Supplementary materials to "Risk Management with Local Least Squares Monte-Carlo"

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1 Options valuation in the Heston model

In Section 4 of the article, butterfly option prices computed with the LSMC and LLSMC are compared to these obtained with a discrete Fourier transform (DFT). We briefly review the procedure to calculate prices by DFT.

The next proposition states that the characteristic function of the stock log-return admits a closed-form expression.

Proposition 1.1. *The characteristic function of* $\ln(S_s/S_0)\,|\mathcal{F}_t$ *under the risk neutral* $\mathbb{Q}$, *for* $s \geq t$ *with* $\omega \in \mathbb{C}$, *is given by the following expression*

$$\mathbb{E}^{\mathbb{Q}}\left(e^{\omega \ln(S_s/S_0)}\,|\,\mathcal{F}_t\right) \quad = \quad \left(\frac{S_t}{S_0}\right)^{\omega} \exp\left(A(\omega,t,s) + B(\omega,t,s)V_t\right). \tag{1}$$

Let us define the following constants:

$$\begin{cases} d = \sqrt{(\rho\sigma\omega-\kappa)^2 + \sigma^2\left(\omega-\omega^2\right)}\,, \\ g = \frac{\kappa-\rho\sigma\omega+d}{\kappa-\rho\sigma\omega-d}\,. \end{cases}$$

The functions $A(\omega,t,s)$ *and* $B(\omega,t,s)$ *in Equation (1) are given by*

$$\begin{aligned} A(\omega,t,s) = {} & r\,\omega\,(s-t) + \\ & \frac{\kappa\gamma}{\sigma^2}\left((\kappa-\rho\sigma\omega+d)(s-t) - 2\ln\left(\frac{1-ge^{d(s-t)}}{1-g}\right)\right), \end{aligned} \tag{2}$$

and

$$B(\omega,t,s) \quad = \quad \frac{\kappa-\rho\sigma\omega+d}{\sigma^2}\,\frac{1-e^{d\,(s-t)}}{1-g\,e^{d\,(s-t)}}\,. \tag{3}$$

For a proof, we refer the reader to Hainaut (2022), chapter 3, p. 65. European call or put options do not have analytical expressions. In order to evaluate these options, we calculate numerically the probability density function of the log-return, $\ln(S_T/S_0)\,|\mathcal{F}_t$, by a discrete Fourier transform (DFT). The characteristic function of a random variable, denoted by $\Upsilon_{t,T}(i\omega) = \mathbb{E}^{\mathbb{Q}}\left(e^{i\,\omega\ln(S_T/S_0)}\,|\,\mathcal{F}_t\right)$ for $\omega \in \mathbb{R}$, is also the inverse Fourier transform of its probability density function (pdf):

$$\begin{aligned} f_{t,T}(u) \quad & = \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Upsilon_{t,T}(i\omega)\,e^{-i\,u\,\omega}d\omega \\ & = \quad \frac{1}{\pi}Re\left(\int_0^{+\infty} \Upsilon_{t,T}(i\omega)e^{-i\,u\,\omega}d\omega\right) \end{aligned} \tag{4}$$

Therefore, we can retrieve the pdf by computing numerically its Fourier transform as stated in the next proposition.

Proposition 1.2. *Let M be the number of steps used in the DFT and $\Delta_u = \frac{2u_{max}}{M-1}$ be this step of discretization. Let us denote $\Delta_\omega = \frac{2\pi}{M\,\Delta_u}$ and*

$$\omega_m = (m-1)\Delta_\omega,$$

for $m = 1...M$. Let $\Upsilon_{t,T}(\omega) = \mathbb{E}^{\mathbb{Q}}\left(e^{\omega \ln(S_T/S_0)}\,|\,\mathcal{F}_t\right)$ be mgf of $\ln(S_T/S_0)$. The values of $f_{t,T}(\cdot)$ the pdf of $\ln(S_T/S_0)\,|\mathcal{F}_t$ at points $u_k = -\frac{M}{2}\Delta_u + (k-1)\Delta_u$ are approached by the sum:

$$f(u_k) \approx \frac{2}{M\,\Delta_u} Re\left(\sum_{m=1}^{M} \varrho_m \Upsilon_{t,T}(i\,\omega_m)(-1)^{m-1} e^{-i\frac{2\pi}{M}(m-1)(k-1)}\right). \tag{5}$$

where $\varrho_m = \frac{1}{2}1_{\{m=1\}} + 1_{\{m\neq 1\}}$.

