Supplementary materials for *optimal commissions and subscriptions in mutual aid platforms*

Yixing Zhao[a], Yan Zeng[b,*]

[a] *School of Finance, Guangdong University of Foreign Studies, Guangzhou, 510006, PR China*

[b] *Lingnan (University) College, Sun Yat-sen University, Guangzhou, 510275, PR China*

Appendix A. Proofs of theorems and equations

The model under limiting case. Considering the limiting case is reasonable in mutual aid platform because most mutual aid plans require at least one million participants. Some top mutual platforms have over 100 million participants (See table 1).

Table 1: Number of participants in mutual aid platforms

Platform	Number of participants (million)
Xianghubao	104.7
Waterdrop Mutual Aid	103.1
Qingsong Mutual Aid	80.0
e Mutual Aid	3.4
Kangai Gongshe	2.6

The variance of the percentage of population suffering the illness is $\frac{(1-p)p}{1{,}000{,}000}$, which is a relatively small number compared to a percentage p. Thus, it is sufficient to consider the continuous model as the limiting case. Therefore, equations in the model can be viewed as the limiting case as well. As discussed above, we consider that our model in under the limiting case. Thus, $\frac{l_i}{m_i}$, which is the percentage of participants who join the plan, is also under the limiting case. Recall that m_i is the total mass of type-i potential participants. If we restrict $\sum_{i=1}^{n} m_i = 1$, m_i can be regarded as the proportion of type-i potential participants among all participants. Suppose $N_{i,T}$ is the population of type-i potential participants and N_T is the population of all potential participants. We further assume that $N_i/N_{i,T} = l_i/m_i$, which means the percentages of participants who join the plan among all type-i participants are the same under both limiting case and non-limiting case. We further have

$$\frac{N_{i,T}}{N_T} \to m_i \quad \text{as all } N_i \to \infty. \tag{1}$$

*Corresponding author. Tel.: +86 20 84110516; fax: +86 20 84114823. E-mail addresses: zengy36@mail.sysu.edu.cn.

Thus, the equation

$$\sum_{j\in\mathcal{N}}(N_i-\boldsymbol{n}_i)S_i=\sum_{j\in\mathcal{N}}\boldsymbol{n}_iI_i$$

can be revised as

$$\sum_{i\in\mathcal{N}}N_{i,T}\frac{l_i}{m_i}\left(1-\frac{\boldsymbol{n}_i}{N_i}\right)S_i=\sum_{i\in\mathcal{N}}N_{i,T}\frac{l_i}{m_i}\frac{\boldsymbol{n}_i}{N_i}I_i.$$

Dividing both sides of the equation by N_T and taking the limit we have

$$\sum_{i\in\mathcal{N}}m_i\frac{l_i}{m_i}\left(1-p_i\right)s_i=\sum_{i\in\mathcal{N}}m_i\frac{l_i}{m_i}p_iI_i,$$

which yields

$$\sum_{i\in\mathcal{N}}l_i\left(1-p_i\right)s_i=\sum_{i\in\mathcal{N}}l_ip_iI_i.$$

Therefore, we can consider our continuous model as the limiting case and expressions such as $(1-p_i)s_i$ and $l_ip_iI_i$ are used in our model. Our results are based on such modeling framework. □

The generalization of the model. Our model is also valid under the limiting case when loss amounts and benefit amounts are random variables rather than constants. Now we suppose that $\boldsymbol{X}_i$ are random variables for all i. Then $\boldsymbol{X}_i$=$\boldsymbol{I}_i$ are also random variables. Let $\boldsymbol{I}_{i,j}$ be the random loss of j-th participant in the group-i. If we further assume that $\boldsymbol{I}_{i,j}$ are iid with finite second moment for all $j=1,\ldots,\boldsymbol{n}_i$. Then the total payment amount from type-i becomes $\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}$ which is a compound rv with

$$\mathbb{E}\left[\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}\right]=E[\boldsymbol{n}_i]E[\boldsymbol{I}_i]=N_ip_i\mathbb{E}[\boldsymbol{I}_i]$$

$$Var\left(\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}\right)=N_ip_iVar(\boldsymbol{I}_i)+N_ip_i(1-p_I)\mathbb{E}[\boldsymbol{I}_i]^2\leq N_ip_i\mathbb{E}[\boldsymbol{I}_i^2].$$

