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Why Is Traditional Optimal Growth Theory Mute? Restoring its Rightful Voice

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Overview

Signed in 1992, the Maastricht Treaty stipulates that the total public debt of each member country should be less than 60% of its GDP and the yearly public deficit smaller than 3% of GDP. Today these criteria apply to 28 countries of the European Union as well as to any potential member. One may infer that those numbers have been determined by some reference to the optimal savings of a nation. In fact, any such link was totally absent, and those rules that have a bearing not only on the present but also on the future welfare of half a billion individuals are not supported in any way by optimal growth theory. Why? This chapter will argue that the theory, as it has been developed, has *never* been able to come up with a reasonable answer to the problem of determining how much a nation should save.

We will show that the traditional approach, based on the systematic use of strictly concave utility functions, never delivered; and when the bold step of modifying the utility function to obtain a reasonable answer was taken, it unfailingly led to nonsensical values for other variables of central importance such as the growth rate of real income per person, the marginal product of capital, or the capital-output ratio.

Our profession should have taken note of those inadequacies long ago. They had been met already by the very originator of the theory, Frank Ramsey[1] (1928) who tried to put numbers on the theory, and whose disappointment when obtaining an "optimal" savings rate of 60% is almost palpable. Thirty years later, Richard Goodwin (1961) obtained even worse results in

[1] Frank Ramsey (1903-1930) was a british philosopher, mathematician and economist who died tragically young. At age 23 he became lecturer in mathematics and then Director of mathematical studies at King's college in Cambridge University. His essay "A mathematical theory of saving" (1928) is a masterpiece.

all models he considered – but contrary to Ramsey, he set out to defend them in a dumbfounding way. Finally, Robert King and Sergio Rebelo (1993) convincingly showed that it was an impossible task to replicate the observed development of an economy by assuming some form of the traditional model. They tried to modify in many ways not only the parameters of the models they were using, but eventually the very nature of the latter – to no avail. Their conclusion was unequivocal.

The central result of this chapter is two-fold: first we demonstrate that the concavity of the utility functions precludes any possibility of a sustained competitive equilibrium; any economy initially in such equilibrium will always veer off from that situation into unwanted trajectories if it is governed by the standard model. We then propose the following solution to the problem of optimal growth: optimal trajectories of the economy, and first and foremost the optimal savings rate, should be determined by the Euler equation resulting from competitive equilibrium. By saving and investing along lines defined by such an equilibrium, society is able to reach simultaneously the following intertemporal optima, additionally to the minimisation of production costs: maximization of the sum of discounted consumption flows, and maximization of the value of society's activity as well as the remuneration of labor. This implies that the utility function is any affine function of consumption, the latter measuring *welfare* flows. We will show that for all parameters in the range of observed or predictable values, as well as for quite different hypotheses regarding the future evolution of population or technical progress, we are always led to very reasonable time paths for all central variables of the economy.

We will proceed as follows. In Section 1 we review evidence of the non applicability of the traditional approach, starting with the Ramsey model. We will show that although the utility function used by Ramsey looked intuitively justifiable, it was very close to a function that implied a 60% savings rate not just at one point (as Ramsey had observed), but at all its points, and that to obtain a more reasonable savings rate – in the range of 10-20% – one should introduce a utility function that could hardly correspond to any individual, and even less to a whole society. We then turn to the second attempt of defining an optimal savings rate, that of Goodwin (1961) where the marginal savings rate could reach 95%. We then remind the reader of the extensive, highly convincing analysis of the problem carried out by King and Rebelo (1993).

In 2009, unaware of the King and Rebelo study (who had used three particular utility functions in their tests), we made a thorough study of *all* possible utility functions belonging to the families (A) $U = (C^{\alpha} - 1)/\alpha$ and (B) $U = -(1/\beta)e^{-\beta C}$. Our aim was to define in the standard model the initial optimal savings rate leading to equilibrium. We showed that for all possible values of parameters in the observed range, equilibrium can be reached in the (A) case with a reasonable initial savings rate only if the coefficient α is extremely low. The situation is even worse with family (B) since equilibrium does not exist any more: what looks like a stable arm in the phase diagram in fact leads to a cusp point invariably followed by disaster in the form of zero consumption reached in finite time. Prompted by the King and Rebelo study, in Section 2 of this paper we extend our analysis to the implied initial growth rate as well as to the limiting value of the marginal productivity of capital, and show that as soon as adjustments are made to the utility function to obtain a reasonable initial savings rate, historically unobserved or unwanted values appear.

We then demonstrate (Section 3) that competitive equilibrium is unsustainable in the traditional model. We suppose that initially the economy is in a situation of competitive equilibrium and that from that point onward it can follow any of two possible kinds of paths:

I) investment is planned in such a way as to maximize intertemporally discounted utility flows, the utility function being the widely used affine transform of a strictly concave power function; this is the traditional approach.

II) investment is made in such a way as to conform to the Euler equation defining competitive equilibrium. This will be our suggested solution to the basic problem of optimal economic growth.

We will show that in the first scenario, although central variables have normal, historically observed *initial* values, *in all cases* their time paths run astray, and explain analytically this behavior.

Section 4 provides our solution: we show that scenario II, while securing the intertemporal optima for society we mentioned earlier, *always* yields reasonable results for the following fundamental variables: the optimal savings rate, the implied growth rate of income per person, and the capital-output ratio; in addition, it secures the most welcome feature of an increasing share of the remuneration of labor in total income.

In Section 5 we take the natural step of checking the robustness of these results not only to changes in the values of the parameters of the model, but to very different evolutions of population and technical progress. Indeed, we hold that a model for short, medium and long run horizons should take into account in particular the quasi-certainty of a non-exponential evolution of population. We will show that despite significantly different hypotheses the time paths of the central variables just mentioned remain within very reasonable, predictable ranges, thus confering a welcome robustness to the model.

1. Three essays that should have been alarm bells: Ramsey (1928), Goodwin (1961), King and Rebelo (1993).

1.1. Ramsey: the first difficulties.

With his highly original, beautifully written essay "A mathematical theory of saving" (1928) Ramsey had considerable merit. On the one hand he set out to tackle a central problem for society: in his own words, he asked "how much of its income should a nation save?" (p. 543). On the other, his exposition was highly interesting from a methodological standpoint; his central result was obtained through three different venues: first by reasoning along purely economic lines; second by applying the calculus of variations; and third, quite surprisingly, through ordinary calculus by making a subtle change of variable in the integral he was minimising.

It is now essential to recall with precision Ramsey's objective and result. (For convenience, we shall use contemporary notation, where $K, L, F(K,L), C$ replace $c, a, f(c,a), x$ and stand for capital, labor, production and consumption respectively). Ramsey looked for the optimal trajectory of saving and investment minimising the integral

$$\int_0^\infty [B - U(C) + V(L)]\, dt$$

where B, standing for "bliss", is an upper bound of utility U reached asymptotically when $C \to \infty$. C is constrained by $F(K,L) = C + \dot{K}$ and $V(L)$ is the disutility of labor. A first order condition for this minimisation, obtained by any of the above-mentioned methods (see Appendix I for two of them), is that saving (or investment) be equal to

$$S^* = \dot{K}^* = \frac{B - [U(C) - V(L)]}{U'(C)}. \tag{1}$$

Then Ramsey set out to put numbers on his formula[2]. For that purpose, he settled with the following (numerical) utility function:

[2] The brilliance of this beautiful essay is somewhat tarnished by a conceptual mistake,

Table 1. *The utility function used by Frank Ramsey*

Family income per annum	Total utility
£150	2
£200	3
£300	4
£500	5
£1000	6
£2000	7
£5000	8 = Bliss

rightly pointed out by Alpha C. Chiang (1992). Upon reaching his formula, Ramsey wrote: "The most remarkable feature of the rule is that it is altogether independent of the production function $F(K,L)$ except in so far this determines bliss". As Chiang notes, this statement is incorrect because $U'(C)$ in the denominator of (1) depends on the production function through the Euler equation $[1/U'(C)]\,\frac{d}{dt}U'(C) + F_K(K,L) = 0$.

We may add that in the numerator $C = F(K,L) - \dot{K}$ the variable L also depends upon the production function; indeed, contrary to most of the literature on optimal economic growth, L is not an exogenous variable, but it is a state variable whose optimal path depends on the system of two Euler equations: first the equation just mentioned; secondly the Euler equation $U'(C)F_L'(K,L) = V'(L)$. (For a derivation of this system of equations as well as for their economic interpretation, see Appendix 1). In fact equation (1) is a second order differential equation in K which is crucially dependent on $F(K,L)$. To make this clear, just consider the case (taken up by Ramsey in his numerical example) where $V(L) = 0$; suppose also that the production function is simply $F(K)$. Then the Ramsey rule implies that the optimal trajectory K^* is governed by the second order, non-linear differential equation

$$\dot{K} = F'(K)\frac{U(C)-B}{U''(C)\left[F'(K)\dot{K}-\ddot{K}\right]}$$

where $C = F(K) - \dot{K}$.

The very fact that in general the Euler equation is a non-linear second order differential equation is far from innocuous. Indeed, unless utility or production are affine functions of their arguments, it is not possible to solve the equation analytically and numerical methods are required. It is our opinion that this fact is at the root of the slow development of the theory of optimal growth and that it explains the quasi inexistence for a long time of actual, computed optimal time paths of capital as well as those of the associated variables such as the savings rate, the marginal productivity of capital, the growth rate of income per person or the capital-output ratio. Indeed, for many years the literature simply focused on the qualitative analysis of the existence and properties of a long-term equilibrium. It is only in the last two decades that actual time paths appeared and comparative dynamics were carried out.

He then chose to determine the optimal saving rate at $C = 200$ (note that his utility function applied to income and consumption flows alike). Since he needed to evaluate $U'(C)$ at that point, he interpolated an arc of parabola between the first three points of his discrete function to get, over the interval $C \in [150, 300]$,

$$U(C) = -C^2/15000 + 13C/300 - 3$$

and therefore $U'(C) = -C/7500 + 13/300$. For some reason, having reached that stage he did not make any hypothesis about the disutility of labor function $V(L)$ other than considering it equal to zero. Applying equation (1), he then obtained an optimal savings flow $S^* = 300$, implying a total income $300 + 200 = 500$, and hence a savings rate equal to 60%.

We might want to know what would have been the optimal saving rate over the whole interval $C \in [150, 300]$, and not just at point $C = 200$. Plugging the arc of parabola and its derivative into (1), we get

$$S^* = (C^{*2}/15000 - 13C^*/300 + 11)/(-C^*/7500 + 13/300),$$

implying an optimal savings rate

$$s^* = S^*/(S^* + C^*) = \left[1 + (-C^{*2}/7500 + 13C^*/300) \,/\, (C^{*2}/15000 - 13C^*/300 + 11)\right]^{-1}$$

whose values are pictured in Figure 1.

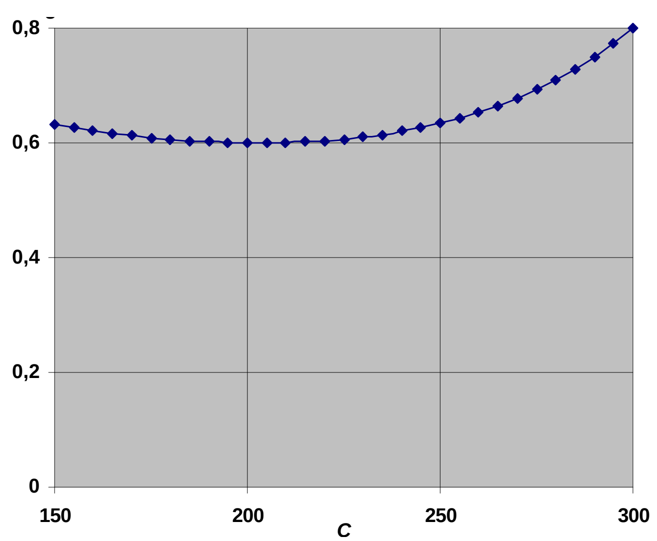

FIGURE 1. *The optimal saving rate s^* implied by the Ramsey model; over the whole interval $C \in [150, 300]$, s^* is equal to or larger than 60%.*

It turns out that the value 60% calculated by Ramsey constitutes in fact the *minimum* optimal saving rate s^* over the whole interval $[150, 300]$; s^* is 63% at $C^* = 150$ and reaches 80% when $C^* = 300$. Note that all these values of the optimal saving rate would have been even higher if Ramsey had taken into account, and given values, to his disutility of labor function $V(L)$, as can be immediately verified from equation (1).

1.1.1 Ramsey's reaction to his result.

One can almost feel Ramsey's disappointment when he wrote: "The rate of saving which the rule requires is greatly in excess of that which anyone would suggest", adding that the utility function he used was "put forward merely as an illustration" (p. 548). His next reaction was quite natural: he wondered whether this excessive optimal saving rate was due to the oversimplification of the production process he had hypothesized, characterized by a constant population and the absence of technical progress. Be it as it may, Ramsey left the matter at that.

1.1.2 A neglected, important question.

We may never know whether Ramsey tried to define another utility function, in the hope of obtaining more reasonable optimal saving rates. But it seems evident that we should ask that very question: on the basis of Ramsey's model, what would be the utility function entailing a reasonable saving rate, for instance a constant rate equal to 10%? We may choose that the utility function goes through one of the points adopted by Ramsey, and want to determine the curve he then should have drawn to obtain that reasonable rate.

The answer can be obtained as follows: first, we equate to a constant the saving rate $s^* = 1/(1 + C^*/S^*)$ and use Ramsey's rule as given by (1) for S^*; second, we integrate the implied differential equation, and identify the constant of integration by using a point in (C, U) space corresponding to Ramsey's utility function (for instance its first point).

From (1), and neglecting $V(L)$ as Ramsey did, the optimal saving rate is:

$$s^* = \left[1 + \frac{CU'(C)}{B - U(C)}\right]^{-1}. \tag{2}$$

Integrating this differential equation results into the utility function

$$U(C) = \kappa C^{1-1/s^*} + B \tag{3}$$

where κ, the constant of integration, can be identified by using any point in (C,U) space, denoted (C_1,U_1).We get

$$U(C) = (U_1 - B)\left(\frac{C}{C_1}\right)^{1-1/s^*} + B. \tag{4}$$

(We verify that with $s^* \in (0,1)$, $\lim_{C\to\infty} U(C) = B$). Setting $s^* = 0.1, B = 8$ and choosing (C_1, U_1) as the first point of the Ramsey curve (150, 2), the resulting function

$$U(C) = -6\,(C/150)^{-9} + 8 \tag{5}$$

is the only utility function going through (150, 2) and yielding, under the Ramsey rule, a constant optimal saving rate equal to 10%. The bad news is that this function, depicted on Figure 2 (red line), makes no sense at all.