This result is proven by discretizing the integral (4). The value of a European option of maturity T and payoff $H(S_T)$ is then approached by the following sum

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)}H(S_T)|\mathcal{F}_t\right) = \sum_{k=1}^{M} f(u_k)\,H(S_0 e^{u_k}). \tag{6}$$

2 Analytical valuation of a participating pure endowment

In Section 5 of the article, LSMC and LLSMC prices of participating pure endowments are compared to analytical ones, obtained with Proposition 5.1 and 5.2 of the paper. We provide below intermediate developments required to prove these propositions.

The next proposition allows us to infer the expressions of $\gamma_r(t)$ and $\gamma_\mu(t)$ matching the initial term structures of interest and mortality rates.

Proposition 2.1. *At time $0 \le t \le T$, the value of the discount bond of maturity T is equal to*

$$\begin{aligned} P(t,T) &= \exp\left(-r_t B_{\kappa_r}(t,T) - \int_t^T \gamma_r(u)\left(1-e^{-\kappa_r(T-u)}\right)du\right) \\ &\quad \times \exp\left(\frac{\sigma_r^2}{2}\int_t^T B_{B_{\kappa_r}}(u,T)^2 du\right), \end{aligned} \tag{7}$$

The survival probability up to time T, is given by

$$\begin{aligned} {}_T p_{x+t} &= \exp\left(-\mu_{x+t} B_{\kappa_\mu}(t,T) - \int_t^T \gamma_x(u)\left(1-e^{-\kappa_\mu(T-u)}\right)du\right) \\ &\quad \times \exp\left(\frac{1}{2}\int_t^T \left(\sigma_x(u)B_{\kappa_\mu}(u,T)\right)^2 du\right), \end{aligned} \tag{8}$$

The pure endowment, ${}_T E_t$, admits the following expression:

$$\begin{aligned} {}_T E_t = \mathbf{1}_{\{\tau \ge t\}} &\exp\left(-r_t B_{\kappa_r}(t,T) - \mu_{x+t} B_{\kappa_\mu}(t,T) + \frac{\sigma_r^2}{2}\int_t^T B_{\kappa_r}(u,T)^2 du\right) \times \\ &\exp\left(-\int_t^T \gamma_r(u)\left(1-e^{-\kappa_r(T-u)}\right)du - \int_t^T \gamma_x(u)\left(1-e^{-\kappa_\mu(T-u)}\right)du\right) \times \\ &\exp\left(\sigma_r \epsilon_{r\mu}\int_t^T \sigma_x(u)B_{\kappa_\mu}(u,T)B_{\kappa_r}(u,T)du + \frac{1}{2}\int_t^T \left(\sigma_x(u)B_{\kappa_\mu}(u,T)\right)^2 du\right), \end{aligned} \tag{9}$$

Sketch of the proof. We can show by direct differentation that interest and mortality rates are equal to

$$\begin{pmatrix} r_s \\ \mu_{x+s} \end{pmatrix} = \begin{pmatrix} e^{-\kappa_r(s-t)} r_t \\ e^{-\kappa_\mu(s-t)} \mu_{x+t} \end{pmatrix} + \begin{pmatrix} \kappa_r \int_t^s \gamma_r(u) e^{-\kappa_r(s-u)} du \\ \kappa_\mu \int_t^s \gamma_x(u) e^{-\kappa_\mu(s-u)} du \end{pmatrix} + \begin{pmatrix} \int_t^s \sigma_r e^{-\kappa_r(s-u)} \epsilon_{rr} dW_u^{(2)} + \int_t^s \sigma_r e^{-\kappa_r(s-u)} \epsilon_{r\mu} dW_u^{(3)} \\ \int_t^s e^{-\kappa_\mu(s-u)} \sigma_x(u) dW_u^{(3)} \end{pmatrix} \tag{10}$$

