Supplementary materials for the article “Risk sharing in equity-linked insurance products: Stackelberg equilibrium between an insurer and a reinsurer”

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Section 1 contains two lemmas that we use to prove the main results of the paper. Section 2 contains plots that complement the numerical studies section from the paper. In particular, Subsection 2.1 contains plots related to the sensitivity analysis of the Stackelberg equilibrium w.r.t. RRA coefficients, r, T, G_T, whereas the description of these plots was included in the article in Section 5.2. In Subsection 2.2, we provide plots that illustrate the dynamic relative-portfolio process of each company over the entire investment horizon. In Section 3 we prove that the reinsurance company is better off when it sells reinsurance with a discounted safety loading $\theta(\alpha) = \alpha \cdot \theta_R^*$ in comparison to not selling reinsurance at all. We also illustrate the corresponding monetary benefit with the help of the wealth-equivalent utility gain, defined in Section 5.3. of the paper. In Section 4, we derive the insurer's optimal strategies in the Stackelberg equilibrium, when the insurance company has a logarithmic-utility and a HARA-utility function. Finally, in Section 5 we discuss ways of adding mortality and surrender risks to the model and the corresponding potential challenges to the derivation of the Stackelberg equilibrium.

1 Auxiliary lemmas

Lemma 1.1 (Put-replicating trading strategies)**.** The replicating strategy $\psi(t)$, $t \in [0,T]$, of the put option P is given by

$$\psi(t) = \left(\frac{P(t) - \pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_0(t)}, 0, \frac{\pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)} \right)^\top,$$

where

$$d_+ := d_+(t, V^{v_I,\pi_B}(t)) := \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r + \frac{1}{2}(\sigma_2 \pi^{CM})^2\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}}.$$

The dynamics of the put option P is given by

$$\begin{aligned} dP(t) =& [V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)\pi^{CM}(\mu_2 - r) + rP(t)]dt \\ &+ V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)\sigma_2\pi^{CM}(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)). \end{aligned}$$

Proof. The price of the put option P at time $t \in [0,T]$ is given by

$$\begin{aligned} P(t) =& \widetilde{Z}(t)^{-1}\mathbb{E}[\widetilde{Z}(T)P(T)|\mathcal{F}_t] \\ =& \exp\left(-r(T-t)\right) G_T \Phi(-d_1(t, V^{v_I,\pi_B}(t))) - V^{v_I,\pi_B}(t)\Phi(-d_2(t, V^{v_I,\pi_B}(t))), \end{aligned}$$

where Φ is the cumulative distribution function of the standard normal distribution and

$$d_1(t, V^{v_I,\pi_B}(t)) := \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r - \frac{1}{2}(\pi^{CM}\sigma_2)^2\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}},$$
$$d_2(t, V^{v_I,\pi_B}(t)) := d_1(t, V^{v_I,\pi_B}(t)) + \pi^{CM}\sigma_2\sqrt{T-t}.$$

The stock price S_2 and the constant-mix portfolio value V^{v_I,π_B} are given by

$$S_2(t) = S_2(0)\exp\left(\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)t + \sigma_2(\rho W_1(t) + \sqrt{1-\rho^2}W_2(t))\right),$$
$$V^{v_I,\pi_B}(t) = v_I\exp\left(\left(r + \pi^{CM}(\mu_2 - r) - \frac{1}{2}(\sigma_2\pi^{CM})^2\right)t + \sigma_2\pi^{CM}(\rho W_1(t) + \sqrt{1-\rho^2}W_2(t))\right).$$

Hence, the relation between S_2 and V^{v_I,π_B} is given by

$$V^{v_I,\pi_B}(t) = \frac{v_I}{S_2(0)}\exp\left(\left(r + \frac{1}{2}\pi^{CM}\sigma_2^2\right)(1-\pi^{CM})t\right)S_2(t)^{\pi^{CM}}.$$

Therefore,

$$\frac{\partial P(t)}{\partial S_1(t)} = 0$$

and

$$\frac{\partial P(t)}{\partial S_2(t)} = \exp\left(-r(T-t)\right)G_T\frac{\partial}{\partial S_2(t)}\Phi(-d_1(t, V^{v_I,\pi_B}(t))) - \frac{\partial V^{v_I,\pi_B}(t)}{\partial S_2(t)}\Phi(-d_2(t, V^{v_I,\pi_B}(t)))$$
$$- V^{v_I,\pi_B}(t)\frac{\partial}{\partial S_2(t)}\Phi(-d_2(t, V^{v_I,\pi_B}(t))).$$

We have for $i = 1, 2$

$$\frac{\partial V^{v_I,\pi_B}(t)}{\partial S_2(t)} = V^{v_I,\pi_B}(t)\pi^{CM}S_2(t)^{-1},$$
$$\frac{\partial(d_i(t, V^{v_I,\pi_B}(t)))}{\partial S_2(t)} = \frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1},$$
$$\frac{\partial\Phi(-d_i(t, V^{v_I,\pi_B}(t)))}{\partial S_2(t)} = -\phi(-d_i(t, V^{v_I,\pi_B}(t)))\frac{\partial(d_i(t, V^{v_I,\pi_B}(t)))}{\partial S_2(t)},$$

