Value-at-Risk to determine the systematic value of the claim. The last column is the hedge-based valuation using a standard deviation principle.

Number of	Esscher transform		VaR	Hedge-Based
policyholders	$\theta^s = -0.95$	$\theta^s = -0.1$		
5	0.729	0.709	0.988	0.694
10	0.717	0.696	0.971	0.679
100	0.696	0. 674	0.947	0.661
1000	0.689	0.667	0.939	0.659
10000	0.687	0.665	0.936	0.659

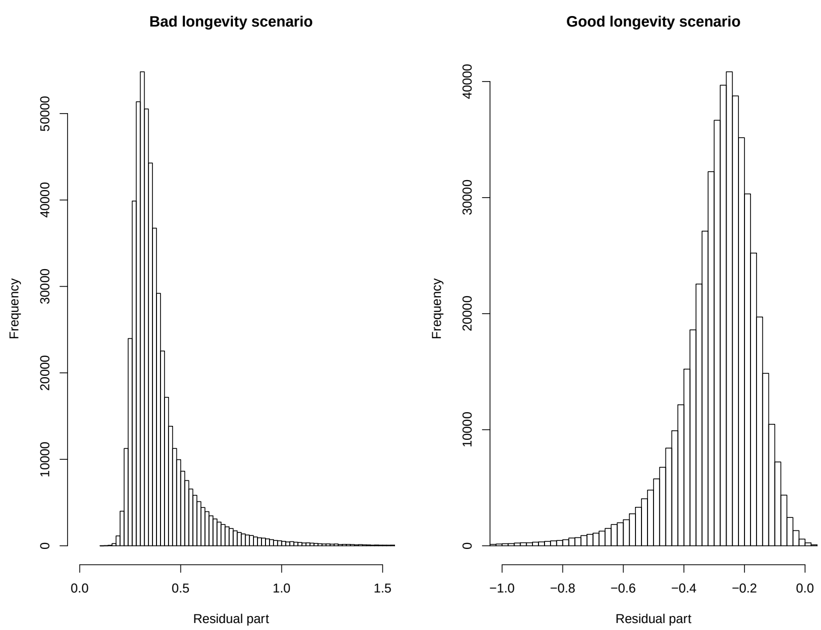


Figure 2: The histogram of the conditional distribution of the residual part $S - H_S^h$ if $Z = -0.1$ (left) and $Z = -0.9$ (right).

3 The additive 3-step valuation

Example 1 (The hedgeable part in a complete financial market) Consider the probability space $(\Omega, \mathbb{P})$ with the universe Ω defined as follows: $\Omega = \{(\omega_{1,i}, \omega_{2,j}) \mid i, j = 1, 2\}$. We have that: $\mathbb{P}[(\omega_{1,i}, \omega_{2,j})] = \frac{1}{4}$, for $i, j = 1, 2$. The financial outcomes are denoted by ω_1^f and ω_2^f and $\omega_i^f = \{(\omega_{1,i}, \omega_{2,j}) \mid j = 1, 2\}$. We assume a stock is traded with time-T value given by Y_1 and $Y_1\left[\omega_1^f\right] = 50$ and $Y_1\left[\omega_2^f\right] = 100$. One can then verify that $\mathbb{E}[Y_1] = 75$ and $\text{Var}[Y_1] = 25^2$. The event ω_1^a corresponds with the survival of the policyholder to time T. We have that $\omega_j^a = \{(\omega_{1,i}, \omega_{2,j}) \mid i = 1, 2\}$. Define the random variable X_1 which takes value one in case the policyholder survives to time T and zero otherwise: $X_1[\omega_1^a] = 1$ and $X_1[\omega_2^a] = 0$. One can then verify that $\mathbb{P}[X_1 = 1] = 1/2$ and, moreover, the random variables X_1 and Y_1 are independent. We also assume that $r = 0$, hence $Y_0 = 1$.

We consider the following claim S:

$$S = X_1 \times (Y_1 - 50)_+ . \tag{3.1}$$

Then the claim S has the same structure as (6.6). To keep the setting as simple as possible and since our focus is here on the hedgeable part of the claim and not the diversifiable part, we assume for this example that $n^a = 1$, however, one can generalize the example and include $n^a > 1$. We can then define the 'hedgeable' part given by (6.7), which we denote by D^h. We have that

$$D^h\left[\omega_1^f\right] = 0 \quad \text{and} \quad D^h\left[\omega_2^f\right] = \frac{50}{2}. \tag{3.2}$$

Note that hedging a call option with strike 50 and maturity T in this market model, requires to buy one unit of the stock (i.e. the delta is 1) and put the amount -50 in the risk-free bank account. The survival probability is 1/2 and therefore, the hedgeable part D^h replicates $1/2$ call option.

The mean-variance hedger $\boldsymbol{\theta}_S$ is given as follows $\boldsymbol{\theta}_S = \left(\frac{1}{2}, \frac{-50}{2}\right)$. The hedgeable part H_S^h defined in (6.1) using the mean-variance hedger is then given by

$$H_S^h\left[\omega_1^f\right] = 0 \quad \text{and} \quad H_S^h\left[\omega_2^f\right] = \frac{50}{2}. \tag{3.3}$$

Comparing (3.2) and (3.3) we conclude that $H_S^h = D^h$. The hedgeable part (6.6) defined in Deelstra et al. (2020) is indeed hedgeable and corresponds with the hedgeable part defined in this paper in (6.1) when the mean-variance hedger is used.

