Population and Geography Do Matter for Sustainable Development

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ONLINE APPENDIX

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1 Proof of Proposition 1

The equilibria of the system of ordinary differential equations (3) - (4) in the main text correspond to the solutions of the following nonlinear system of algebraic equations:

$$0 = L(t)\left[\frac{\Omega}{1+\theta P} - L(t)\right] \tag{A1}$$

$$0 = P(t)\left[\frac{\beta}{1+\tau A L(t)^{1-\alpha}} - \delta_P\right]. \tag{A2}$$

It is straightforward to verify that the origin is an equilibrium $E_1 = (0,0)$. Straightforward algebra leads to the other two equilibria, $E_2 = (\Omega_x, 0)$ and $E_3 = (\overline{L}, \overline{P})$, with:

$$\overline{L} = \left(\frac{\beta - \delta_P}{\tau A \delta_P}\right)^{\frac{1}{1-\alpha}} \tag{A3}$$

$$\overline{P} = \frac{1}{\theta}\left[\frac{\Omega}{\left(\frac{\beta-\delta_P}{\tau A \delta}\right)^{\frac{1}{1-\alpha}}} - 1\right]. \tag{A4}$$

The parametric restriction in Proposition 1 guarantees that $\overline{L} > 0$ and $\overline{P} > 0$; indeed, $\beta - \delta_P > 0$ ensures that $\overline{L} > 0$, while $\Omega^{1-\alpha}\tau A \delta_P > \beta - \delta_P$ that $\overline{P} > 0$.

The stability property of the origin E_1 cannot be analyzed via the traditional linearization method. It is however possible to show that the trajectories are eventually escaping from a circular sector surrounding the origin, provided that the radius of this sector is small enough. For this purpose, let us express the vector of the initial condition as:

$$L(0) = L_0 \tag{A5}$$

$$P(0) = P_0 = vL_0, \tag{A6}$$

where $v = \tan(\theta)^{-1}$ implicitly defines the direction of the vector of initial conditions (L_0, P_0), whose angle with respect to the L axis is θ. The idea is to show that, $\forall v \in (0, +\infty)$, the following vector field:

$$\frac{dL}{dt} = L_0\left[\frac{\Omega}{1+\theta v L_0} - L_0\right] \tag{A7}$$

$$\frac{dP}{dt} = vL_0\left[\frac{\beta}{1+\tau A L_0^{1-\alpha}} - \delta_P\right] \tag{A8}$$

has both positive components eventually, when $L_0 \to 0$. When L_0 tends to zero, equations (A7) - (A8) can be written as:

$$\frac{dL}{dt} \simeq L_0\left[\Omega - L_0\right] \tag{A9}$$

$$\frac{dP}{dt} \simeq vL_0(\beta - \delta_P). \tag{A10}$$

Given that $\Omega > 0$, the quantity on the RHS of equation (A9) will eventually become positive, no matter the value of v. As for equation (A10), the RHS is always positive $\forall v \in (0, +\infty)$, provided that the parametric restriction required by Proposition 1 is met. It remains to explore the extreme case where $v = 0$ or $v = +\infty$, that is the axes $P = 0$ and $L = 0$ respectively. When $P = 0$, the RHS of equation (3) in the main text is eventually positive, in the limit $L_0 \to 0$, as shown before, while in equation (4) in the main text, the RHS is identically null. When $L = 0$, the RHS in equation (3) is null, while the RHS in equation (4) is positive, because $\beta - \delta_P > 0$ by assumption. The trajectories are eventually escaping from a circular sector in the positive orthant around the origin.

For what concerns the other two equilibria, linearization can be applied. The associated Jacobian matrix is given by:

$$J(L,P) = \begin{bmatrix} \frac{\Omega}{1+\theta P} - 2L & -\frac{\theta \Omega L}{(1+\theta P)^2} \\ -\frac{\beta P(1-\alpha)\tau A L^{1-\alpha}}{(1+\tau A L^{1-\alpha})^2 L} & \frac{\beta}{1+\tau A L^{1-\alpha}} - \delta_P \end{bmatrix}. \tag{A11}$$

Let us start with $E_2 = (\Omega, 0)$. The Jacobian matrix (A11) evaluated at E_2 reads as follows:

$$J(\Omega, 0) = \begin{bmatrix} -\Omega & -\theta\Omega^2 \\ 0 & \frac{\beta}{1+\tau A \Omega^{1-\alpha}} - \delta_P \end{bmatrix}.$$

Since the term $a_{2,2}$ is negative, due to the parametric restriction required by Proposition 1, the determinant is positive and the trace is negative, thus the equilibrium E_2 is asymptotically stable. Finally we consider $E_3 = (\overline{L}, \overline{P})$.

Simple inspection of the Jacobian matrix (A11) shows that the terms $a_{1,2}$ and $a_{2,1}$ are both positive under Proposition 1. The Jacobian matrix (A11) evaluated at E_3 in this case becomes:

$$J(\overline{L}, \overline{P}) = \begin{bmatrix} -\left(\frac{\beta-\delta}{\tau A \delta_P}\right)^{\frac{1}{1-\alpha}} & < 0 \\ < 0 & 0 \end{bmatrix}.$$

Both the determinant and the trace are negative, thus the equilibrium E_3 is saddle point stable.