Furthermore,

$$\mathbb{E}\left[\frac{1}{N_ip_i}\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}\right]=E[\boldsymbol{I}_i],\qquad Var\left(\frac{1}{N_ip_i}\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}\right)\leq\frac{\mathbb{E}[\boldsymbol{I}_i^2]}{N_ip_i}\to0,\quad N_i\to\infty.$$

By Chebyshev's inequality, we have

$$\Pr\left(\left|\frac{1}{N_ip_i}\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}-E[\boldsymbol{I}_i]\right|\geq c\right)\leq c^{-2}Var\left(\frac{1}{N_ip_i}\sum_{j=1}^{\boldsymbol{n}_i}\boldsymbol{I}_{i,j}\right)\to0,\quad\forall c>0.$$

Therefore, in one risk group case, we still have

$$S_i = \frac{1}{N_i - \boldsymbol{n}_i} \sum_{j=1}^{\boldsymbol{n}_i} \boldsymbol{I}_{i,j} = \frac{N_i p_i}{N_i - \boldsymbol{n}_i} \frac{1}{N_i p_i} \sum_{j=1}^{\boldsymbol{n}_i} \boldsymbol{I}_{i,j} = \frac{p_i}{1 - \frac{\boldsymbol{n}_i}{N_i}} \frac{1}{N_i p_i} \sum_{j=1}^{\boldsymbol{n}} \boldsymbol{I}_{i,j} \to \frac{p\mathbb{E}[\boldsymbol{I}_i]}{1 - p_i}, \quad N_i \to \infty.$$

Similar argument can also be applied to the cases with multiple risk groups. Therefore, our model is valid under the limiting case when loss amounts are either constants or random variables. □

The limit s_i of S_i. Note that S_i is a function of random variables r_i, which is the random percentage of participants suffering the illness, given by the following equation

$$S_i = \frac{w_i \sum_{j \in \mathcal{N}} l_j r_j I_j}{\sum_{j \in \mathcal{N}} l_j (1 - r_j) w_j}, \quad \forall i \in \mathcal{N}.$$

Moreover, we have $r_i = \frac{\boldsymbol{n}_i}{N_i}$, where $\boldsymbol{n}_i$ is the random number of illed participants, as illustrated in the Appendix. When the number of participants N_i goes infinity, $r_i = \dfrac{\boldsymbol{n}_i}{N_i}$ converges to p_i almost surely for all i. That is

$$\Pr\left(\lim_{N_i \to \infty} r_i = p_i\right) = 1.$$

We can also note that s_i is a function of p_i, which is expressed as

$$s_i = \frac{w_i \sum_{j \in \mathcal{N}} l_j p_j I_j}{\sum_{j \in \mathcal{N}} l_j (1 - p_j) w_j}, \quad \forall i \in \mathcal{N}.$$

We recall the continuous mapping theorem. Let $\{X_n\}$ and X be random variables and g be a continuous function with $\Pr(X \in D_g) = 0$, where D_g is the set of discontinuity points. The continuous mapping theorem states that if $X_n \xrightarrow{a.s.} X$ then $g(X_n) \xrightarrow{a.s.} g(X)$. Consider the function $g_i(x_1, \ldots, x_n) = \dfrac{w_i \sum_{j \in \mathcal{N}} l_j x_j I_j}{\sum_{j \in \mathcal{N}} l_j (1 - x_j) w_j}$. We have $S_i = g_i(r_1, \ldots, r_n)$ and $s_i = g_i(p_1, \ldots, p_n)$. By the continuous mapping theorem, we have $S_i \xrightarrow{a.s.} s_i$,that is

$$\Pr\left(\lim_{\forall N_i \to \infty} S_i = s_i\right) = 1.$$

□

Proof of Equation (6). We assume that w_i is the weight factor of the type-i participants' payments. Thus, we have

$$\frac{S_i}{S_j} = \frac{w_i}{w_j}, \quad \forall i, j \in \mathcal{N}.$$

Then, we let $S_i = \frac{w_i}{w_1} S_1$ and plug it into Equation (5) and obtain

$$S_1 = \frac{w_1 \sum_{j \in \mathcal{N}} l_j r_j I_j}{\sum_{j \in \mathcal{N}} l_j (1 - r_j) w_j}.$$

Similarly, for all $i \in \mathcal{N}$, we have

$$S_i = \frac{w_i \sum_{j\in\mathcal{N}} l_j r_j I_j}{\sum_{j\in\mathcal{N}} l_j (1-r_j) w_j}.$$

