Technical Appendix of "Trade Openness, Government Size and Factor Intensities"

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This online appendix provides more details about the model discussed in the main paper, and lists the proofs for all propositions. Section 1 characterizes the steady state discussed in section 3 of the main paper and proves proposition 1. Section 2 adds capital to the production of public goods. Sections 3 to 5 explore the impacts of productive government expenditures according to different output elasticities of capital and labor and prove proposition 2 to 4.

1 Steady State and Proof of Proposition 1

Equation (1) to (6) and equation (11) to (21) characterize the competitive equilibrium for the model economy introduced in section 2. We remove all variables' time subscript to reach the steady state equilibrium. Assume symmetry across industries in the tradable sector and across industries in the non-tradable sector and we obtain a 17-equation system with 17 unknown endogenous variables. Solve the equation system and the unique steady state is reported in (22).

We first show that, the capital-specificity assumption allows the existence of the steady state equilibrium. The steady state version of equations (2), (3), (5), (6), (13), (14) is copied here

$$\frac{W}{p_i^T} = (1-\alpha)\left(K_i^T\right)^{\alpha}\left(L_i^T\right)^{-\alpha} \tag{A1}$$

$$\frac{R_i^T}{p_i^T} = \alpha\left(K_i^T\right)^{\alpha-1}\left(L_i^T\right)^{1-\alpha} \tag{A2}$$

$$\frac{W}{p_j^N} = (1-\gamma)\left(K_j^N\right)^{\gamma}\left(L_j^N\right)^{-\gamma} \tag{A3}$$

$$\frac{R_j^N}{p_j^N} = \gamma\left(K_j^N\right)^{\gamma-1}\left(L_j^N\right)^{1-\gamma} \tag{A4}$$

$$\frac{R_i^T}{P_i^T} = \frac{1/\beta-1}{1-\tau} \tag{A5}$$

$$\frac{R_j^N}{P_j^N} = \frac{1/\beta-1}{1-\tau} \tag{A6}$$

Based on the above equations, one can easily show that

$$\frac{R_i^T}{P_i^T} = \frac{R_j^N}{P_j^N} \tag{A7}$$

$$\frac{W}{P_i^T} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{1/\beta-1}{1-\tau}\right)^{-\frac{\alpha}{1-\alpha}} \tag{A8}$$

$$\frac{W}{P_i^T} = \gamma^{\frac{\gamma}{1-\gamma}}(1-\gamma)\left(\frac{1/\beta-1}{1-\tau}\right)^{-\frac{\gamma}{1-\gamma}} \tag{A9}$$

Equation (A7) indicates the resulting equalization of returns on different varieties of capital in steady state equilibrium in the presence of industry-specific capital, which is obtained through agents' intertemporal substitution and capital adjustment in each sector. If capital is homogenous and can move freely across sectors, then $R_i^T = R_j^N$ must hold in any equilibrium, which implies that $P_i^T = P_j^N$ from (A7). The price equalization also leads to the equalization of the left hand sides of equation (A8) and (A9), which contradicts with the assumed differences in the factor intensities on the right hand sides of these equations. Therefore the steady state equilibrium does not exist without industry-specific capital.

Below we prove proposition 1 presented in the text. With the same symmetry assumptions across industries in the tradable sector and non-tradable sector, respectively, the steady state version of (9) reduces to

$$
\begin{aligned}
U &= \frac{\left[\left(\exp\int_0^1 \log C_s ds\right)^{\eta} G^{1-\eta}\right]^{1-\rho}}{1-\rho} \\
&= \frac{\left[\left(C_i^{\theta} C_j^{1-\theta}\right)^{\eta} G^{1-\eta}\right]^{1-\rho}}{1-\rho} \equiv \frac{\widetilde{U}^{1-\rho}}{1-\rho}
\end{aligned}
$$

The function $F(\theta,\tau)$ is obtained by taking the derivative of $\log \widetilde{U}$ with respect to τ. The proof of proposition 1 includes three steps.

Step 1: Government size τ is determined by the first order necessary condition from the benevolent government's maximization problem (23):

$$
\begin{aligned}
F(\theta,\tau) &= \frac{-\eta\left[\theta(\alpha-\gamma)+\gamma-\alpha\gamma\right]}{(1-\alpha)(1-\gamma)(1-\tau)} - \frac{\eta}{1-\tau} + \frac{1-\eta}{\tau} \\
&\quad - \frac{\gamma-\theta(\gamma-\alpha)}{1-\gamma(1-\tau)-\theta(\alpha-\gamma)(1-\tau)} \\
&= 0
\end{aligned} \tag{A10}
$$

Equation (A10) is an implicit function of openness θ and government size τ. We resort to the implicit function theorem to uncover the steady state relationship between these two variables:

$$
\frac{\partial \tau}{\partial \theta} = -\frac{\partial F(\theta,\tau)/\partial\theta}{\partial F(\theta,\tau)/\partial\tau} \tag{A11}
$$

From (A10), we have

$$
\begin{aligned}
\frac{\partial F(\theta,\tau)}{\partial\theta} &= \frac{-\eta(\alpha-\gamma)}{(1-\alpha)(1-\gamma)(1-\tau)} - \frac{\alpha-\gamma}{\left[1-\gamma(1-\tau)-\theta(\alpha-\gamma)(1-\tau)\right]^2} \\
&= (\gamma-\alpha)\left[\frac{\eta}{(1-\alpha)(1-\gamma)(1-\tau)} + \frac{1}{\left[1-\gamma(1-\tau)-\theta(\alpha-\gamma)(1-\tau)\right]^2}\right]
\end{aligned} \tag{A12}
$$

Given $\alpha,\gamma,\eta \in (0,1)$, for any $\theta,\tau \in (0,1)$, equation (A12) implies that

$$
sign\left(\frac{\partial F(\theta,\tau)}{\partial\theta}\right) = sign(\gamma-\alpha) \tag{A13}
$$

Step 2: Given $\alpha,\gamma,\eta \in (0,1)$ and $\theta,\tau \in (0,1)$, if we could show $\frac{\partial F(\theta,\tau)}{\partial\tau} < 0$ when $max\{\alpha,\gamma\} < \sqrt{1-\eta}$, then equation (A11) and (A13) imply that $sign\left(\frac{\partial\tau}{\partial\theta}\right) = sign(\gamma-\alpha)$.

To start, from (A10), we have

$$
\begin{aligned}
\frac{\partial F(\theta,\tau)}{\partial\tau} &= \frac{-\eta\left[\theta(\alpha-\gamma)+\gamma-\alpha\gamma\right]}{(1-\alpha)(1-\gamma)(1-\tau)^2} - \frac{\eta}{(1-\tau)^2} \\
&\quad - \left\{\frac{1-\eta}{\tau^2} - \frac{\left[\gamma-\theta(\gamma-\alpha)\right]^2}{\left[1-\gamma(1-\tau)-\theta(\alpha-\gamma)(1-\tau)\right]^2}\right\}
\end{aligned} \tag{A14}
$$

It is obvious to show that the first two terms in equation (A14) are both negative. We need to show that the third term is also negative when $max\{\alpha, \gamma\} < \sqrt{1-\eta}$.

