Online Appendix - Optimal VIX-linked Structure for the Target Benefit Pension Plan

Appendix A. Notations

Here is the list of all notations used in this paper, and the benchmarks parameters values used in the analysis. The model parameters are based on the "SV0" and "SVJ0" calibrations in Pan (2002).

Economic variables

$S_0(t)$	time-t value of the risk-free asset
$S_1(t)$	time-t value of the risky asset
$v(t)$	time-t value of the instantaneous variance of the risky asset
N_t	time-t value of the Poisson process represents the jump component
Z_n	return of the n-th jump
$L(t)$	time-t value of the salary
VIX_t	time-t value of the volatility index
$\overline{\mathrm{VIX}^2}$	benchmark squared volatility index ($\lim_{t\to\infty} \mathbb{E}^{\mathbb{Q}}[\mathrm{VIX}_t^2]$)
$\pi(t)$	amount invested in the risky asset at time t
$X(t)$	asset level at time t

Dynamic Parameters

r	0.02	0.02	risk-free rate
λ	4.4	3.1	risk premia for the return
θ	0	27.1	jump intensity
μ_z	-	-0.003	mean of the jump return ($\mu_z^{\mathbb{Q}} = -0.18$)
σ_z	-	0.0325	standard deviation of the jump return
ρ_v	-0.57	-0.52	correlation between the diffusion terms of volatility and equity
κ_v	5.3	7.1	speed of mean reversion for the variance process
$\bar{v}$	0.0242	0.0134	unconditional mean of the variance process
σ_v	0.38	0.28	variance of the variance process

κ_l	-0.027	-	speed of mean reversion for the salary process
$\bar{L}(t)$	0	$\exp(\psi t)$	unconditional mean of the salary process
ψ	-	0.027	expected growth rate of the average wage index
σ_l	0	-	volatility of the salary process
ρ_{lv}	0	-	correlation between the diffusion terms of salary and volatility
ρ_{lS}	0	-	correlation between the diffusion terms of salary and equity
Pension Plan			
$c(t)$	0.1	-	contribution rate at time t
$b(t)$			benefit (replacement rate) at time t
$\mathcal{R}(t)$	$\mathcal{R} = 1865$	-	# of retirees at time t
$\mathcal{A}(t)$	$\mathcal{A} = 3851$	-	# of active workers at time t
$n_y(t)$	$n_A(t) = 100$	-	# of employees aged y
$\mathcal{H}(t)$	$25,379$		actuarial liability value at time t in the real term
ϕ	0.025	-	discount rate for liability valuation
$\hat{b}$	0.65	-	benchmark replacement rate for liability valuation
Preference Parameters			
γ_r	50	-	risk aversion parameter for the retirement income
γ_T	$\frac{0.03\times\gamma_r}{\mathcal{R}(T)}$	-	risk aversion parameter for the terminal asset level
ϱ	1	-	weight given to the preference of the terminal asset level
ζ	0	-	discount rate for time preference

Table A.1: List of mathematical symbols, superscript $\mathbb{Q}$ represents the parameter under the risk neutral measure, the numbers in the second column represent the benchmark scenario, and the numbers in the third column are used for sensitivity tests.

Appendix B. State-price Density, Admissible Set, and Derivation of Theorem 3.1

Appendix B.1. State-price Density

This part provides the state-price density that links the data-generating process (1) with the risk-neutral dynamics (3).

Consider a candidate state-price density $\mathcal{K}$ of the form

$$\mathcal{K}(t) = \exp(-rt)\,\varepsilon\left(-\int_0^t \varphi^T(u)\mathrm{d}\mathbf{W}(u)\right)\exp\left(\sum_{n=1}^{N_t} Z_n^{\mathcal{K}}\right),$$

where $\varepsilon(\cdot)$ denotes the stochastic exponential i.e., $\varepsilon(X_t) = \exp(X_t - [X_t, X_t]/2)$ with $[X_t, X_t]$ being the total quadratic-variation process, $\mathbf{W}(t) = [W_S(t), W_v(t)]^T$, and $\varphi(t) = [\varphi_1(t), \varphi_2(t)]^T$ are the market prices of the Brownian shocks in the price and volatility defined by

$$\varphi_1(t) = \lambda\sqrt{v(t)}, \quad \varphi_2(t) = -\frac{\lambda\rho_v}{\sqrt{1-\rho_v}}\sqrt{v(t)},$$

with λ being a constant coefficient. For this specification of the market price of risk, the time-t instantaneous risk premium associated with the diffusive price shock is $\lambda \cdot v(t)$.