where ϕ is the density of the standard normal distribution. Hence,

$$\frac{\partial P(t)}{\partial S_2(t)} = \exp\left(-r(T-t)\right)G_T\frac{\partial}{\partial S_2(t)}\Phi(-d_1(t, V^{v_I,\pi_B}(t))) - \frac{\partial V^{v_I,\pi_B}(t)}{\partial S_2(t)}\Phi(-d_2(t, V^{v_I,\pi_B}(t)))$$
$$- V^{v_I,\pi_B}(t)\frac{\partial}{\partial S_2(t)}\Phi(-d_2(t, V^{v_I,\pi_B}(t)))$$
$$= -\exp\left(-r(T-t)\right)G_T\phi(-d_1(t, V^{v_I,\pi_B}(t)))\frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1}$$
$$- \frac{\pi^{CM}V^{v_I,\pi_B}(t)\Phi(-d_2(t, V^{v_I,\pi_B}(t)))}{S_2(t)}$$
$$+ V^{v_I,\pi_B}(t)\phi(-d_2(t, V^{v_I,\pi_B}(t)))\frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1}$$
$$= -\frac{\pi^{CM}V^{v_I,\pi_B}(t)\Phi(-d_2(t, V^{v_I,\pi_B}(t)))}{S_2(t)}.$$

We define $d_2(t, V^{v_I,\pi_B}(t)) =: d_+$. Since $\Phi(-x) = 1 - \Phi(x)$, we get

$$\frac{\partial P(t)}{\partial S_2(t)} = -\frac{\pi^{CM}V^{v_I,\pi_B}(t)\Phi(-d_2(t, V^{v_I,\pi_B}(t)))}{S_2(t)} = \frac{\pi^{CM}V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)}.$$

□

Lemma 1.2. Let $\xi^{\max} = \bar{\xi} < \frac{v_I}{(1+\theta_R^{\max})P(0)}$. Then the function ν from Proposition 3.1 is

$$\nu(\xi) = \mathbb{E}[U_I(I_I(y_\lambda^*(\xi)\widetilde{Z}_{\lambda^*}(T)))]$$

for $\xi \in [0, \xi^{\max}]$, where I_I is the inverse of U_I' and $y_\lambda^*(\xi)$ is the Lagrange multiplier given by

$$\mathbb{E}[\widetilde{Z}_\lambda(T)\hat{I}_I(y_\lambda^*(\xi)\widetilde{Z}_\lambda(T))] = v_I - \xi(1+\theta_R)P(0).$$

Proof. Recall from Proposition 3.1:

$$\nu(\xi) := \mathbb{E}[U_I(\max\{I_I(y_\lambda^*(\xi)\widetilde{Z}_{\lambda^*}(T)), \xi P(T)\})],$$

where the Lagrange multiplier $y_\lambda^*(\xi)$ is given by

$$\mathbb{E}[\widetilde{Z}_\lambda(T)\hat{I}_I(y_\lambda^*(\xi)\widetilde{Z}_\lambda(T))] = v_I - \xi(1+\theta_R)P(0).$$

$\hat{I}_I$ is the inverse function of $\hat{U}_I'$. This function is bijective on $(0, U_I'(\xi P(T))]$ and equals

$$\hat{I}_I(y) = I_I(y) - \xi P(T)$$

for $y \in (0, U_I'(\xi P(T))]$. From this we have the following:

$$\hat{I}_I(y) > 0 \Leftrightarrow y \in (0, U_I'(\xi P(T))). \tag{1.1}$$

For the Lagrange multiplier $y_\lambda^*(\xi)$ it follows that

$$\begin{aligned}\mathbb{E}[\widetilde{Z}_\lambda(T)\hat{I}_I(y_\lambda^*(\xi)\widetilde{Z}_\lambda(T))] = \underbrace{v_I - \xi(1+\theta_R)P(0)}_{>0} &\Leftrightarrow \hat{I}_I(y_\lambda^*(\xi)\widetilde{Z}_\lambda(T)) > 0 \ \mathbb{Q}\text{-a.s.}\\ \Leftrightarrow y_\lambda^*(\xi)\widetilde{Z}_\lambda(T) < U_I'(\xi P(T)) \ \mathbb{Q}\text{-a.s.} &\Leftrightarrow I_I(y_\lambda^*(\xi)\widetilde{Z}_\lambda(T)) > \xi P(T) \ \mathbb{Q}\text{-a.s.},\end{aligned}$$

where the second equivalence holds from (1.1) and the third from the fact that I_I is strictly decreasing. Hence, we get

$$\nu(\xi) = \mathbb{E}[U_I(\max\{I_I(y_\lambda^*(\xi)\widetilde{Z}_{\lambda^*}(T)), \xi P(T)\})] = \mathbb{E}[U_I(I_I(y_\lambda^*(\xi)\widetilde{Z}_{\lambda^*}(T)))].$$

□

2 Additional figures

This section contains two subsections. In Subsection 2.1, we provide plots related to the sensitivity analysis of the Stackelberg equilibrium at the start of the product. In Subsection 2.2, we plot the optimal relative-portfolio processes of the players.

2.1 Sensitivity analysis of Stackelberg equilibrium at $t = 0$

Figures 2.1, 2.2, 2.3 and 2.4 graphically illustrate the sensitivity of the Stackelberg equilibrium at $t = 0$ w.r.t. the RRA coefficients of parties, the interest rate, the time to product maturity and the level of the capital guarantee. The description of these plots can be found in Section 5.2 of the article.