Let $M = \gamma - \theta(\gamma - \alpha) \in (0, 1)$. In order for the third term to be negative, we need

$$\frac{[\gamma - \theta(\gamma - \alpha)]^2}{[1 - \gamma(1-\tau) - \theta(\alpha - \gamma)(1-\tau)]^2} < \frac{1-\eta}{\tau^2}$$

or

$$\frac{M^2}{[1-(1-\tau)M]^2} < \frac{1-\eta}{\tau^2}$$

It implies

$$\frac{M}{1-(1-\tau)M} < \frac{\sqrt{1-\eta}}{\tau}$$

since $M \in (0, 1)$. Rewriting this condition gives

$$\tau < \frac{\sqrt{1-\eta}\,(1-M)}{M\left(1-\sqrt{1-\eta}\right)} \tag{A15}$$

If the RHS of (A15) is bigger than one

$$\frac{\sqrt{1-\eta}\,(1-M)}{M\left(1-\sqrt{1-\eta}\right)} > 1$$

then inequality (A15) holds for all $\tau \in (0, 1)$. This implies

$$\theta(\alpha - \gamma) < \sqrt{1-\eta} - \gamma \tag{A16}$$

In order for (A16) to hold for all $\theta \in (0, 1)$, the following two conditions need to be satisfied, depending on the relative size of α and γ:

$$\begin{aligned} \alpha &< \sqrt{1-\eta} \text{ when } \alpha > \gamma \\ \gamma &< \sqrt{1-\eta} \text{ when } \alpha < \gamma \end{aligned}$$

These two conditions could be summarized as

$$max\{\alpha, \gamma\} < \sqrt{1-\eta}$$

Step 3: The last step is to check that, under all conditions in the proposition, the second order condition of the government's maximization problem (23) is satisfied, *i.e.*, indeed we reach the maximum. This requires the following condition:

$$\frac{\partial^2 \log U}{\partial \tau^2} = \frac{\partial F(\theta, \tau)}{\partial \tau} < 0 \tag{A17}$$

We notice that this condition (A17) coincides with (A14), which has already been shown to hold in step 2. This completes the proof of proposition 1. *Q.E.D.*

2 Adding Capital to Government Production

In this appendix, we show that, when capital is added into the production of the public goods, the long-run relation between trade openness and government size is still affected by the relative factor-intensities in the tradable and non-tradable sectors.

2.1 Households and Firms

The consumer's problem is the same as in the text. The accumulated capital in the tradable sector K_{it}^{T}, however, is divided into two parts: K_{it}^{PT}, to be used in the production of private goods and K_{it}^{GT}, to be used in the production of public goods. The same division applies to the non-tradable sector

$$K_{it}^{T} = K_{it}^{PT} + K_{it}^{GT}, \quad i \in [0,\theta] \tag{B1}$$

$$K_{jt}^{N} = K_{jt}^{PN} + K_{jt}^{GN}, \quad j \in [\theta,1] \tag{B2}$$

Firm's problem is also the same except that the capital used in the private sectors is now denoted by K_{it}^{PT} and K_{jt}^{PN}, respectively

$$y_{it}^{T} = \left(K_{it}^{PT}\right)^{\alpha}\left(L_{it}^{T}\right)^{1-\alpha}, \quad i \in [0,\theta] \tag{B3}$$

$$y_{jt}^{N} = \left(K_{jt}^{PN}\right)^{\gamma}\left(L_{jt}^{N}\right)^{1-\gamma}, \quad j \in [\theta,1] \tag{B4}$$

2.2 Government

Government produces public consumption goods using both capital and labor. Since there exists a continuum of varieties of capital in our model economy, we assume that all varieties of capital are necessary in the production of public goods. K_{st}^{G} denotes the capital from industry $s \in (0,1)$ utilized by the government; in particular, K_{it}^{GT} and K_{jt}^{GN} represent the amounts of capital used by the government from industry $i \in (0,\theta)$ in the tradable sector and from industry $j \in (\theta,1)$ in the non-tradable sector. Government production function is

$$G_t = \left(\exp\int_0^1 \log K_{st}^{G} ds\right)^m L_{gt}^{1-m} \tag{B5}$$

$$= \left[\exp\left(\int_0^{\theta} \log K_{it}^{GT} di + \int_{\theta}^{1} \log K_{jt}^{GN} dj\right)\right]^m L_{gt}^{1-m} \tag{B6}$$

where $m \in (0,1)$. Government collects tax revenue to pay the wage bill as well as the capital rent and hence is subject to the following budget constraint

$$W_t L_{gt} + \int_0^\theta R_{it}^T K_{it}^{GT} di + \int_\theta^1 R_{jt}^N K_{jt}^{GN} dj = \tau_t \left(W_t L + \int_0^\theta R_{it}^T K_{it}^T di + \int_\theta^1 R_{jt}^N K_{jt}^N dj \right) \quad \text{(B7)}$$

Since the government takes wage rate and rental rate as given, a constant return to scale production function (B5) implies that

$$R_{it}^T K_{it}^{GT} = m p_{gt} G_t \quad \text{(B8)}$$

$$R_{jt}^N K_{jt}^{GN} = m p_{gt} G_t \quad \text{(B9)}$$

$$W_t L_{gt} = (1-m) p_{gt} G_t \quad \text{(B10)}$$

2.3 Steady State Equilibrium

We define a competitive equilibrium and solve for the unique steady state equilibrium. A similar function $F(\theta, \tau)$ is derived by the benevolent government that characterizes the long-run relation between trade openness and government size

$$\begin{aligned} F(\theta,\tau) &\equiv \frac{\partial \log U}{\partial \tau} = \frac{1}{U}\frac{\partial U}{\partial \tau} \\ &= \left\{ \frac{\eta}{1-\alpha} + \frac{\eta(1-\theta)(\gamma-\alpha)}{(1-\alpha)(1-\gamma)} + \frac{m(1-\eta)\left[\theta(\alpha-\gamma)+1-\alpha\right]}{(1-\alpha)(1-\gamma)} \right\} \frac{-1}{1-\tau} \\ &\quad + \frac{1-\eta}{\tau} - \frac{\theta(\alpha-\gamma)+\gamma-m}{\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau-m\tau} \\ &= 0 \end{aligned}$$

It turns out that the comovement of trade openness and government size is determined by the relative factor intensities in the tradable and non-tradable sectors under very loose sufficient conditions.

Proposition B1

Given the above model with capital in the production process of public goods and $0 < \alpha, \gamma, \eta, m < 1$, *we have*

$$sign\left(\frac{\partial \tau}{\partial \theta}\right) = sign\left(\gamma - \alpha\right)$$

for any $\tau \in (0,1)$, $\theta \in (0,1)$ *if the following condition holds*

$$
\begin{aligned}
& 8\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-m)\cdot \min\{1-\alpha, 1-\gamma\} \\
> \quad & \max\left\{\sqrt{1-\alpha}, \sqrt{1-\gamma}\right\}\cdot \max\left\{(\gamma-m)^2, (\alpha-m)^2\right\}
\end{aligned}
$$

Proof: Due to the introduction of capital in the production of public goods, the proof is more involved. We still resort to the implicit function theorem to obtain the sufficient condition. The proof has three steps.

Step 1: show that $sign\left(\frac{\partial F(\theta,\tau)}{\partial\theta}\right) = sign(\gamma-\alpha)$

Take the partial derivative of $F(\theta,\tau)$ with respect to θ and rewrite the resulting equation

$$
\frac{\partial F(\theta,\tau)}{\partial\theta} = (\gamma-\alpha)\left\{\frac{\eta+m(1-\eta)}{(1-\alpha)(1-\gamma)(1-\tau)} + \frac{1-m}{[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau-m\tau]^2}\right\}
$$

Given that $\alpha,\gamma,m,\tau \in (0,1)$, the term in the above curly bracket is positive. Therefore $sign\left(\frac{\partial F(\theta,\tau)}{\partial\theta}\right) = sign(\gamma-\alpha)$.