□

Proof of (9). Under the equivalence principle, $p_i I_i = (1-p_i)s_i$, $i = 1,2$. Under the uniform distribution, $F^{-1}(1-q) = (1-q)\bar{v}$. Then, Equation (1) can be reduced to

$$p_i I_i \alpha + \beta = (1-l_i)\bar{v}, \quad i = 1,2.$$

This reduction yields

$$l_{1,1} = 1 - \frac{p_1 I\alpha + \beta}{\bar{v}}, \quad l_{1,2} = 1 - \frac{p_2 I\alpha + \beta}{\bar{v}}.$$

□

Proof of Equations (10). Note that $s_1 = s_2 = \frac{p_1 I_1 l_1 + p_2 I_2 l_2}{(1-p_1)l_1 + (1-p_2)l_2}$. We consider two cases. In the first case, $(1-p_2)s_2(1+\alpha) + \beta - p_2 I_2 \le \bar{v}$. In this case, not all type-2 participants participate in the plan. Thus, Equation (1) still holds. From Equation (1), we have

$$\begin{aligned}
(1-p_1)(1+\alpha)\frac{p_1 I_1 l_1 + p_2 I_2 l_2}{(1-p_1)l_1 + (1-p_2)l_2} + \beta - p_1 I_1 = (1-l_1)\bar{v},\\
(1-p_2)(1+\alpha)\frac{p_1 I_1 l_1 + p_2 I_2 l_2}{(1-p_1)l_1 + (1-p_2)l_2} + \beta - p_2 I_2 = (1-l_2)\bar{v}.
\end{aligned}$$

After simplification, we have

$$\begin{aligned}
&\frac{(1-p_1)^2 + (1-p_2)^2}{(1-p_1)}\bar{v}{l_1}^2 + \\
&\left[(1-p_1)(p_1 I\alpha + \beta - \bar{v}) + (1-p_2)\Big(p_2 I(1+\alpha) + \bar{v} - \beta + p_2 I - \frac{1-p_2}{1-p_1}(\bar{v} - \beta + p_1 I)\Big)\right] l_1 + \\
&\frac{(1-p_1)}{\bar{v}}\Big(p_2 I(1+\alpha) - \frac{1-p_2}{1-p_1}(\bar{v} - \beta + p_1 I)\Big)\Big(p_2 I(1+\alpha) + \bar{v} - \beta + p_2 I - \frac{1-p_2}{1-p_1}(\bar{v} - \beta + p_1 I)\Big) = 0, \\
&\frac{(1-p_1)^2 + (1-p_2)^2}{(1-p_2)}\bar{v}{l_2}^2 + \\
&\left[(1-p_2)(p_2 I\alpha + \beta - \bar{v}) + (1-p_1)\Big(p_1 I(1+\alpha) + \bar{v} - \beta + p_1 I - \frac{1-p_1}{1-p_2}(\bar{v} - \beta + p_2 I)\Big)\right] l_2 + \\
&\frac{(1-p_2)}{\bar{v}}\Big(p_1 I(1+\alpha) - \frac{1-p_1}{1-p_2}(\bar{v} - \beta + p_2 I)\Big)\Big(p_1 I(1+\alpha) + \bar{v} - \beta + p_1 I - \frac{1-p_1}{1-p_2}(\bar{v} - \beta + p_2 I)\Big) = 0.
\end{aligned} \tag{2}$$

Let

$$a_1 = \frac{(1-p_1)^2 + (1-p_2)^2}{(1-p_1)}\bar{v},$$

$$b_1 = (1-p_1)(p_1 I\alpha + \beta - \bar{v}) + (1-p_2)\Big(p_2 I(1+\alpha) + \bar{v} - \beta + p_2 I - \frac{1-p_2}{1-p_1}(\bar{v} - \beta + p_1 I)\Big),$$

$$c_1 = \frac{(1-p_1)}{\bar{v}}\Big(p_2 I(1+\alpha) - \frac{1-p_2}{1-p_1}(\bar{v} - \beta + p_1 I)\Big)\Big(p_2 I(1+\alpha) + \bar{v} - \beta + p_2 I - \frac{1-p_2}{1-p_1}(\bar{v} - \beta + p_1 I)\Big),$$