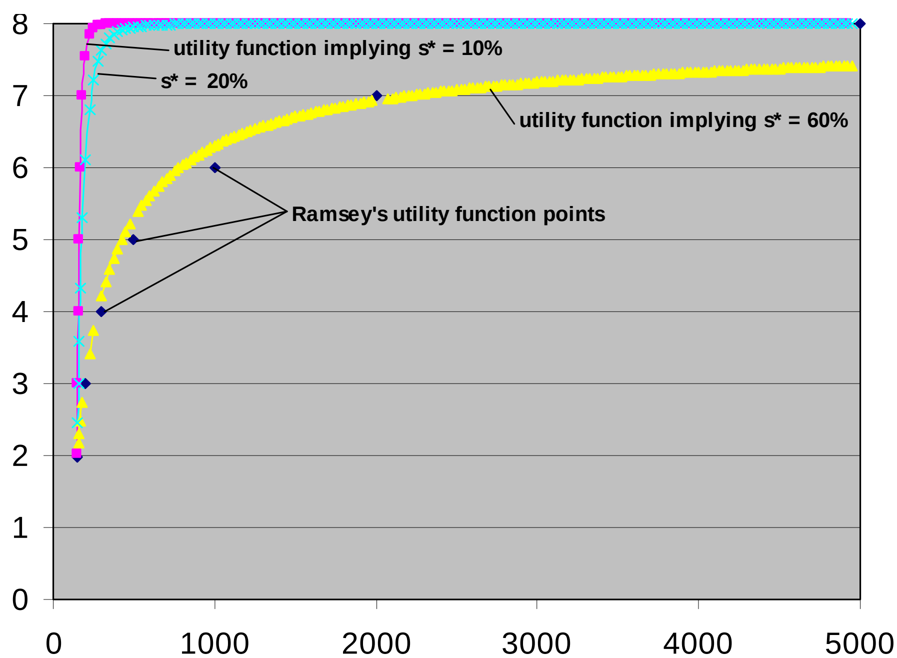

Figure 2. *The blue dots depict Ramsey's utility function; they are close to a utility curve entailing a constant savings rate equal to 60% at all its points (yellow curve). For the savings rate to be equal to 10%, the utility function should correspond to the red curve. The $s^* = 20\%$ light blue curve is hardly distinguishable from the red curve.*

To paraphrase Ramsey, its extreme properties are also "greatly in excess of that which anyone would suggest". Indeed the curve is close to a vertical, almost immediately followed by a horizontal; the bliss level is practically attained at $C = 300$ already ($U(300) = 7.99$). The marginal utility is $U'(C) = 0.36(C/150)^{-10}$; this implies that multiplying C by a factor λ divides the marginal utility by λ^{10}. An example illustrates the oddity of such a construct. Consider any country whose real income per person, over a very long time span, was multiplied by $\lambda = 10^{9/10} \approx 7.943$. Thanks to the work of Louis Johnston and Samuel H. Williamson (2013), we can estimate that such an increase took about 115 years to be achieved in the United States (on a time frame ending in 2012) and 150 years in the United Kingdom. Applying the above-mentioned utility function would mean that at the beginning of the 20th century the marginal utility of consumption in the U.S. was by $\lambda^{10} = 10^{(9/10)10} =$ *one billion* times higher than it is today, certainly an indefensible proposition.

One may think that choosing a larger optimal saving rate might improve the situation. That is not the case: a 20% saving rate entails a utility curve (blue line) hardly distinguishable from the preceding one. On the same figure we have also depicted the curve corresponding to the constant rate $s^* = 60\%$ (yellow line). It can be seen that Ramsey chose a utility function that seemed reasonable to him (and probably to most of his readers) that was very close to a function implying an optimal saving level equal to 60% at *all* its points.

We might also attribute the observed antinomy between what appears as a reasonable utility function and a reasonable optimal savings rate to the very model that Ramsey put forward (in which, for instance, the future utility flows are not discounted). This is not the case either. We will show that, time and again, for whatever model we might consider, not only such bland opposition is maintained, but it extends to irrealistic values of other variables of fundamental importance such as the marginal productivity of capital, the growth rate of income per person, or the capital-output ratio.

1.2. The second warning bell: Goodwin (1961)

1.2.1 Context and results.

The problem of the optimal savings rate came again to the forefront with the paper by Richard Goodwin "The optimal growth path for an underdeveloped economy" (*The Economic Journal,* 1961). Before describing Goodwin's models and results, it may be useful to consider the times at which the author was writing. Although his paper was published in 1961, its substance was originally presented to the Oxford-London-Cambridge Seminar on November 10, 1956. Those years were marked by the widely shared belief, even in countries like the United Kingdom or France, that planning was the answer to all possible economic woes, from shortages to inflation and unemployment. We therefore should hardly be surprised when Goodwin boldly wrote: "The planners may determine the marginal utility curve in any way or may accept any sort of directive about it" (p. 763) – a statement that would seem quite extraordinary today, to say the least, but that explains the reaction he would have when confronted to his results.

The author used three types of utility functions: the first was derived numerically through the United Kingdom marginal income-tax schedule, 1953-54, for a married couple with two children; the second was $\ln(C - \bar{C})$ where $\bar{C}$ is a subsistence level; the third was $\left[(C - \bar{C})^{1-\epsilon} - 1\right]/(1-\epsilon)$[3]. Production was supposed to be a linear function of capital.

His results should have been startling for anybody, including the author himself. In model I (corresponding to the first utility function) the optimal saving rate grew to 62% after 28 years, with an implied marginal saving rate of 79% at year 20. In model II, the optimal saving rate was 59% at year 24, with a marginal saving rate equal to 68% at year 12. Model III (where Goodwin chose $\epsilon = 0.2$) was even more disastrous, leading to an optimal saving rate equal to 83% at year 36, and marginal rates of at least 95% between years 28 and 32.

1.2.2 Goodwin's reaction.

Contrary to Ramsey's natural reaction to such excessive saving rates, Goodwin found those numbers perfectly justifiable. Already after getting model I results, he explained them by the gains of productivity that might be bestowed onto future generations; those gains would be so big that they would justify huge sacrifices made by present generations; in his own words: "So great are the gains that we are fully justified in robbing the poor to give to the rich!" (p. 765). With such a conviction, it is not surprising that when all results of his three models were in, Goodwin wrote:

> "Some violent process of capital accumulation of the type illustrated is the ideal. The simplifications of the model give an unduly sharp outline of the ideal policy, but its general character is surely a sound guide to policy" (p. 772-773).

[3] Goodwin's second utility function is the particular case of the third one when $\epsilon \to 1$.

It is difficult to gauge what the general feeling of the profession has been after the publication of those strong statements, but no doubt some members must have had serious reservations. It seems appropriate to mention that in a conference given in 2006, Robert Solow said he vividly remembered having read Goodwin's paper just before or just after its publication, and to have been "very worried" about its excessive optimal saving rates.

1.3 The paper that should have been the final alarm bell: King and Rebelo (1993).

Twenty years ago, King and Rebelo published an important, illuminating study on the transition paths for a neo-classical economy with intertemporally optimizing households. Basically, they worked with three utility functions: (*i*) $\log C$; (*ii*) a transform of the log function of the Stone-Geary type; and (*iii*) $(-1/9)(C^{-9}-1)$. The production function was of the Cobb-Douglas type, with a 1/3 capital share and labor-augmenting progress. In a second part of their paper they also considered a CES function with an elasticity of substitution between 0.9 and 1.25, and finally introduced a large array of variants to the basic model.

It is worth pausing here an instant to consider to what extremes King and Rebelo had recourse regarding the utility functions in order to give the traditional model maximum chances of reflecting historical experience. Indeed, taking C to the power -9 is no trivial matter. It entails exactly the same, *extreme* properties of the function as those we had found earlier had we wanted the Ramsey model to yield a 10% or 20% optimal savings rate rather than 60% or more: a graph practically undistinguishable from a vertical line immediately followed by a horizontal[4], and a "bliss" level, here at height $\lim_{C\to\infty} = 0.\bar{1}$, practically reached at $C = 1.7$. Exactly as before, the marginal utility ($U'(C) = C^{-10}$) has the implausible property that *from any initial level*, multiplying consumption by a factor λ reduces the marginal utility by a factor λ^{10}. But even these extreme assumptions do not prevent the model of yielding ill-fated time paths, as King and Rebelo clearly demonstrate.

The authors describe with great precision the implications of each of those intertemporal preferences on the time paths of the following variables: output, consumption, investment, output growth rate, saving–investment rate,

[4] It is worth noting that the graph of $U = (-1/9)(C^{-9} - 1)$ is extremely close to that of the limiting curve $U = \lim_{\alpha\to-\infty} (1/\alpha)(C^{\alpha} - 1)$, defined by the negative part of the vertical $C = 1$, followed by the horizontal $U = 0$ for $C > 1$.

real interest rate and wage rate. A first striking result is that for $U = \log C$ the initial saving rate is nearly 50%, converging toward 25% after 30 years. For the initial saving rate to be in a more reasonable range (about 10%), one has to turn to the two remaining utility functions; but then the saving rate has the disturbing property of *increasing* toward 25%, while one would expect technical progress on the contrary to alleviate the sacrifices incurred by society's investment. (Later on, in the model we propose, we will observe the optimal saving rate to be permanently decreasing, if ever slowly).

Even more worrying is the catastrophic behavior of the *real* interest rate. Whatever the utility function chosen, it starts at 105%; it remains above 20% for 5, 11 and 19 years with each of the utility functions mentioned above. The worst behavior is associated with utility function (iii) whose extreme properties were just described: starting at 105%, the real rate of interest remains above 14% for the whole time span (30 years) examined by the authors.

Keeping then the $\log C$ utility function only, King and Rebelo tried other values for the capital share (0.5 and 0.9). With a 50% capital share, the initial real interest rate was still as high as 34%; but then the initial saving rate jumped to 53%, and converged to 38%. Only a 90% capital share produced an acceptable interest rate, but at the expense of a "wildly counterfactual" (in their own words) saving rate close to 68% *at any point of time*.

Highly interestingly, the authors tell us how their audiences reacted when presented these results, and what steps they then took:

> "There was a recurrent reaction from audiences. A particular modification of our basic model would be suggested as a means of avoiding the very high marginal product of capital in the early stage of development. Then, other supporting evidence for this modification would be introduced and debated. In thinking through modifications suggested by a number of seminar audiences and others of our own design, we divided them into two groups. First, there are alternative parameter choices when we work within the basic neoclassical model's production function. Second, there are modifications of other attributes (such as vintage capital, investment adjustment costs, separate production functions for consumption and investment goods, or international capital flows)" (p. 920).

The corresponding, important results are presented in their Tables 1 and

2 (real interest rate implications of different hypotheses regarding the production function, including varying the elasticity of substitution from 0.9 to 1.25), and figure 6 which presents the outcomes of the close, above-mentioned, relatives of the mainstream neo-classical model. The authors' conclusion is unambiguous:

> "In exploring some plausible alterations of the basic model, we found that it was impossible to explain important components of economic growth in terms of transition dynamics without introducting some related implication that strongly contradicted historical experience" (p. 929).

2. The ill-fated role of utility functions.

2.1 A first analysis.

There are some common, striking features in all essays that either aimed at determining the optimal saving rate or, more generally, tried to replicate historical patterns under the assumption of intertemporal maximisation. One of them is that they always made use of strictly concave utility functions. If these functions were not numerically defined, they were either an affine transform of the power function (including the log function as a particular case) or were of the Stone-Geary form – we have never seen any numerical application of the negative exponential form $(-1/\beta)e^{-\beta C}, \beta > 0$, nevertheless often declared fit for service.

Whenever counterfactual results appeared, authors seemed to be forced in the same direction: changing the values of parameters used in their models (sometimes even changing the very significance of those parameters – a bold step, to say the least) or changing the models altogether. But they never contemplated the possibility that the root of the serious, repeatedly encountered problems laid in the very concavity of the utility functions.

In 2009, unaware of the contribution by King and Rebelo, we carried out a systematic study of the consequences of that concavity on the optimal trajectories of consumption and capital (La Grandville, 2009, pp. 234-261). Recall that King and Rebelo had tested three specific utility functions, mentioned above. For our part we put to the test *all* possible values of α in the function $U(C) = (C^{\alpha} - 1)/\alpha$, as well as all possible values of β in the negative exponential form $U(C) = (-1/\beta)e^{-\beta C}$, $\beta > 0$. We first briefly summarize our results.

We used the central aims and hypotheses of the neo-classical model: maximisation of $\int_0^{\infty} L_t U(C_t/L_t) e^{-it} dt$ under the constraint $C_t = F(K_{t,} L_t, t) -$

$\dot{K}_t = F(K_{t,}e^{gt}L_t) - \dot{K}_t$, where $\dot{K}_t$ is *net* investment; $F(K_{t,}L_t, t)$ is of CES form with labor-augmenting progress at constant rate g, and L_t grows at constant rate n. We were able to obtain a complete picture of the relationship between the utility functions and their resulting optimal paths thanks to the generosity of our colleague Ernst Hairer who built a program to determine the initial optimal saving rate leading to the equilibrium point – and not to a collapse of the economy – not only for $\alpha \in (-\infty, 1)$ but for a whole range of values of the following parameters: the elasticity of substitution, the discounting rate of the utility flows, the population growth rate and the growth rate of technical progress.

Regarding the power function, the conclusion is as plain as it is dramatic: for all acceptable values of the parameters, a reasonable initial saving rate, in the order of 10%, can be obtained only if the power in the utility function is in the neighborhood of -5. Apart from other serious drawbacks to be underlined in the next Section, it requires a utility function such that whenever consumption is multiplied by λ, the marginal utility is divided by λ^6, with a "bliss level" very quickly reached. It would be very difficult to find an individual whose attitude toward consumption would fit that pattern, and certainly impossible to convince a whole society that such utility function is just hers.

The situation is even more disastrous when considering the negative exponential $U(C) = (-1/\beta)e^{-\beta C}$, because in that case *no equilibrium point exists any more*. Whatever the initial saving value, the economy will collapse either because of excessive consumption or over-accumulation of capital. Indeed, what might look, in the phase diagram, as a stable arm leading asymptotically toward an equilibrium point, is in fact quite deceptive: it will not lead to an equilibrium point, but to a *cusp point, reached in finite time*. From that point onward, inexorably the economy will be led to overinvest until consumption becomes nil. The path leading to the cusp point is not even separating two families of divergent curves (one of those leading to zero consumption, the other to zero capital). For a range of initial C_0 values above that leading to the cusp point, the trajectories pass to the left of the cusp, then curl upon themselves to fatally bring down consumption (see figures 10.12 and 10.13, pp. 255-256).

2.2 A further examination.

In this preceding (2009) analysis we had been concerned about two issues: the existence of a steady state, and the initial value of the optimal savings rate leading to a steady state. This analysis was carried out for all possible values of the parameters relevant to the utility functions, and a wide array of values for the discounting factor and the parameters reflecting the production process. But we had not determined what would be the consequences of these optimal savings rates on two fundamental features of the economy: the implied initial growth rate of real income per person $\dot{y}_0^*/y_0^*$, and the long-term, ultimate value of the marginal productivity of capital, $\lim_{t\to\infty} F_K(K_t, L_t, t)$.

We now address these issues; the results are in Table 2. The initial optimal

TABLE 2. *The ill-fated implications of the traditional approach for any power utility function*
$U(C) = (C^\alpha - 1)/\alpha$; ($\delta = 1/3$; $i = 0.04$; $n = 0.01$; $H = 0.02$).

	initial optimal savings rate s_0^*			initial optimal growth rate $\dot{y}_0^*/y_0^*$			$\lim_{t\to\infty} F_K^*$
α	$\sigma = 0.5$	$\sigma = 0.8$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0.8$	$\sigma = 1$	(indep. of σ)
0.5	0.58	0.71	0.81	0.20	0.25	0.28	0.05
0	0.41	0.50	0.64	0.15	0.17	0.22	0.06
-1	0.26	0.30	0.45	0.10	0.11	0.16	0.08
-2	0.19	0.21	0.35	0.07	0.08	0.13	0.10
-3	0.15	0.16	0.28	0.06	0.06	0.10	0.12
-4	0.12	0.13	0.23	0.05	0.05	0.09	0.14
-5	0.1	0.11	0.19	0.04	0.05	0.07	0.16
-6	0.08	0.09	0.16	0.04	0.04	0.06	0.18
-7	0.07	0.07	0.13	0.03	0.03	0.05	0.20
-8	0.06	0.06	0.11	0.03	0.03	0.05	0.22
-9	0.05	0.06	0.09	0.03	0.03	0.04	0.24

savings rate s_0^* is determined numerically thanks to Ernst Hairer's program; for the analytic derivation of $\dot{y}_0^*/y_0^*$ and $\lim_{t\to\infty} F_K^*$, see La Grandville, 2009, pp. 237-239).

If α is in a seemingly acceptable range (say when $0 < \alpha < 1$), it leads to abnormal initial optimal savings and growth rates (note that the worst scenarios correspond to $\sigma = 1$, nevertheless a value still often used. For instance, if $\alpha = 0.5$, $s_0^* = 81\%$ and $\dot{y}_0^*/y_0^* = 28\%$). When α becomes negative, entailing a bliss level very quickly reached, we are led to levels of marginal productivity of capital that were never observed. Whatever characteristic we are willing to attribute to the utility function, we cannot escape the same kind of implications contradictory to historical experience – or, plainly, to common sense – as those forcefully set forth by King and Rebelo.