The jump risks are priced by the jump component in the state-price density. Whenever the underlying price jumps, the state-price density also jumps. The jump sizes $Z_n^{\mathcal{K}}$ are assumed to be i.i.d. normal with mean $\mu_{\mathcal{K}}$ and variance $\sigma_{\mathcal{K}}^2$, and assumed to be independent of W_S and W_v and inter-jump times. The random jump sizes $Z_n^{\mathcal{K}}$ and Z_n are allowed to be correlated with constant $\rho_{\mathcal{K}}$, but assumed to be independent across different jump times. Treating $\mu_{\mathcal{K}}$, $\sigma_{\mathcal{K}}$, and $\rho_{\mathcal{K}}$ as free parameters, the most general form of jump-risk premia is obtained. We constrain the mean relative jump size in the state-price density to be zero, i.e., $\mu_{\mathcal{K}} + \frac{\sigma_{\mathcal{K}}^2}{2} = 0$. This constraint is translated to a zero jump-timing risk premium. If we turn off the correlation between $Z_n^{\mathcal{K}}$ and Z_n by letting $\rho_{\mathcal{K}} = 0$, the jump-size risk premium is zero.

For $\mathcal{K}$ to be a state-price density, the deflated processes $S_0^{\mathcal{K}} = \mathcal{K} \cdot S_0$ and $S^{\mathcal{K}} = \mathcal{K} \cdot S$ are required to be a local martingale. Applying Ito's formula, we have

$$\begin{aligned}
\mathrm{d}S^{\mathcal{K}}(t) &= \left(\sqrt{v(t)} - \varphi_1(t)\right) S^{\mathcal{K}}(t)\,\mathrm{d}W_S(t) - \varphi_2(t) S^{\mathcal{K}}(t)\,\mathrm{d}W_v(t) + [\exp(Z_{N_t}^{\mathcal{K}} + Z_{N_t}) - 1] S^{\mathcal{K}}(t-)\,\mathrm{d}N_t \\
&\quad - \theta v(t)\mu^{\mathbb{Q}} S^{\mathcal{K}}(t)\,\mathrm{d}t, \\
\mathrm{d}S_0^{\mathcal{K}}(t) &= -\varphi_1(t) S_0^{\mathcal{K}}(t)\,\mathrm{d}W_S(t) - \varphi_2(t) S_0^{\mathcal{K}}(t)\,\mathrm{d}W_v(t) + [\exp(Z_{N_t}^{\mathcal{K}}) - 1] S_0^{\mathcal{K}}(t-)\,\mathrm{d}N_t.
\end{aligned}$$

Similar to Pan (2002), we have that $S^{\mathcal{K}}(t)$ and $S_0^{\mathcal{K}}(t)$ are local martingales. If $S_0^{\mathcal{K}}(t)$ is actually a martingale, then $S_0^{\mathcal{K}}(t)$ uniquely defines an equivalent martingale measure $\mathbb{Q}$. Letting

$$\mathbf{W}^{\mathbb{Q}}(t) = [W_S^{\mathbb{Q}}(t), W_v^{\mathbb{Q}}(t)]^T = \mathbf{W}(t) + \int_0^t \varphi(u)\mathrm{d}u,$$

one can show that the dynamics of (S, v) under $\mathbb{Q}$ are indeed in the form of the risk-neutral dynamics (3). Moreover, the Poisson process N_t has the same distribution under both the risk-neutral measure $\mathbb{Q}$ and the physical measure $\mathbb{P}$.