$$a_2 = \frac{(1-p_1)^2 + (1-p_2)^2}{(1-p_2)}\bar{v},$$

$$b_2 = (1-p_2)(p_2 I\alpha + \beta - \bar{v}) + (1-p_1)\Big(p_1 I(1+\alpha) + \bar{v} - \beta + p_1 I - \frac{1-p_1}{1-p_2}(\bar{v} - \beta + p_2 I)\Big)$$

and

$$c_2 = \frac{(1-p_2)}{\bar{v}}\Big(p_1 I(1+\alpha) - \frac{1-p_1}{1-p_2}(\bar{v} - \beta + p_2 I)\Big)\Big(p_1 I(1+\alpha) + \bar{v} - \beta + p_1 I - \frac{1-p_1}{1-p_2}(\bar{v} - \beta + p_2 I)\Big).$$

If $b_1^2 < 4a_1c_1$ or $b_2^2 < 4a_2c_2$, Equation (2) does not have real roots, indicating that population equilibrium cannot be reached. Thus, suppose that $b_1^2 \geq 4a_1c_1$ and $b_2^2 \geq 4a_2c_2$. By solving the above equations and omitting the negative roots, we obtain

$$l_{2,1} = \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}, \quad l_{2,2} = \frac{-b_2 + \sqrt{b_2^2 - 4a_2c_2}}{2a_3}.$$

In the second case, $(1-p_2)s_2(1+\alpha) + \beta - p_2 I_2 > \bar{v}$. In this case, all type-2 participants participate in the plan, i.e., $l_{2,2} = 1$. Then, we have

$$(1-p_1)\bar{v}{l_1}^2 + ((1-p_1)(p_1 I\alpha - \bar{v} + \beta) + (1-p_2)\bar{v})l_1 +$$
$$(1-p_1)p_2 I(1+\alpha) - (1-p_2)(\bar{v} + p_1 I - \beta) = 0.$$

Let

$$a_3 = (1-p_1)\bar{v},$$

$$b_3 = (1-p_1)(p_1 I\alpha - \bar{v} + \beta) + (1-p_2)\bar{v}$$

and

$$c_3 = (1-p_1)p_2 I(1+\alpha) - (1-p_2)(\bar{v} + p_1 I - \beta).$$

We obtain

$$l_{2,1} = \frac{-b_3 + \sqrt{b_3^2 - 4a_3c_3}}{2a_3}, \quad l_{2,2} = 1.$$

□

Proof of Theorem 1. According to problem (P1), we write the corresponding Lagrangian function $\mathcal{L}$ as

$$\mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{i \in \mathcal{N}} \alpha_i l_i p_i I_i + \sum_{i \in \mathcal{N}} \beta_i l_i + \sum_{i \in \mathcal{N}} \lambda_i \alpha_i + \sum_{i \in \mathcal{N}} \mu_i \beta_i,$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are KKT multipliers with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. According to the stationary condition, we calculate the derivatives of $\mathcal{L}$ with respect to α_i, β_i and have

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = l_i p_i I_i + \alpha_i p_i I_i \frac{\partial l_i}{\partial \alpha_i} + \beta_i \frac{\partial l_i}{\partial \alpha_i} + \lambda_i = 0, \quad \forall i \in \mathcal{N}, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \beta_i} = \alpha_i p_i I_i \frac{\partial l_i}{\partial \beta_i} + \beta_i \frac{\partial l_i}{\partial \beta_i} + l_i + \mu_i = 0, \quad \forall i \in \mathcal{N}. \tag{4}$$

From the complimentary slackness conditions, we have

$$\lambda_i \alpha_i = 0, \quad \mu_i \beta_i = 0, \quad \forall i \in \mathcal{N}, \tag{5}$$

and

$$\lambda_i \geq 0, \quad \mu_i \geq 0, \quad \forall i \in \mathcal{N}.$$

Under the fair risk exchange scheme, we have $p_i I_i = (1 - p_i) s_i$. The expression for l_i can be rewritten as

$$l_i = m_i \big[1 - F_i \big(\alpha_i p_i I_i + \beta_i \big) \big].$$

Thus,

$$\frac{\partial l_i}{\partial \alpha_i} = -m_i p_i I_i F_i' \big(\alpha_i p_i I_i + \beta_i \big) \quad \frac{\partial l_i}{\partial \beta_i} = -m_i F_i' \big(\alpha_i p_i I_i + \beta_i \big) \quad \forall i \in \mathcal{N},$$

where F_i' is the first-order derivative of F_i. Plugging into Equations (3) and (4) yields

$$l_i p_i I_i - m_i p_i I_i (\alpha_i p_i I_i + \beta_i) F_i' \big(\alpha_i p_i I_i + \beta_i \big) + \lambda_i = 0, \quad \forall i \in \mathcal{N} \tag{6}$$

and

$$-m_i (\alpha_i p_i I_i + \beta_i) F_i' \big(\alpha_i p_i I_i + \beta_i \big) + l_i + \mu_i = 0, \quad \forall i \in \mathcal{N}. \tag{7}$$