Step 2: find the sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial\tau} < 0$

Take the partial derivative of $F(\theta,\tau)$ with respect to τ and rewrite the resulting equation

$$
\begin{aligned}
\frac{\partial F(\theta,\tau)}{\partial\tau} \quad = \quad & \left\{\frac{\eta}{1-\alpha} + \frac{\eta(1-\theta)(\gamma-\alpha)}{(1-\alpha)(1-\gamma)} + \frac{m(1-\eta)\left[\theta(\alpha-\gamma)+1-\alpha\right]}{(1-\alpha)(1-\gamma)}\right\}\frac{-1}{(1-\tau)^2} \\
& -\frac{1-\eta}{\tau^2} + \frac{\left[\theta(\alpha-\gamma)+\gamma-m\right]^2}{\left[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau-m\tau\right]^2}
\end{aligned}
$$

$\frac{\partial F(\theta,\tau)}{\partial\tau} < 0$ is obtained when

$$
\begin{aligned}
& \left\{\frac{\eta}{1-\alpha} + \frac{\eta(1-\theta)(\gamma-\alpha)}{(1-\alpha)(1-\gamma)} + \frac{m(1-\eta)\left[\theta(\alpha-\gamma)+1-\alpha\right]}{(1-\alpha)(1-\gamma)}\right\}\frac{1}{(1-\tau)^2} + \frac{1-\eta}{\tau^2} \\
> \quad & \frac{\left[\theta(\alpha-\gamma)+\gamma-m\right]^2}{\left[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau-m\tau\right]^2}
\end{aligned}
$$

or

$$
\begin{aligned}
& \frac{\eta(1-\gamma)+m(1-\eta)(1-\alpha)+(\gamma-\alpha)\left[\eta(1-\theta)-m(1-\eta)\theta\right]}{(1-\alpha)(1-\gamma)(1-\tau)^2} + \frac{1-\eta}{\tau^2} \\
> \quad & \frac{\left[\theta(\alpha-\gamma)+\gamma-m\right]^2}{\left[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau-m\tau\right]^2}
\end{aligned}
\tag{B11}
$$

First, we begin with the LHS of the inequality (B11). Denote the numerator of the first term as x: $x \equiv \eta(1-\gamma)+m(1-\eta)(1-\alpha)+(\gamma-\alpha)\left[\eta(1-\theta)-m(1-\eta)\theta\right]$. It is easily shown that $x \in (0,1)$ for any $\alpha, \gamma, m, \tau \in (0,1)$ and

$$\begin{aligned} 0 &< (1-\gamma)\left[\eta+m(1-\eta)\right] < x < (1-\alpha)\left[\eta+m(1-\eta)\right] < 1, \text{ when } \alpha<\gamma \\ 0 &< (1-\alpha)\left[\eta+m(1-\eta)\right] < x < (1-\gamma)\left[\eta+m(1-\eta)\right] < 1, \text{ when } \alpha>\gamma \end{aligned}$$

With the above relations

$$\begin{aligned} LHS &> \frac{(1-\gamma)\left[\eta+m(1-\eta)\right]}{(1-\alpha)(1-\gamma)(1-\tau)^2}+\frac{1-\eta}{\tau^2} \\ &\geqslant \frac{2\sqrt{\eta+m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\alpha}\tau(1-\tau)}, \text{ when } \alpha<\gamma \\ LHS &> \frac{(1-\alpha)\left[\eta+m(1-\eta)\right]}{(1-\alpha)(1-\gamma)(1-\tau)^2}+\frac{1-\eta}{\tau^2} \\ &\geqslant \frac{2\sqrt{\eta+m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\gamma}\tau(1-\tau)}, \text{ when } \alpha>\gamma \end{aligned}$$

The second part in the above two expressions is based on the inequality relations between the arithmetic and geometric means.

Second, we focus on the RHS of inequality (B11). Due to the occurrence of m in the production of the public goods, we need to discuss all possible cases for relative factor intensities in the tradable and non-tradable sectors, as well as the relative factor intensity in the production of the public goods. Denote $RHS = \frac{TOP^2}{BOT^2}$, where $TOP \equiv \theta(\alpha-\gamma)+\gamma-m$ and $BOT \equiv \theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau-m\tau$.

Case 1.1: when $m<\alpha<\gamma$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned} 0 &< \alpha-m < TOP < \gamma-m < 1 \\ 0 &< 1-\gamma+\gamma\tau-m\tau < BOT < 1-\alpha+\alpha\tau-m\tau < 1 \\ RHS &< \frac{(\gamma-m)^2}{(1-\gamma+\gamma\tau-m\tau)^2} \end{aligned}$$

Case 1.2.1: when $\alpha < m < \gamma$ and $m > \frac{\alpha+\gamma}{2}$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned}
-1 &< \alpha - m < TOP < \gamma - m < 1 \\
0 &< \gamma - m < m - \alpha \\
0 &< 1 - \gamma + \gamma\tau - m\tau < BOT < 1 - \alpha + \alpha\tau - m\tau < 1 \\
RHS &< \frac{(\alpha - m)^2}{(1 - \gamma + \gamma\tau - m\tau)^2}
\end{aligned}$$

Case 1.2.2: when $\alpha < m < \gamma$ and $m < \frac{\alpha+\gamma}{2}$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned}
-1 &< \alpha - m < TOP < \gamma - m < 1 \\
0 &< m - \alpha < \gamma - m \\
0 &< 1 - \gamma + \gamma\tau - m\tau < BOT < 1 - \alpha + \alpha\tau - m\tau < 1 \\
RHS &< \frac{(\gamma - m)^2}{(1 - \gamma + \gamma\tau - m\tau)^2}
\end{aligned}$$

Case 1.3: when $\alpha < \gamma < m$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned}
-1 &< \alpha - m < TOP < \gamma - m < 0 \\
0 &< 1 - \gamma + \gamma\tau - m\tau < BOT < 1 - \alpha + \alpha\tau - m\tau < 1 \\
RHS &< \frac{(\alpha - m)^2}{(1 - \gamma + \gamma\tau - m\tau)^2}
\end{aligned}$$

Case 2.1: when $m < \gamma < \alpha$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned}
0 &< \gamma - m < TOP < \alpha - m < 1 \\
0 &< 1 - \alpha + \alpha\tau - m\tau < BOT < 1 - \gamma + \gamma\tau - m\tau < 1 \\
RHS &< \frac{(\alpha - m)^2}{(1 - \alpha + \alpha\tau - m\tau)^2}
\end{aligned}$$

Case 2.2.1: when $\gamma < m < \alpha$ and $m > \frac{\alpha+\gamma}{2}$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned}
-1 &< \gamma - m < TOP < \alpha - m < 1 \\
0 &< \alpha - m < m - \gamma \\
0 &< 1 - \alpha + \alpha\tau - m\tau < BOT < 1 - \gamma + \gamma\tau - m\tau < 1 \\
RHS &< \frac{(\gamma - m)^2}{(1 - \alpha + \alpha\tau - m\tau)^2}
\end{aligned}$$

Case 2.2.2: when $\gamma < m < \alpha$ and $m < \frac{\alpha+\gamma}{2}$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned} -1 &< \gamma - m < TOP < \alpha - m < 1 \\ 0 &< m - \gamma < \alpha - m \\ 0 &< 1 - \alpha + \alpha\tau - m\tau < BOT < 1 - \gamma + \gamma\tau - m\tau < 1 \\ RHS &< \frac{(\alpha - m)^2}{(1 - \alpha + \alpha\tau - m\tau)^2} \end{aligned}$$

Case 2.3: when $\gamma < \alpha < m$. Since $\theta \in (0,1)$, one can show that

$$\begin{aligned} -1 &< \gamma - m < TOP < \alpha - m < 0 \\ 0 &< 1 - \alpha + \alpha\tau - m\tau < BOT < 1 - \gamma + \gamma\tau - m\tau < 1 \\ RHS &< \frac{(\gamma - m)^2}{(1 - \alpha + \alpha\tau - m\tau)^2} \end{aligned}$$

A sufficient condition for (B11) to hold is obtained when $LHS > RHS$ for any $\alpha, \gamma, m, \tau, \theta \in (0,1)$. Given the above discussions on the LHS and RHS, let us move on to the comparison under different cases.

(1) when $\alpha < \gamma$ and $m < \frac{\alpha+\gamma}{2}$ (case 1.1 and case 1.2.2)

$$\begin{aligned} LHS &> \frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\alpha}\tau(1-\tau)} \\ RHS &< \frac{(\gamma - m)^2}{(1 - \gamma + \gamma\tau - m\tau)^2} \end{aligned}$$

A sufficient condition for $LHS > RHS$ is given by

$$\frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\alpha}\tau(1-\tau)} > \frac{(\gamma - m)^2}{(1 - \gamma + \gamma\tau - m\tau)^2}$$

or

$$A_1\tau^2 + B_1\tau + C_1 > 0 \tag{B12}$$

where

$$\begin{aligned} A_1 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(\gamma - m)^2 + \sqrt{1-\alpha}(\gamma - m)^2 > 0 \\ B_1 &= 4\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)(\gamma - m) - \sqrt{1-\alpha}(\gamma - m)^2 \\ C_1 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)^2 > 0 \end{aligned}$$

Given $A_1 > 0$ and $C_1 > 0$, a sufficient condition that guarantees (B12) to hold for all $\tau \in (0,1)$ is either $B_1 > 0$ or the discriminant $\Delta = B_1^2 - 4A_1C_1 < 0$. The resulting conditions are derived as follows

$$4\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)(\gamma - m) > \sqrt{1-\alpha}(\gamma - m)^2$$

or

$$8\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)(1 - m) > \sqrt{1-\alpha}(\gamma - m)^2 \tag{B13}$$

It turns out that the second inequality permits a larger range of model parameters. A sufficient condition for (B11) to hold is given by (B13) when $\alpha < \gamma$ and $m < \frac{\alpha+\gamma}{2}$.