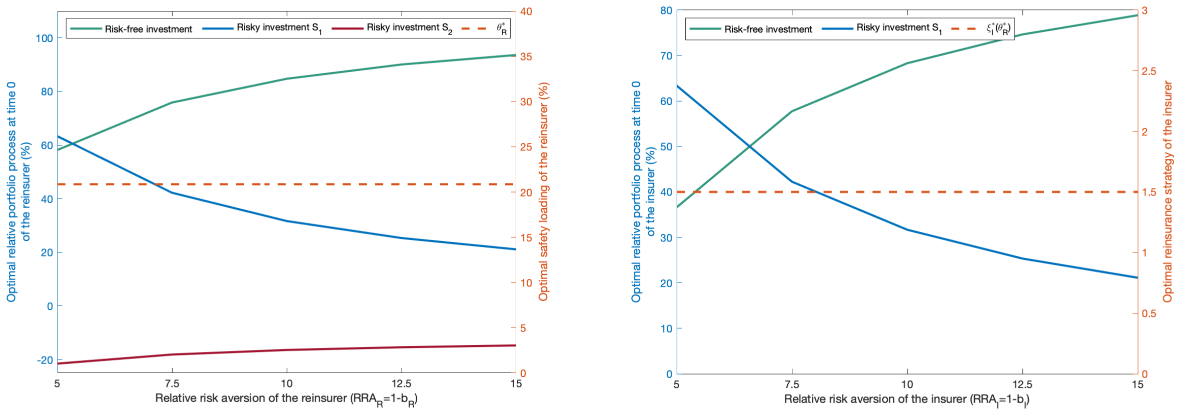

Figure 2.1: Sensitivity of the Stackelberg equilibrium w.r.t. RRA_R and RRA_I

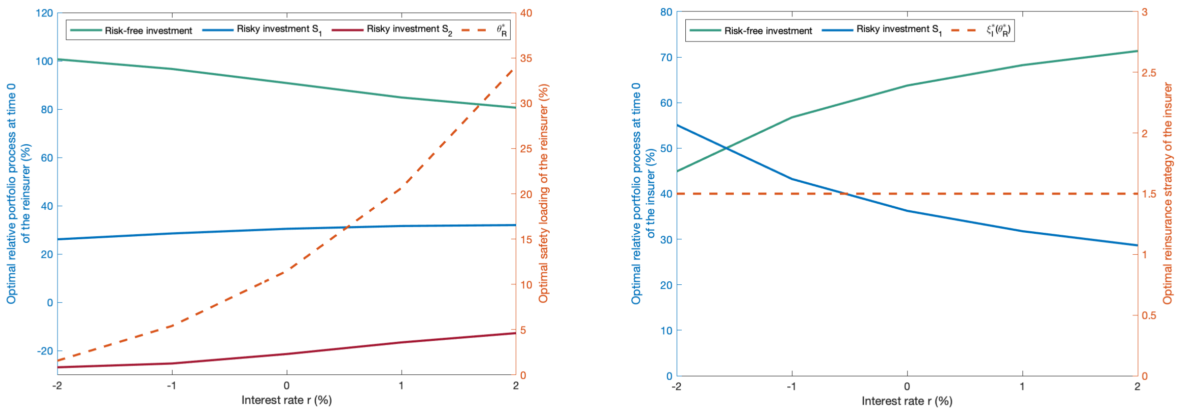

Figure 2.2: Sensitivity of the Stackelberg equilibrium w.r.t. r

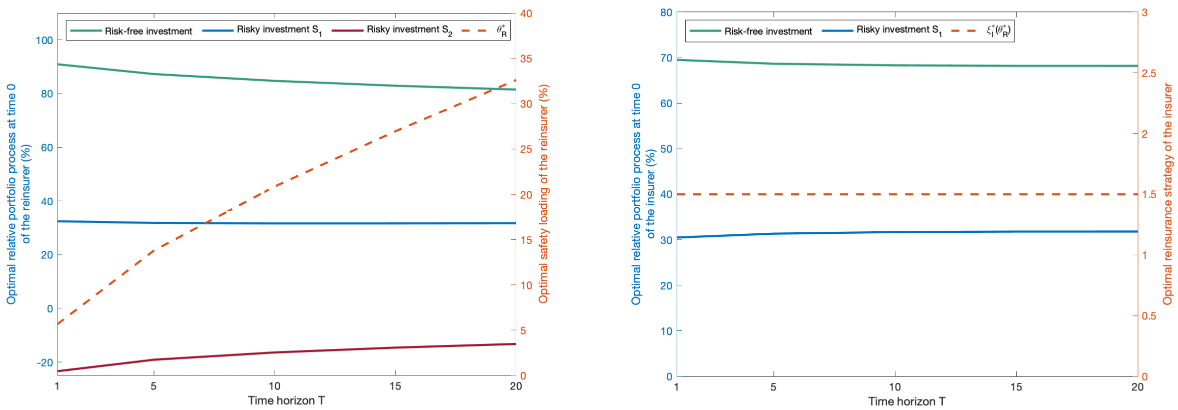

Figure 2.3: Sensitivity of the Stackelberg equilibrium w.r.t. T

2.2 Investment strategies over whole time horizon

In this subsection, we illustrate how the investment strategies of the reinsurer and the insurer can develop over the entire time horizon $[0, T]$. In Figures 2.5 and 2.6, we show 100 exemplary paths of the relative-portfolio process (semi-transparent lines) as well as the average of the investment strategies (thick nontransparent lines) over the whole time horizon. The average values are calculated using 10000 simulated paths.