Case (i): If $\alpha_i > 0$ and $\beta_i > 0$, we have $\lambda_i = \mu_i = 0$. Both equations (6) and (7) generate

$$-m_i (\alpha_i p_i I_i + \beta_i) F_i' \big(\alpha_i p_i I_i + \beta_i \big) + l_i = 0. \tag{8}$$

Note that $\alpha_i p_i I_i + \beta_i = F_i^{-1}\left(1 - \frac{l_i}{m_i}\right)$ and

$$\begin{aligned}\frac{\mathrm{d}F_i^{-1}\left(1-\frac{l_i}{m_i}\right)}{\mathrm{d}l_i} &= -\frac{1}{m_i}\left[F_i^{-1}\left(1-\frac{l_i}{m_i}\right)\right]' \\ &= -\frac{1}{m_i F_i'\left(F_i^{-1}\left(1-\frac{l_i}{m_i}\right)\right)} \\ &= -\frac{1}{m_i F_i'(\alpha_i p_i I_i + \beta_i)}.\end{aligned}$$

Thus, from Equation (8), we have

$$F_i^{-1}\left(1-\frac{l_i}{m_i}\right) + l_i \frac{\mathrm{d}F_i^{-1}\left(1-\frac{l_i}{m_i}\right)}{\mathrm{d}l_i} = 0. \tag{9}$$

Thus, the optimal participant population $l_i{}^*$ can be determined by

$$F_i^{-1}\left(1-\frac{l_i{}^*}{m_i}\right) + l_i{}^* \frac{\mathrm{d}F_i^{-1}\left(1-\frac{l_i{}^*}{m_i}\right)}{\mathrm{d}l_i{}^*} = 0 \tag{10}$$

, and the optimal commission rate α_i^* and subscription β_i^* are provided by

$$\alpha_i^* p_i I_i + \beta_i^* = F_i^{-1}\left(1-\frac{l_i{}^*}{m_i}\right). \tag{11}$$

Case (ii): If $\alpha_i = 0$ and $\beta_i > 0$, then $\lambda_i \geq 0$ and $\mu_i = 0$. From Equations (6) and (7), we have

$$-m_i \beta_i F_i'(\beta_i) + l_i = -\frac{\lambda_i}{p_i I_i} \tag{12}$$

and

$$-m_i \beta_i F_i'(\beta_i) + l_i = 0 \tag{13}$$

By comparing Equations (12) and (13), we have $\lambda_i = 0$. Note that Equation (8) degenerates to Equation (13) when $\alpha_i = 0$. Thus, Equation (10) still applies in this case, and Equation (11) is rewritten as

$$\beta_i^* = F_i^{-1}\left(1-\frac{l_i{}^*}{m_i}\right). \tag{14}$$

Case (iii): If $\alpha_i > 0$ and $\beta_i = 0$, we can still have $\lambda_i = 0$ and $\mu_i = 0$ by following similar steps as in case (ii). Therefore, we can have

$$\alpha_i^* = \frac{F_i^{-1}\left(1-\frac{l_i{}^*}{m_i}\right)}{p_i I_i}. \tag{15}$$

□

Proof of Proposition 1. If $i = j$, the derivative of s_i with respect to l_i is given by

$$\frac{\partial s_i}{\partial l_i} = w_i \frac{p_i I_i (\sum_{i\in\mathcal{N}} l_i(1-p_i)w_i) - (1-p_i)w_i(\sum_{i\in\mathcal{N}} l_i p_i I_i)}{(\sum_{i\in\mathcal{N}} l_i(1-p_i)w_i)^2}.$$