(2) when $\alpha < \gamma$ and $m > \frac{\alpha+\gamma}{2}$ (case 1.2.1 and case 1.3)

$$\begin{aligned} LHS &> \frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\alpha}\tau(1-\tau)} \\ RHS &< \frac{(\alpha - m)^2}{(1-\gamma+\gamma\tau - m\tau)^2} \end{aligned}$$

A sufficient condition for $LHS > RHS$ is given by

$$\frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\alpha}\tau(1-\tau)} > \frac{(\alpha - m)^2}{(1-\gamma+\gamma\tau - m\tau)^2}$$

or

$$A_2\tau^2 + B_2\tau + C_2 > 0 \tag{B14}$$

where

$$\begin{aligned} A_2 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(\gamma - m)^2 + \sqrt{1-\alpha}(\alpha - m)^2 > 0 \\ B_2 &= 4\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)(\gamma - m) - \sqrt{1-\alpha}(\alpha - m)^2 \\ C_2 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)^2 > 0 \end{aligned}$$

Given $A_2 > 0$ and $C_2 > 0$, a sufficient condition that guarantees (B14) to hold for all $\tau \in (0,1)$ is either $B_2 > 0$ or the discriminant $\Delta = B_2^2 - 4A_2C_2 < 0$. The resulting conditions are derived as follows

$$8\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\gamma)(1 - m) > \sqrt{1-\alpha}(\gamma - m)^2 \tag{B15}$$

The sufficient condition for (B11) to hold is given by (B15) when $\alpha < \gamma$ and $m > \frac{\alpha+\gamma}{2}$.

(3) when $\alpha > \gamma$ and $m < \frac{\alpha+\gamma}{2}$ (case 2.1 and case 2.2.2)

$$\begin{aligned} LHS &> \frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\gamma}\tau(1-\tau)} \\ RHS &< \frac{(\alpha - m)^2}{(1-\alpha+\alpha\tau-m\tau)^2} \end{aligned}$$

A sufficient condition for $LHS > RHS$ is given by

$$\frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\gamma}\tau(1-\tau)} > \frac{(\alpha - m)^2}{(1-\alpha+\alpha\tau-m\tau)^2}$$

or

$$A_3\tau^2 + B_3\tau + C_3 > 0 \tag{B16}$$

where

$$\begin{aligned} A_3 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(\alpha - m)^2 + \sqrt{1-\gamma}(\alpha - m)^2 > 0 \\ B_3 &= 4\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\alpha)(\alpha - m) - \sqrt{1-\gamma}(\alpha - m)^2 \\ C_3 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\alpha)^2 > 0 \end{aligned}$$

Given $A_3 > 0$ and $C_3 > 0$, a sufficient condition that guarantees (B16) to hold for all $\tau \in (0,1)$ is either $B_3 > 0$ or the discriminant $\Delta = B_3^2 - 4A_3C_3 < 0$. The resulting conditions are derived as follows

$$8\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\alpha)(1-m) > \sqrt{1-\gamma}(\alpha - m)^2 \tag{B17}$$

The sufficient condition for (B11) to hold is given by (B17) when $\alpha > \gamma$ and $m < \frac{\alpha+\gamma}{2}$.

(4) when $\alpha > \gamma$ and $m > \frac{\alpha+\gamma}{2}$ (case 2.2.1 and case 2.3)

$$\begin{aligned} LHS &> \frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\gamma}\tau(1-\tau)} \\ RHS &< \frac{(\gamma - m)^2}{(1-\alpha+\alpha\tau-m\tau)^2} \end{aligned}$$

A sufficient condition for $LHS > RHS$ is given by

$$\frac{2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}}{\sqrt{1-\gamma}\tau(1-\tau)} > \frac{(\gamma - m)^2}{(1-\alpha+\alpha\tau-m\tau)^2}$$

or

$$A_4\tau^2 + B_4\tau + C_4 > 0 \tag{B18}$$

where

$$\begin{aligned} A_4 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(\alpha - m)^2 + \sqrt{1-\gamma}(\gamma - m)^2 > 0 \\ B_4 &= 4\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\alpha)(\alpha - m) - \sqrt{1-\gamma}(\gamma - m)^2 \\ C_4 &= 2\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\alpha)^2 > 0 \end{aligned}$$

Given $A_4 > 0$ and $C_4 > 0$, a sufficient condition that guarantees (B18) to hold for all $\tau \in (0,1)$ is either $B_4 > 0$ or the discriminant $\Delta = B_4^2 - 4A_4C_4 < 0$. The resulting conditions are derived as follows

$$8\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-\alpha)(1-m) > \sqrt{1-\gamma}(\gamma - m)^2 \tag{B19}$$

The sufficient condition for (B11) to hold is given by (B19) when $\alpha > \gamma$ and $m > \frac{\alpha+\gamma}{2}$.

Finally, inequalities (B13), (B15), (B17), and (B19) imply that, for any $\alpha, \gamma, m, \tau, \theta \in (0,1)$, a sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ to hold can be summarized as

$$\begin{aligned} & 8\sqrt{\eta + m(1-\eta)}\sqrt{1-\eta}(1-m) \cdot \min\{1-\alpha, 1-\gamma\} \\ > \; & \max\left\{\sqrt{1-\alpha}, \sqrt{1-\gamma}\right\} \cdot \max\left\{(\gamma - m)^2, (\alpha - m)^2\right\} \end{aligned} \tag{B20}$$

Step 3: show the proposition

Given that $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right) = sign(\gamma - \alpha)$ and $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ under the sufficient condition (B20), the implicit function theorem implies that

$$\frac{\partial \tau}{\partial \theta} = -\frac{\partial F(\theta,\tau)/\partial \theta}{\partial F(\theta,\tau)/\partial \tau}$$

Hence

$$sign\left(\frac{\partial \tau}{\partial \theta}\right) = sign(\gamma - \alpha)$$

provided that (B20) is satisfied. This completes the proof of the proposition. *Q.E.D.*

3 Productive Government Expenditure with $\chi_1 = \chi_2$

To save space, we did not repeat the problems of the household, firms, and government in this appendix. They all follow in a similar way as derived in the text. With the unique steady state equilibrium solved, the function $F(\theta,\tau)$ characterizing the size-openness relation is given by

$$\begin{aligned} F(\theta,\tau) &\equiv \frac{\partial \log U}{\partial \tau} \\ &= \frac{1-\alpha+\theta(\alpha-\gamma)}{(1-\alpha)(1-\gamma)}\frac{-1}{1-\tau}+\frac{\chi(1-\alpha+\alpha\theta-\gamma\theta)}{(1-\alpha)(1-\gamma)}\frac{1}{\tau} \\ &\quad -\frac{[(1-\alpha)(1-\gamma)+\chi(1-\alpha+\alpha\theta-\gamma\theta)]\,[\alpha\theta+(1-\theta)\gamma]}{(1-\alpha)(1-\gamma)\,[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)]} \\ &= 0 \end{aligned} \tag{C1}$$

Proof of Proposition 2: The proof of the proposition has three steps.[1]