Appendix B.2. Admissible Set

Definition Appendix B.1. *A strategy (π, b) is called an admissible strategy, i.e., $(\pi, b) \in \Pi$, if it satisfies the following conditions:*

(i) (π, b) *is* $\mathbb{F}$*-progressively measurable;*

(ii) $E_{t,x,v,l}\left[\int_0^T \pi^2(t)v(t)\mathrm{d}t\right] < \infty$*;*

(iii) Equation (4) has a unique strong solution for any $(t, x, v, l) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Appendix B.3. Derivation of Theorem 3.1

The associate HJB equation for the value function $H(t, x, v, l)$ is

$$\sup_{(b,\pi)\in\Pi}\Big\{-\zeta H + H_t + [rx + \pi\lambda v - \pi\theta\mu^{\mathbb{Q}}v + cl\mathcal{A}(t) - bl\mathcal{R}(t)]H_x + \kappa_v(\bar{v} - v)H_v + \kappa_l(\bar{L}(t) - l)H_l$$
$$+\frac{1}{2}\pi^2 vH_{xx} + \frac{1}{2}\sigma_v^2 vH_{vv} + \frac{1}{2}\sigma_l^2 vH_{ll} + \pi\sigma_v v\rho_v H_{xv} + \pi\sigma_l v\rho_{lS} H_{xl}$$
$$+\sigma_l\sigma_v v(\rho_{lS}\rho_v + \rho_{lv}\sqrt{1-\rho_v^2})H_{lv} + \theta v\mathbb{E}[H(t, x + \pi(\mathrm{e}^{Z_n} - 1), v, l) - H(t, x, v, l)] + \mathcal{R}(t)U(bl; \gamma_r)\Big\} = 0. \tag{B.1}$$

With the conjecture that

$$H(t, x, v, l) = -\frac{\varrho}{\gamma_T}\mathrm{e}^{-\gamma_T[A(t)x+\bar{A}(t)v+\hat{A}(t)l+\tilde{A}(t)]}$$

and with the terminal conditions $A(T) = 1$, $\bar{A}(T) = \hat{A}(T) = \tilde{A}(T) = 0$, the first-order condition implies that the optimal replacement rate b^* is

$$b^*(t) = -\frac{1}{\gamma_r l}\ln(\varrho A(t)) + \frac{\gamma_T}{\gamma_r l}[A(t)x + \bar{A}(t)v + \hat{A}(t)l + \tilde{A}(t)], \tag{B.2}$$

and the optimal investment strategy π^* satisfies

$$-(\lambda - \theta\mu^{\mathbb{Q}}) + \pi^*\gamma_T A(t) + \sigma_v\rho_v\gamma_T\bar{A}(t) + \sigma_l\rho_{lS}\gamma_T\hat{A}(t) - \theta\mathbb{E}[\mathrm{e}^{-\gamma_T A(t)\pi^*(\mathrm{e}^{Z_n}-1)}(\mathrm{e}^{Z_n} - 1)] = 0. \tag{B.3}$$

Substitute the optimal replacement rate and the investment strategy into the HJB equation and by separating the variables, we obtain the following ODE system,