3. How the strict concavity of utility functions makes competitive equilibrium unsustainable.

We will now show that the traditional approach, in its attempt to optimize the evolution of an economy by positing a strictly concave utility function, is simply incompatible with competitive equilibrium. To do so, we will assume that an economy is initially in a state of competitive equilibrium.We will then suppose that two different courses can be pursued:

I) investment is planned in such a way as to maximize intertemporally discounted utility flows, the utility function being the widely used affine transform of a *strictly* concave power function; this is the traditional approach.

II) investment is made in such a way as to conform the Euler equation defining competitive equilibrium. This will be our suggested solution to the basic problem of optimal economic growth.

We will also widen our hypothesis regarding the structure of the production process by allowing, in both scenarios, technical progress to be not only labor-augmenting but capital-augmenting as well. In the traditional literature on the neo-classical model, only labor-augmenting technical progress is allowed, apparently for the following reason: that restricting hypothesis is considered necessary for the growth rate of income per person to converge asymptotically toward the rate of labor-augmenting progress, the only exception applying in the Cobb-Douglas case. We have recently shown this assumption to be wrong by demonstrating a new property of general means of order p when p is negative – precisely the case where $0 < \sigma < 1$ (La Grandville 2011) and we will check that indeed, in both scenarios I and II the growth rate of income per person does converge toward the rate of labor-augmenting progress although progress is capital-enhancing as well.

In each scenario I and II we will depict the evolution of the economy represented by the following variables: the optimal saving rate, the growth rate of income per person, the marginal product of capital and the capital-output coefficient.

3.1 Initial conditions as determined from competitive equilibrium.

We suppose that at the initial time competitive equilibrium prevails in the economy. This implies that the capital stock is in such an amount that its marginal productivity is equal to the rate of interest. Total output (*net* of depreciation), denoted by Y_t, is given by a production function of CES form featuring both labor- and capital-augmenting technical progress; to this aim we define factor-enhancing functions of time G_t and H_t such that their growth rates $\dot{G}_t/G \equiv g(t)$ and $\dot{H}_t/H_t \equiv h(t)$ are positive; G_0 and H_0 are normalized to 1. Labor is the exogeneous increasing function of time L_t, with $L_0 = 1$. In a first step we will consider that the functions G_t, H_t and L_t are the exponentials $G_t = e^{gt}$, $H_t = e^{ht}$ and $L_t = e^{gt}$;in Section 5, to test the robustness of the model we suggest, we will suppose that those exponentials are replaced by S–shaped functions. The production function is the general mean of order p of the enhanced inputs G_tK_t and H_tL_t :

$$Y_t = F(G_tK_t, H_tL_t) = Y_0\{\delta[G_tK_t/K_0]^p + (1-\delta)[H_tL_t/L_0]^p\}^{1/p}\ ,\ p \neq 0 \quad (6)$$

where the order p is the increasing function of the elasticity of substitution σ: $p = 1 - 1/\sigma$. Note that p will always be negative because σ is supposed to be in the range where it has been most often observed, i.e. between 0 and 1. However, for comparison purposes we will also give results corresponding to the $p = 0,\ \sigma = 1$ Cobb-Douglas case

$$Y_t = Y_0(G_tK_t/K_0)^{\delta}(H_tL_t/L_0)^{1-\delta}. \quad (7)$$

In the case $0 < \sigma < 1$, the fundamental competitive equilibrium equality $F_{K_t} = i$ leads to the following equation in K_t:

$$F_{K_t}(G_tK_t, H_tL_t) = Y_0\{\delta[G_tK_t/K_0]^p + (1-\delta)[H_tL_t/L_0]^p\}^{(1/p)\ -1}\ .$$

$$.\delta {K_t}^{p-1}(G_t/K_0)^p = i, \quad p < 0,\ 0 < \sigma < 1 \quad (8)$$

which can be solved to yield the optimal time path K_t^* :

$$K_t^* = \frac{K_0}{L_0}\left(\frac{1-\delta}{\delta}\right)^{\sigma/(\sigma-1)} \frac{L_tH_tG_t^{-1}}{\left[i^{\sigma-1}\delta^{-\sigma}\left(Y_0/K_0\right)^{1-\sigma}G_t^{1-\sigma} - 1\right]^{\sigma/(\sigma-1)}},\quad 0 < \sigma < 1. \quad (9)$$

K_0 and Y_0 are identified by setting $t = 0$ in (9); we obtain $K_0/Y_0 = \delta/i$. We now can normalize Y_0 to one; thus $K_0 = \delta/i$; finally, the optimal time path of capital is

$$K_t^* = \frac{\delta}{i}\left(\frac{1-\delta}{G_t^{1-\sigma}-\delta}\right)^{\sigma/(\sigma-1)} L_t H_t G_t^{-1}, \quad 0 < \sigma < 1. \tag{10}$$

The optimal trajectory of output and income Y_t^* follows from replacing (10) into (6), using the same identifications. We obtain

$$Y_t^* = L_t H_t \left[\delta\left(\frac{1-\delta}{G_t^{1-\sigma}-\delta}\right)+1-\delta\right]^{1/p} = L_t H_t \left(\frac{1-\delta G_t^{\sigma-1}}{1-\delta}\right)^{\sigma/(1-\sigma)}, \ 0 < \sigma < 1. \tag{11}$$

We can verify that $K_0 = \delta/i$ and $Y_0 = 1$.

An important observation is now in order. Note that when $\sigma \neq 1$ the time path K_t^* is defined *for all t if and only if* $\sigma < 1$[5]. Indeed such is the condition for the denominator $G_t^{1-\sigma} - \delta$ to be positive for all t . Since G_t is larger than 1 as well as increasing and unbounded, if $\sigma > 1$ there always exist a time $\bar{t}$ from which $G_t^{1-\sigma} - \delta$ becomes zero and then negative. The economic reason for this is the following: we know that σ is a powerful engine of growth; this is due to its considerable enhancement of the marginal productivity of capital; but it cannot become too powerful, because to maintain the equality $F_K = i$ capital should then increase extremely fast, entailing explosive growth: it can be verified that $lim_{t\to\bar{t}} K_t^* = \infty$. It is also a good place to remember that time and again the empirical estimates of σ have been strictly lower than one, and, on the other hand, that $\sigma > 1$ would make no sense at all, since it would imply that any amount of output could be produced either without capital or without labor (indeed, in that case the isoquants cut the axes).

[5]In the $\sigma = 1$ Cobb-Douglas case, formulas (10)–(12) have to be reworked from $F_K = i$ using this time (7) for $F(.)$. The results are

$$K_t^* = \frac{\delta}{i} L_t H_t G_t^{\delta/(1-\delta)},$$

$$Y_t^* = L_t H_t G_t^{\delta/(1-\delta)},$$

and

$$K_t^*/Y_t^* = \frac{\delta}{i}.$$

As mentioned before, we give these results for complete reference only, because time and again σ has been observed as smaller than one.

K_t^* and Y_t^*, given by (9) and(11), lead to the the following optimal evolution of the capital-output ratio

$$K_t^*/Y_t^* = \frac{\delta}{i} G_t^{-(1-\sigma)}, \quad 0 \leq \sigma \leq 1. \tag{12}$$

Innocuous as this last formula may seem, it carries a wealth of good news. The first is that, contrary to what we saw just before where *all* concave power utility functions made the capital–output ratio increase to absurd values, here the ratio *always diminishes* – it is good news: no one would want an economy where the stock of capital increases more rapidly than its output when it is expected, on the contrary, that technological progress will enable to use relatively *less* capital for a given product. The second good news is that since the remuneration of capital is fixed at $F_{K^*} = i$, the share of capital in total income $F_{K^*} K_t^*/Y_t^* = i K_t^*/Y_t^* = \delta G_t^{-(1-\sigma)}$ will always diminish to the benefit of the share of labor.

There are now several ways to determine the optimal savings and investment rate. One of the simplest is to first evaluate the optimal growth rate of Y_t^*. Denoting the growth rates of G_t, H_t and L_t by g_t, h_t and n_t respectively, we get

$$\dot{Y}_t^*/Y_t^* = n_t + h_t + \sigma\delta \left(G_t^{1-\sigma} - \delta\right)^{-1} g_t, \quad 0 \leq \sigma \leq 1 \tag{13}$$

Applying (12), the growth rate of capital is $\dot{K}_t^*/K_t^* = \dot{Y}_t^*/Y_t^* - (1-\sigma) g_t$ and therefore, after simplifications,

$$\dot{K}_t^*/K_t^* = n_t + h_t + g_t \left[\frac{\sigma}{1 - \delta G_t^{\sigma-1}} - 1\right], \quad 0 \leq \sigma \leq 1. \tag{14}$$

The optimal savings rate s_t^* is equal to $\dot{K}_t^*/Y_t^* = \left(\dot{K}_t^*/K_t^*\right)\left(K_t^*/Y_t^*\right)$; so we have

$$s_t^* = \frac{\delta}{i} \left\{ n_t + h_t + g_t \left[\frac{\sigma}{1 - \delta G_t^{\sigma-1}} - 1\right] \right\} G_t^{-(1-\sigma)}, \quad 0 \leq \sigma \leq 1. \tag{15}$$

We are now in a position to identify the optimal initial level of consumption $C_0^* = (1 - s_0^*) Y_0^*$; since $Y_0^* = 1$, we have

$$C_0^* = 1 - \frac{\delta}{i} \left\{ n_0 + h_0 + g_0 \left[\frac{\sigma}{1-\delta} - 1\right] \right\}, \quad 0 \leq \sigma \leq 1^6. \tag{16}$$

[6] Note that formulas (13) to (16) apply directly in the $\sigma = 1$ case. One gets $\dot{Y}_t^*/Y_t^* = \dot{K}_t^*/K_t^* = n_t + h_t + \frac{\delta}{1-\delta} g_t$; $s_t^* = \frac{\delta}{i}(n_t + h_t + \frac{\delta}{1-\delta} g_t)$ and $C_0^* = 1 - \frac{\delta}{i}(n_0 + h_0 + \frac{\delta}{1-\delta} g_0)$.

Hence the optimal initial conditions K_0^*, C_0^* corresponding to $F_{K_t} = i$ are given by $K_0^* = \delta/i$ and equation (16). They define the common starting point shared by scenarios I and II.

Before we describe the evolution of the economy in each of those settings, let us consider the values taken by the common initial savings rate s_0^* and the common initial growth rate of real income per person $\dot{y}_0^*/y_0^*$. Indeed, we need to ascertain in particular that intricate as formula (15) for s_t^* may look, it always yields very reasonable numbers already at time $t = 0$. We will take $\delta = 0.25$ and $n = 0.01$; the factor enhancing growth rates g and h will be those measured by Sato (2006, p. 60) for the United States over the period 1909-1989: $g = 0.004$ and $h = 0.02$. Tables 3 and 4 indicate s_0^* and $\dot{y}_0^*/y_0^*$ for σ in the range 0.5 to 0.8 (most observed) and i between 0.04 and 0.06.

TABLE 3. *The initial savings rate s_0^* implied by competitive equilibrium, as a function of the elasticity of substitution σ and the rate of interest i ; in percent* ($\delta = 0.25$; $n = 0.01$; $h = 0.02$; $g = 0.004$).

σ / i	0.5	0.55	0.6	0.65	0.7	0.75	0.8
0.04	17.9	18.1	18.3	18.4	18.6	18.8	18.9
0.05	14.3	14.5	14.6	14.7	14.9	15.0	15.1
0.06	11.9	12.1	12.2	12.3	12.4	12.5	12.6

The initial values of the growth rate of income $\dot{y}_0^*/y_0^*$, independent of i, are given in the following table.

TABLE 4. *The initial values of the growth rate of income per person $\dot{y}_0^*/y_0^*$ implied by competitive equilibrium, as a function of the elasticity of substitution σ; in percent* ($\delta = 0.25$; $n = 0.01$; $h = 0.02$; $g = 0.004$).

σ	0.5	0.55	0.6	0.65	0.7	0.75	0.8
$\dot{y}_0^*/y_0^*$	2.07	2.07	2.08	2.09	2.09	2.10	2.11

It can be seen that the initial savings and growth rates implied by competitive equilibrium are in a very reasonable range, historically observed. They stay in stark contrast to the results presented earlier, corresponding to all possible concave power utility functions. Consider for instance the case of the logarithmic utility function, corresponding to $\alpha = 0$ (second line in Table 2, with $i = 0.04$), and take $\sigma = 0.5$. It can be seen that the initial "optimal" savings rate necessary to put the economy on the stable branch in the phase diagram is 41%, implying also a never observed *real* growth rate equal to 15%. If σ had been equal to 0.8, the results would have been even more disastrous: the initial saving rate would have climbed to 50% and the growth rate to 17%.

Thus equipped with initial conditions corresponding to competitive equilibrium, we can describe what will happen to the economy if either scenario I or II is pursued; in scenario I, investment is planned on the basis not just of one, but *all* possible concave power utility functions. Its fateful consequences are laid out in Section 3.2; the inability of scenario I to maintain trajectories that would replicate competitive equilibrium is explained in Section 3.3.

Scenario II is our solution to the problem of optimal economic growth: investing in such a way that competitive equilibrium is maintained through time. We will show that it entails no less than 5 maximisation objectives for society, apart from the minimization of production costs. We lay out the resulting, very reasonable time paths in section 4. The robustness of these results is finally tested in section 5 by considering quite different evolutions of population and technical progress.

3.2 Planning with strictly concave utility functions from an initial situation of competitive equilibrium: a disaster in the making.

Given the above-defined initial conditions reflecting competitive equilibrium, we now maximize $\int_0^\infty U(C_t)e^{-it}dt$ under the constraint $C_t = F(K_t, t) - \dot{K}_t$ where $F(.)$ is defined by (6), and where $U(C) = (C^\alpha - 1)/\alpha$. Our functional is $\int_0^\infty U[F(K,t) - \dot{K}]e^{-it}dt = \int_0^\infty V(K, \dot{K}, t]dt$; the Euler equation

$$\frac{\partial V}{\partial K} - \frac{d}{dt}\frac{\partial V}{\partial \dot{K}} = 0$$

together with the constraint leads to the system of first order non-linear equations

$$\dot{C} = \frac{C}{1-\alpha}\left\{\left[\delta(e^{gt}iK/\delta)^p + (1-\delta)e^{p(n+h)t}\right]^{(1/p)-1} \delta^{1-p} i^p e^{pgt} K^{p-1} - i]\right\} \quad (17)$$

$$\dot{K} = [\delta(e^{gt}iK/\delta)^p + (1-\delta)e^{p(n+h)t}]^{1/p} \; - C \tag{18}$$

The concavity of the integrand with respect to K and $\dot{K}$ and the transversality conditions (shown to be met at the end of this section) ensure that this system leads to a unique maximum, given the above defined initial conditions K_0^*, C_0^*.

We started the tests of the utility function by using the parameter values mentioned above: $n = 0.01$; $\delta = 0.25$; $i = 0.04$; $\sigma = 0.8$; $h = 0.02$; $g = 0.004$. Solving numerically system (17, 18) and plugging the solution K_t^* into (6) enables to determine the evolution of the growth rate of the real income per person $\dot{y}_t^*/y_t^*$ (see Figure 3) for 25 values of the parameter α of the utility function, ranging from 0.8 (upper curve in the left part of the diagram) to -8.8 by steps of -0.4.