Let $\frac{\partial s_i}{\partial l_i} > 0$, and we have

$$p_i I_i \sum_{i\in\mathcal{N}} l_i(1-p_i)w_i - (1-p_i)w_i \sum_{i\in\mathcal{N}} l_i p_i I_i > 0,$$

which yields

$$\frac{p_i I_i}{(1-p_i)} > \frac{w_i \sum_{i\in\mathcal{N}} l_i p_i I_i}{\sum_{i\in\mathcal{N}} l_i(1-p_i)w_i} = s_i.$$

If $i \neq j$, the derivative of s_i with respect to l_i is given by

$$\frac{\partial s_i}{\partial l_j} = w_i \frac{p_j I_j (\sum_{i\in\mathcal{N}} l_i(1-p_i)w_i) - (1-p_j)w_j(\sum_{i\in\mathcal{N}} l_i p_i I_i)}{(\sum_{i\in\mathcal{N}} l_i(1-p_i)w_i)^2}, \quad \text{for } i \neq j.$$

Let $\frac{\partial s_i}{\partial l_j} > 0$, and we have

$$p_j I_j \sum_{i\in\mathcal{N}} l_i(1-p_i)w_i - (1-p_j)w_j \sum_{i\in\mathcal{N}} l_i p_i I_i > 0,$$

which yields

$$\frac{p_j I_j}{(1-p_j)} > \frac{w_j \sum_{i\in\mathcal{N}} l_i p_i I_i}{\sum_{i\in\mathcal{N}} l_i(1-p_i)w_i} = s_j.$$

□

Proof of Proposition 2. We consider the following function:

$$G_i(\alpha_i, \beta_i, \boldsymbol{l}) = (1-p_i)(1+\alpha_i)w_i \sum_{j\in\mathcal{N}} l_j p_j I_j - \left(F_i^{-1}\left(1 - \frac{l_i}{m_i}\right) - \beta_i + p_i I_i\right) \sum_{j\in\mathcal{N}} l_j(1-p_j)w_j = 0.$$

Calculating the derivative of G_i with respect to α_i yields

$$\frac{\partial G_i}{\partial \alpha_i} = (1-p_i)w_i \sum_{j\in\mathcal{N}} l_j p_j I_j > 0.$$

Calculating the derivative of G_i with respect to β_i gives

$$\frac{\partial G_i}{\partial \beta_i} = \sum_{j\in\mathcal{N}} l_j(1-p_j)w_j > 0.$$

Calculating the derivative of G_i with respect to l_i yields

$$\begin{aligned}\frac{\partial G_i}{\partial l_i} &= (1-p_i)(1+\alpha_i)w_i p_i I_i - \left(F_i^{-1}\left(1-\frac{l_i}{m_i}\right) - \beta_i + p_i I_i\right)(1-p_i)w_i - F_i^{-1}\left(1-\frac{l_i}{m_i}\right)' \sum_{j\in\mathcal{N}} l_j(1-p_j)w_j \\ &= (1-p_i)(1+\alpha_i)w_i(p_i I_i - (1-p_i)s_i) - F_i^{-1}\left(1-\frac{l_i}{m_i}\right)' \sum_{j\in\mathcal{N}} l_j(1-p_j)w_j.\end{aligned}$$

Note that F_i^{-1} is an increasing function; thus, $F_i^{-1}\left(1-\frac{l_i}{m_i}\right)$ is a decreasing function of l_i. Therefore, we have $F_i^{-1}\left(1-\frac{l_i}{m_i}\right)' \leq 0$, which leads to $\frac{\partial G_i}{\partial l_i} > 0$, and we have

$$\frac{\partial l_i}{\partial \alpha_i} = -\frac{\partial G_i/\partial \alpha_i}{\partial G_i/\partial l_i} < 0 \quad \text{and} \quad \frac{\partial l_i}{\partial \beta_i} = -\frac{\partial G_i/\partial \beta_i}{\partial G_i/\partial l_i} < 0.$$

Calculating the derivative of G_i with respect to l_j $(j \neq i)$ yields

$$\begin{aligned}\frac{\partial G_i}{\partial l_j} &= (1-p_i)(1+\alpha_i)w_i p_j I_j - \left(F_i^{-1}\left(1-\frac{l_i}{m_i}\right) - \beta_i + p_i I_i\right)(1-p_j)w_j \\ &= (1-p_i)(1+\alpha_i)w_i(p_j I_j - (1-p_j)s_j).\end{aligned}$$