$$A_t(t) + rA(t) - \frac{\mathcal{R}(t)\gamma_T A^2(t)}{\gamma_r} = 0,$$
$$\bar{A}_t(t) + \pi^*(\lambda - \theta\mu^{\mathbb{Q}})A(t) - \kappa_v\bar{A}(t) - \frac{1}{2}(\pi^*)^2\gamma_T A^2(t) - \frac{1}{2}\sigma_v^2\gamma_T\bar{A}^2(t) - \frac{1}{2}\sigma_l^2\gamma_T\hat{A}^2(t)$$
$$-\pi^*\sigma_v\rho_v\gamma_T A(t)\bar{A}(t) - \pi^*\sigma_l\rho_{lS}\gamma_T A(t)\hat{A}(t) - \sigma_l\sigma_v(\rho_{lS}\rho_v + \rho_{lv}\sqrt{1-\rho_v^2})\gamma_T\bar{A}(t)\hat{A}(t)$$
$$-\frac{\theta}{\gamma_T}\mathbb{E}[\mathrm{e}^{-\gamma_T A(t)\pi^*(\mathrm{e}^{Z_n}-1)} - 1] - \frac{\mathcal{R}(t)\gamma_T A(t)\bar{A}(t)}{\gamma_r} = 0,$$
$$\hat{A}_t(t) + c\mathcal{A}(t)A(t) - \kappa_l\hat{A}(t) - \frac{\mathcal{R}(t)\gamma_T A(t)\hat{A}(t)}{\gamma_r} = 0,$$
$$\frac{\zeta}{\gamma_T} + \tilde{A}_t(t) + \kappa_v\bar{v}\bar{A}(t) + \kappa_l\bar{L}(t)\varrho\hat{A}(t) + \frac{\mathcal{R}(t)A(t)}{\gamma_r}\ln(\varrho A(t)) - \frac{\mathcal{R}(t)\gamma_T A(t)\tilde{A}(t)}{\gamma_r} - \frac{\mathcal{R}(t)A(t)}{\gamma_r} = 0, \tag{B.4}$$

with $A(T) = 1$, $\bar{A}(T) = \hat{A}(T) = \tilde{A}(T) = 0$.

Appendix C. Explicit Forms of $A(t)$, $\hat{A}(t)$ and $\bar{A}(t)$

Appendix C.1. Solution of $A(t)$ and $\hat{A}(t)$

Since $A(t)$ is the solution of a Riccati equation, its explicit form can be derived as

$$A(t) = \frac{e^{(T-t)r}}{1 + \frac{\gamma_T}{\gamma_r}\int_t^T e^{(T-s)r}\mathcal{R}(s)\,\mathrm{d}s}. \tag{C.1}$$

In addition, the explicit solution for $\hat{A}(t)$ can be derived as

$$\hat{A}(t) = \int_t^T c\frac{\mathcal{A}(s)}{\mathcal{R}(s)} \times (\beta_A(s) + \kappa_l) \times \frac{\gamma_r}{\gamma_T} e^{-\int_t^s \beta_A(u)+\kappa_l du}\mathrm{d}s. \tag{C.2}$$

Appendix C.2. Solution of $\bar{A}$, when jump risk and salary risk are ignored

Ignoring the jump and the salary risks, the ODE equation for $\bar{A}(t)$ can be simplified to

$$\bar{A}_t(t) - (\lambda\sigma_v\rho_v + \kappa_v + \beta_A(t))\bar{A}(t) - \frac{1}{2}\sigma_v^2(1-\rho_v^2)\gamma_T\bar{A}^2(t) + \frac{\lambda^2}{2\gamma_T} = 0. \tag{C.3}$$

The explicit solution of $\bar{A}(t)$ can be derived as:

$$\bar{A}(t) = \begin{cases} \dfrac{\nu_1 - \nu_1 \mathrm{e}^{-\frac{\sigma_v^2(1-\rho_v^2)\gamma_T(\nu_1-\nu_2)}{2}(T-t)}}{1 - \frac{\nu_1}{\nu_2}\mathrm{e}^{-\frac{\sigma_v^2(1-\rho_v^2)\gamma_T(\nu_1-\nu_2)}{2}(T-t)}}, & \rho_v \neq \pm 1, \\ \dfrac{\lambda^2}{2\gamma_T}\displaystyle\int_t^T \mathrm{e}^{-\int_0^w(\lambda\sigma_v+\kappa_v+\beta_A(s))\mathrm{d}s}\mathrm{d}w \cdot \mathrm{e}^{-\int_0^t(\lambda\sigma_v+\kappa_v+\beta_A(s))\mathrm{d}s}, & \rho_v = 1,\ \ \lambda\sigma_v\rho_v + \kappa_v \neq -\beta_A(t), \\ \dfrac{\lambda^2}{2\gamma_T}\displaystyle\int_t^T \mathrm{e}^{-\int_0^w(-\lambda\sigma_v+\kappa_v+\beta_A(s))\mathrm{d}s}\mathrm{d}w \cdot \mathrm{e}^{-\int_0^t(-\lambda\sigma_v+\kappa_v+\beta_A(s))\mathrm{d}s}, & \rho_v = -1,\ \ \lambda\sigma_v\rho_v + \kappa_v \neq -\beta_A(t), \\ \dfrac{\lambda^2}{2\gamma_T}(T-t), & \rho_v = \pm 1,\ \ \lambda\sigma_v\rho_v + \kappa_v = -\beta_A(t), \end{cases} \tag{C.4}$$