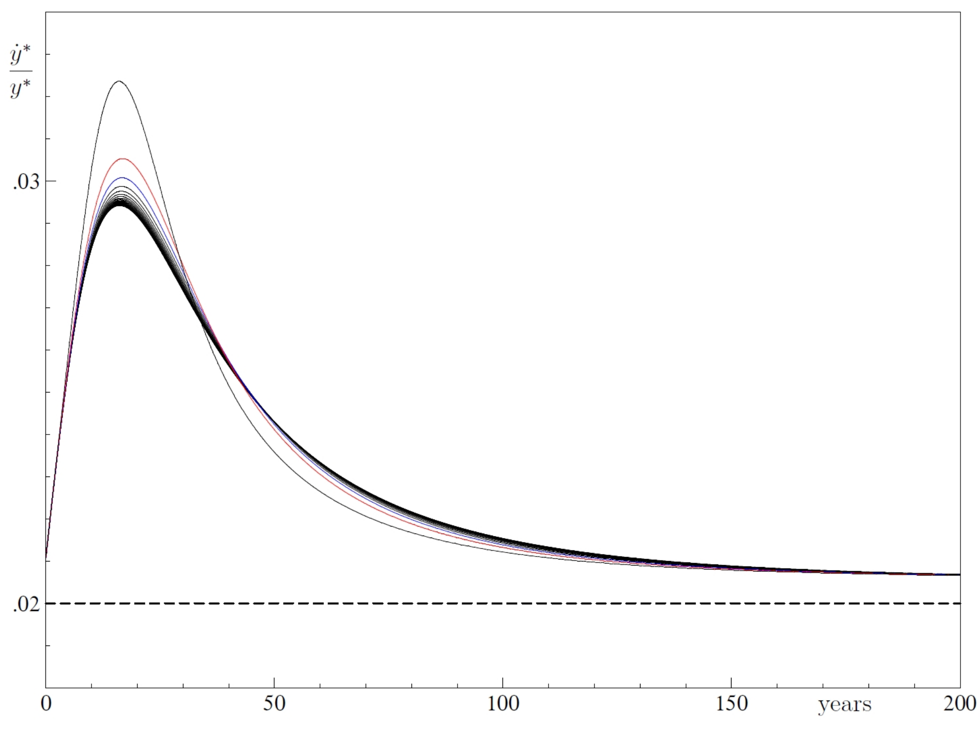

FIGURE 3. *The growth rate of real income per person for 25 values of α in the utility function $(C^\alpha - 1)/\alpha$ ranging from $\alpha = 0.8$ (upper curve in the left part of the diagram) to $\alpha = -8.8$ by steps of -0.4;* $n = .01$; $\delta = 0.25$; $i = 0.04$; $\sigma = 0.8$; $H = .02$; $g = 0.004$.

The third curve from above corresponds to $\alpha = 0$ (the case $U = \log C$). The lower curve in the left-hand side – becoming the upper curve on the right – corresponds to $\alpha = -8.8$; it practically gives the limiting curve when $\alpha \to -\infty$ (a property shared in the next diagrams). The reason is the following: when $\alpha = -8.8$ the utility curve almost reaches its asymptotic limit $(\lim_{\alpha \to -\infty} (1/\alpha)(C^{\alpha} - 1))$ defined by the vertical $C = 1$ in negative space followed by the horizontal $U = 0$ for $C > 1$.

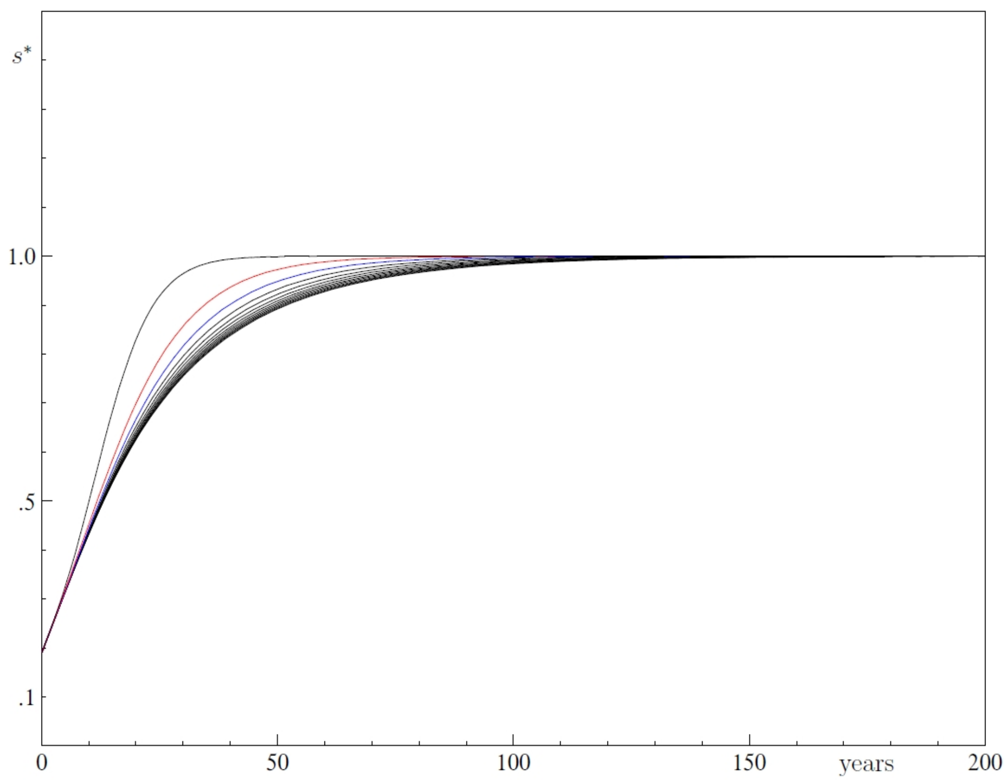

FIGURE 4. *The inordinate behavior of the savings rate for 25 values of α in the utility function $(C^\alpha - 1)/\alpha$ ranging from $\alpha = 0.8$ (upper curve) to $\alpha = -8.8$ (lower curve) by steps of -0.4.*

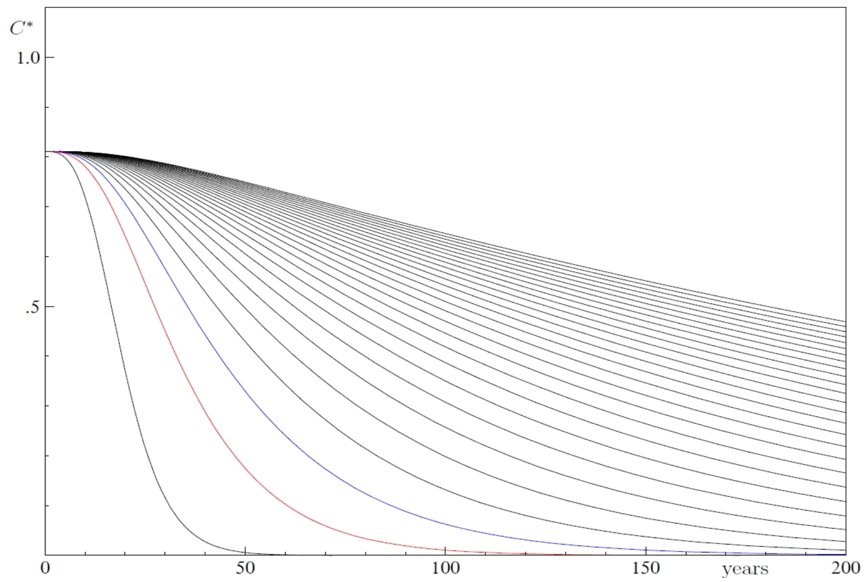

FIGURE 5. *The unwanted behavior of consumption for 25 values of* α *in the utility function* $(C^{\alpha}-1)/\alpha$ *ranging from* $\alpha = 0.8$ *(first curve on the left) to* $\alpha = -8.8$ *(last curve on the right).*

It can be seen that for all α values the growth rate increases in a first phase toward a maximum close to 3%. Whatever the value of α, the growth rate then decreases towards $\lim_{t\to\infty} \dot{y}_t^*/y_t^* = h = 0.02$. While this evolution does not seem improbable – although a growth rate of real income per $\dot{y}_t^*/y_t^*$person higher than 2.8% for 20 consecutive years whatever the utility function is a bit suspicious – it corresponds in fact to disaster: it parallels an ever-growing savings rate, tending very fast, asymptotically, toward 100%, as can be seen in Figure 4. [**Insert here Figure 4**] *For all alpha values*, the savings rate becomes equal to or larger than 50 % before 14 years. This absurd situation is confirmed by the permanently declining consumption from its initial value, as shown in figure 5.

Such an excessive saving rate is naturally conducive to an inappropriately high growth rate of the capital stock. Its evolution is depicted in figure 6. Notice that for any α value in the utility function, this rate exceeds 4% per year for about 80 years.[**Insert here Figures 5 and 6**]

This inordinate growth rate of capital explains the inability of concave utility functions to sustain competitive equilibrium expressed by the equality $F_K(K,t) = i$. Indeed, the excess investment pushes down the marginal productivity of capital at levels lower than the rate of interest; this is illustrated in figure 7, and paralleled by the non-sensical evolution of the capital-output ratio (figure 8). From an initial, reasonable value equal to $K_0^*/Y_0^* = \delta/i = 6.25$ – corresponding to competitive equilibrium – the capital-output ratio increases and tends asymptotically toward an absurd value equal to 32 for *any* utility function. [**Insert here Figures 7 and 8**]. One would expect, of course, that technical progress enhancing capital would *reduce,* not increase, the need of fixed capital for one unit of net output. On the other hand, in the competitive equilibrium model we suggest hereafter, we will see that the capital-output ratio decreases, if ever slowly.

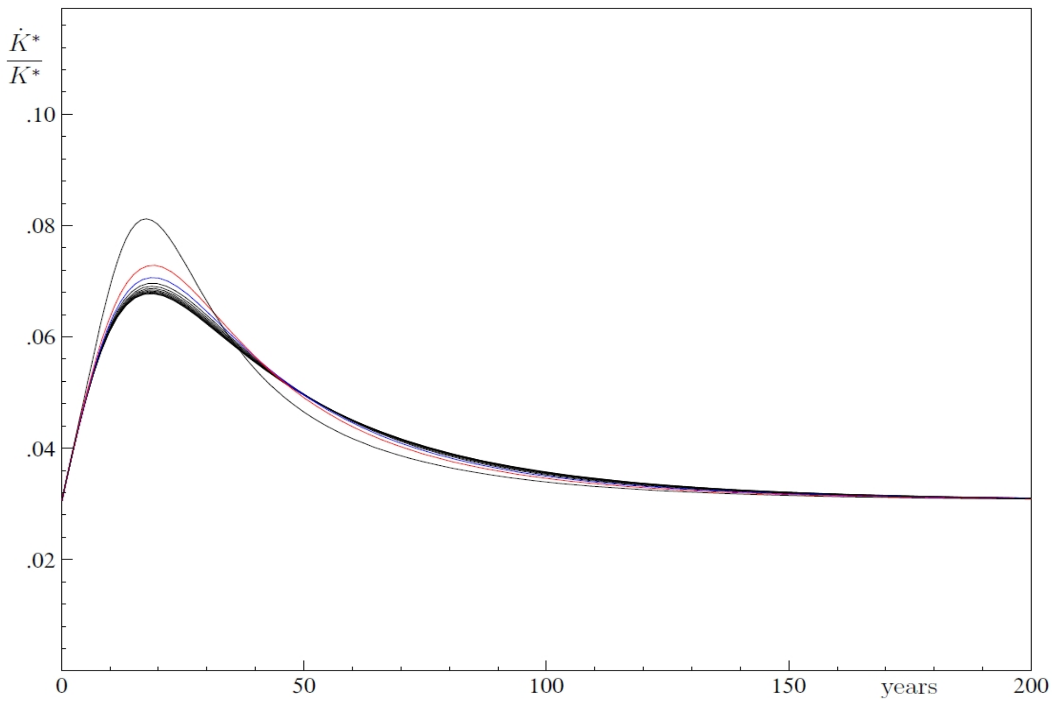

FIGURE 6. *The inordinate behavior of the growth rate of capital for 25 values of* α *in the utility function* $(C^\alpha - 1)/\alpha$ *ranging from* $\alpha = 0.8$ *(upper curve in the left part of the diagram) to* $\alpha = -8.8$.

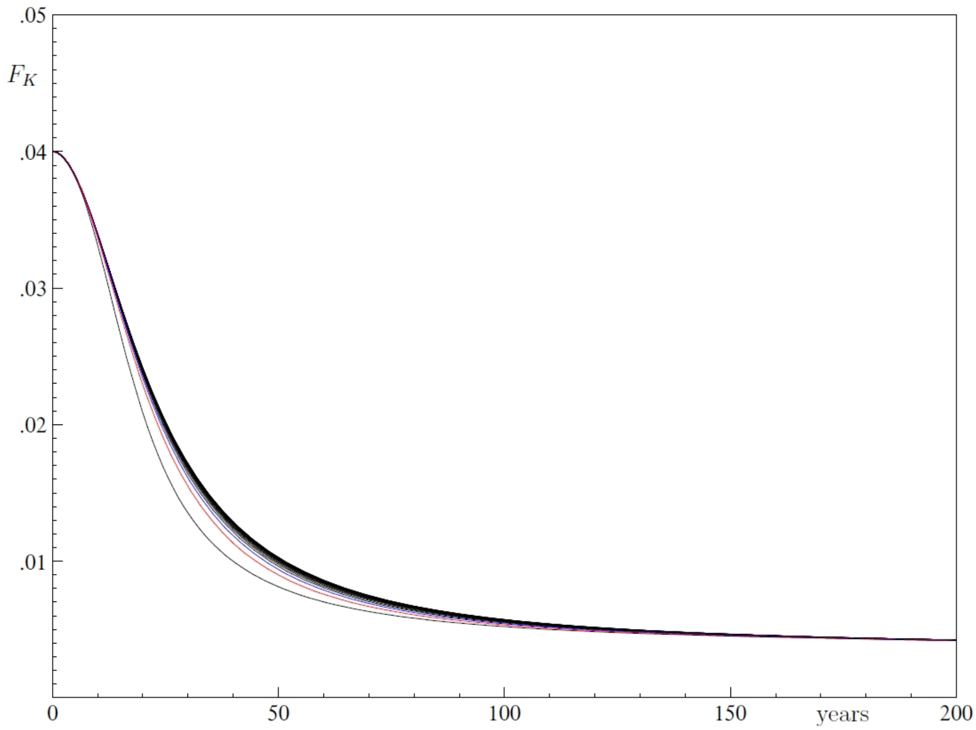

FIGURE 7. *An illustration of the unability of concave utility functions to sustain competitive equilibrium. Excess investment unfailingly lowers too quickly the marginal productivity of capital.*

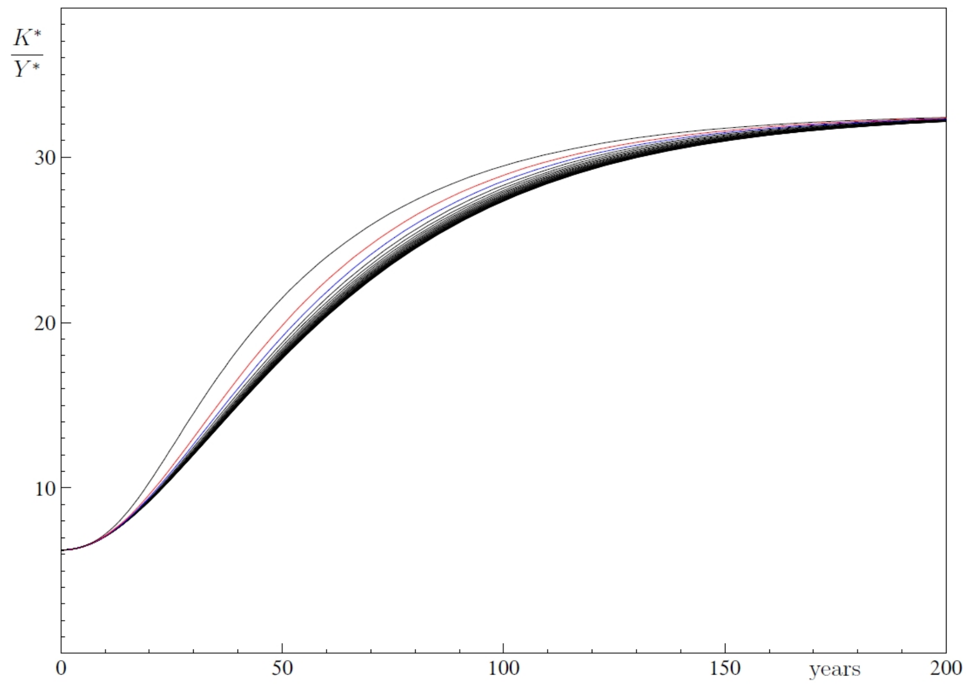

FIGURE 8. *The inordinate behavior of the capital-output ratio for 25 values of α in the utility function $(C^\alpha - 1)/\alpha$.*

We still have to show that the transversality conditions at infinity are met. With

$$\int_0^\infty U[F(K,t) - \dot{K}]e^{-it}dt = \int_0^\infty V(K, \dot{K}, t]e^{-it}dt, \tag{19}$$

the first condition is $\lim_{t\to\infty} \partial V/\partial \dot{K} = \lim_{t\to\infty} -U'(C)e^{-it} = 0$, always met. The second one is $\lim_{t\to\infty} V = 0$ and is enforced by the convergence of the integral; this last condition is met by the fact that the integrand is positive and that e^{-it} converges to zero faster than U^{-1}.