If $\frac{p_j I_j}{(1-p_j)} \geq s_j$, we have $\frac{\partial G_i}{\partial l_j} \geq 0$, which leads to

$$\frac{\partial l_j}{\partial \alpha_i} = -\frac{\partial G_i/\partial \alpha_i}{\partial G_i/\partial l_j} \leq 0 \quad \text{and} \quad \frac{\partial l_j}{\partial \beta_i} = -\frac{\partial G_i/\partial \beta_i}{\partial G_i/\partial l_j} \leq 0.$$

If $\frac{p_j I_j}{(1-p_j)} < s_j$, we have $\frac{\partial G_i}{\partial l_j} < 0$, which leads to

$$\frac{\partial l_j}{\partial \alpha_i} = -\frac{\partial G_i/\partial \alpha_i}{\partial G_i/\partial l_j} > 0 \quad \text{and} \quad \frac{\partial l_j}{\partial \beta_i} = -\frac{\partial G_i/\partial \beta_i}{\partial G_i/\partial l_j} > 0.$$

□

Proof of Corollary 1. Calculating derivatives of both sides of Equation (19) with respect to p_i and I_i, respectively, yields

$$\frac{\partial \alpha_i^*}{\partial p_i} = -\frac{F^{-1}\left(1-\frac{l_i{}^*}{m_i}\right)}{p_i^2 I_i}, \quad \frac{\partial \alpha_i^*}{\partial I_i} = -\frac{F^{-1}\left(1-\frac{l_i{}^*}{m_i}\right)}{p_i I_i^2}.$$

Note that $F^{-1}\left(1-\frac{l_i{}^*}{m_i}\right) \geq 0$ since F^{-1} is a distribution function. Thus, we have

$$\frac{\partial \alpha_i^*}{\partial p_i} \leq 0, \quad \frac{\partial \alpha_i^*}{\partial I_i} \leq 0.$$

Calculating derivatives of both sides of Equation (20) with respect to p_i and I_i, respectively, yields

$$\frac{\partial \beta_i^*}{\partial p_i} = 0, \quad \frac{\partial \beta_i^*}{\partial I_i} = 0.$$

□

Proof of Theorem 2. From Equation (21), we have

$$l_1 = m_1 \left(1 - \frac{\alpha p_1 I_1 + \beta}{\bar{v}_1}\right), \quad l_2 = m_2 \left(1 - \frac{\alpha p_2 I_2 + \beta}{\bar{v}_2}\right). \tag{16}$$

Plugging them into Equation (22) yields

$$\mathcal{L} = -\left(\frac{m_1 p_1^2 I_1^2}{\bar{v}_1} + \frac{m_2 p_2^2 I_2^2}{\bar{v}_2}\right)\alpha^2 - 2\left(\frac{m_1 p_1 I_1}{\bar{v}_1} + \frac{m_2 p_2 I_2}{\bar{v}_2}\right)\alpha\beta - \left(\frac{m_1}{\bar{v}_1} + \frac{m_2}{\bar{v}_2}\right)\beta^2 + (m_1 p_1 I_1 + m_2 p_2 I_2)\alpha + (m_1 + m_2)\beta.$$

Calculating the derivative of $\mathcal{L}$ with respect to α and setting it to 0 yields

$$\frac{\partial \mathcal{L}}{\partial \alpha} = -2\left(\frac{m_1 p_1^2 I_1^2}{\bar{v}_1} + \frac{m_2 p_2^2 I_2^2}{\bar{v}_2}\right)\alpha - 2\left(\frac{m_1 p_1 I_1}{\bar{v}_1} + \frac{m_2 p_2 I_2}{\bar{v}_2}\right)\beta + (m_1 p_1 I_1 + m_2 p_2 I_2) = 0. \tag{17}$$

Calculating the derivative of $\mathcal{L}$ with respect to β and setting it to 0 yields

$$\frac{\partial \mathcal{L}}{\partial \beta} = -2\left(\frac{m_1 p_1 I_1}{\bar{v}_1} + \frac{m_2 p_2 I_2}{\bar{v}_2}\right)\alpha - 2\left(\frac{m_1}{\bar{v}_1} + \frac{m_2}{\bar{v}_2}\right)\beta + (m_1 + m_2) = 0. \tag{18}$$