where

$$\nu_{1,2} = \frac{-\lambda\sigma_v\rho_v - \kappa_v - \beta_A(t) \pm \sqrt{(\lambda\sigma_v\rho_v + \kappa_v + \beta_A(t))^2 + \lambda^2\sigma_v^2(1-\rho_v^2)}}{\sigma_v^2(1-\rho_v^2)\gamma_T}.$$

Appendix D. Contribution Rate as a Control Variable

Define the value function $G(t,x,v,l)$ as

$$\begin{aligned} G(t,x,v,l) = \sup_{(\pi,c,b)\in\bar{\Pi}} \mathbb{E}_{t,x,v,l}\Big[\int_t^T e^{-\zeta(s-t)} \times [\mathcal{A}(s) \times U((1-c(s)) \times L(s);\gamma_a) \\ + \mathcal{R}(s) \times U(b(s) \times L(s);\gamma_r)]\,\mathrm{d}s + \varrho \times e^{-\zeta(T-t)} \times U(X(T);\gamma_T)\Big], \end{aligned} \tag{D.1}$$

where γ_a is the risk aversion parameters for the active members, $\bar{\Pi}$ is the corresponding admissible set and the pension asset now has the following dynamics

$$\begin{aligned}\mathrm{d}X(t) = &\Big[rX(t) + \pi(t)\nu(t)(\lambda - \theta\mu^{\mathbb{Q}}) + (c(t)\mathcal{A}(t) - b(t)\mathcal{R}(t)) \times L(t)\Big]\mathrm{d}t \\ &+ \pi(t)\sqrt{\nu(t)}\,\mathrm{d}W_S(t) + \pi(t)\,\mathrm{d}\left(\sum_{n=1}^{N_t}(e^{Z_n} - 1)\right).\end{aligned}$$

The associate HJB equation for the objective function (D.1) is

$$\begin{aligned}\sup_{(\pi,c,b)\in\bar{\Pi}} \Big\{ &-\zeta G + G_t + [rx + \pi\lambda\nu - \pi\theta\mu^{\mathbb{Q}}\nu + cl\mathcal{A}(t) - bl\mathcal{R}(t)]G_x + \kappa_\nu(\bar{\nu} - \nu)H_\nu + \kappa_l(\bar{L}(t) - l)G_l \\ &+ \frac{1}{2}\pi^2\nu G_{xx} + \frac{1}{2}\sigma_\nu^2\nu G_{\nu\nu} + \frac{1}{2}\sigma_l^2\nu G_{ll} + \pi\sigma_\nu\nu\rho_\nu G_{x\nu} + \pi\sigma_l\nu\rho_{lS}G_{xl} \\ &+ \sigma_l\sigma_\nu\nu(\rho_{lS}\rho_\nu + \rho_{l\nu}\sqrt{1-\rho_\nu^2})H_{l\nu} + \theta\nu\mathbb{E}[G(t, x + \pi(\mathrm{e}^{Z_n} - 1), \nu, l) - G(t, x, \nu, l)] \\ &+ \mathcal{A}(t)U((1-c)l; \gamma_a) + \mathcal{R}(t)U(bl; \gamma_r)\Big\} = 0.\end{aligned} \tag{D.2}$$