3.3 The incompatibility of the traditional approach and competitive equilibrium: an analytic explanation.

We illustrated numerically the fact that competitive equilibrium could not be sustained by any utility function of the type $U(C) = (C^\alpha - 1)/\alpha$, the traditional approach leading to definitely unwarranted time paths for variables of central importance: fast declining consumption and over-accumulation of the capital stock. We can show that the same conclusion applies to *any* strictly concave utility function and explain the unwanted behavior of our central variables.

Consider the Euler second-order differential equation, written as follows:

$$i = F_K(K, L, t) + \frac{\dot{U}'_C}{U'_C} = F_K(K, L, t) + (\frac{U''_C}{U'_C}C)\frac{\dot{C}}{C}. \tag{20}$$

If $U''(C) < 0$ and $U'(C) > 0$, and unless we are in a stationary state, where $\dot{C} = 0$, the Euler equation will always preclude the competitive equilibrium equality $F_K(K, L, t) = i$.

We now explain the strange, definitely unwanted, behavior of the main variables that we just witnessed. As soon as investment is carried out, the capital stock increases, leading to a decrease of the marginal productivity of capital. To rebalance the Euler equation (whose left-hand side is the constant i), the last term of the equation *must be positive, and increase.* That last term is the product of $U''_C C/U'_C$, always negative, and $\dot{C}/C$. Therefore $\dot{C}/C$ *must be negative*, implying that consumption must decrease and that savings must increase. In our case, the elasticity of the marginal utility of consumption $U''_C C/U'_C$ is equal to $\alpha - 1$, a *negative constant.* Such a decrease in consumption will need to be the more pronounced as the absolute value of $\alpha - 1$ is low. This explains why, in figure 5, the curve on the far left that describes the fastest drop in consumption corresponds to $\alpha = 0.8$ (and $|0.8 - 1| = 0.2$) while the curve on the far right, exhibiting a less severe decrease, pertains to $\alpha = -8.8$ (and $|-8.8 - 1| = 9.8$). In Figures 4 and 8 the same observations apply to explain why the curves representing the highest savings rate and capital-output ratio at any point of time are generated by $\alpha = 0.8$.

4. A suggested solution

In the intertemporal optimization problem considered above, the only way to enforce $i(t) = F_K(K, L, t)$ is to have $U''(C) = 0$, i.e. $U(C) = aC + b$, where a and b are constants, a particular case being ours[7], $U(C) = C$. It is a good place to remember that utility functions, at the macroeconomic level

[7] As a referee has pointed out, it is not the first time a linear objective has been used: Intriligator (1971) and Kamikigashi and Roy (2006) are examples, albeit in different contexts.

were simple, direct transpositions of functions considered at the micro-level. We take the liberty of suggesting that before bending down into a concave curve the relationship between consumption and society's representation of welfare, we first take away from net national income the huge amount of expenditures that have simply no relationship with any present or future well-being.

We thus should be tending toward a measure of the quality of life that offers much less reason to be transformed into a concave function than what was the case previously. There are sound reasons not to introduce such transformations. Consider, for instance, medical discoveries that enhance both the length and the quality of life of a large part of the population, either in rich or poor countries. Wouldn't we then conclude that those health services generate linear or even *convex*, rather than concave utility flows? Also, contrary to what is assumed at the individual, micro level, *the very knowledge that not only some given person but the rest of society as well as all future generations are able to benefit from those discoveries* can hardly induce to penalize them with a transformation into some concave function. Exactly the same reasoning would apply to the all-important expenditures on education.

Consequently, in what follows we take the step of considering that C stands for *welfare* flows, F representing output net of a) physical and natural capital depreciation and b) all goods and services reducing welfare. In the same way $\dot{K}$ is standing only for investment in goods and services improving society's present and future well-being.

We are thus led to maximize $W = \int_0^\infty C_t \exp(-\int_0^t i(z)dz)dt$ under the constraint $C_t = F(K_t, L_t, t) - \dot{K}_t$. The Euler equation leads of course to the competitive equilibrium condition $i(t) = F_K(K, L, t)$ and, if $i(t)$ is constant, to equations (9) to (16) introduced in section 3 to determine the optimal time paths and their initial values corresponding to such equilibrium.

We should stress here an important point about the exact nature of i, the discount rate of consumption flows. It represents society's rate of preference for the present, and it is equal to an interest rate made out of the two usual components as they appear in the financial literature: a risk-free rate *and a risk premium*, both evaluated on long horizons. We have emphasized the words "*risk premium*" to underline the fact that the search for the optimal savings-investment rate definitely applies in an uncertain world, risk being taken into account. What should be the order of magnitude of i? We can safely say that it has definitely decreased in the last millenium; indeed, the rate of preference for the present was definitely higher in the middle ages when life expectancy was less than half of what it is today in advanced societies (As the French historian Pierre Gaxotte famously wrote: "the man of the middle ages does not know of time and numbers"). In the last centuries, this rate has certainly fluctuated mainly because of wars and the fear of wars. But in the long run, it has almost certainly decreased, and we feel entitled to think of it as an average of expected real rates of return on all all types of investments that have been made in the economy in the past decennies. The difficulty here hinges on the calculation of the weights to be given to each category of investments. To circumvent partially this problem, we can think of i in a range, exactly in the same way as we choose the other parameters of the model (for instance the parameters reflecting the production process or the evolution of technical progress pertaining to capital or labor). A final caveat is in order: the rate of preference should be at least equal to the equilibrium (long-term) growth rate of consumption; this is a necessary condition for the integral $\int_0^\infty C_t \exp(-it)dt$ to converge.

We now want to show that all equations (9) to (16) will *always* yield reasonable initial values and future time paths for the following fundamental variables: the optimal savings rate, the implied growth rate of income per person, and the capital-output ratio.

Before proceeding we should point out how appropriate the adjective "optimal" in this context is, since all time paths described hereafter correspond to no less than *five* simultaneous optima, in addition to the minimization of production costs.

4.1 The intertemporal optimality of competitive equilibrium: its multiple facets in one theorem.

We will show how investing in such a way that the marginal productivity of capital stays equal to the rate of interest generates for society five benefits of considerable importance; those benefits may be very surprising in the sense that they can be – and most probably are – far removed from the initial objective of investors – which might simply have been the minimisation of their production costs. We will prove the following:

THEOREM 1. *Let the production function $F(K_t, L_t, t)$ be concave and homogeneous of degree one in K, L; technical progress may be labor- and capital-augmenting. If investment is carried out through time in such a way* *that the marginal productivity of capital is maintained equal to the rate of interest $i(t)$, and if capital is remunerated by $i(t)K(t)$, society simultaneously maximizes five magnitudes:*

1. the sum of the discounted consumption flows society can acquire from now to infinity $\int_0^\infty C(t)e^{-\int_0^t i(z)dz}dt$;

2. the value of society's activity at any point of time t, defined by the consumption flow received at time t plus the rate of increase in the value of the capital stock at that time. In present value this sum is equal to $C_t e^{-\int_0^t i(z)dz} + \frac{d}{dt}[\lambda(t)K(t)]$ where $\lambda(t)$ is the discounted price of capital;

3. the total value of society's activity over an infinite time span $\int_0^\infty \{C_t e^{-\int_0^t i(z)dz}dt + \frac{d}{dt}[\lambda(t)K(t)]\}dt$;

4. the remuneration of labor at any point of time $F(K_t, L_t, t) - i(t)K(t)$;

5. the total remuneration of labor over an infinite time span $\int_0^\infty e^{-\int_0^t i(z)dz}[F(K_t, L_t, t) - i(t)K(t)]dt$.

Proof of 1. Maximizing $\int_0^\infty C_t e^{-\int_0^t i(z)dz}dt$ under the constraint $C_t = F(K_t, L_t, t) - \dot{K}(t)$ amounts to maximizing

$$W = \int_0^\infty \left[F(K_t, L_t, t) - \dot{K}(t)\right] e^{-\int_0^t i(z)dz}dt, \tag{21}$$

denoted $\int_0^\infty \varphi(K, \dot{K}, t)dt$. Applying the Euler equation $\varphi_K - \frac{d}{dt}\varphi_{\dot{K}} = 0$ results in the condition

$$F_K(K_t, L_t, t) = i(t). \tag{22}$$

Due to the concavity of $\varphi(K, \dot{K}, t)$ in the variables K and $\dot{K}$, we may apply Takayama's theorem to ascertain that (22) is a necessary and sufficient condition for a global maximum of W, provided that the transversality conditions at infinity are met. Those conditions are $\lim_{t\to\infty} \partial\varphi/\partial\dot{K} = 0$ and $\lim_{t\to\infty} \varphi(K, \dot{K}, t) = 0$. The first condition can be immediately checked: it implies $\lim_{t\to\infty} \partial\varphi/\partial\dot{K} = \lim_{t\to\infty}(-e^{-\int_0^t i(z)dz}) = 0$, always verified. The second condition is met as long as $\int_0^\infty \varphi(K, \dot{K}, t)dt$ converges, a property easily obtained due to the fast convergence of the exponential $e^{-\int_0^t i(z)dz}$. □

Proof of 2. As defined above, the value of society's activity is measured by the Dorfmanian $D(K, \dot{K}, t)$; using the constraint introduced before, it can be expressed as

$$\begin{aligned} D(K, \dot{K}, t) &= C_t e^{-\int_0^t i(z)dz} + \frac{d}{dt}[\lambda(t)K(t)] \\ &= [F(K_t, L_t, t) - \dot{K}_t]e^{-\int_0^t i(z)dz} + \lambda(t)\dot{K}_t + \dot{\lambda}(t)K_t. \quad (23) \end{aligned}$$

where $\lambda(t)$ is the price of one unit of capital at time t, in present value. Setting the gradient of D with respect to K and $\dot{K}$ to 0 gives

$$\frac{\partial D}{\partial K_t} = F_{K_t}(K_t, L_t, t)e^{-\int_0^t i(z)dz} + \dot{\lambda}(t) = 0 \quad (24)$$

and

$$\frac{\partial D}{\partial \dot{K}_t} = -e^{-\int_0^t i(z)dz} + \lambda(t) = 0; \quad (25)$$

eliminating $\lambda(t)$ yields $F_K(K_t, L_t, t) = i(t)$ which, together with the concavity of the Dorfmanian with respect to K and $\dot{K}$ gives a necessary and sufficient condition for a global maximum of D.

An alternative method woud have been to write a Dorfmanian written in terms of K, C, t, i.e. by considering consumption rather than investment as the control variable. Denoted $E(K, C, t)$, it is equal to

$$\begin{aligned} E(K, C, t) &= C_t e^{-\int_0^t i(z)dz} + \lambda(t)\dot{K}_t + \dot{\lambda}(t)K_t \\ &= C_t e^{-\int_0^t i(z)dz} + \lambda(t)[F(K_t, L_t, t) - C_t] + \dot{\lambda}(t)K_t. \quad (26) \end{aligned}$$

Setting to zero the gradient of E with respect to K and C leads to $F_K(K_t, L_t, t) = i(t)$ as well.

We still have to prove that $\lambda(t)$, given by equation (25) as equal to $e^{-\int_0^t i(z)dz}$, is indeed the present value of one unit of capital used at time t. This will be true if and only if at any time t the rate of increase of the *optimal* value of the functional W^* with respect to capital is equal to one. We have, from

$$W = \int_t^\infty \left[F(K_\tau, L_\tau, \tau) - \dot{K}(\tau)\right] e^{-\int_t^\tau i(z)dz} d\tau$$

$$\frac{\partial W_t}{\partial K_t} = \frac{\partial}{\partial K}\int_t^\infty [F(K_\tau, L_\tau, \tau) - \dot{K}_\tau] e^{-\int_t^\tau i(z)dz} d\tau$$
$$= \int_t^\infty F_K(K_\tau, L_\tau, \tau) e^{-\int_t^\tau i(z)dz} d\tau. \qquad (27)$$

Replacing in the last term of (27) $F_K(K_\tau, L_\tau, \tau)$ by $i(\tau)$ gives

$$\frac{\partial W^*}{\partial K_t} = \int_t^\infty i(\tau) e^{-\int_t^\tau i(z)dz} d\tau = \left[-e^{-\int_t^\tau i(z)dz}\right]_t^\infty = 1; \qquad (28)$$

hence $\lambda(t) = e^{-\int_0^t i(z)dz}$ is indeed the present value of $\partial W^*/\partial K_t$ and therefore the discounted price of one unit of capital set in use at time t, as was to be shown for the Dorfmanian to measure the value of society's activity. □

Proof of 3. Maximizing at any point of time a function $f(t)$ will generate a maximum of the integral $\int_0^\infty f(t)dt$ as long as the integral converges, which is the case here. We can verify that $F_K(K_t, L_t, t) = i(t)$ optimizes the total value of society's activity over $t \in [0, \infty)$ by maximising the indefinite integral of the Dorfmanian

$$\int_0^\infty D(K, \dot{K}, t)dt = \int_0^\infty \{C_t e^{-\int_0^t i(z)dz} dt + \frac{d}{dt}[\lambda(t)K(t)]\} dt$$

$$= \int_0^\infty \{[F(K_t, L_t, t) - \dot{K}_t] e^{-\int_0^t i(z)dz} + \lambda(t)\dot{K}_t + \dot{\lambda}(t)K_t\} dt. \qquad (29)$$

The Euler equation can be shown to be equal to

$$\frac{\partial D}{\partial K_t} - \frac{d}{dt}\frac{\partial D}{\partial \dot{K}_t} = e^{-\int_0^t i(z)dz}[F_K(K_t, L_t, t) - i(t)] = 0, \qquad (30)$$

leading to $F_K(K_t, L_t, t) = i(t)$.

An alternative approach would have been to integrate the Dorfmanian expressed as a function of the two arguments $K(t)$ and $C(t)$ as introduced above, denoted $E(K, C, t)$. Maximizing

$$\int_0^\infty E(K, C, t)dt = \int_0^\infty \{C_t e^{-\int_0^t i(z)dz} + \lambda(t)[F(K_t, L_t, t) - C_t] + \dot{\lambda}(t)K_t\} dt \qquad (31)$$

now implies solving the system of Euler equations corresponding to each function $K(t)$ and $C(t)$; these simplify to

$$\frac{\partial E}{\partial C_t} = e^{-\int_0^t i(z)dz} - \lambda(t) = 0 \tag{32}$$

$$\frac{\partial E}{\partial K_t} = \lambda(t) F_K(K_t, L_t, t) + \dot{\lambda}(t)] = 0, \tag{33}$$

giving after simplification $F_K(K_t, L_t, t) = i(t)$ as well. □

Proof of 4. When maximizing the value of society's activity at any point of time, we have determined the value of $\lambda(t)$ as $e^{-\int_0^t i(z)dz}$; replacing this value into the Dorfmanian expressed either as $D(K, \dot{K}, t)$ or as $E(K, C, t)$ gives the Dorfmanian evaluated at its maximum value, denoted D^* :

$$D^* = C_t^* e^{-\int_0^t i(z)dz} + \lambda(t)\dot{K}_t^* + \dot{\lambda}(t)K_t^*$$

$$= C_t^* e^{-\int_0^t i(z)dz} + e^{-\int_0^t i(z)dz}\dot{K}_t^* - i(t)e^{-\int_0^t i(z)dz}K_t^*$$

$$= e^{-\int_0^t i(z)dz}\left[C_t^* + \dot{K}_t^* - i(t)K_t^*\right] = e^{-\int_0^t i(z)dz}\left[F(K_t^*, L_t, t) - i(t)K_t^*\right] \tag{34}$$

Since $i(t)K_t$ is the remuneration of capital, the bracketed term is the remuneration of labor which has been maximized with D. □

Proof of 5. The maximization of the total remuneration of labor over $[0, \infty)$, the integral $\int_0^\infty e^{-\int_0^t i(z)dz}[F(K_t, L_t, t) - i(t)K(t)]dt$, follows immediately either from a differential or a variational argument as those used in the proof of 3. □

Taken individually, any of those five outcomes of competitive equilibrium admittedly constitute surprises. One of the most startling is that the equality $F_K(K_t, L_t, t) = i(t)$ not only maximizes intertemporally consumption as well as the value of society's activity but that it also maximizes the remuneration of labor and, additionally, that the last two quantities are equal.