Step 1: find a sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$

Given (C1), take the partial derivative with respect to τ

$$\begin{aligned} \frac{\partial F(\theta,\tau)}{\partial \tau} &= \frac{1-\alpha+\theta(\alpha-\gamma)}{(1-\alpha)(1-\gamma)}\frac{-1}{(1-\tau)^2}+\frac{\chi(1-\alpha+\alpha\theta-\gamma\theta)}{(1-\alpha)(1-\gamma)}\frac{-1}{\tau^2} \\ &\quad +\frac{[(1-\alpha)(1-\gamma)+\chi(1-\alpha+\alpha\theta-\gamma\theta)]\,[\alpha\theta+(1-\theta)\gamma]^2}{(1-\alpha)(1-\gamma)\,[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)]^2} \end{aligned}$$

$\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ implies

$$\begin{aligned} & \frac{1-\alpha+\theta(\alpha-\gamma)}{(1-\tau)^2}+\frac{\chi(1-\alpha+\alpha\theta-\gamma\theta)}{\tau^2} \\ > \; & \frac{[(1-\alpha)(1-\gamma)+\chi(1-\alpha+\alpha\theta-\gamma\theta)]\,[\alpha\theta+(1-\theta)\gamma]^2}{[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)]^2} \end{aligned} \tag{C2}$$

Case 1: when $\alpha < \gamma$

Since $\theta \in (0,1)$,

$$\begin{aligned} LHS &\equiv \frac{1-\alpha+\theta(\alpha-\gamma)}{(1-\tau)^2}+\frac{\chi(1-\alpha+\alpha\theta-\gamma\theta)}{\tau^2} \\ &> \frac{1-\gamma}{(1-\tau)^2}+\frac{\chi(1-\gamma)}{\tau^2} \geqslant \frac{2\sqrt{\chi}(1-\gamma)}{\tau(1-\tau)} \\ RHS &\equiv \frac{[(1-\alpha)(1-\gamma)+\chi(1-\alpha+\alpha\theta-\gamma\theta)]\,[\alpha\theta+(1-\theta)\gamma]^2}{[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)]^2} \\ &< \frac{[(1-\alpha)(1-\gamma)+\chi(1-\alpha)]\,\gamma^2}{[1-(1-\tau)\gamma]^2} \end{aligned}$$

[1] More detailed derivations are skipped in many places but are available upon request.

A sufficient condition for (C2) or $LHS > RHS$ is given by

$$\frac{2\sqrt{\chi}(1-\gamma)}{\tau(1-\tau)} > \frac{\left[(1-\alpha)(1-\gamma)+\chi(1-\alpha)\right]\gamma^2}{\left[1-(1-\tau)\gamma\right]^2}$$

or

$$A_1\tau^2 + B_1\tau + C_1 > 0 \tag{C3}$$

where

$$\begin{aligned}
A_1 &= \gamma^2\left[2\sqrt{\chi}(1-\gamma)+(1-\alpha)(1-\gamma+\chi)\right] > 0 \\
B_1 &= 4\sqrt{\chi}\gamma(1-\gamma)^2 - (1-\alpha)(1-\gamma+\chi)\gamma^2 \\
C_1 &= 2\sqrt{\chi}(1-\gamma)^3 > 0
\end{aligned}$$

For any $\tau \in (0,1)$, a sufficient condition for (C3) to holds is $B_1 > 0$ or the discriminant $\Delta = B_1^2 - 4A_1C_1 < 0$, which can be summarized by the following expression

$$\chi - \frac{8(1-\gamma)^2}{(1-\alpha)\gamma^2}\sqrt{\chi} + 1 - \gamma < 0 \tag{C4}$$

Case 2: when $\alpha > \gamma$

Since $\theta \in (0,1)$,

$$\begin{aligned}
LHS &\equiv \frac{1-\alpha+\theta(\alpha-\gamma)}{(1-\tau)^2} + \frac{\chi(1-\alpha+\alpha\theta-\gamma\theta)}{\tau^2} \\
&> \frac{1-\alpha}{(1-\tau)^2} + \frac{\chi(1-\alpha)}{\tau^2} \geqslant \frac{2\sqrt{\chi}(1-\alpha)}{\tau(1-\tau)} \\
RHS &\equiv \frac{\left[(1-\alpha)(1-\gamma)+\chi(1-\alpha+\alpha\theta-\gamma\theta)\right]\left[\alpha\theta+(1-\theta)\gamma\right]^2}{\left[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)\right]^2} \\
&< \frac{\left[(1-\alpha)(1-\gamma)+\chi(1-\gamma)\right]\alpha^2}{\left[1-(1-\tau)\alpha\right]^2}
\end{aligned}$$

A sufficient condition for (C2) or $LHS > RHS$ is given by

$$\frac{2\sqrt{\chi}(1-\alpha)}{\tau(1-\tau)} > \frac{\left[(1-\alpha)(1-\gamma)+\chi(1-\gamma)\right]\alpha^2}{\left[1-(1-\tau)\alpha\right]^2}$$

or

$$A_2\tau^2 + B_2\tau + C_2 > 0 \tag{C5}$$

where

$$\begin{aligned}
A_2 &= \alpha^2 \left[2\sqrt{\chi}(1-\alpha) + (1-\gamma)(1-\alpha+\chi)\right] > 0 \\
B_2 &= 4\sqrt{\chi}\alpha(1-\alpha)^2 - (1-\gamma)(1-\alpha+\chi)\alpha^2 \\
C_2 &= 2\sqrt{\chi}(1-\alpha)^3 > 0
\end{aligned}$$

For any $\tau \in (0,1)$, a sufficient condition for (C5) to holds is $B_2 > 0$ or the discriminant $\Delta = B_2^2 - 4A_2C_2 < 0$, which can be summarized by the following expression

$$\chi - \frac{8(1-\alpha)^2}{(1-\gamma)\alpha^2}\sqrt{\chi} + 1 - \alpha < 0 \tag{C6}$$

To summarize both cases in (C4) and (C6), for any $\alpha, \gamma, \chi, \tau, \theta \in (0,1)$, a sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ is

$$\chi - M\sqrt{\chi} + N < 0 \tag{C7}$$

where

$$\begin{aligned}
M &= \min\left\{\frac{8(1-\alpha)^2}{(1-\gamma)\alpha^2}, \frac{8(1-\gamma)^2}{(1-\alpha)\gamma^2}\right\} \\
N &= \min\{1-\alpha, 1-\gamma\}
\end{aligned}$$

Step 2: find a sufficient condition for $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right) = sign(\gamma - \alpha)$

Given (C1), take the partial derivative with respect to θ and simplify the expression

$$\begin{aligned}
\frac{\partial F(\theta,\tau)}{\partial \theta} = & \frac{(\gamma-\alpha)}{(1-\alpha)(1-\gamma)}\frac{1}{1-\tau} + \frac{\chi(\alpha-\gamma)}{(1-\alpha)(1-\gamma)}\frac{1}{\tau} \\
& + \frac{(\gamma-\alpha)(1-\alpha)(1-\gamma)}{(1-\alpha)^2(1-\gamma)^2\left[1-(1-\tau)(\gamma+\theta\alpha-\theta\gamma)\right]^2} \\
& \times \left\{\begin{array}{c} \chi(\gamma+\theta\alpha-\theta\gamma)\left[1-(1-\tau)(\gamma+\theta\alpha-\theta\gamma)\right] \\ +\chi(1-\alpha+\theta\alpha-\theta\gamma) + (1-\alpha)(1-\gamma) \end{array}\right\}
\end{aligned}$$

With $\alpha, \gamma, \chi, \tau, \theta \in (0,1)$, it can be shown that the sign of the first and third term above is determined by $sign(\gamma - \alpha)$ and the sign of the second term is determined by $sign(\alpha - \gamma)$.