The integrals of interest and mortality rates are obtained by direct integration

$$\begin{pmatrix} \int_t^T r_s ds \\ \int_t^T \mu_{x+s} ds \end{pmatrix} = \begin{pmatrix} r_t B_{\kappa_r}(t,T) \\ \mu_{x+t} B_{\kappa_\mu}(t,T) \end{pmatrix} + \begin{pmatrix} \int_t^T \gamma_r(u) \left(1 - e^{-\kappa_r(T-u)}\right) du \\ \int_t^T \gamma_x(u) \left(1 - e^{-\kappa_\mu(T-u)}\right) du \end{pmatrix} + \begin{pmatrix} \sigma_r \epsilon_{rr} \int_t^T B_{\kappa_r}(u,T)\, dW_u^{(2)} + \sigma_r \epsilon_{r\mu} \int_t^T B_{\kappa_r}(u,T) dW_u^{(3)} \\ \int_t^T \sigma_x(u) B_{\kappa_\mu}(u,T)\, dW_u^{(3)} \end{pmatrix}. \tag{11}$$

The results follow from the log-normality of $e^{-\int_t^T r_s ds}$ or $e^{-\int_t^T r_s + \mu_{x+s} ds}$ and of $\epsilon_{rr}^2 + \epsilon_{r\mu}^2 = 1$.

end

From Equation (7), we deduce that the function $\gamma_r(u)$ must satisfy the next relation to match the initial yield curve of zero-coupon bond:

$$\int_0^T \gamma_r(u) \left(1 - e^{-\kappa_r(T-u)}\right) du = -\ln P(0,T) - r_0 B_{\kappa_r}(0,T) \tag{12}$$

$$+\frac{\sigma_r^2}{2} \int_0^T B_{\kappa_r}(u,T)^2 du\,. \tag{13}$$

Deriving twice this expression leads to the following useful reformulation of $\gamma_r(T)$:

$$\gamma_r(T) = -\frac{1}{\kappa_r} \partial_T^2 \ln P(0,T) - \partial_T \ln P(0,T) + \frac{\sigma_r^2}{2\kappa_r^2} \left(1 - e^{-2\kappa_r T}\right), \tag{14}$$

where $-\partial_T \ln P(0,T)$ is the instantaneous forward rate. For a given initial mortality curve ${}_T p_x$, we show in a similar manner that the function $\gamma_x(u)$ satisfies the relation

$$\begin{aligned} \gamma_x(T) &= -\frac{1}{\kappa_\mu} \partial_T^2 \ln {}_T p_x - \partial_T \ln {}_T p_x + \frac{1}{\kappa_\mu} \int_0^T \sigma_x(u)^2 \left(e^{-2\kappa_\mu(T-u)}\right) du \\ &= -\frac{1}{\kappa_\mu} \partial_T^2 \ln {}_T p_x - \partial_T \ln {}_T p_x + \frac{\alpha^2 e^{2\beta x}}{2\kappa_\mu(\kappa_\mu + \beta)} \left(e^{2\beta T} - e^{-2\kappa_\mu T}\right). \end{aligned} \tag{15}$$

Equations (14) and (15) allows us to rewrite bond prices, survival probabilities and endowments as function of initial term structures of mortality and interest rates. Analytical expressions are provided in Proposition 5.1 the proof is summarized below.

Proposition 5.1 : sketch of the proof By direct integration of Equations (14) and (15), we obtain that

$$\begin{aligned} \int_t^T \gamma_r(u) \left(1 - e^{-\kappa_r(T-u)}\right) du &= \left(\partial_t \ln P(0,t)\right) B_{\kappa_r}(t,T) - \ln \frac{P(0,T)}{P(0,t)} + \frac{\sigma_r^2}{2\kappa_r^2}(T-t) \\ &\quad -\frac{\sigma_r^2}{2\kappa_r^2} B_{\kappa_r}(t,T) - \frac{\sigma_r^2}{4\kappa_r} e^{-2\kappa_r t} B_{\kappa_r}(t,T)^2\,, \end{aligned}$$

and

$$
\int_t^T \gamma_x(u)\left(1-e^{-\kappa_\mu(T-u)}\right) du = \left(\partial_t \ln\, {}_tp_x\right) B_{\kappa_\mu}(t,T) - \ln \frac{{}_Tp_x}{{}_tp_x} + \frac{\alpha^2 e^{2\beta x}}{2\kappa_\mu(\kappa_\mu+\beta)}
$$
$$
\times\left(e^{2\beta T} B_{2\beta}(t,T) - e^{-2\kappa_\mu t} B_{2\kappa_\mu}(t,T) - e^{2\beta T} B_{2\beta+\kappa_\mu}(t,T) + e^{-\kappa_\mu(T+t)} B_{\kappa_\mu}(t,T)\right)
$$

Combining these expressions with these of Proposition 2.1.

end

The next result presents the dynamics of the discount bond and endowment under the pricing measure. This is a direct consequence of the Itô's lemma applied to Proposition 2.1.