We will now show that, surprising as this last equality may be, it perfectly squares with a basic principle of national accounting, namely that at any time t the total remuneration of factors must be equal to consumption plus investment. The Dorfmanian, denoted D^* at its maximal value, has just been shown to be equal to the present value of the remuneration of labor;

so the *current* value at time t of that remuneration is $e^{\int_0^t i(z)dz}\ D^*$; on the other hand the remuneration of capital is $i(t)K_t^*$. We must now verify that the sum of those factor payments is equal to consumption plus investment at their optimum values. We have indeed

$$i(t)K_t^* + e^{\int_0^t i(z)dz}\ D^*(C, K, \dot{K}, t)$$
$$= i(t)K_t^* + e^{\int_0^t i(z)dz}\left[C_t^* e^{-\int_0^t i(z)dz} + \tfrac{d}{dt}\left(\lambda_t^* K_t^*\right)\right]$$
$$= i(t)K_t^* + e^{\int_0^t i(z)dz}\left[C_t^* e^{-\int_0^t i(z)dz} + e^{-\int_0^t i(z)dz}\dot{K}_t^* - i(t)e^{-\int_0^t i(z)dz}K_t^*\right] = C_t^* + \dot{K}_t^* \tag{35}$$

as was to be ascertained.

4.2 The optimal evolution of the economy under competitive equilibrium.

We now want to assess the values taken by central variables of the economy, namely the investment-savings rate, the growth rate of real income per person and the capital-output ratio if we manage to save and invest in such a way as to maintain competitive equilibrium.

In a first approach, we assume constant growth rates for L_t, G_t and H_t, denoted n, g and h (in the next section we will assume very different time paths for those variables). We choose $n = 0.01$; for g, h and σ we took the estimates made by Sato for the U.S. economy over an 80-year time-span. Thus $\sigma = 0.8$; $h = 0.02$ and $g = 0.004$ as a first series of values for those parameters.

4.2.1 The optimal time path of the savings rate.

We are now in a position to undertake the comparative dynamics of the optimal savings rate, and answer in particular the nagging question asked by Frank Ramsey and certainly by anybody who would take up the subject of optimal growth: will technical progress increase or decrease the optimal savings rate? We will now use our central equation (6) not only, as we did before to determine the initial conditions prevailing in a competitive economy, but to study its whole time-path

$$s_t^* = \frac{\delta}{i}\left\{n + h + g\left[\frac{\sigma}{1-\delta e^{-g(1-\sigma)t}} - 1\right]\right\}e^{-g(1-\sigma)t}, \quad 0 \le \sigma \le 1. \tag{36}$$

Examination of (46) immediately reveals that s_t^* is an increasing function of the elasticity of substitution σ and a decreasing function of the rate of interest. It will also decrease through time for any given value of the parameters.

Those dependencies are very natural. For instance the property $\partial s_t^*/\partial\sigma > 0$ is easily understood if we think of σ as a powerful engine of growth; the reason is that income per person, as a general mean of order p ($p = 1 - 1/\sigma$), is an increasing function of its order and therefore of σ, with an inflection point close to $p = 0$, i.e. when σ is in the observed range, considered here $(0.5 < \sigma < 0.8)$[8].

Also, it would be disastrous if the sacrifice made by society through time in the form of its savings rate were increasing or constant despite technical progress. Tables 5 and 6 present first results for the values of the parameters indicated above.

TABLE 5. *The optimal savings rate $s^*(t,i)$ as a function of the rate of preference for the present, and as a slowly decreasing function of time;*

$\sigma = 0.8$

$n = .01$; $\delta = 0.25$; $g = .004$; $h = 0.02$

t \ i	0.04	0.045	0.05	0.055	0.06
0	18.9	16.8	15.1	13.8	12.6
30	18.5	16.4	14.8	13.4	12.3
60	18.0	16.0	14.4	13.1	12.0

$\sigma = 0.5$

The good news is that the optimal savings rate is always in very reasonable ranges. From (38) it can be seen that its welcome decrease through time is solely due to the presence of capital-augmenting technical. (If g were equal to zero, the optimal savings rate would remain at the constant $\frac{\delta}{i}(n+h)$). Also, whatever values of g and h, a value $\sigma = 1$ would make s^* remain constant, at level $(\delta/i)\left[n + h + g(\frac{\delta}{1-\delta})\right]$. However we should underline that, time and again, the elasticity of substitution has been observed as *lower than,* not

[8]In La Grandville (1989) we conjectured that the spectacular growth in East-Asian countries was due less to technical progress than a higher elasticity of substitution. The conjecture was successfully tested by Ky Hyang Yuhn(1991) in the case of South Korea. For the existence of a unique inflection point in the general mean, see the conjecture offered in La Grandville and Solow (2006); the proof is due to Thanh, Nam Phan and Mach Nguyet Minh (2008).

equal to one, and that, as we had observed in Section 3.1, $\sigma = 1$ constitutes the *upper limit* for which a competitive equilibrium can be sustained.

The positive dependency between s_t^* and the rate of labor-augmenting technical progress h is immediately established from (36), but an assesment of the effect of changing g on s_t^* cannot be easily made analytically due to the complexity of $\partial s_t^*/\partial g$.However, a clear pattern can be established with numerical representations. At any time t, whether an increase in g will impart an increase in s_t^* will depend on the value of the elasticity of substitution. There will always exist a value $\bar{\sigma}$ for which $\partial s_t^*/\partial g = 0$.For instance at $t = 0$, it can be seen from (36) that the coefficient multiplying g, equal to $\frac{\sigma\delta}{1-\delta} - (1-\sigma)$, reduces to zero for $\sigma = 1 - \delta$. above this critical value $\partial s_t^*/\partial g > 0$, and below $\partial s_t^*/\partial g < 0$.In our example, this value is $\bar{\sigma} = 0.75$. as time progresses, this value increases (for instance, with $t = 30$, $\bar{\sigma} = 0.82$). We can conclude that for the values of σ usually observed an increase in g will have an effect identical to that of h: to diminish s_t^*.

4.2.2 The optimal growth rate of income per person

From (13), the optimal growth rate of income per person is

$$\dot{y}_t^*/y_t^* = h + \sigma g \frac{\delta}{G^{(1-\sigma)} - \delta}, \qquad \sigma \leq 1. \tag{37}$$

It immediately appears that the growth rate $\dot{y}_t^*/y_t^*$, an increasing function of the elasticity of substitution, is higher than h and very slowly decreases asymptotically towards h, as illustrated in Table 7. (Notice once more that the ultimate growth rate of income per person may converge toward the rate of labor-augmenting technical progress even in the presence of capital-augmenting progress – this is due to the property of general means with negative order we mentioned earlier).

TABLE 7. *The optimal growth rate of income per person $r^*(t,i) = \dot{y}_t^*/y_t^*$ as a function of the elasticity of substitution*
$n = .01$; $\delta = 0.25$; $g = .004$; $h = 0.02$

t \ σ	0.5	0.55	0.6	0.65	0.7	0.75	0.8
0	2.07	2.07	2.08	2.09	2.09	2.10	2.11
30	2.06	2.07	2.08	2.08	2.08	2.09	2.10
60	2.06	2.06	2.07	2.08	2.08	2.09	2.10

4.2.3 The optimal time path of the capital-output ratio.

In a reassuring way, the capital-output ratio K^*/Y^*, determined from (10) and (11) as

$$K^*/Y^* = \frac{\delta}{i} e^{-(1-\sigma)gt}, \qquad \sigma \leq 1 \tag{38}$$

is a slowly decreasing function of time. It would be indeed bad news if this ratio were to stay constant (the case $\sigma = 1$, with $K^*/Y^* = \delta/i$), meaning that society would have to match any growth rate of its standard of living by the same growth of fixed capital; and it would be absurd news if, as seen above in the traditional approach (section 3.2 above and Figure 8), from a competitive equilibrium value the capital-output ratio were to increase six-fold whatever the $\alpha < 1$ value in the utility function, despite the presence of capital-augmenting technical progress! Here the ratio's rate of decline is $(1-\sigma)\,g$, depending positively on g and negatively on σ, which makes good economic sense.

TABLE 8. *The capital-output ratio K^*/Y^* as a function of time and the rate of preference for the present*; $n = .01$; $\delta = 1/3$; $\sigma = 0.8$; $h = .02$; $g = 0.004$.

	$\sigma = 0.5$			$\sigma = 0.8$		
t \ i	0.04	0.05	0.06	0.04	0.05	0.06
0	6.25	5.00	4.17	6.25	5	4.17
30	5.89	4.71	3.92	6.10	4.88	4.07
60	5.54	4.43	3.70	5.96	4.76	3.97

4.2.4 The optimal evolution of the labor share in competitive equilibrium.

From Section 4, we know that the remuneration of labor, equal to the value of society's activity, is maximised at any point of time, and therefore intertemporally. But what is the evolution of the share of labor through time? From (38), we can determine the share of capital as $iK^*/Y^* = \delta e^{-(1-\sigma)gt}, \quad \sigma \leq 1$. Therefore, the share of labor, denoted θ_t^*, is equal to

$$\theta_t^* = 1 - \delta e^{-(1-\sigma)gt}, \; 0 \leq \sigma \leq 1. \tag{39}$$

It can be seen that initially this share is independent of σ and g, and that it slowly increases asymptotically from $1-\delta$ towards 1. For instance, with $\delta = 0.25$, $\sigma = 0.5$ and $g = 0.04$, $\theta_0^* = 0.75$ and $\theta_{30}^* = 0.76$. In this 30-year time span, it can be calculated from (9) and (11) (and the same other parameters as in section 4.4) that the total remuneration of labor, $Y - iK$, has been multiplied by 2.56, implying an increase of the wage rate equal to 2.1 percent per year[9].

5. The robustness of the optimal savings rate. The normal impact of different scenarios.

A natural question to ask at this point is: what would be the impact on the optimal savings rate and other central variables of very different scenarios pertaining to the population evolution and to technical progress? In our 2011 paper, we had observed that those scenarios had little effect on the order of magnitude of s_t^*. For instance, we modeled the population evolution in such a way that its growth rate would ultimately decrease, population tending toward a plateau. However, we had made the hypothesis that the growth rates of the technical progress coefficients, while decreasing in time would still tend to a positive limit; for instance we supposed that in the limit h would tend toward $h = 1.3\%$. Acceptable for the medium time horizons as this hypothesis may be, it makes very little sense in the very long run because of the obvious unsustainability of exponentials. We should now consider the possibility that the economy could converge toward a stationary state, possibly corresponding to a very high income per person, and see what this implies.

To that effect we replace the exponentials by S-shaped functions with the following properties. Let $G(t)$ designate a generic function of time whose growth rate $g(t)$ is also a fonctiun of time. Suppose that $G(0) = G_0$, and that the growth rate $g(t)$, with an initial value g_0 (observed today) is decreasing at a rate $(1/g)dg/dt = \gamma$ ($\gamma < 0$). We thus have $g(t) = g_0 e^{\gamma t}$, $\gamma < 0$; this implies $(1/G)dG/dt = g(t) = g_0 e^{\gamma t}$ and therefore

$$G(t) = G_0 e^{\int_0^t g(z)dz} = G_0 \exp \int_0^t g_0 e^{\gamma z} dz = G_0 \exp[g_0(e^{\gamma t} - 1)/\gamma]. \qquad (40)$$

[9]Notice that the growth rate of the wage rate being slightly above the growth rate of income per person is due to the (positive) instantaneous growth rate of the labor share, equal to the difference between the growth rates of the wage rate and of income per person. With $\theta = (wL/Y) = w/y$, $\dot{\theta}/\theta = \dot{w}/w - \dot{y}/y$.

(Note from the last term that $\lim_{\gamma \to 0} G_0 \exp[g_0(e^{\gamma t} - 1)/\gamma] = G_0 \exp(g_0 t)$, as it should). As $t \to \infty$, $G(t)$ tends toward the asymptote[10] $G(\infty) = G_0 e^{-g_0/\gamma}$. Let A designate the asymptotic factor defined by the ratio $G(\infty)/G_0$; we have, setting $G_0 = 1$

$$A \equiv G(\infty) = e^{-g_0/\gamma}. \tag{41}$$

To visualize and model easily the S-shaped $G(t)$ curve, it is convenient to express it by reference to the asymptote A rather than γ, the (negative) growth rate of $g(t)$. From (41),

$$\gamma = -g_0/\ln A \tag{42}$$

and therefore

$$G(t, A) = A^{[1 - \exp(-g_0 t/\ln A)]}. \tag{43}$$

as a function of A. If $A > e$, $G(t, A)$ is S-shaped with an inflection point at $\hat{t} = (1/\gamma)\ln(-\gamma/g_0) = (1/g_0)\ln A \ln(\ln A)$. If $1 < A \leq e$, $G(t)$ is strictly concave throughout, with same asymptote $A = e^{-g_0/\gamma}$. Figure 9 illustrates the evolution of $G(t, A)$ for $g_0 = 0.01$ and $A = e, 5, 10, 20$ and ∞ (the last case corresponding to the exponential $e^{0.01t}$).

[10] Since $\gamma < 0, \lim_{\gamma \to 0} G_0 e^{-g_0/\gamma} = \infty$, as it should.

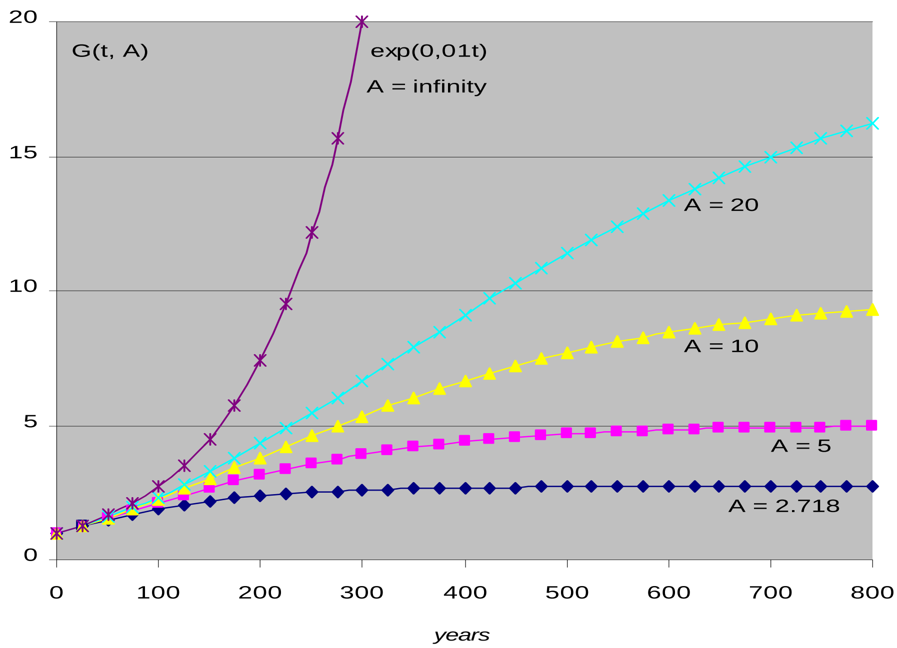

FIGURE 9. *The evolution of the* $G(t, A) = A^{[1-\exp(-g_0 t/\ln A)]}$ *function for various asymptotes* A; *initial growth rate:* $g_0 = 0.01$.