As long as τ is not sufficiently close to zero, $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right) = sign(\gamma - \alpha)$.[2] Rewrite $\frac{\partial F(\theta,\tau)}{\partial \theta}$

$$\frac{\partial F(\theta,\tau)}{\partial \theta} = \frac{\gamma - \alpha}{(1-\alpha)(1-\gamma)}\left(\frac{1}{1-\tau} - \frac{\chi}{\tau} + term3\right)$$

where

$$term3 = \frac{\chi(\gamma + \theta\alpha - \theta\gamma)}{1-(1-\tau)(\gamma+\theta\alpha-\theta\gamma)} + \frac{(1-\alpha)(1-\gamma) + \chi\left[1-\alpha+\theta(\alpha-\gamma)\right]}{\left[1-(1-\tau)(\gamma+\theta\alpha-\theta\gamma)\right]^2} \tag{C8}$$

We will find a minimum $\underline{\tau}$ such that for $\tau \in (\underline{\tau}, 1)$, $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right) = sign(\gamma - \alpha)$; that is

$$\frac{1}{1-\tau} - \frac{\chi}{\tau} + term3 > 0 \tag{C9}$$

Case 1: when $\alpha < \gamma$

Since $\theta \in (0,1)$

$$\begin{aligned} term3 &> \frac{\chi\alpha}{1-(1-\tau)\alpha} + \frac{(1-\alpha+\chi)(1-\gamma)}{\left[1-(1-\tau)\alpha\right]^2} \\ &> \frac{\chi\alpha}{1-(1-\tau)\alpha} + \frac{(1-\alpha+\chi)(1-\gamma)}{1-(1-\tau)\alpha} \end{aligned}$$

A sufficient condition for (C9) is

$$\frac{1}{1-\tau} - \frac{\chi}{\tau} + \frac{\chi\alpha}{1-(1-\tau)\alpha} + \frac{(1-\alpha+\chi)(1-\gamma)}{1-(1-\tau)\alpha} > 0$$

or

$$A_3\tau^2 + B_3\tau + C_3 > 0 \tag{C10}$$

where

$$\begin{aligned} A_3 &= \alpha - (1-\gamma)(1-\alpha+\chi) \\ B_3 &= \chi(2-\alpha-\gamma) + (1-\alpha)(2-\gamma) > 0 \\ C_3 &= -\chi(1-\alpha) < 0 \end{aligned}$$

[2] Since we are only deriving a sufficient condition for the relation between trade openness and government size to be affected by the relative factor intensities, the derived results are not necessary conditions. Our numerical analysis shows that this proposition may still hold under some parameters when these sufficient conditions are violated.

According to the shape of the parabola, the minimum $\underline{\tau}$ is derived as

$$\begin{aligned}
\underline{\tau} &= \frac{-B_3 + \sqrt{B_3^2 - 4A_3C_3}}{2A_3}, \text{ the larger root when } A_3 > 0 \\
\underline{\tau} &= \frac{\chi(1-\alpha)}{(1-\alpha)(2-\gamma) + \chi(2-\alpha-\gamma)}, \text{ when } A_3 = 0 \\
\underline{\tau} &= \frac{-B_3 + \sqrt{B_3^2 - 4A_3C_3}}{2A_3}, \text{ the smaller root when } A_3 < 0
\end{aligned} \tag{C11}$$

Case 2: when $\alpha > \gamma$

Since $\theta \in (0, 1)$

$$\begin{aligned}
term3 &> \frac{\chi\gamma}{1-(1-\tau)\gamma} + \frac{(1-\gamma+\chi)(1-\alpha)}{[1-(1-\tau)\gamma]^2} \\
&> \frac{\chi\gamma}{1-(1-\tau)\gamma} + \frac{(1-\gamma+\chi)(1-\alpha)}{1-(1-\tau)\gamma}
\end{aligned}$$

A sufficient condition for (C9) is

$$\frac{1}{1-\tau} - \frac{\chi}{\tau} + \frac{\chi\gamma}{1-(1-\tau)\gamma} + \frac{(1-\gamma+\chi)(1-\alpha)}{1-(1-\tau)\gamma} > 0$$

or

$$A_4\tau^2 + B_4\tau + C_4 > 0 \tag{C12}$$

where

$$\begin{aligned}
A_4 &= \gamma - (1-\alpha)(1-\gamma+\chi) \\
B_4 &= \chi(2-\alpha-\gamma) + (2-\alpha)(1-\gamma) > 0 \\
C_4 &= -\chi(1-\gamma) < 0
\end{aligned}$$

According to the shape of the parabola, the minimum $\underline{\tau}$ is derived as

$$\begin{aligned}
\underline{\tau} &= \frac{-B_4 + \sqrt{B_4^2 - 4A_4C_4}}{2A_3}, \text{ the larger root when } A_4 > 0 \\
\underline{\tau} &= \frac{\chi(1-\gamma)}{(2-\alpha)(1-\gamma) + \chi(2-\alpha-\gamma)}, \text{ when } A_4 = 0 \\
\underline{\tau} &= \frac{-B_4 + \sqrt{B_4^2 - 4A_4C_4}}{2A_4}, \text{ the smaller root when } A_4 < 0
\end{aligned} \tag{C13}$$

To summarize both cases, let $A = \min\{A_3, A_4\}$, $B = \max\{B_3, B_4\}$, and $C = \min\{C_3, C_4\}$

$$\underline{\tau} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \text{ if } A \neq 0 \text{ and otherwise} \tag{C14}$$

$$\underline{\tau} = \max\left\{\frac{\chi(1-\gamma)}{(2-\alpha)(1-\gamma) + \chi(2-\alpha-\gamma)}, \frac{\chi(1-\alpha)}{(1-\alpha)(2-\gamma) + \chi(2-\alpha-\gamma)}\right\}$$

With $\alpha, \gamma, \chi, \theta \in (0,1)$ and $\tau \in (\underline{\tau}, 1)$, $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right) = sign(\gamma - \alpha)$.

Step 3: show the proposition

Given that $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ under (C7) and $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right) = sign(\gamma - \alpha)$ under (C14), the implicit function theorem implies that

$$\frac{\partial \tau}{\partial \theta} = -\frac{\partial F(\theta,\tau)/\partial \theta}{\partial F(\theta,\tau)/\partial \tau}$$

Hence

$$sign\left(\frac{\partial \tau}{\partial \theta}\right) = sign(\gamma - \alpha)$$

provided that (C7) and (C14) are satisfied. This completes the proof of proposition 2. *Q.E.D.*

4 Productive Government Expenditure with $\alpha = \gamma$

When $\alpha = \gamma$, production of tradable and non-tradable sectors are differentiated by the output elasticities of public expenditure: $\chi_1 \neq \chi_2$ in (25) and (26). The function $F(\theta,\tau)$ characterizing the size-openness relation is given by

$$\begin{aligned} F(\theta,\tau) &\equiv \frac{1}{1-\alpha}\frac{-1}{1-\tau} + \frac{\chi_1\theta + \chi_2(1-\theta)}{1-\alpha}\frac{1}{\tau} \\ &\quad - \frac{\chi_1\theta + \chi_2(1-\theta) + 1 - \alpha}{1-\alpha}\frac{\alpha}{1-\alpha+\alpha\tau} \\ &= 0 \end{aligned} \tag{D1}$$

Proof of Proposition 3: The proof of the proposition has two steps.

Step 1: find a sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$

Given (D1), take the partial derivative with respect to τ

$$\frac{\partial F(\theta,\tau)}{\partial \tau} = \frac{-1}{(1-\alpha)(1-\tau)^2} - \frac{\chi_1\theta + \chi_2(1-\theta)}{(1-\alpha)\tau^2} + \frac{\chi_1\theta + \chi_2(1-\theta) + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2$$

$\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ implies that

$$\frac{1}{(1-\alpha)(1-\tau)^2} + \frac{\chi_1\theta + \chi_2(1-\theta)}{(1-\alpha)\tau^2} > \frac{\chi_1\theta + \chi_2(1-\theta) + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2 \tag{D2}$$

Case 1: when $\chi_1 > \chi_2$

Since $\theta \in (0.1)$

$$\begin{aligned} LHS &\equiv \frac{1}{(1-\alpha)(1-\tau)^2} + \frac{\chi_1\theta + \chi_2(1-\theta)}{(1-\alpha)\tau^2} \\ &> \frac{1}{(1-\alpha)(1-\tau)^2} + \frac{\chi_2}{(1-\alpha)\tau^2} \geqslant \frac{2\sqrt{\chi_2}}{(1-\alpha)\tau(1-\tau)} \\ RHS &\equiv \frac{\chi_1\theta + \chi_2(1-\theta) + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2 < \frac{\chi_1 + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2 \end{aligned}$$