Corollary 2.2. *Under the risk neutral measure $\mathbb{Q}$, the dynamics of the zero-coupon bond and of the pure endowment at time $t \leq T$ are given by*

$$
\begin{cases}
dP(t,T) &= r_t P(t,T)dt - P(t,T) B_{\kappa_r}(t,T)\sigma_r\left(\epsilon_{rr} dW_t^{(2)} + \epsilon_{r\mu} dW_t^{(3)}\right), \\
d\,{}_TE_t &= {}_TE_t\left(r_t + \mu_{x+t}\right)dt - {}_TE_t \sigma_r \epsilon_{rr} B_{\kappa_r}(t,T) dW_t^{(2)} \\
& \quad - {}_TE_t\left(B_{\kappa_\mu}(t,T)\sigma_x(t) + \sigma_r \epsilon_{r\mu} B_{\kappa_r}(t,T)\right) dW_t^{(3)} + {}_T E_t\, d\mathbf{1}_{\{\tau \geq t\}}\,.
\end{cases} \tag{16}
$$

As $\mathbb{E}^{\mathbb{Q}}\left(d\mathbf{1}_{\{\tau\geq t\}}\right) = -\mu_{x+t}dt$, we check that the pure endowment has a return equal to the risk free rate: $\mathbb{E}^{\mathbb{Q}}\left(d\,{}_TE_t\right) = {}_TE_t\, r_t\, dt$. In order to obtain a closed form expression of the saving contract, we perform a change of measure using as Radon-Nykodym derivative:

$$
\left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_T = \mathbb{E}^{\mathbb{Q}}\left(\tfrac{d\mathbb{F}}{d\mathbb{Q}} | \mathcal{F}_T\right) = \frac{e^{-\int_0^T (r_s+\mu_{x+s})ds}}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T (r_s+\mu_{x+s})ds} | \mathcal{F}_0\right)}\,. \tag{17}
$$

From Equations (11), this change of measure is rewritten as follows:

$$
\begin{aligned}
\left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_T &= \exp\left(-\frac{\sigma_r^2\epsilon_{rr}^2}{2}\int_0^T B_{\kappa_r}(u,T)^2 du - \sigma_r\epsilon_{rr}\int_0^T B_{\kappa_r}(u,T)\, dW_u^{(2)}\right) \\
&\times \exp\left(-\int_0^T \left(\sigma_r\epsilon_{r\mu} B_{\kappa_r}(u,T) + \sigma_x(u) B_{\kappa_\mu}(u,T)\right) dW_u^{(3)}\right) \\
&\times \exp\left(-\frac{1}{2}\int_0^T \left(\sigma_r\epsilon_{r\mu} B_{\kappa_r}(u,T) + \sigma_x(u) B_{\kappa_\mu}(u,T)\right)^2 du\right).
\end{aligned}
$$

We recognize a Doleans-Dade exponential and then under the measure $\mathbb{F}$, $W_t^{(2)\mathbb{F}}$ and $W_t^{(3)\mathbb{F}}$ defined by

$$
\begin{cases}
dW_t^{(2)\mathbb{F}} &= dW_t^{(2)} + \sigma_r\epsilon_{rr} B_{\kappa_r}(t,T)dt\,, \\
dW_t^{(3)\mathbb{F}} &= dW_t^{(3)} + \sigma_r\epsilon_{r\mu} B_{\kappa_r}(t,T)dt + \sigma_x(t) B_{\kappa_\mu}(t,T)dt\,,
\end{cases} \tag{18}
$$

are Brownian motions. The dynamic of the stock indice is modified as follows under $\mathbb{F}$,

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \left(r_t - \sigma_S\left(\epsilon_{Sr}\sigma_r\epsilon_{rr} + \epsilon_{S\mu}\sigma_r\epsilon_{r\mu}\right) B_{\kappa_r}(t,T) - \sigma_S\epsilon_{S\mu}\sigma_x(t) B_{\kappa_\mu}(t,T)\right) dt \\
&\quad + \sigma_S\epsilon_{SS} dW_t^{(1)} + \sigma_S\epsilon_{Sr} dW_t^{(2)\mathbb{F}} + \sigma_S\epsilon_{S\mu} dW_t^{(3)\mathbb{F}}\,.
\end{aligned} \tag{19}
$$