Similar to $H(t, x, \nu, l)$, with the conjecture on $G(t, x, \nu, l)$ as

$$G(t, x, \nu, l) = -\frac{\varrho}{\gamma_T}\mathrm{e}^{-\gamma_T[B(t)x + \bar{B}(t)\nu + \hat{B}(t)l + \tilde{B}(t)]}$$

and the boundary conditions $B(T) = 1$, $\bar{B}(T) = \hat{B}(T) = \tilde{B}(T) = 0$, we can obtain the optimal contribution rate c^*, the optimal replacement rate b^* and the optimal investment strategy π^* as

$$c^*(t) = 1 + \frac{1}{\gamma_a l}\ln(\varrho B(t)) - \frac{\gamma_T}{\gamma_a l}[B(t)x + \bar{B}(t)\nu + \hat{B}(t)l + \tilde{B}(t)], \tag{D.3}$$

$$b^*(t) = -\frac{1}{\gamma_r l}\ln(\varrho B(t)) + \frac{\gamma_T}{\gamma_r l}[B(t)x + \bar{B}(t)\nu + \hat{B}(t)l + \tilde{B}(t)], \tag{D.4}$$

$$-(\lambda - \theta\mu^{\mathbb{Q}}) + \pi^*\gamma_T B(t) + \sigma_\nu\rho_\nu\gamma_T\bar{B}(t) + \sigma_l\rho_{lS}\gamma_T\hat{A}(t) - \theta\mathbb{E}[\mathrm{e}^{-\gamma_T B(t)\pi^*(\mathrm{e}^{Z_n}-1)}(\mathrm{e}^{Z_n} - 1)] = 0. \tag{D.5}$$

Substituting back to the HJB equation, and by separating the variables, we have the following

ODE system,

$$
\begin{aligned}
&B_t(t) + rB(t) - \frac{\mathcal{A}(t)\gamma_T B^2(t)}{\gamma_a} - \frac{\mathcal{R}(t)\gamma_T B^2(t)}{\gamma_r} = 0,\\
&\bar{B}_t(t) + \pi^*(\lambda - \theta\mu^{\mathbb{Q}})B(t) - \kappa_v\bar{B}(t) - \frac{1}{2}(\pi^*)^2\gamma_T B^2(t) - \frac{1}{2}\sigma_v^2\gamma_T\bar{B}^2(t) - \frac{1}{2}\sigma_l^2\gamma_T\hat{B}^2(t)\\
&-\pi^*\sigma_v\rho_v\gamma_T B(t)\bar{B}(t) - \pi^*\sigma_l\rho_{lS}\gamma_T B(t)\hat{B}(t) - \sigma_l\sigma_v(\rho_{1_1}\rho_v + \rho_{lv}\sqrt{1-\rho_v^2})\gamma_T\bar{B}(t)\hat{B}(t)\\
&-\frac{\theta}{\gamma_T}\mathbb{E}[\mathrm{e}^{-\gamma_T B(t)\pi^*(\mathrm{e}^{Z_n}-1)} - 1] - \frac{\mathcal{A}(t)\gamma_T B(t)\bar{B}(t)}{\gamma_a} - \frac{\mathcal{R}(t)\gamma_T B(t)\bar{B}(t)}{\gamma_r} = 0,\\
&\hat{B}_t(t) - \kappa_l\hat{B}(t) + \mathcal{A}(t)B(t) - \frac{\mathcal{A}(t)\gamma_T B(t)\hat{B}(t)}{\gamma_a} - \frac{\mathcal{R}(t)\gamma_T B(t)\hat{B}(t)}{\gamma_r} = 0,\\
&\frac{\zeta}{\gamma_T} + \tilde{B}_t(t) + \kappa_v\bar{v}\bar{B}(t) + \kappa_l\bar{L}(t)\varrho\hat{B}(t) + \frac{\mathcal{A}(t)B(t)}{\gamma_a}\ln(\varrho B(t)) - \frac{\mathcal{A}(t)\gamma_T B(t)\tilde{B}(t)}{\gamma_a} - \frac{\mathcal{A}(t)B(t)}{\gamma_a}\\
&+\frac{\mathcal{R}(t)B(t)}{\gamma_r}\ln(\varrho B(t)) - \frac{\mathcal{R}(t)\gamma_T B(t)\tilde{B}(t)}{\gamma_r} - \frac{\mathcal{R}(t)B(t)}{\gamma_r} = 0
\end{aligned}
\tag{D.6}
$$

with $B(T) = 1$, $\bar{B}(T) = \hat{B}(T) = \tilde{B}(T) = 0$.