A sufficient condition for (D2) to hold is

$$\frac{2\sqrt{\chi_2}}{(1-\alpha)\tau(1-\tau)} > \frac{\chi_1 + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2$$

or

$$A_5\tau^2 + B_5\tau + C_5 > 0 \tag{D3}$$

where

$$\begin{aligned} A_5 &= 2\alpha^2\sqrt{\chi_2} + (\chi_1 + 1 - \alpha)\alpha^2 > 0 \\ B_5 &= 4\alpha(1-\alpha)\sqrt{\chi_2} - (\chi_1 + 1 - \alpha)\alpha^2 \\ C_5 &= 2(1-\alpha)^2\sqrt{\chi_2} > 0 \end{aligned}$$

According to the shape of the parabola, a sufficient condition for (D3) to hold for all $\tau \in (0,1)$ is either $B_5 > 0$ or $\Delta = B_5^2 - 4A_5C_5 < 0$, which can be summarized by the following expression

$$8(1-\alpha)\sqrt{\chi_2} > (\chi_1 + 1 - \alpha)\alpha^2 \tag{D4}$$

Case 2: when $\chi_1 < \chi_2$

Since $\theta \in (0.1)$

$$\begin{aligned} LHS &\equiv \frac{1}{(1-\alpha)(1-\tau)^2} + \frac{\chi_1\theta + \chi_2(1-\theta)}{(1-\alpha)\tau^2} \\ &> \frac{1}{(1-\alpha)(1-\tau)^2} + \frac{\chi_1}{(1-\alpha)\tau^2} \geqslant \frac{2\sqrt{\chi_1}}{(1-\alpha)\tau(1-\tau)} \\ RHS &\equiv \frac{\chi_1\theta + \chi_2(1-\theta) + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2 < \frac{\chi_2 + 1 - \alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2 \end{aligned}$$

A sufficient condition for (D2) to hold is

$$\frac{2\sqrt{\chi_1}}{(1-\alpha)\tau(1-\tau)} > \frac{\chi_2+1-\alpha}{(1-\alpha)(1-\alpha+\alpha\tau)^2}\alpha^2$$

or

$$A_6\tau^2 + B_6\tau + C_6 > 0 \tag{D5}$$

where

$$\begin{aligned} A_6 &= 2\alpha^2\sqrt{\chi_1} + (\chi_2+1-\alpha)\alpha^2 > 0 \\ B_6 &= 4\alpha(1-\alpha)\sqrt{\chi_1} - (\chi_2+1-\alpha)\alpha^2 \\ C_6 &= 2(1-\alpha)^2\sqrt{\chi_1} > 0 \end{aligned}$$

According to the shape of the parabola, a sufficient condition for (D5) to hold for all $\tau \in (0,1)$ is either $B_6 > 0$ or $\Delta = B_6^2 - 4A_6C_6 < 0$, which can be summarized by the following expression

$$8(1-\alpha)\sqrt{\chi_1} > (\chi_2+1-\alpha)\alpha^2 \tag{D6}$$

To summarize both cases in (D4) and (D6), a sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial\tau} < 0$ to be true for any $\tau, \theta \in (0,1)$ is

$$8(1-\alpha)\min\left\{\sqrt{\chi_1}, \sqrt{\chi_2}\right\} > \alpha^2 \max\left\{\chi_1+1-\alpha, \chi_2+1-\alpha\right\} \tag{D7}$$

Step 2: show the proposition

Given (D1), take the partial derivative with respect to θ

$$\frac{\partial F(\theta,\tau)}{\partial\theta} = \frac{(\chi_1-\chi_2)(1-\alpha)}{(1-\alpha)\tau(1-\alpha+\alpha\tau)}$$

It is clear that $sign\left(\frac{\partial F(\theta,\tau)}{\partial\theta}\right) = sign(\chi_1-\chi_2)$. According the implicit function theorem

$$\frac{\partial\tau}{\partial\theta} = -\frac{\partial F(\theta,\tau)/\partial\theta}{\partial F(\theta,\tau)/\partial\tau}$$

Hence

$$sign\left(\frac{\partial\tau}{\partial\theta}\right) = sign(\chi_1-\chi_2)$$

provided that (D7) holds. This completes the proof of this proposition. *Q.E.D.*

5 Productive Government Expenditure with $\alpha \neq \gamma$ and $\chi_1 \neq \chi_2$

In a more general case when $\alpha \neq \gamma$ and $\chi_1 \neq \chi_2$, function $F(\theta, \tau)$ that describes the relation between size and openness is

$$\begin{aligned} F(\theta,\tau) &\equiv \frac{\partial \log U}{\partial \tau} \\ &= \frac{1-\alpha+\theta(\alpha-\gamma)}{(1-\alpha)(1-\gamma)}\frac{-1}{1-\tau} + \frac{\chi_1\theta(1-\gamma)+\chi_2(1-\theta)(1-\alpha)}{(1-\alpha)(1-\gamma)}\frac{1}{\tau} \\ &\quad - \frac{[\chi_1\theta(1-\gamma)+\chi_2(1-\theta)(1-\alpha)+(1-\alpha)(1-\gamma)]\,[\alpha\theta+(1-\theta)\gamma]}{(1-\alpha)(1-\gamma)\,[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)]} \\ &= 0 \end{aligned} \tag{E1}$$

Proof of Proposition 4: The proof has three steps.

Step 1: find a sufficient condition for $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$

Given (E1), take the partial derivative with respect to τ and simplify the expression

$$\begin{aligned} \frac{\partial F(\theta,\tau)}{\partial \tau} &= \frac{\theta(\alpha-\gamma)+1-\alpha}{(1-\alpha)(1-\gamma)}\frac{-1}{(1-\tau)^2} - \frac{\theta[\chi_1(1-\gamma)-\chi_2(1-\alpha)]+\chi_2(1-\alpha)}{(1-\alpha)(1-\gamma)\tau^2} \\ &\quad + \frac{\{\theta[\chi_1(1-\gamma)-\chi_2(1-\alpha)]+(1-\alpha)(1-\gamma+\chi_2)\}\,[\theta(\alpha-\gamma)+\gamma]^2}{(1-\alpha)(1-\gamma)\,[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau]^2} \end{aligned}$$

$\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ implies that

$$\begin{aligned} &\frac{\theta(\alpha-\gamma)+1-\alpha}{(1-\alpha)(1-\gamma)(1-\tau)^2} + \frac{\theta[\chi_1(1-\gamma)-\chi_2(1-\alpha)]+\chi_2(1-\alpha)}{(1-\alpha)(1-\gamma)\tau^2} \\ > &\frac{\{\theta[\chi_1(1-\gamma)-\chi_2(1-\alpha)]+(1-\alpha)(1-\gamma+\chi_2)\}\,[\theta(\alpha-\gamma)+\gamma]^2}{(1-\alpha)(1-\gamma)\,[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau]^2} \end{aligned} \tag{E2}$$

Case 1: when $\alpha < \gamma$ and $\chi_1(1-\gamma) > \chi_2(1-\alpha)$

Since $\theta \in (0,1)$

$$\begin{aligned} LHS &\equiv \frac{\theta(\alpha-\gamma)+1-\alpha}{(1-\alpha)(1-\gamma)(1-\tau)^2} + \frac{\theta[\chi_1(1-\gamma)-\chi_2(1-\alpha)]+\chi_2(1-\alpha)}{(1-\alpha)(1-\gamma)\tau^2} \\ &> \frac{1}{(1-\alpha)(1-\tau)^2} + \frac{\chi_2}{(1-\gamma)\tau^2} \geqslant \frac{2\sqrt{\chi_2}}{\sqrt{(1-\alpha)(1-\gamma)}\tau(1-\tau)} \\ RHS &\equiv \frac{\{\theta[\chi_1(1-\gamma)-\chi_2(1-\alpha)]+(1-\alpha)(1-\gamma+\chi_2)\}\,[\theta(\alpha-\gamma)+\gamma]^2}{(1-\alpha)(1-\gamma)\,[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau]^2} \\ &< \frac{1-\alpha+\chi_1}{1-\alpha}\frac{\gamma^2}{(1-\gamma+\gamma\tau)^2} \end{aligned}$$