If we remember that $\epsilon_{SS}^2 + \epsilon_{Sr}^2 + \epsilon_{S\mu}^2 = 1$, applying the Itô's lemma to $\ln S_t$ leads to the following expression for the stock indice under $\mathbb{F}$:

$$
\begin{aligned}
S_t &= S_0 \exp\left(\int_0^t r_u du - \frac{\sigma_S^2}{2}t - \sigma_S \epsilon_{S\mu} \int_0^t \sigma_x(u) B_{\kappa_\mu}(u,T) du\right) \\
&\quad \times \exp\left(-\sigma_S \left(\epsilon_{Sr}\sigma_r\epsilon_{rr} + \epsilon_{S\mu}\sigma_r\epsilon_{r\mu}\right) \int_0^t B_{\kappa_r}(u,T) du\right) \\
&\quad \times \exp\left(\sigma_S \epsilon_{SS} W_t^{(1)} + \sigma_S \epsilon_{Sr} W_t^{(2)\mathbb{F}} + \sigma_S \epsilon_{S\mu} W_t^{(3)\mathbb{F}}\right),
\end{aligned} \tag{20}
$$

Taking advantage the log-normality of S_T under the $\mathbb{F}$-measure, we deduce the closed-form expression of Proposition 5.2 for call options embedded in the participating pure endowment.

Proposition 5.2, sketch of the proof.

As $S_T = \frac{S_T}{P(T,T)}$, we focus on on the dynamics of $d\frac{S_t}{P(t,T)}$. From the Itô's lemma, we have that

$$
d\frac{1}{P(t,T)} = -\frac{r_t}{P(t,T)}dt + \frac{B_{\kappa_r}(t,T)^2\sigma_r^2}{P(t,T)}dt + \frac{B_{\kappa_r}(t,T)}{P(t,T)}\sigma_r\left(\epsilon_{rr}dW_u^{(2)} + \epsilon_{r\mu}dW_u^{(3)}\right).
$$

The dynamic of $\frac{S_t}{P(t,T)}$ is therefore equal to

$$
\begin{aligned}
d\left(\frac{S_t}{P(t,T)}\right) &= \frac{S_t}{P(t,T)}\left[B_{\kappa_r}(t,T)^2\sigma_r^2 + B_{\kappa_r}(t,T)\left(\sigma_r\sigma_S\epsilon_{rr}\epsilon_{Sr} + \sigma_r\sigma_S\epsilon_{r\mu}\epsilon_{S\mu}\right)\right]dt \\
&\quad + \frac{S_t}{P(t,T)}\sigma_S\epsilon_{SS}dW_t^{(1)} + \frac{S_t}{P(t,T)}\left(\sigma_S\epsilon_{Sr} + \epsilon_{rr}\sigma_r B_{\kappa_r}(t,T)\right)dW_t^{(2)} \\
&\quad + \frac{S_t}{P(t,T)}\left(\sigma_S\epsilon_{S\mu} + \epsilon_{r\mu}\sigma_r B_{\kappa_r}(t,T)\right)dW_t^{(3)}.
\end{aligned}
$$

Using again the Itô's lemma, we find $d\ln\left(\frac{S_t}{P(t,T)}\right)$ under $\mathbb{Q}$ and from Equation (18), obtain the differential under $\mathbb{F}$:

$$
\begin{aligned}
d\ln\left(\frac{S_t}{P(t,T)}\right) &= -\left[\frac{\sigma_r^2}{2}B_{\kappa_r}(t,T)^2 + \sigma_r\epsilon_{r\mu}\sigma_x(t)B_{\kappa_r}(t,T)B_{\kappa_\mu}(t,T)\right]dt \\
&- \left[\frac{1}{2}\sigma_S^2 + \sigma_S\sigma_r\left(\epsilon_{rr}\epsilon_{Sr} + \epsilon_{S\mu}\epsilon_{r\mu}\right)B_{\kappa_r}(t,T) + \sigma_S\epsilon_{S\mu}\sigma_x(t)B_{\kappa_\mu}(t,T)\right]dt \\
&+ \sigma_S\epsilon_{SS}dW_t^{(1)} + \left(\sigma_S\epsilon_{Sr} + \sigma_r\epsilon_{rr}B_{\kappa_r}(t,T)\right)dW_t^{(2)\mathbb{F}} \\
&+ \left(\sigma_S\epsilon_{S\mu} + \sigma_r\epsilon_{r\mu}B_{\kappa_r}(t,T)\right)dW_t^{(3)\mathbb{F}}.
\end{aligned}
$$