If $(\bar{v}_1 - \bar{v}_2)(p_1 I_1 - p_2 I_2) > 0$ and $(p_1 I_1/\bar{v}_1 - p_2 I_2/\bar{v}_2)(p_1 I_1 - p_2 I_2) > 0$, by solving Equations (17) and (18), we have

$$\alpha^* = \frac{1}{2}\frac{\bar{v}_1 - \bar{v}_2}{p_1 I_1 - p_2 I_2}, \quad \beta^* = \frac{1}{2}\frac{p_1 I_1 \bar{v}_2 - p_2 I_2 \bar{v}_1}{p_1 I_1 - p_2 I_2}.$$

If $(\bar{v}_1 - \bar{v}_2)(p_1 I_1 - p_2 I_2) \leq 0$ or $(p_1 I_1/\bar{v}_1 - p_2 I_2/\bar{v}_2)(p_1 I_1 - p_2 I_2) \leq 0$ $(p_1 I_1 \neq p_2 I_2)$, the optimal solution is on the boundary. We consider two cases: $\alpha = 0$ or $\beta = 0$. We find that the revenue reaches its maximum when $\alpha = 0$ and have the optimal solution

$$\alpha^* = 0, \quad \beta^* = \frac{\bar{v}_1 \bar{v}_2}{2}\frac{m_1 + m_2}{m_1 \bar{v}_2 + m_2 \bar{v}_1}.$$

If $p_1 I_1 = p_2 I_2$, we plug it into Equation (17) and have

$$\alpha^* p_1 I_1 + \beta^* = \frac{\bar{v}_1 \bar{v}_2}{2} \frac{m_1 + m_2}{m_1 \bar{v}_2 + m_2 \bar{v}_1}.$$

If $(\bar{v}_1 - \bar{v}_2)(p_1 I_1 - p_2 I_2) > 0$ and $(p_1 I_1/\bar{v}_1 - p_2 I_2/\bar{v}_2)(p_1 I_1 - p_2 I_2) > 0$, plugging Equation (26) into Equation (16) yields

$$l^*{}_1 = \frac{m_1}{2}, \quad l^*{}_2 = \frac{m_2}{2}.$$

Otherwise, plugging Equation (28) into Equation (16) yields

$$l^*{}_1 = \frac{m_1}{2} \left(1 + \frac{m_2(\bar{v}_1 - \bar{v}_2)}{m_1 \bar{v}_2 + m_2 \bar{v}_1} \right), \quad l^*{}_2 = \frac{m_2}{2} \left(1 + \frac{m_1(\bar{v}_2 - \bar{v}_1)}{m_1 \bar{v}_2 + m_2 \bar{v}_1} \right).$$

If $(\bar{v}_1 - \bar{v}_2)(p_1 I_1 - p_2 I_2) > 0$ and $(p_1 I_1/\bar{v}_1 - p_2 I_2/\bar{v}_2)(p_1 I_1 - p_2 I_2) > 0$, plugging Equation (29) into Equation (22) yields

$$V_{opt,1} = \frac{m_1 \bar{v}_1}{4} + \frac{m_2 \bar{v}_2}{4}.$$

Otherwise, plugging Equation (30) into Equation (22) yields

$$V_{opt,2} = \frac{\bar{v}_1 \bar{v}_2}{4} \frac{(m_1 + m_2)^2}{m_1 \bar{v}_2 + m_2 \bar{v}_1}.$$

We have

$$\begin{aligned} V_{opt,1} - V_{opt,2} =& \frac{m_1 \bar{v}_1}{4} + \frac{m_2 \bar{v}_2}{4} - \frac{\bar{v}_1 \bar{v}_2}{4} \frac{(m_1 + m_2)^2}{m_1 \bar{v}_2 + m_2 \bar{v}_1} \\ =& \frac{1}{4} \frac{m_1^2 \bar{v}_1 \bar{v}_2 + m_1 m_2 \bar{v}_1^2 + m_1 m_2 \bar{v}_2^2 + m_2^2 \bar{v}_1 \bar{v}_2 - m_1^2 \bar{v}_1 \bar{v}_2 - 2 m_1 m_2 \bar{v}_1 \bar{v}_2 - m_2^2 \bar{v}_1 \bar{v}_2}{m_1 \bar{v}_2 + m_2 \bar{v}_1} \\ =& \frac{m_1 m_2}{4} \frac{(\bar{v}_1 + \bar{v}_2)^2}{m_1 \bar{v}_2 + m_2 \bar{v}_1} > 0. \end{aligned}$$

□