Consider now that evolutions of population $L(t)$ and factors reflecting technical progress $G(t)$ and $H(t)$ share the properties of that generic function. Their growth rates $n(t)$, $g(t)$ and $h(t)$ are declining at constant rates $\dot{n}(t)/n(t) = \nu < 0$, $\dot{g}(t)/g(t) = \gamma_K < 0$ and $\dot{h}(t)/h(t) = \gamma_L < 0$; their initial values are the observed n, g and h previously mentioned.

The competititive equilibrium condition $F_K(K_t, L_t, t) = i$ now implies new optimal trajectories: indeed, all equations (9) to (15) now incorporate the $S-$shaped curves $L(t)$, $H(t)$ and $G(t)$

$$L(t) = A_L^{[1-\exp(-n_0 t/\ln A_L)]} \tag{44}$$

$$H(t) = A_H^{[1-\exp(-h_0 t/\ln A_H)]} \tag{45}$$

$$G(t) = A_G^{[1-\exp(-g_0 t/\ln A_G)]}, \tag{46}$$

as well as their their growth rates

$$n(t) = n_0 \exp(-n_0 t / \ln A_L) \tag{47}$$

$$h(t) = h_0 \exp(-h_0 t / \ln A_H) \tag{48}$$

$$g(t) = g_0 \exp(-g_0 t / \ln A_G). \tag{49}$$

We now make a strong hypothesis: we suppose – arbitrarily – that $L(t)$, $H(t)$ and $G(t)$ will never exceed either 5 times or 10 times their initial value (we set all asymptotes A first to 5, and then to 10). We will compare the 30-year and 60-year horizon values in the exponential case (hypothesis 1) to those obtained in the entirely different setting given by the S-shaped curves described above (hypothesis 2). To that purpose, we choose as before the case of the U.S. economy for which we have estimates of h, g and σ, those made by Sato (2006) over an 80-year time-span. Thus $\sigma = 0.8$; also $h = .02$ and $g = 0.004$ will serve as the initial values h_0 and g_0 in the functions $h(t) = h_0 \exp(\gamma_L t)$ and $g(t) = g_0 \exp(\gamma_K t)$, γ_L, $\gamma_K < 0$; for the initial population growth rate n_0 we choose 0.01.

Before giving the results corresponding to those hypotheses, it may be useful to get a sense of the characteristics of the stationary state of an economy in competitive equilibrium when the population $L(t)$ and the technical progress coefficients $G(t)$ and $H(t)$ have asymptotic factors A_L, A_G, $A_{H.}$. Indeed, it seems very difficult to answer intuitively the following question: given those asymptotes, what could be the asymptotic factors of central variables of the economy ? For instance, if all A's were 10, would consumption per person in the stationary state have increased more than 10-fold or less, in in what proportion?

From equations (9) to (16) and (44) to (49) the asymptotic factor of consumption per person c^*_∞ / c^*_0 is

$$c^*_\infty / c^*_0 = \frac{(1 - s^*_\infty) y^*_\infty}{(1 - s^*_0) y^*_0} = \frac{A_H \left[\frac{1 - \delta A_G^{\sigma - 1}}{1 - \delta}\right]^{\sigma/(1-\sigma)}}{1 - \frac{\delta}{j}\left\{n_0 + h_0 + g_0 \left[\frac{\sigma\delta}{1-\delta} - (1-\sigma)\right]\right\}}; \tag{50}$$

The asymptotic factors of real income per person y^*_∞ / y^*_0 and of the capital-output ratio are

$$y^*_\infty / y^*_0 = A_H \left[\frac{1 - \delta A_G^{\sigma - 1}}{1 - \delta}\right]^{\sigma/(1-\sigma)} \tag{51}$$

and

$$(K^*_\infty/Y^*_\infty)/(K^*_0/Y^*_0) = A_G^{-(1-\sigma)}. \tag{52}$$

respectively. Table (9) gives the values taken by those growth factors when the stationary state is reached.

TABLE 9. *The asymptotic factors of consumption per person, income per person and capital coefficient after the economy has reached the stationary state*($i = 0.04$; $n_0 = .01$; $\delta = 0.25$; $g_0 = .02$; $h_0 = 0.004$).

	A_L, A_G, $A_H = 5$				A_L, A_G, $A_H = 10$			
σ	0.5	0.6	0.7	0.8	0.5	0.6	0.7	0.8
c^*_∞/c^*_0	7.2	7.6	8.1	8.8	15.0	16.1	17.6	19.6
y^*_∞/y^*_0	5.9	6.2	6.6	7.1	12.3	13.2	14.3	15.9
$(K^*_\infty/Y^*_\infty)/(K^*_0/Y^*_0)$	0.32	0.40	0.50	0.63	0.45	0.52	0.62	0.72

If all asympotic factors range between 5 and 10, when the stationary has been reached consumption per person will have increased by a factor between approximately 7 and 20, which makes good sense. Note that this factor is an increasing function of the elasticity of substitution, always an efficiency parameter. Note finally the welcome decrease through time of the capital coefficient, naturally more marked when the asymptotes A_G, A_H are lower.

Of great importance of course is what happens in what can be considered today as a medium horizon (30 years) and a long one (60 years). Table (10) gives the evolution of the optimal savings rate, under the hypothesis 1: exponential evolutions of L, G and H, and hypothesis 2: $S-$shaped evolutions.

TABLE 10. *The optimal savings rate $s^*(t,i)$ as a function of the rate of preference for the present, and as a slowly decreasing function of time;* $\sigma = 0.8$; $n_0 = .01$; $\delta = 0.25$; $g_0 = .004$; $h_0 = 0.02$.

i / t	0.04	0.045	0.05	0.055	0.06
0	18.9	16.8	15.1	13.8	12.6
Hypothesis 1: exponential evolutions of L, G and H					
30	18.5	16.4	14.8	13.4	12.3
60	18.0	16.0	14.4	13.1	12.0
Hyp. 2: S-shaped evolutions; toward a stationary state; $A = 5$					
30	13.6	12.1	10.9	9.9	9.1
60	10.0	8.9	7.9	7.2	6.6
Hyp. 2: S-shaped evolutions; toward a stationary state; $A = 10$					
30	14.9	13.2	11.9	10.8	9.9
60	11.8	14.5	9.4	8.6	7.9

The results exactly square with what we expect: when L, G and H are $S-$shaped the savings rates are lower than in the exponential case and, as in the other case, slowly decrease toward zero, thus driving the economy to the stationary state (note again that s^* is the *net* savings and investment rate, net of depreciation; in the stationary state, there still is gross savings and investment, equal to the depreciation of capital).

The same type of observations can be made regarding the evolution of the growth of real income per person, as evidenced in Table 11.

Table 11. *The optimal growth rate of income per person* $r^*(t,i) = \dot{y}_t^*/y_t^*$ *as a function of the elasticity of substitution* ($i = 0.04$; $n_0 = .01$; $\delta = 0.25$; $g_0 = .02$; $h_0 = 0.004$).

t \ σ	0.5	0.55	0.6	0.65	0.7	0.75	0.8
0	2.07	2.07	2.08	2.09	2.09	2.10	2.11
Hypothesis 1: exponential evolutions of L, G and H							
30	2.06	2.07	2.08	2.08	2.08	2.09	2.10
60	2.06	2.06	2.07	2.08	2.08	2.09	2.10
Hyp. 2: S-shaped evolutions of L, G and H; toward a stationary state; $A = 5$							
30	1.43	1.44	1.45	1.45	1.46	1.47	1.47
60	1.00	1.00	1.01	1.02	1.02	1.03	1.04
Hyp. 2: S-shaped evolutions of L, G and H; toward a stationary state; $A = 10$							
30	1.60	1.61	1.62	1.62	1.63	1.63	1.64
60	1.24	1.25	1.25	1.26	1.26	1.27	1.28

TABLE 12. *The optimal optimal capital-output ratio K^*/Y^* as a function of the elasticity of substitution* ($i = 0.04$; $n_0 = .01$; $\delta = 0.25$; $g_0 = .02$; $h_0 = 0.004$).

σ / t	0.5	0.55	0.6	0.65	0.7	0.75	0.8
0	6.25	6.25	6.25	6.25	6.25	6.25	6.25
Hypothesis 1: exponential evolutions of L, G and H							
30	5.89	5.92	5.96	5.99	6.03	6.07	6.10
60	5.54	5.61	5.68	5.75	5.82	5.89	5.96
Hyp. 2: S-shaped evolutions of L, G and H; toward a stationary state; $A = 5$							
30							
60	5.91	5.94	5.98	6.01	6.04	6.08	6.11
	5.63	5.69	5.75	5.81	5.87	5.94	6.00
Hyp. 2: S-shaped evolutions of L, G and H; toward a stationary state; $A = 10$							
30	5.90	5.94	5.97	6.01	6.04	6.07	6.11
60	5.61	5.67	5.73	5.79	5.86	5.92	5.99

While the optimal savings rate and the growth rate are given by expressions (49) and (48), the capital-output ratio is given by the much simpler expression $K^*/Y^* = \left(\frac{\delta}{i}\right) G(t)^{-(1-\sigma)}$, $\sigma \leq 1$; therefore we can expect hypothesis b to have little effect if the initial part of the $S-$shaped curve for G(t) differs little from that of the exponential; indeed, this is the case, and the capital-output coefficients in hypothesis b are within one decimal of those presented earlier in table

More generally, we can attribute this robustness of all results to two factors: first, although the evolutions of population and technical progress are dramatically different in the long run, they remain relatively close in the medium term (half a century); indeed, for $A = 5$ their *growth rates* diminish at rates $\nu = -n_0/\ln A = -0.62\%$ per year, $\gamma_K = -g_0/\ln A = -0.25\%$, and $\gamma_L = -h_0/\ln A = -1.24\%$; for $A = 10$, the rates are -0.43%, -0.17% and -0.87% respectively.

The second reason stems from the quick convergence of the indefinite integral of the discounted consumtion flows

$$W^*(i,\delta,\sigma,n,h,g) = \int_0^\infty C^*(t)e^{-it}dt = \int_0^\infty (1-s_t^*)Y_t^* e^{-it}dt =$$
$$\int_0^\infty \left\{1 - \tfrac{\delta}{i}e^{-g_t(1-\sigma)t}\left\{n_t + h_t + g_t\left[\tfrac{\sigma}{1-\delta G_t^{\sigma-1}} - 1\right]\right\}\right\}$$
$$\cdot \left\{L(t)H(t)\left[\tfrac{1-\delta G(t)^{\sigma-1}}{1-\delta}\right]^{\sigma/(1-\sigma)}\right\} e^{-it}dt, \quad 0<\sigma<1.$$

This integral does not have a closed form, but its value is easily determined numerically. It can also be proven to converge. Let $\hat{c}_t^* \equiv \dot{C}_t^*/C_t^*$ designate the growth rate of consumption. With parameters in the ranges considered here, it can be shown that the initial rate growth rate $\hat{c}_0^*$ is always lower than i, and that the rate $\hat{c}_t^*$ continuously decreases, tending asymptotically toward $n+h$ (this is confirmed by the property indicated in Section 3: $\lim_{t\to\infty} s_t^* = 0$ and $\lim_{t\to\infty} \dot{Y}_t^*/Y_t^* = n+h$). Therefore the integral will have an upper bound $W_+^* = \int_0^\infty C_0^* e^{\hat{c}_0^* t} e^{-it}dt = C_0^*/\left[i - \hat{c}_0^*\right]$.On the other hand, since consumption always rises, tending asymptotically toward A, the integral will be larger than $W_-^* = \int_0^\infty C_0^* e^{-it}dt = C_0^*/i$. Therefore it will converge to a value W^* between a lower bound $W_-^* = C_0^*/i$ and an upper bound $\bar{W}^* = C_0^*/\left[i-\hat{c}_0^*\right]$.We thus have

$$C_0^*/i < W^* < C_0^*/\left[i-\hat{c}_0^*\right]$$

(with C_0^* given by (16)) as a proof of convergence. Furthermore, the fact that the integrand is contained within negative exponentials is an indication that the convergence of the integral will be fast. Consider for instance horizons T_1 and T_2 corresponding to 50% and 99% of W^* respectively (T_1 and T_2 are defined by $\int_0^{T_1} C^*(t)e^{-it}dt = 0.5W^*$ and $\int_0^{T_2} C^*(t)e^{-it}dt = 0.99W^*$). If the parameters are $\sigma = 0.5$; $n_0 = .01$; $\delta = 0.25$; $g_0 = .02$; $h_0 = 0.004$, T_1 is as low as 26 years and $T_2 = 139$ years with $i = 0.05$; if $i = 0.05$, $T_1 = 37$ and $T_2 = 180$ years. This implies that the optimal paths will be little sensitive to quite different scenarios pertaining to the population evolution as well as to the evolution of technical progress.

7. Conclusion.

Extending the concept of a concave utility function from micro representations to macroeconomics was an intuitive, apparently defensible idea, but it led optimal growth theory into a blind alley, precluding any possibility of solving its central problem: simultaneously determining meaningful time paths for the optimal savings rate and for other central variables of an economy. Ever since Ramsey's first experiment, it has been repeatedly demonstrated that such a function, whatever extreme properties it was imparted with, led to at least one evolution of a fundamental variable that was either strongly contradictory to historical experience, or simply unacceptable by society.

For our part, we have on the one hand confirmed the serious warning signs sent to our profession in the writings of Ramsey and Goodwin, and most forcibly by King and Rebelo; in fact we confirmed what would be concluded by anyone who would care to solve numerically the differential equations implied by the theory. On the other hand we offered an explanation to those dire results: the traditional approach prevents competitive equilibrium to be sustained. In particular we showed that if the economy was initially in a state close to competititive equilibrium, any attempt to define an optimal investment time path along traditional lines inevitably led to a catastrophic evolution of the economy, marked by a permanent decrease in consumption accompanied by an inordinate accumulation of capital.

Our solution to the problem of optimal growth is then the following: first, rather than bending all consumption into a concave function as it has been done until now, we retain in consumption what can be considered as *welfare flows* for society. This approach leads in a natural way to the following objective, probably conforming to the desires of most individuals: maximizing the sum of discounted *welfare* flows (contrast this with the traditional approach: imposing a utility function on every individual, with the *certainty* that it will lead to unwanted time paths for the economy). Then define with i the rate of preference of society for the present, that naturally incorporates a risk premium. We believe it will definitely be easier to obtain a consensus on such a rate than on some utility function, even if society is completely unaware of the impracticability of such functions. That rate could be an average of historically observed real rates of return on capital. Then, as a rule, saving and investment decisions should conform to the equation of competitive equilibrium $i = F_K(K, L, t)$. This is the Euler equation for the maximization just defined; with the general, historically observed hypotheses of the neoclassical model, the equation will always have a solution $K_t^* = F_K^{-1}(i, L, t)$, leading to a meaningful saving-investment rate $s_t^* = \dot{K}_t^*/F(K_t^*, L_t, t)$.

This proposal offers three advantages:

1. The time path K_t^* is optimal in more than one way: in addition to minimizing production costs, it maximizes intertemporally the following

magnitudes: the sum of discounted consumption flows; the total value of society's activity, equal to the sum of consumption and the increase in the value of capital; and, finally, the total remuneration of labor, shown to be equal to this sum.

2. K_t^* always leads to reasonable time paths of the economy. In addition, the optimal savings rate and the capital-output ratio – that reflect a sacrifice made by society – both exhibit the most welcome feature of being slowly decreasing over time.

3. All implied time paths are extremely robust to variations in the parameters of the model, as well as to highly different predictions regarding the future evolutions of population and technical progress. Even drastic predictions – for instance assuming that the variables reflecting those evolutions will soon reach a plateau – are unable to make central variables of the economy deviate from reasonable, predictable ranges.