By direct integration, we reformulate S_T as follows:

$$
\begin{aligned}
S_T = \frac{S_t}{P(t,T)} &\exp\left(-\frac{\sigma_r^2}{2}\int_t^T B_{\kappa_r}(u,T)^2 du - \sigma_r\epsilon_{r\mu}\int_t^T \sigma_x(u)B_{\kappa_r}(u,T)B_{\kappa_\mu}(u,T)du\right) \\
&\times \exp\left(-\frac{\sigma_S^2(T-t)}{2} - \sigma_S\sigma_r\left(\epsilon_{Sr}\epsilon_{rr} + \epsilon_{S\mu}\epsilon_{r\mu}\right)\int_t^T B_{\kappa_r}(u,T)du\right) \\
&\times \exp\left(-\sigma_S\epsilon_{S\mu}\int_t^T \sigma_x(u)B_{\kappa_\mu}(u,T)du + \int_t^T \sigma_S\epsilon_{SS}dW_u^{(1)}\right) \\
&\times \exp\left(\int_t^T \left(\sigma_S\epsilon_{Sr} + \sigma_r\epsilon_{rr}B_{\kappa_r}(u,T)\right)dW_u^{(2)\mathbb{F}}\right) \\
&\times \exp\left(\int_t^T \left(\sigma_S\epsilon_{S\mu} + \sigma_r\epsilon_{r\mu}B_{\kappa_r}(u,T)\right)dW_u^{(3)\mathbb{F}}\right)
\end{aligned} \tag{21}
$$

Equation (21) emphasizes that $\ln \frac{S_T/P(T,T)}{S_t/P(t,T)} \sim N\left(\mu_{\mathbb{F}}, v_{\mathbb{F}}\right)$ is log-normal with a mean and variance given by equation (28) in the paper. Using standard calculations, we can show that if $\Phi(.)$ is the cdf of a $N(0,1)$ and

$$\begin{aligned} d_2 &= \frac{\ln\left(\frac{C}{S_t/P(t,T)}\right) - \mu_{\mathbb{F}}}{v_{\mathbb{F}}}, \\ d_1 &= d_2 - v_{\mathbb{F}} \end{aligned}$$

then the expected postive diffence between S_T and C_T under the forward measure is given by

$$\mathbb{E}^{\mathbb{F}}\left(\left(S_T - C\right)_+ | \mathcal{F}_t\right) = \frac{S_t}{P(t,T)} e^{\mu_{\mathbb{F}} + \frac{v_{\mathbb{F}}^2}{2}} \Phi\left(-d_1\right) - C\Phi\left(-d_2\right) . \tag{22}$$

This last result allows us to infer Equation (29) in the paper.

end

3 Interest rate assumptions

In Sub-section 5.2 of the article, we model the initial yield curve with the Nelson-Siegel (NS) model. In this framework, initial instantaneous forward rates are provided by the following function:

$$f(0,t) := -\partial_t \ln P(0,t) = b_0^{(r)} + \left(b_{10}^{(r)} + b_{11}^{(r)} t\right) \exp\left(-c_1^{(r)} t\right) .$$

Parameters $\{b_0, b_{10}, b_{11}, c_1\}$ are estimated by minimizing the quadratic spread between market and model zero-coupon yields:

$$P(0,t) = \exp\left(b_0^{(r)} + \frac{1}{t}\frac{b_{10}^{(r)}}{c_1^{(r)}}\left(1 - e^{-c_1^{(r)} t}\right) + \frac{1}{t}\frac{b_{11}^{(r)}}{\left(c_1^{(r)}\right)^2}\left(1 - \left(c_1^{(r)} t + 1\right) e^{-c_1^{(r)} t}\right)\right) .$$

We fit the NS model to the yield curve of Belgian state bonds observed on the 23^{th} of November 22 and obtain estimates reported in Table 1.

Parameter	Value
$b_0^{(r)}$	0.0308
$b_{10}^{(r)}$	-0.0008
$b_{11}^{(r)}$	-0.0212
$c_1^{(r)}$	0.6594

Table 1: Nelson-Siegel parameters, Belgian state bonds, 23/11/22.