Appendix E. Optimal Target Benefit Design without Stochastic Volatility

Assume the risky asset follows the Geometric Brownian Motion, which the volatility is a constant and the jump risk is excluded,

$$\frac{\mathrm{d}S(t)}{S(t)} = (r + \mu)\,\mathrm{d}t + \sigma_S\,\mathrm{d}W_S(t).$$

Then, the pension asset has the following SDE,

$$\mathrm{d}X(t) = \left[rX(t) + \pi(t)\mu + (c\mathcal{A}(t) - b(t)\mathcal{R}(t))L(t)\right]\mathrm{d}t + \pi(t)\sigma_S\,\mathrm{d}W_S(t).$$

With the same objective function as Problem 5, and define the value function as $H(t, X_t = x)$, then following the same procedure as before, we have

$$H(t, x) = -\frac{\varrho}{\gamma_T}\mathrm{e}^{-\gamma_T[A(t)x + \tilde{A}(t)]},$$

where $A(t)$ and $\tilde{A}(t)$ satisfy the following ODE system,

$$
\begin{aligned}
&A_t(t) + rA(t) - \frac{\mathcal{R}(t)\gamma_T A^2(t)}{\gamma_r} = 0,\\
&\frac{\zeta}{\gamma_T} + \tilde{A}_t(t) + cL(t)\mathcal{A}(t)A(t) + \frac{\mu^2}{2\sigma_S^2\gamma_T} + \frac{\mathcal{R}(t)A(t)}{\gamma_r}\ln(\varrho A(t)) - \frac{\mathcal{R}(t)\gamma_T A(t)\tilde{A}(t)}{\gamma_r} - \frac{\mathcal{R}(t)A(t)}{\gamma_r} = 0,
\end{aligned}
\tag{E.1}
$$

with $A(T) = 1$, $\tilde{A}(T) = 0$. By abuse of notation, $A(t)$ and $\tilde{A}(t)$ defined in this section are unrelated to $A(t)$ and $\tilde{A}(t)$ in other sections.

The optimal replacement rate b^* is

$$b^*(t) = -\frac{1}{\gamma_r L(t)} \ln(\varrho A(t)) + \frac{\gamma_T}{\gamma_r L(t)}[A(t)X(t) + \tilde{A}(t)], \tag{E.2}$$

and the optimal investment strategy π^* is

$$\pi^*(t) = \frac{\mu}{\sigma_S^2 \gamma_T A(t)}. \tag{E.3}$$

Multiply b^* with the salary index $L(t)$, and with some rearrangement, we can show that

$$\text{benefit}(t) = \bar{b} \times L(t) + \beta_A(t) \left(\frac{X(t) - \text{AL}(t)}{\mathcal{R}(t)} \right),$$

where

$$\begin{aligned} \beta_A(t) &= \frac{\gamma_T}{\gamma_r} A(t)\mathcal{R}(t), \\ \bar{b}(t) &= -\frac{1}{\gamma_r} \ln(\varrho A(t)) + \frac{\gamma_T}{\gamma_r}\tilde{A}(t) + \frac{\gamma_T}{\gamma_r}\text{AL}(t). \end{aligned}$$

Immediately we observe that the ODE for $A(t)$ is the same with and without the stochastic volatility and the jump risk, and therefore the value of $\beta_A(t)$ is the same under these models.

References and Notes

Pan J (2002). "The jump-risk premia implicit in options: Evidence from an integrated time-series study." *Journal of Financial Economics*, **63**(1), 3–50.