In Figure 2.5, we see that the reinsurer invests on average around 30% of its wealth in the first risky asset and sells between 5% and 15% of the second risky asset. The closer the maturity of the equity-linked product, the more the reinsurer invests in S_1 and the less it sells S_2. The share of capital invested in the risk-free asset S_0 decreases on average.

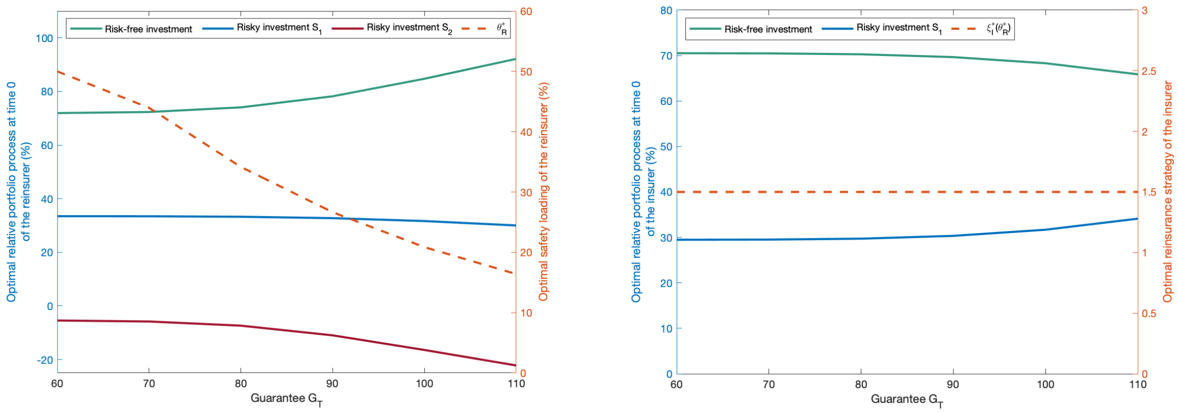

Figure 2.4: Sensitivity of the Stackelberg equilibrium w.r.t. G_T

In contrast, the insurer's fraction of wealth invested in the first risky asset decreases on average over the time horizon and therefore the fraction invested in the risk-free asset increases, as Figure 2.6 indicates.

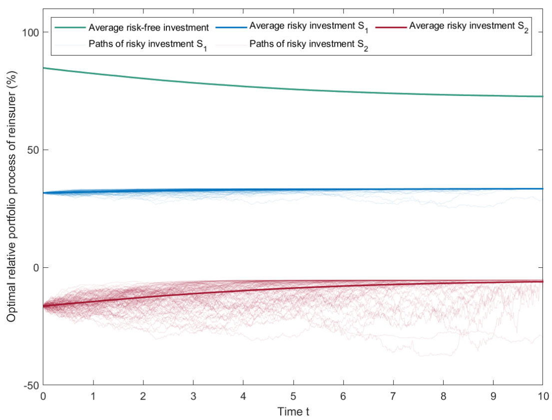

Figure 2.5: Reinsurer's investment strategy over the time horizon $[0, T]$

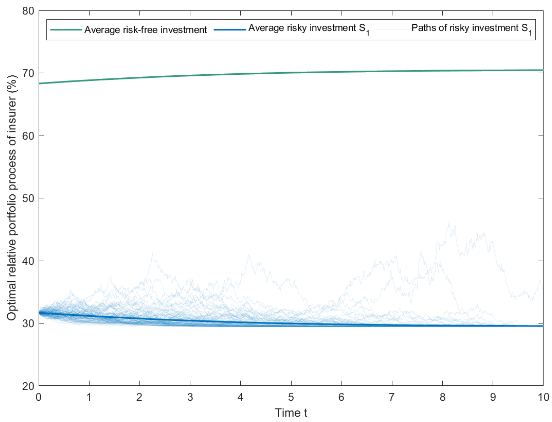

Figure 2.6: Insurer's investment strategy over the time horizon $[0, T]$

3 Reinsuirer's incentive to charge a discounted safety loading

In this section, we answer the question whether the expected utility of the reinsurer is higher when it charges a discounted safety loading in comparison to the situation when it does not sell any reinsurance. In other words, we explore whether the participation constraint is satisfied for the reinsurer when it charges a lower (discounted) safety loading than the equilibrium one.