4 Mortality rate assumptions

In Sub-section 5.2 of the article, the volatility of mortality rates is fitted by least square minimization of spreads between $\sigma_x(.)$ and empirical deviations of variations of mortality rates by cohort (ages between 20 and 90 years from 1950 to 2020). If $\mu_x^{(y)}$ is the observed mortality rates at age x during the calendar year y, we denote by $\Delta\mu_x^{(y)} = \mu_x^{(y)} - \mu_{x-1}^{(y-1)}$

and by S_x the standard deviation of $\Delta\mu_x^{(y)}$ for y=1950 to 2020. The α and β are obtained by minimizing the sum

$$\alpha, \beta = \arg\min \sum_{x=20}^{90} \left(S_x - \alpha e^{\beta x} \right)^2 .$$

On the other hand, the initial curve of survival probabilities is described by a Makeham's model, i.e.

$$\begin{aligned} {}_tp_x &= \exp - \int_x^{x+t} \left(a^{(\mu)} + b^{(\mu)} \left(c^{(\mu)} \right)^s \right) ds \\ &= \exp(-a^{(\mu)} t) \exp \left(-\frac{b^{(\mu)}}{\ln c^{(\mu)}} \left(\left(c^{(\mu)} \right)^{x+t} - \left(c^{(\mu)} \right)^x \right) \right) . \end{aligned}$$

where $a^{(\mu)}, b^{(\mu)}, c^{(\mu)} \in \mathbb{R}^+$. These parameters and the reversion speed κ_u are obtained by least square minimization of spreads between prospective and model survival probabilities. Prospective survival probabilities are computed with a Lee-Carter model fitted to Belgian mortality rates from 1950 to 2020 for 0 to 105 years, male population. Model ${}_tp_x$ are computed with Equation (25) for $x = 20$ years old. Estimated parameters are provided in Table 2.

Parameters			
$a^{(\mu)}$	1.006349e-03	κ_μ	0.83925
$b^{(\mu)}$	2.790903e-07	α	8.5277e-7
κ_μ	0.83925	β	0.11094

Table 2: Mortality parameters, Belgian male mortality rates, year 2020.

5 LSMC and LLSMC, 100 000 simulations

Tables 3 and 4 compare goodness of fit statistics and runtimes of the LSMC and LLSMC, computed with 10 000 and 10 000 simulations, in the case study developped in Section 4.

d_h	R^2	$\sqrt{\text{MSE}(\mathcal{V})}$	$\sqrt{\text{MSE}}$	d.f.	Time (sec).
10 000 simulations					
2	0.0397	0.36	2.10	6	1.95
3	0.0451	0.57	2.10	10	1.65
4	0.0499	1.07	2.09	15	1.39
5	0.0522	2.39	2.09	21	2.00
6	0.0536	1.93	2.09	28	1.71
100 000 simulations					
2	0.0408	0.40	2.12	6	13.39
3	0.0463	0.58	2.11	10	13.47
4	0.0514	1.12	2.11	15	13.03
5	0.0531	1.24	2.10	21	13.77
6	0.0543	1.33	2.10	28	12.32

Table 3: R^2, MSE and MSE($\mathcal{V}$) of regressions of Y_t on $\boldsymbol{X}_t$ in the LSMC model. d.f. is the number of parameters.

K	d_γ	d_h	R^2	$\sqrt{\text{MSE}(\mathcal{V})}$	$\sqrt{\text{MSE}}$	d.f.	R^2_{loc}	Time (sec).
10 000 simulations								
3	2	3	0.0526	0.19	2.09	42	0.95	4.66
2	2	3	0.0525	0.20	2.09	26	0.87	4.48
5	2	3	0.0527	0.20	2.10	74	0.98	5.14
6	2	4	0.0524	0.20	2.10	120	0.99	5.59
2	2	2	0.0521	0.21	2.09	18	0.87	5.75
3	2	2	0.0524	0.21	2.09	30	0.95	4.28
4	2	2	0.0525	0.21	2.09	42	0.98	4.40
5	2	2	0.0525	0.21	2.10	54	0.98	5.05
5	2	4	0.0527	0.21	2.10	99	0.99	4.87
6	2	2	0.0525	0.22	2.10	66	0.99	5.50
100 000 simulations								
3	1	3	0.0106	0.85	2.14	36	0.95	4.31
3	1	2	0.0102	0.87	2.14	24	0.95	4.35
4	1	4	0.0082	0.88	2.15	69	0.98	4.61
5	1	3	0.0078	0.90	2.15	62	0.98	4.36
4	1	2	0.0073	0.93	2.14	33	0.98	4.56
6	1	2	0.0074	0.93	2.14	51	0.99	4.53
5	1	2	0.0073	0.95	2.14	42	0.98	4.39
6	1	3	0.0074	0.95	2.15	75	0.99	4.57
4	1	3	0.0072	0.97	2.14	49	0.98	4.53
6	1	4	0.0079	1.16	2.15	105	0.99	3.29