In his introduction to the 1975 edition of Adam Smith's magnum opus, William Letwin wrote: "Far from being a hymn in praise of anarchic greed, the ´Wealth of Nations´ is a reasoned argument for justice, order, liberty and *prudent plenty*" (our italics). It is definitely arguable that with optimal growth theory we are looking for rules enabling society to achieve this last objective. It is our hope that the numbers suggested here, based on competitive equilibrium with its associate optima, contribute to that rightful purpose.

Appendix 1: Two methods for obtaining the Ramsey optimal savings-investment rate rate.

$$S^* = \dot{K}^* = \frac{B - [U(C) - V(L)]}{U'(C)}. \tag{53}$$

The first method for obtaining the optimal saving rate in the Ramsey model, suggested by Ramsey himself, does not take the most direct route; however, it is highly interesting in its own sake because it is based upon two Euler equations that have a direct economic interpretation. The second one is much simpler and relies on the Beltrami equation; it can also be directly interpreted along economic lines. We present both.

1. First method; Ramsey's own derivation.

Remember that Ramsey wanted to find the optimal paths of capital and labor minimizing the functional

$$W(K,L)=\int_0^\infty [B-U(C)+V(L)]\,dt. \tag{54}$$

subject to the constraint

$$C=F(K,L)-\dot{K}; \tag{55}$$

where B denotes "bliss", defined as the saturation level of the utility function; V is the disutility function of labor. This amounts to minimizing

$$W(K,L)=\int_0^\infty \left\{B-U\left[F(K,L)-\dot{K}\right]+V(L)\right\}dt. \tag{56}$$

We denote this functional $W(K,L)$ to underline the fact that, contrary to the usual models of economic growth, labor is not an exogenous function of time: here it is a function that needs to be optimally chosen. Since we have two functions of time to be determined, we need to apply the result shown in Chapter 7, Section 4: denoting $J(K,\dot{K},L)$ the integrand of the above functional (56), a first-order condition for minimizing $W(K,L)$ is that $K(t)$ and $L(t)$ solve the system of the two following Euler equations:

$$\frac{\partial G}{\partial K}-\frac{d}{dt}\frac{\partial G}{\partial \dot{K}}=0 \tag{57}$$

and

$$\frac{\partial G}{\partial L}-\frac{d}{dt}\frac{\partial G}{\partial \dot{L}}=0. \tag{58}$$

These equations lead to

$$\dot{U}'(C)=-U'(C)F_K(K,L) \tag{59}$$

and

$$U'(C)F_L(K,L)=V'(L) \tag{60}$$

respectively. Each could have been arrived at through the following economic reasoning. Consider first (59). An optimal path for $C(t)$ must be such that the sacrifice generated by a small amount of consumption dC and transformed into capital $\Delta K=dC\Delta t$, and thus postponed by Δt, is exactly compensated by the rewards at time $t+\Delta t$; those rewards are the additional output harvested thanks to ΔK plus the benefit of transforming back capital into consumption at time $t+\Delta t$. We must have

$$\text{marginal sacrifice} = \text{marginal rewards}$$

$$U'_C(t)dC = U'_C(t+\Delta t)F_K(t+\Delta t)dC\Delta t + U'_C(t+\Delta t)dC. \qquad (61)$$

Dividing (61) by $dC\Delta t$ and rearranging:

$$\frac{U'_C(t+\Delta t) - U'_C(t)}{\Delta t} = -U'_C(t+\Delta t)F_K(t+\Delta t) \qquad (62)$$

Taking the limit of (62) when $t \to 0$ yields (59). Notice that this reasoning and the resulting equation is just the particular case of what we did in Chapter ... Section... where we had considered that society had a positive rate of preference for the present $i(t)$ (while here, in this Ramsey model, $i(t)$ is identically zero for all t). If we had considered a positive rate of interest, we would have written the equality between sacrifice and rewards either in present value (by dividing the right hand side by $1+i(t)$) or in future value (by multiplying the left-hand side by by $1+i(t)$). Whatever our approach, in the limit we would have obtained the traditional equation $i(t) = F(K) + \dot{U}'(C)/U'(C)$ whose particular case corresponding to $i(t) = 0$ is just (59).

Equation (60) is even more quicker to derive: indeed, if an optimal time path for labor is attained, it must be such that the sacrifice made by working an additional hour, $V'(L)$, is exactly matched by the compensating rewards, equal to $U'(C)F'(L)$; this is just equation (60).

Deriving now $S^* = \dot{K}^* = \{B - [U(C) - V(L)]\}/U'(C)$ from $(55), (59)$ and (60) took from Ramsey considerable ingenuity. It goes through the derivation of an expression apparently not directly linked to the final result, followed by the integration of the same expression.

Here is Ramsey's remarkable demonstration. Take the derivative with respect to C of the product $U'(C).F[K(C), L(C)]$ where both K and L are considered as functions of C:

$$\frac{d}{dC}U'(C)F(K,L)] = U''(C)F(K,L) + U'(C)[F_K\frac{dK}{dC} + F_L\frac{dL}{dC}]$$

$$= U''(C)F(K,L) + U'(C)F_K\frac{dK}{dC} + U'(C)F_L\frac{dL}{dC}. \qquad (63)$$

In the right-hand side of (63) multiply the second term by dt/dt, and in the third term use (60) to replace $U'(C)F_L$ by $V'(L)$, we get

$$\frac{d}{dC}U'(C)F(K,L)] = U''(C)F(K,L) + U'(C)F_K\frac{dK}{dt}\frac{dt}{dC} + V'(L)\frac{dL}{dC} = \qquad (64)$$

Use the constraint (55) to replace dK/dt by $F(K,L)-C$; from (59), write $U'(C)F_K = -U''(C)dC/dt$ to finally obtain

$$\frac{d}{dC}U'(C)F(K,L)] = U''(C)F(K,L)+U'(C)F_K\left[F(K,L)-C\right]\frac{dt}{dC}+V'(L)\frac{dL}{dC}$$

$$= U''(C)F(K,L) - U''(C)\frac{dC}{dt}\left[F(K,L)-C\right]\frac{dt}{dC} + V'(L)\frac{dL}{dC}$$

$$= CU''(C) + V'(L)\frac{dL}{dC}. \qquad (65)$$

Integrating (65) – by parts $CU''(C)$ – yields

$$U'(C)F(K,L)] = CU'(C) - U(C) + \ V(L) + \beta \qquad (66)$$

where β is a constant of integration which can be identified as follows: when time tends to infinity, $U(C)-V(L)$ tends to bliss (denoted B); on the other hand $\lim_{t\to\infty} U'(C) = 0$, and we may suppose that $\lim_{t\to\infty} F(K,L)$ is finite; thus $\beta = B$ and we have

$$U'(C)\left[F(K,L)-C\right] = U'(C)\dot{K} = B - U(C) + \ V(L); \qquad (67)$$

finally yielding

$$S^* = \dot{K}^* = \frac{B - [U(C) - V(L)]}{U'(C)}. \qquad (68)$$

2. Second method: using the Beltrami equation.

It is quite possible that Ramsey was not aware of the Beltrami equation, because it would have almost immediately led him to his final result. Indeed, we can notice that Ramsey's functional $\int_0^\infty \left\{B - U\left[F(K,L)-\dot{K}\right] + V(L)\right\}dt$ does not depend explicitely upon the variable t; therefore we can apply the Beltrami equation we had seen in Chapter 7, which transforms the second-order Euler equation into a first-order one.

If the integrand of the functional is denoted $J(K,\dot{K},L)$, the Beltrami equation is

$$J(K,\dot{K},L) - \dot{K}J_{\dot{K}}(K,\dot{K},L) = M, \qquad (69)$$

where M is a constant of integration to be identified. Applied to our problem the Beltrami equation gives

$$B - [U(C) - V(L)] - \dot{K}U'(C) = M. \qquad (70)$$

The constant of integration M can be identified as follows: as $t \to \infty$, $U(C) - V(L)$ tends to the bliss level B; also, $U'(C)$ tends to zero. Therefore, we can identify M as zero and (70) becomes

$$B - [U(C) - V(L)] - \dot{K}U'(C) = 0, \tag{71}$$

yielding immediately the Ramsey equation (68).

A final question is: does this equation allow a direct economic interpretation? It does: write (68) as

$$U'(C) = \frac{B - [U(C) - V(L)]}{S^*} = \frac{B - [U(C) - V(L)]}{\dot{K}^*} \tag{72}$$

The numerator of the fraction on the right-hand side is, at any point of time, the sacrifice borne by society, measured by that part of bliss not achieved (the difference between bliss and net utility); so at any point of time the sacrifice per unit of investment or per unit of savings should be equal to the marginal utility of consumption.

Appendix 2. Derivation of the optimal time path of capital in competitive equilibrium.

The production function is the general mean of order p of the enhanced inputs G_tK_t and H_tL_t :

$$Y_t = F(G_tK_t, H_tL_t) = Y_0\{\delta[G_tK_t/K_0]^p + (1-\delta)[H_tL_t/L_0]^p\}^{1/p} \ , \ p \neq 0 \tag{73}$$

where the order p is the increasing function of the elasticity of substitution σ: $p = 1 - 1/\sigma$. The fundamental competitive equilibrium equality $F_{K_t} = i$ leads to the following equation in K_t:

$$F_{K_t}(G_tK_t, H_tL_t) = Y_0\{\delta[G_tK_t/K_0]^p + (1-\delta)[H_tL_t/L_0]^p\}^{(1-p)/p} \ . \\ .\delta {K_t}^{p-1}(G_t/K_0)^p = i, \quad p < 0, \ 0 \leq \sigma < 1 \tag{74}$$

which can be solved to yield the optimal time path K_t^*. For convenience denote $\delta[G_t/K_0]^p \equiv a$ and $(1-\delta)[H_tL_t/L_0]^p \equiv b$.Also, write ${K_t}^{p-1}$ as ${K_t}^{p-1\frac{1-p}{p}\frac{p}{1-p}} = {K_t}^{-p\frac{1-p}{p}}$.Therefore (73) simplifies to

$$Y_0\{a{K_t}^p + b\}^{(1-p)/p} \ .a{K_t}^{-p\frac{1-p}{p}} = aY_0\{a + b{K_t}^{-p}\}^{\frac{1-p}{p}} \ = i$$

from which

$$K_t^* = \left[\frac{b}{\left(\frac{i}{aY_0}\right)^{p/(1-p)} - a}\right]^{1/p} = \left[\frac{b/a}{\frac{1}{a}\left(\frac{i}{aY_0}\right)^{p/(1-p)} - 1}\right]^{1/p}; \tag{75}$$

Reverting to $a = \delta[G_t/K_0]^p$ and $b \equiv (1-\delta)[H_tL_t/L_0]^p$ yields

$$K_t^* = \frac{K_0}{L_0}\left(\frac{1-\delta}{\delta}\right)^{\sigma/(\sigma-1)} \frac{L_tH_tG_t^{-1}}{\left[i^{\sigma-1}\delta^{-\sigma}\left(Y_0/K_0\right)^{1-\sigma} G_t^{1-\sigma} - 1\right]^{\sigma/(\sigma-1)}}, \quad 0 \leq \sigma < 1. \tag{76}$$

K_0 and Y_0 are identified by setting $t = 0$ in (75); we obtain $K_0/Y_0 = \delta/i$. We now can normalize Y_0 to one; thus $K_0 = \delta/i$; finally, the optimal time path of capital is

$$K_t^* = \frac{\delta}{i}\left(\frac{1-\delta}{G_t^{1-\sigma} - \delta}\right)^{\sigma/(\sigma-1)} L_tH_tG_t^{-1}, \qquad 0 \leq \sigma < 1. \tag{77}$$

The optimal trajectory of output and income Y_t^* follows from replacing (76) into (72), using the same identifications. We obtain

$$Y_t^* = L_tH_t\left[\delta\left(\frac{1-\delta}{G_t^{1-\sigma} - \delta}\right) + 1 - \delta\right]^{\sigma/(\sigma-1)} = L_tH_t\left(\frac{1-\delta G_t^{\sigma-1}}{1-\delta}\right)^{\sigma/(1-\sigma)}, \ 0 \leq \sigma < 1. \tag{78}$$

Appendix 3. Derivation of the optimal time path of capital-output ratio in competitive equilibrium.

To determine the optimal time path of the capital-output ratio, it is easier to use the first part of equation (77). Denoting $\frac{1-\delta}{G_t^{1-\sigma}-\delta} \equiv m$, we have from (76) and (77)

$$\frac{K_t^*}{Y_t^*} = \frac{\delta}{i}\left(\frac{1-\delta}{G_t^{1-\sigma} - \delta}\right)^{\sigma/(\sigma-1)} L_tH_tG_t^{-1}/\left\{L_tH_t\left[\delta\left(\frac{1-\delta}{G_t^{1-\sigma} - \delta}\right) + 1 - \delta\right]^{\sigma/(\sigma-1)}\right\} =$$

$$\frac{\delta}{i}G_t^{-1}\left[\frac{m}{\delta m + 1 - \delta}\right]^{\sigma/(\sigma-1)} = \frac{\delta}{i}G_t^{-1}\left[\frac{1}{\delta + (1-\delta)m^{-1}}\right]^{\sigma/(\sigma-1)} =$$

$$\frac{\delta}{i}G_t^{-1}\left[\frac{1}{\delta+(1-\delta)\frac{G_t^{1-\sigma}-\delta}{1-\delta}}\right]^{\sigma/(\sigma-1)}=\frac{\delta}{i}G_t^{-(1-\sigma)}. \tag{79}$$

References

Alpha C. Chiang, *Elements of Dynamic Optimization*, McGraw-Hill International Series, 1992.

Dorfman, Robert (1969), "An Economic Interpretation of Optimal Control Theory", *American Economic Review*, December 1969, Vol. 59, No 5, pp. 817-831.

Goodwin, Richard M. (1961), "The Optimal Growth Path for an Underdeveloped Economy", *The Economic Journal,*, 71 (284), pp. 756-774.

Intriligator, M.D (1971), *Mathematical Optimisation and economic theory, Prentice hall,* pp. 413 ff.

Johnston, Louis and Samuel H. Williamson (2013). "What Was the U.S. GDP Then?", http://www.measuringworth.com/datasets/usgdp/result.php

Kamikigashi, T. and S. Roy (2006), Dynamic Optimisation with a Nonsmooth, Non-convex technology: the Case of a Linear Objective Function, *Journal of Economic Theory,* 29, 325-340.

King, Robert G. and Rebelo, Sergio T. (1993), "Transitional Dynamics and Economic Growth in the Neoclassical Model", *The American Economic review,* Vol. 83, No 4. pp. 908-931.

La Grandville, Olivier de (1989), "In Quest of the Slutsky Diamond" , *The American Economic Review,* Vol. 79, No 3, pp. 468-481.

—– (2009), *Economic Growth - A Unified Approach,* with two special contributions by Robert M. Solow, Cambridge University Press, Cambridge.

—— (2011), "A New Property of General Means of Order p, with Applications to Economic Growth", *The Australian Journal of Mathematical Analysis with Applications*, Volume 8, Issue 1, Article 3.

—— (2012), "How Much Should a Nation Save? A New Answer", *Studies in Non-linear Dynamics and Econometrics*", Vol. 16, no 2, pp. 1–35.

—— (2014), "Optimal Growth Theory: Challenging Problems and Suggested Solutions, *Economic Modelling*, 36, pp. 608-611.

—— and Robert M. Solow (2006), "A Conjecture on General Means", *Journal of Inequalities in Pure and Applied Mathematics,* Volume 7, No 1, Article 3.

Ramsey, Frank (1928), "A Mathematical Theory of Saving", *The Economic Journal,* Vol. 38, No. 152 , pp. 543-559.

Sato, Ryuzo (2006), *Biased Technical Change and Economic Conservation Laws,* Springer, New York.

Thanh, Nam Phan and Mach Nguyet Minh (2008), "Proof of a Conjecture on General Means", *Journal of Inequalities in Pure and Applied Mathematics,*Volume 9, No 3, Article 86, 2008.

Yuhn, Ky Hyang (1991), "Economic Growth, Technical Change Biases, and the Elasticity of Substitution: a Test of the de La Grandville Hypothesis", *The Review of Economics and Statistics,* LXIII, No 2, 1991, p. 340-6.