When $\theta_R \in (0, \theta_R^*)$, the reinsurer benefits from offering reinsurance in comparison to the case of not selling reinsurance at all. The benefit increases as $\theta_R > 0$ increases and $\theta_R < \theta_R^*$. To see this, recall that the reinsurer's optimal total terminal wealth without reinsurance is given by

$$V_R^{v_R,0(0,0),\pi_R^*}(T) = I_R(y_R^* \widetilde{Z}(T)),$$

where y_R^* solves $\mathbb{E}[\widetilde{Z}(T) I_R(y_R^* \widetilde{Z}(T))] = v_R$. In contrast, the reinsurer's optimal total terminal wealth with reinsurance that is priced with a safety loading that is slightly lower than the equilibrium one is given by

$$V_R^{v_R,0(\xi_I^*(\theta_R(\alpha)),\theta_R(\alpha)),\pi_R^*}(T) - \xi_I^*(\theta_R(\alpha))P(T) = I_R(y_R^*(\theta_R(\alpha))\widetilde{Z}(T)),$$

where $y_R^*(\theta_R(\alpha))$ solves $\mathbb{E}[\widetilde{Z}(T) I_R(y_R^*(\theta_R(\alpha))\widetilde{Z}(T))] = v_R + \xi_I^*(\theta_R(\alpha))\theta_R(\alpha)P(0)$. As we can see, the reinsurer's optimal total terminal wealth without reinsurance is the same as with reinsurance with full discount (i.e., $\alpha = 0$ and, therefore, $\theta_R = 0$). If $\xi_I^*(\theta_R(\alpha))\theta_R(\alpha)$ strictly increases when α increases, then $I_R(y_R^*(\theta_R(\alpha)\widetilde{Z}(T))$ increases strictly up to $I_R(y_R^*(\theta_R^*)\widetilde{Z}(T))$. Since the reinsurer's expected utility is strictly concave in the terminal wealth and $I_R(y_R^*(\theta_R^*)\widetilde{Z}(T))$ maximizes it, it is strictly increasing on the interval $[0, I_R(y_R^*(\theta_R^*)\widetilde{Z}(T))]$. Hence, it holds for $\alpha > 0$ that

$$\begin{aligned}\mathbb{E}\left[U_R\left(V_R^{v_{R,0}(0,0),\pi_R^*}(T)\right)\right] &= \mathbb{E}\left[U_R\left(V_R^{v_{R,0}(\xi_I^*(\theta_R(0)),\theta_R(0)),\pi_R^*}(T) - \xi_I^*(\theta_R(0))P(T)\right)\right] \\ &= \mathbb{E}\left[U_R\left(I_R(y_R^*(\theta_R(0)\widetilde{Z}(T))\right)\right] < \mathbb{E}\left[U_R\left(I_R(y_R^*(\theta_R(\alpha)\widetilde{Z}(T))\right)\right] \\ &= \mathbb{E}\left[U_R\left(V_R^{v_{R,0}(\xi_I^*(\theta_R(\alpha)),\theta_R(\alpha)),\pi_R^*}(T) - \xi_I^*(\theta_R(\alpha))P(T)\right)\right].\end{aligned}$$

In our case, we have $\xi_I^*(\theta_R(\alpha))\theta_R(\alpha) = \bar{\xi}\alpha\theta_R^*$, which is increasing in $\alpha \in [0, 1]$.

Figure 3.1 illustrates in monetary terms the benefit of a discounted safety loading in comparison to the case of not selling reinsurance at all.

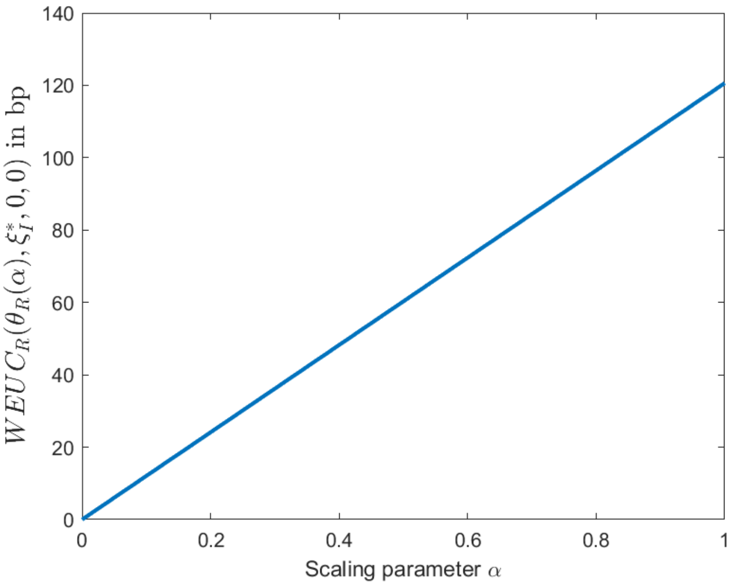

Figure 3.1: Impact of α on the reinsurer's WEUC with the reference combination $(\pi_R^*(\cdot|\theta_R(\alpha)), \theta_R(\alpha))$ and the alternative combination $(\pi_R^*(\cdot|0), 0)$

4 Discussion on other utility functions

This section consists of two subsections. In Subsection 4.1, we show that an insurer with a logarithmic or with a HARA-utility function becomes indifferent in the amount of reinsurance when the reinsurance company charges the equilibrium safety loading. In Subsection 4.2, we derive the optimal relative-portfolio process in the Stackelberg equilibrium for an insurer whose preferences are described by a logarithmic-utility or a HARA-utility function.

4.1 Insurer's best response to the equilibrium safety loading

In this subsection, we show that for utility functions, other than the power-utility function, the insurer is also indifferent in the amount of reinsurance to purchase in the equilibrium.