Table 4: R^2, MSE, MSE($\mathcal{V}$) and R^2_{loc} for the LLSMC model. d.f. is the number of parameters.

Tables 5 and 6 compare goodness of fit statistics and runtimes of the LSMC and LLSMC, computed with 10 000 and 10 000 simulations, in the case study of Section 5.

d_h	R^2	$\sqrt{\text{MSE}(\mathcal{V})}$	$\sqrt{\text{MSE}}$	$\sqrt{\text{EMSE}}$	d.f.	Time (sec.)
10 000 simulations						
2	0.3815	1.98	11.48	1.14	10	12.83
3	0.3874	2.09	11.35	0.97	20	11.58
4	0.3864	2.14	11.29	0.96	35	11.42
5	0.3875	2.90	11.40	0.80	56	11.56
6	0.3955	3.93	11.25	0.91	84	11.58
100 000 simulations						
2	0.3779	2.03	11.43	1.35	10	321.14
3	0.3822	1.81	11.37	0.96	20	547.36
4	0.3795	1.63	11.41	0.90	35	493.71
5	0.3878	0.80	11.33	0.63	56	557.72
6	0.3885	0.77	11.35	0.42	84	613.17

Table 5: R^2, MSE, MSE($\mathcal{V}$) of regressions of Y_t on $\boldsymbol{X}_t$ in the LSMC model. $\sqrt{\text{EMSE}}$ is the MSE valued with analytical prices. d.f. is the number of parameters.

K	d_γ	d_h	R^2	$\sqrt{\text{MSE}(\mathcal{V})}$	$\sqrt{\text{MSE}}$	$\sqrt{\text{EMSE}}$	d.f.	R^2_{loc}	Time (sec.)
10 000 simulations									
5	3	2	0.3952	0.65	11.28	0.47	130	0.97	76.44
4	3	3	0.3952	0.69	11.28	0.41	140	0.96	46.72
3	3	2	0.3952	0.76	11.25	0.37	70	0.93	71.14
4	3	2	0.3952	0.77	11.26	0.41	100	0.95	59.80
5	2	2	0.3918	0.79	11.29	0.70	90	0.97	68.32
5	3	3	0.3953	0.81	11.31	0.47	180	0.97	68.00
2	3	2	0.3946	0.84	11.23	0.37	40	0.88	72.97
5	3	1	0.3949	0.86	11.26	0.49	100	0.97	78.56
4	2	3	0.3923	0.87	11.30	0.71	110	0.96	55.33
4	2	2	0.3923	0.88	11.27	0.71	70	0.95	68.57
100 000 simulations									
5	3	2	0.3868	0.31	11.36	0.29	130	0.97	972.62
4	3	3	0.3866	0.32	11.36	0.31	140	0.96	1028.79
3	3	2	0.3868	0.40	11.36	0.28	70	0.93	819.64
4	3	2	0.3866	0.33	11.36	0.33	100	0.96	854.90
5	2	2	0.3842	0.56	11.38	0.78	90	0.97	808.62
5	3	3	0.3868	0.35	11.36	0.27	180	0.97	985.53
2	3	2	0.3865	0.89	11.36	0.35	40	0.89	836.63
5	3	1	0.3864	0.81	11.36	0.41	100	0.97	969.00
4	2	3	0.3844	0.59	11.38	0.75	110	0.96	1095.46
4	2	2	0.384	0.63	11.38	0.81	70	0.96	799.51

Table 6: R^2, MSE, MSE($\mathcal{V}$) and R^2_{loc} for the LLSMC model. $\sqrt{\text{MSE}}$, exact is the MSE valued with analytical prices. d.f. is the number of parameters.