Example 2 (The hedgeable part in an incomplete financial market) In this example we repeat the previous example, but we now consider a financial market which is incomplete. More concrete, we define the universe as follows: $\Omega = \{(\omega_{1,i}, \omega_{2,j}) \mid i = 1, 2, 3 \text{ and } j = 1, 2\}$. We have that: $\mathbb{P}[(\omega_{1,i}, \omega_{2,j})] = \frac{1}{6}$, for $i = 1, 2, 3$ and $j = 1, 2$. The time-T stock price is denoted by Y_1 and we now have 3 possible outcomes for the stock price: $\omega_i^f = \{(\omega_{1,i}, \omega_{2,j}) \mid j = 1, 2\}$. We assume that $Y_1\left[\omega_1^f\right] = 0$, $Y_1\left[\omega_2^f\right] = 50$ and $Y_1\left[\omega_3^f\right] = 100$. One verify that $\mathbb{P}[Y_1 = x] = 1/3$ for $x = 0, 50, 100$. Moreover, we have that $\mathbb{E}[Y_1] = 50$ and $\text{Var}[Y_1] = 50^2 \times 2/3$. The random variable X_1 is defined similarly as in the previous example and gives the value 1 in case the policyholder survives to time T and zero otherwise. We have that $\mathbb{E}[X_1] = 1/2$. The hybrid claim S is defined by (3.1).

The 'hedgeable' part (6.7) defined in Deelstra et al. (2020) is again denoted by D^h and we have that

$$D^h\left[\omega_1^f\right] = D^h\left[\omega_2^f\right] = 0 \quad \text{and} \quad D^h\left[\omega_3^f\right] = 50. \tag{3.4}$$

Note that this results in 3 equations which cannot be solved if we can only invest in the stock Y_1 and the risk-free bank account. We conclude that the 'hedgeable' part D^h cannot be replicated by traded assets. Therefore, the valuation of the 'hedgeable' part is not solely based on traded prices and requires a choice of the risk neutral probability measure.

The mean-variance hedger $\boldsymbol{\theta}_S$ for the claim S is given by $\boldsymbol{\theta}_S = \left(-\frac{25}{6}, \frac{1}{4}\right)$. The hedgeable part H_S^h defined in (6.1) when using the mean-variance hedger is then given by

$$H_S^h\left[\omega_1^f\right] = -\frac{25}{6}, \quad H_S^h\left[\omega_2^f\right] = \frac{25}{3} \quad \text{and} \quad H_S^h\left[\omega_3^f\right] = 25 \times \frac{5}{6}. \tag{3.5}$$

Observing Expressions (3.4) and (3.5), we conclude that the hedgeable part of a product claim used in Deelstra et al. (2020) differs from the hedgeable part H_S^h we use in this paper. Moreover, the hedgeable part H_S^h can, by construction, be replicated using traded assets, whereas the claim D^h is not replicable.

4 Appendix: Details on the numerical example in Section 2

4.1 The financial risks

We assume that the stock Y_1 follows a geometric Brownian motion:

$$\mathrm{d}Y_1 = P\left(\mu^f \mathrm{d}t + \sigma^f \mathrm{d}W^f\right),$$

where $\left\{W^f(t) \mid t \in [0,T]\right\}$ is a Brownian motion. Then:

$$Y_1 = P\mathrm{e}^{\left(\mu^f - \frac{1}{2}\left(\sigma^f\right)^2\right)T + \sigma^f\sqrt{T}\Phi^{-1}(U)},$$

where Φ is the distribution function of a standard normal distribution. We have that

$$\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[S^f\right] = \mathrm{e}^{-rT} + \alpha\left(P\Phi(d_v) - P(1+i)^T\mathrm{e}^{-rT}\Phi(d_2)\right), \tag{4.1}$$

where

$$d_1 = \frac{\left(r + \frac{1}{2}\left(\sigma^f\right)^2\right)T - \log(1+v)^T}{\sigma^f\sqrt{T}}, \text{ and } d_2 = d_1 - \sigma^f\sqrt{T}.$$

4.2 Force of mortality

We assume that the force of mortality of the remaining lifetime of the policyholders is a stochastic process. The force of mortality at time t is denoted by λ_t and follows an Ornstein-Uhlenbeck process:

$$\mathrm{d}\lambda = \mu_\lambda \mathrm{d}t + \sigma_\lambda \mathrm{d}W^s, \tag{4.2}$$

where $\{W^s(t)|\, t \in [0,T]\}$ is a Brownian motion independent from the Brownian motion W^f. Define the random variable Z as follows: $Z = \mathrm{e}^{-\int_0^T \lambda_s \mathrm{d}s}$. One can prove that Z has a normal distribution: $Z \stackrel{d}{=} N\left(\mu^s, (\sigma^s)^2\right)$, where

$$\begin{aligned} -\mu^s &= \lambda_0 \zeta \\ (\sigma^s)^2 &= \frac{\sigma_\lambda^2}{\mu_\lambda^2}\left(\frac{\mu_\lambda}{2}\zeta^2 - \zeta + T\right) \\ \zeta &= \frac{\mathrm{e}^{\mu_\lambda T} - 1}{\mu_\lambda}. \end{aligned}$$

References

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