As in the paper, we assume $\xi^{\max} = \bar{\xi} < \frac{v_I}{(1+\theta_R^{\max})P(0)}$ and denote by λ^* the dual process characterizing the optimal auxiliary market. Recall that by Lemma 1.2:

$$\nu(\xi) = \mathbb{E}[U_I(I_I(y_\lambda^*(\xi)\widetilde{Z}_{\lambda^*}(T)))].$$

Logarithmic utility function. We have $U_I(x) := \ln(x)$ and $I_I(x) = \frac{1}{x}$. Hence, it holds

$$\nu(\xi) = -\mathbb{E}[\ln(\widetilde{Z}_{\lambda^*}(T))] - \ln(y^*_{\lambda^*}(\xi))$$

with the Lagrange multiplier (determined by the budget constraint)

$$y^*_{\lambda^*}(\xi) = (v_I - \xi(1+\theta_R)P(0) + \xi\mathbb{E}[P(T)\widetilde{Z}_{\lambda^*}(T)])^{-1}.$$

Hence, ν is of the form

$$\nu(\xi) = -C + \ln(C_1 - \xi C_2)$$

where $C, C_1, C_2 \in \mathbb{R}$ are constants with $C_2 = (1+\theta_R)P(0) - \mathbb{E}[\widetilde{Z}_{\lambda^*}(T)P(T)]$. For $C_2 > 0$, ν strictly decreases if ξ increases, i.e., $\xi_I^* = 0$. For $C_2 = 0$, ν is independent of ξ, i.e., $\xi_I^* = \tilde{\xi}$ for any $\tilde{\xi} \in [0, \bar{\xi}]$. For $C_2 < 0$, ν is strictly increasing in ξ, i.e., $\xi_I^* = \bar{\xi}$.

For the equilibrium safety loading θ_R^*, $C_2 = 0$, thus, the insurer's best response in terms of the reinsurance amount is $\xi_I^* = \tilde{\xi}$ for any $\tilde{\xi} \in [0, \bar{\xi}]$, i.e., the insurance company is indifferent in the amount of reinsurance in equilibrium.

HARA-utility function. We have $U_I(x) := \frac{(x+a)^{b_I}}{b_I}$ with $a \in \mathbb{R}$ s.t. $x + a > 0$ and $b_I \in (-\infty, 1)\backslash\{0\}$ and $I_I(x) = x^{\frac{1}{b_I - 1}} - a$. The function ν is given by

$$\nu(\xi) = \frac{1}{b_I}\mathbb{E}\left[(y^*_{\lambda^*}(\xi)\widetilde{Z}_{\lambda^*}(T))^{\frac{b_I}{b_I-1}}\right]$$

with the Lagrange multiplier (determined by the budget constraint)

$$y^*_{\lambda^*}(\xi) = \left(\frac{v_I - \xi(1+\theta_R)P(0) + a\mathbb{E}[\widetilde{Z}_{\lambda^*}(T)] + \xi\mathbb{E}[P(T)\widetilde{Z}_{\lambda^*}(T)]}{\mathbb{E}[\widetilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}]}\right)^{b_I - 1}.$$

Hence, ν is of the form

$$\nu(\xi) = \frac{1}{b_I}\mathbb{E}\left[\widetilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}\right](C_1 - \xi C_2)^{b_I},$$

where $C_1, C_2 \in \mathbb{R}$ are constants with $C_2 = (1+\theta_R)P(0) - \mathbb{E}[\widetilde{Z}_{\lambda^*}(T)P(T)]$, which is the same as in the case of a logarithmic-utility function. Thus, we conclude that for θ_R^*, the insurer's best response is $\xi_I^* = \tilde{\xi}$ for any $\tilde{\xi} \in [0, \bar{\xi}]$.

4.2 Insurer's relative portfolio process in the equilibrium

In this subsection, we derive the insurer's relative portfolio process in the Stackelberg equilibrium. We show that for insurers with logarithmic-utility and with HARA-utility functions the optimal auxiliary market is the same as the one for the insurer with a corresponding power-utility function. We do not derive here the reinsurer's optimal relative portfolio process, because its derivation is much simpler than the derivation for the insurer and boils down to applying Proposition 3.3 to get the optimal trading strategy and to converting it to the relative portfolio process via Relation (3.4) from the paper.

Logarithmic utility function. We have that $U_I(x) := \ln(x)$ and $\hat{I}_I(x) = \frac{1}{x} - \xi P(T)$. We prove that $\hat{I}_I$ and $\frac{d\hat{I}_I(x)}{dx}$ are polynomially bounded at 0 and ∞: For $x > 0$ it holds

$$|\hat{I}_I(x)| = \left|\frac{1}{x} - \xi P(T)\right| \overset{(i)}{\leq} \frac{1}{x} + \xi P(T) \overset{(ii)}{\leq} (1 + \xi P(T))\left(\frac{1}{x} + x\right);$$

$$\left|\frac{d\hat{I}_I(x)}{dx}\right| = \left|-\frac{1}{x^2}\right| \overset{x \geq 0}{=} \left(\frac{1}{x}\right)^2 \overset{x>0}{\leq} \left(x + \frac{1}{x}\right)^2,$$

where in (i) we use the Cauchy-Schwartz inequality and in (ii) $\frac{1}{x} + x \geq 1$ for all $x > 0$. By Lemma 1.2, we have

$$I_I(y^*(\xi)\widetilde{Z}_\lambda(T)) > \xi P(T).$$

From this result and by Proposition 3.1, the optimal terminal wealth (before reinsurance payment, i.e., not the total terminal wealth) is given by

$$V_\lambda^*(T) = I_I(y^*(\xi)\widetilde{Z}_\lambda(T)) - \xi P(T) = (y^*(\xi)\widetilde{Z}_\lambda(T))^{-1} - \xi P(T). \tag{4.1}$$