A sufficient condition for (E2) to hold is

$$\frac{2\sqrt{\chi_2}}{\sqrt{(1-\alpha)(1-\gamma)}\tau(1-\tau)} > \frac{1-\alpha+\chi_1}{1-\alpha}\frac{\gamma^2}{(1-\gamma+\gamma\tau)^2}$$

or

$$A_7\tau^2 + B_7\tau + C_7 > 0 \tag{E3}$$

where

$$\begin{aligned} A_7 &= 2\sqrt{\chi_2}(1-\alpha)\gamma^2 + \sqrt{(1-\alpha)(1-\gamma)}(1-\alpha+\chi_1)\gamma^2 > 0 \\ B_7 &= 4\sqrt{\chi_2}(1-\alpha)(1-\gamma)\gamma - \sqrt{(1-\alpha)(1-\gamma)}(1-\alpha+\chi_1)\gamma^2 \\ C_7 &= 2\sqrt{\chi_2}(1-\alpha)(1-\gamma)^2 > 0 \end{aligned} \tag{E4}$$

A sufficient condition for (E3) to hold for any $\tau \in (0,1)$ is $B_7 > 0$ or $\Delta = B_7^2 - 4A_7C_7 < 0$, which can be summarized by the following expression

$$(1-\alpha+\chi_1)\gamma^2 < 8\sqrt{\chi_2(1-\alpha)(1-\gamma)} \tag{E5}$$

Case 2: when $\alpha > \gamma$ **and** $\chi_1(1-\gamma) < \chi_2(1-\alpha)$

Since $\theta \in (0,1)$

$$\begin{aligned} LHS &\equiv \frac{\theta(\alpha-\gamma)+1-\alpha}{(1-\alpha)(1-\gamma)(1-\tau)^2} + \frac{\theta\left[\chi_1(1-\gamma)-\chi_2(1-\alpha)\right]+\chi_2(1-\alpha)}{(1-\alpha)(1-\gamma)\tau^2} \\ &> \frac{1}{(1-\gamma)(1-\tau)^2} + \frac{\chi_1}{(1-\alpha)\tau^2} \geqslant \frac{2\sqrt{\chi_1}}{\sqrt{(1-\alpha)(1-\gamma)}\tau(1-\tau)} \\ RHS &\equiv \frac{\{\theta\left[\chi_1(1-\gamma)-\chi_2(1-\alpha)\right]+(1-\alpha)(1-\gamma+\chi_2)\}\left[\theta(\alpha-\gamma)+\gamma\right]^2}{(1-\alpha)(1-\gamma)\left[\theta(\gamma-\alpha)(1-\tau)+1-\gamma+\gamma\tau\right]^2} \\ &< \frac{1-\gamma+\chi_2}{1-\gamma}\frac{\alpha^2}{(1-\alpha+\alpha\tau)^2} \end{aligned}$$

A sufficient condition for (E2) to hold is

$$\frac{2\sqrt{\chi_1}}{\sqrt{(1-\alpha)(1-\gamma)}\tau(1-\tau)} > \frac{1-\gamma+\chi_2}{1-\gamma}\frac{\alpha^2}{(1-\alpha+\alpha\tau)^2}$$

or

$$A_8\tau^2 + B_8\tau + C_8 > 0 \tag{E6}$$

where

$$\begin{aligned}
A_8 &= 2\sqrt{\chi_1}(1-\gamma)\alpha^2 + \sqrt{(1-\alpha)(1-\gamma)}(1-\gamma+\chi_2)\alpha^2 > 0 \\
B_8 &= 4\sqrt{\chi_1}(1-\alpha)(1-\gamma)\alpha - \sqrt{(1-\alpha)(1-\gamma)}(1-\gamma+\chi_2)\alpha^2 \\
C_8 &= 2\sqrt{\chi_1}(1-\gamma)(1-\alpha)^2 > 0
\end{aligned} \tag{E7}$$

A sufficient condition for (E6) to hold for any $\tau \in (0,1)$ is $B_8 > 0$ or $\Delta = B_8^2 - 4A_8C_8 < 0$, which can be summarized by the following expression

$$(1-\gamma+\chi_2)\alpha^2 < 8\sqrt{\chi_1(1-\alpha)(1-\gamma)} \tag{E8}$$

To summarize, $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ holds when either of the following two groups of conditions is satisfied:

Group one: (1) $\alpha < \gamma$, (2) $\chi_1(1-\gamma) > \chi_2(1-\alpha)$, (3) $(1-\alpha+\chi_1)\gamma^2 < 8\sqrt{\chi_2(1-\alpha)(1-\gamma)}$;

Group two: (1) $\alpha > \gamma$, (2) $\chi_1(1-\gamma) < \chi_2(1-\alpha)$, (3) $(1-\gamma+\chi_2)\alpha^2 < 8\sqrt{\chi_1(1-\alpha)(1-\gamma)}$.

Step 2: find a sufficient condition for $sign\left(\frac{\partial F(\theta,\tau)}{\partial \theta}\right)$ to be determined by the relative factor intensities

Given (E1), take the partial derivative with respect to θ and simplify the expression

$$\begin{aligned}
\frac{\partial F(\theta,\tau)}{\partial \theta} = & \frac{\gamma-\alpha}{(1-\alpha)(1-\gamma)(1-\tau)} \\
& + \frac{(\gamma-\alpha)\left\{\theta\left[\chi_1(1-\gamma)-\chi_2(1-\alpha)\right]+(1-\alpha)(1-\gamma+\chi_2)\right\}}{(1-\alpha)(1-\gamma)\left[\theta(1-\alpha+\alpha\tau)+(1-\theta)(1-\gamma+\gamma\tau)\right]^2} \\
& + \frac{\left[\chi_1(1-\gamma)-\chi_2(1-\alpha)\right]\left[\theta(1-\alpha)+(1-\theta)(1-\gamma)\right]}{(1-\alpha)(1-\gamma)\left[\tau\theta(1-\alpha+\alpha\tau)+\tau(1-\theta)(1-\gamma+\gamma\tau)\right]}
\end{aligned}$$

Given $\alpha, \gamma, \chi_1, \chi_2 \in (0,1)$, the sign of $\frac{\partial F(\theta,\tau)}{\partial \theta}$ is determined by $\gamma-\alpha$ and $\chi_1(1-\gamma)-\chi_2(1-\alpha)$. When $\alpha < \gamma$ and $\chi_1(1-\gamma) > \chi_2(1-\alpha)$, $\frac{\partial F(\theta,\tau)}{\partial \theta} > 0$; when $\alpha > \gamma$ and $\chi_1(1-\gamma) < \chi_2(1-\alpha)$, $\frac{\partial F(\theta,\tau)}{\partial \theta} < 0$.

Step 3: show the proposition

According the implicit function theorem

$$\frac{\partial \tau}{\partial \theta} = -\frac{\partial F(\theta,\tau)/\partial \theta}{\partial F(\theta,\tau)/\partial \tau}$$

We have shown that:

when (1) $\alpha < \gamma$, (2) $\chi_1(1-\gamma) > \chi_2(1-\alpha)$, and (3) $(1-\alpha+\chi_1)\gamma^2 < 8\sqrt{\chi_2(1-\alpha)(1-\gamma)}$, $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ and $\frac{\partial F(\theta,\tau)}{\partial \theta} > 0$: hence $\frac{\partial \tau}{\partial \theta} > 0$;
when (1) $\alpha > \gamma$, (2) $\chi_1(1-\gamma) < \chi_2(1-\alpha)$, and (3) $(1-\gamma+\chi_2)\alpha^2 < 8\sqrt{\chi_1(1-\alpha)(1-\gamma)}$, $\frac{\partial F(\theta,\tau)}{\partial \tau} < 0$ and $\frac{\partial F(\theta,\tau)}{\partial \theta} < 0$: hence $\frac{\partial \tau}{\partial \theta} < 0$. This completes the proof of this proposition.

Q.E.D.