Thus, using $\frac{d\hat{I}(x)}{dx} = -x^{-2}$ and Proposition 3.1, we get that the insurer's optimal relative portfolio process in $\mathcal{M}_\lambda$ satisfies:

$$\begin{aligned}
\pi_\lambda^*(t)V_\lambda^*(t) =& -(\sigma^\top)^{-1}\gamma_\lambda\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)y^*(\xi)\widetilde{Z}_\lambda(T)(-y^*(\xi)\widetilde{Z}_\lambda(T))^{-2}|\mathcal{F}_t] \\
=&(\sigma^\top)^{-1}\gamma_\lambda\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)(y^*(\xi)\widetilde{Z}_\lambda(T))^{-1}|\mathcal{F}_t] \\
\overset{(4.1)}{=}&(\sigma^\top)^{-1}\gamma_\lambda\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)(V_\lambda^*(T) + \xi P(T))|\mathcal{F}_t] \\
=&(\sigma^\top)^{-1}\gamma_\lambda(V_\lambda^*(t) + \xi\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)P(T)|\mathcal{F}_t]).
\end{aligned}$$

Therefore:

$$\pi_\lambda^*(t) = (\sigma^\top)^{-1}\gamma_\lambda(1 + \xi\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)P(T)|\mathcal{F}_t]/V_\lambda^*(t)).$$

The optimal dual process λ^* is the same as in the case of a power-utility function, since $\pi_\lambda^*(t) \in K \Leftrightarrow (\sigma^\top)^{-1}\gamma_\lambda \in K \Leftrightarrow \pi_\lambda^M \in K$ with π_λ^M defined in Corollary 4.1 in the paper.

HARA-utility function. We have that $U_I(x) := \frac{(x+a)^{b_I}}{b_I}$ with $a \in \mathbb{R}$ s.t. $x + a > 0$ and $b_I \in (-\infty, 1)\backslash\{0\}$ and $\hat{I}_I(x) = x^{\frac{1}{b_I-1}} - a - \xi P(T)$. Now we prove that $\hat{I}_I$ and $\frac{d\hat{I}_I(x)}{dx}$ are polynomially bounded at 0 and ∞.

For $x > 0$ it holds

$$\begin{aligned}
|\hat{I}_I(x)| =& \left|x^{\frac{1}{b_I-1}} - a - \xi P(T)\right| \underset{x>0}{\overset{(i)}{\leq}} x^{\frac{1}{b_I-1}} + |a + \xi P(T)| = \left(\frac{1}{x}\right)^{\frac{1}{1-b_I}} + |a + \xi P(T)| \\
&\underset{(iii)}{\overset{(ii)}{\leq}} \left(x + \frac{1}{x}\right)^{\frac{1}{1-b_I}} + |a + \xi P(T)| \overset{(iv)}{\leq} (1 + |a + \xi P(T)|)\left(x + \frac{1}{x}\right)^{\frac{1}{1-b_I}};
\end{aligned}$$

$$\left|\frac{d\hat{I}_I(x)}{dx}\right| = \left|\frac{1}{b_I - 1}x^{\frac{2-b_I}{b_I-1}}\right| \underset{x>0}{\overset{b_I-1<0}{=}} \frac{1}{1-b_I}x^{\frac{2-b_I}{b_I-1}} = \frac{1}{1-b_I}\left(\frac{1}{x}\right)^{\frac{2-b_I}{1-b_I}} \underset{(iii)}{\overset{(ii)}{\leq}} \frac{1}{1-b_I}\left(x + \frac{1}{x}\right)^{\frac{2-b_I}{1-b_I}},$$

where we use in

- (i) Cauchy-Schwartz inequality;
- (ii) $x + \frac{1}{x} > \frac{1}{x}$ since $x > 0$;
- (iii) $x \mapsto x^k$ increases for $k > 0$ with $k = \frac{1}{1-b_I} > 0$ or $k = \frac{2-b_I}{1-b_I} > 0$ for all $b_I \in (-\infty, 1)\backslash\{0\}$;
- (iv) $1 \leq \left(x + \frac{1}{x}\right)^{\frac{1}{1-b_I}}$ for all $x > 0$.

By Lemma 1.2, we have

$$I_I(y^*(\xi)\widetilde{Z}_\lambda(T)) > \xi P(T).$$

From this result and by Proposition 3.1, the optimal terminal wealth (before reinsurance payment, i.e., not the total terminal wealth) is given by

$$V_\lambda^*(T) = I_I(y^*(\xi)\widetilde{Z}_\lambda(T)) - \xi P(T) = (y^*(\xi)\widetilde{Z}_\lambda(T))^{\frac{1}{b_I-1}} - a - \xi P(T). \tag{4.2}$$

Thus, using $\frac{d\hat{I}(x)}{dx} = \frac{1}{b_I - 1} x^{\frac{2-b_I}{b_I-1}}$ and Proposition 3.1, we get that the insurer's optimal relative portfolio process in $\mathcal{M}_\lambda$ satisfies:

$$\begin{aligned}
\pi^*_\lambda(t)V^*_\lambda(t) =& -(\sigma^\top)^{-1}\gamma_\lambda \widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)y^*(\xi)\widetilde{Z}_\lambda(T)\frac{1}{b_I-1}(y^*(\xi)\widetilde{Z}_\lambda(T))^{\frac{2-b_I}{b_I-1}}|\mathcal{F}_t] \\
=& \frac{1}{1-b_I}(\sigma^\top)^{-1}\gamma_\lambda \widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)(y^*(\xi)\widetilde{Z}_\lambda(T))^{\frac{1}{b_I-1}}|\mathcal{F}_t] \\
\overset{(4.2)}{=}& \frac{1}{1-b_I}(\sigma^\top)^{-1}\gamma_\lambda \widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)(V^*_\lambda(T)+a+\xi P(T))|\mathcal{F}_t] \\
=& \frac{1}{1-b_I}(\sigma^\top)^{-1}\gamma_\lambda (V^*_\lambda(t)+\xi\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)P(T)|\mathcal{F}_t]+a).
\end{aligned}$$

Thus:

$$\pi^*_\lambda(t) = \pi^M_\lambda(1+\xi\widetilde{Z}_\lambda(t)^{-1}\mathbb{E}[\widetilde{Z}_\lambda(T)P(T)|\mathcal{F}_t]/V^*_\lambda(t) + a/V^*_\lambda(t)).$$

The optimal dual process λ^* is the same as in the case of a power-utility function, since $\pi^*_\lambda(t) \in K \Leftrightarrow \pi^M_\lambda \in K$ with π^M_λ defined in Corollary 4.1 in the paper.

5 Discussion on the surrender and mortality risk

Mortality and/or surrender risks could be added in various ways to the insurer's (follower's) optimization problem. If these risks are independent of the financial risks and the insurer pays money to the buyer of the equity-linked product at time T only if the buyer does not surrender and does not die before T, then the Stackelberg equilibrium does not change. However, once the dependence between financial risks and surrender/mortality risks is introduced and/or the payoff of the insurer is more intertwined with surrender/mortality risks, the insurer's optimization problem becomes much more complicated and deserves a separate treatment.

Consider a random variable τ that indicates the time of death or surrender of the policy-holder. We could consider the following problem of the insurer:

$$\max_{(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}\left[U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\mathbb{1}_{\{\tau\geq T\}}\right], \tag{5.1}$$

where the set of admissible strategies of the insurer is:

$$\begin{aligned}
\Lambda_I := \{(\pi_I,\xi_I)|\ & \pi_I \text{ self-financing}, \pi_I(t)\in K\ \mathbb{Q}\text{-a.s. } \forall t\in[0,T],\ \xi_I\in[0,\xi^{\max}], \\
& V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(t)\geq 0\ \mathbb{Q}\text{-a.s. } \forall t\in[0,T] \text{ and} \\
& \mathbb{E}[(U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\mathbb{1}_{\{\tau\geq T\}})^-]<\infty\}.
\end{aligned}$$

If τ is independent of financial risks, then:

$$\begin{aligned}
&\mathbb{E}\left[U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\mathbb{1}_{\{\tau\geq T\}}\right] \\
&\overset{(i)}{=} \mathbb{E}\left[\mathbb{E}\left[U_I((V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\mathbb{1}_{\{\tau\geq T\}}|\mathcal{F}_T\right]\right] \\
&\overset{(ii)}{=} \mathbb{E}\left[U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\mathbb{E}\left[\mathbb{1}_{\{\tau\geq T\}}|\mathcal{F}_T\right]\right] \\
&\overset{(iii)}{=} \mathbb{E}\left[U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\right]\mathbb{E}\left[\mathbb{1}_{\{\tau\geq T\}}\right]
\end{aligned}$$

where we use in (i) we use a tower rule of conditional expectations, in (ii) the $\mathcal{F}_T$-measurability of the financial risks, in (iii) the independence between τ and $\mathcal{F}_T$. Thus, the insurer's optimization problem we considered in the paper has the same solution as the solution to (5.1), because they differ only by a positive constant multiplier in the objective function. As a result, the Stackelberg equilibrium remains the same as before.

The situation when the policyholder also receives some money at the time of death or surrender can be modelled by incorporating it into the insurer's objective function as follows:

$$\max_{(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}\left[U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(\tau)+\xi_I P(\tau))\mathbb{1}_{\{\tau<T\}}+U_I(V_I^{v_{I,0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))\mathbb{1}_{\{\tau\geq T\}}\right]. \tag{5.2}$$

This way of modeling implies that the put option is of an American type, not a European one, as in our article. This introduces several additional levels of complexity. The first challenge is pricing this option. The second issue is solving the insurer's problem (5.2) with two terms in the insurer's problem with uncertainty about both the time and the amount of the payoff. The third challenge is solving the optimization problem of the reinsurance company, as it has to dynamically hedge its short position in the put option with an uncertain exercise time. The Stackelberg equilibrium would most probably change under this model, but we do not see a direct way of getting the intuition about the direction of change.

A different approach to modeling surrender and death benefit can be found in Kronborg and Steffensen (2015). There, the researchers take a standpoint of a policyholder who dynamically controls the death benefit and the investment strategy. The controlled process that models the death benefit appears in the drift part of the wealth process. Following their approach in our article would require new tools to solving the bi-level optimization problem that models the overall Stackelberg game.

References

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