A Appendices

A.1 Detailed Properties of IEES Functions

A.1.1 IEES(Π)

In the following we shall consider four specific cases of IEES(Π) functions, delineated by the assumptions made with respect to ψ and σ_0.

Case with $\psi > 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k. Since also the elasticity of substitution increases with the capital share, it follows that the elasticity of substitution increases with k as well. The production function is well-defined, increasing and concave in its entire domain $k \in [0, +\infty)$. We obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}} > 0, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \tag{A.1}$$

$$\lim_{k\to 0} \varphi(k) = 0, \qquad \lim_{k\to\infty} \varphi(k) = \varphi_0 \left(\frac{\sigma_0}{\sigma_0 - 1}\right)^{\frac{1}{\psi}} < 0, \tag{A.2}$$

$$\lim_{k\to 0} \sigma(k) = 1, \qquad \lim_{k\to\infty} \sigma(k) = +\infty. \tag{A.3}$$

Case with $\psi < 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k. Since the elasticity of substitution, on the other hand, decreases with the capital share, it follows that the elasticity of substitution decreases with k as well. The production function is well-defined, increasing and concave in its entire domain $k \in [0, +\infty)$. We obtain the following limits:

$$\lim_{k\to 0}\frac{\pi(k)}{1-\pi(k)}=0, \qquad \lim_{k\to\infty}\frac{\pi(k)}{1-\pi(k)}=\frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}}>0, \tag{A.4}$$

$$\lim_{k\to 0}\varphi(k)=\varphi_0\left(\frac{\sigma_0}{\sigma_0-1}\right)^{\frac{1}{\psi}}<0, \qquad \lim_{k\to\infty}\varphi(k)=-\infty, \tag{A.5}$$

$$\lim_{k\to 0}\sigma(k)=+\infty, \qquad \lim_{k\to\infty}\sigma(k)=1. \tag{A.6}$$

Case with $\psi > 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio k. Since the elasticity of substitution, on the other hand, increases with the capital share, it follows that the elasticity of substitution falls with k. The production function is well-defined, increasing and concave only for $k \in [0, k_{max}]$, where $k_{max} = k_0(1-\sigma_0)^{-1/\psi}$. We obtain the following limits:

$$\lim_{k\to 0}\frac{\pi(k)}{1-\pi(k)}=\frac{\pi_0}{1-\pi_0}\sigma_0^{-\frac{1}{\psi}}>0, \qquad \lim_{k\to k_{max}}\frac{\pi(k)}{1-\pi(k)}=0, \tag{A.7}$$

$$\lim_{k\to 0}\varphi(k)=0, \qquad \lim_{k\to k_{max}}\varphi(k)=-\infty, \tag{A.8}$$

$$\lim_{k\to 0}\sigma(k)=1, \qquad \lim_{k\to k_{max}}\sigma(k)=0. \tag{A.9}$$

Case with $\psi < 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio k. Since also the elasticity of substitution is inversely related to the capital share, it follows that the elasticity of substitution increases with k. The production function is well-defined, increasing and concave only for $k \in [k_{min}, +\infty)$, where $k_{min} = k_0(1-$

$\sigma_0)^{-1/\psi}$. We obtain the following limits:

$$\lim_{k \to k_{min}} \frac{\pi(k)}{1-\pi(k)} = +\infty, \qquad \lim_{k \to \infty} \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0} \sigma_0^{-\frac{1}{\psi}} > 0, \tag{A.10}$$

$$\lim_{k \to k_{min}} \varphi(k) = 0, \qquad \lim_{k \to \infty} \varphi(k) = -\infty, \tag{A.11}$$

$$\lim_{k \to k_{min}} \sigma(k) = 0, \qquad \lim_{k \to \infty} \sigma(k) = 1. \tag{A.12}$$

Our empirical analysis (Section 8) suggests that this case of IEES(Π) functions is preferred by the data on aggregate production in the post-war US economy.

A.1.2 IEES(MRS)

In the following we shall consider two specific cases of IEES(MRS) functions, delineated by the assumptions made with respect to ψ.

Case with $\psi > 0$. In this case, the elasticity of substitution decreases with the marginal rate of substitution ($\varphi_0 < 0$) and thus increases with the factor ratio k (recall that by concavity and constant returns to scale, the MRS necessarily decreases with k). The production function is well-defined, increasing and concave only for $k \in [k_{min}, +\infty)$, where $k_{min} = k_0 e^{-\sigma_0/\psi}$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, declining if $k \in (k_{min}, \tilde{k})$ and increasing for $k > \tilde{k}$. The minimum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{min} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0 - 1}{\psi}} \sigma_0^{\frac{1}{\psi}}. \tag{A.13}$$

We also obtain the following limits:

$$\lim_{k\to k_{min}} \frac{\pi(k)}{1-\pi(k)} = +\infty, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \tag{A.14}$$

$$\lim_{k\to k_{min}} \varphi(k) = 0, \qquad \lim_{k\to\infty} \varphi(k) = -\infty, \tag{A.15}$$

$$\lim_{k\to k_{min}} \sigma(k) = 0, \qquad \lim_{k\to\infty} \sigma(k) = +\infty. \tag{A.16}$$

Our empirical analysis (Section 8) suggests that this case of IEES(MRS) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, the elasticity of substitution increases with the marginal rate of substitution and thus falls with the factor ratio k. The production function is well-defined, increasing and concave only for $k \in [0, k_{max}]$, where $k_{max} = k_0 e^{-\sigma_0/\psi}$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, increasing when $k \in (0, \tilde{k})$ and falling for $k \in (\tilde{k}, k_{max})$. The maximum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{max} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0-1}{\psi}} \sigma_0^{\frac{1}{\psi}}. \tag{A.17}$$

We also obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = 0, \qquad \lim_{k\to k_{max}} \frac{\pi(k)}{1-\pi(k)} = 0, \tag{A.18}$$

$$\lim_{k\to 0} \varphi(k) = 0, \qquad \lim_{k\to k_{max}} \varphi(k) = -\infty, \tag{A.19}$$

$$\lim_{k\to 0} \sigma(k) = +\infty, \qquad \lim_{k\to k_{max}} \sigma(k) = 0. \tag{A.20}$$

A.1.3 IEES(k)

In the following we shall consider two specific cases of IEES(k) functions, delineated by the assumptions made with respect to ψ.

Case with $\psi > 0$. In this case, we assume that the elasticity of substitution increases with the factor ratio k. The production function is well-defined, increasing and concave in its domain $k \in [0, +\infty)$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, declining if $k \in (0, \tilde{k})$ and increasing for $k > \tilde{k}$. The minimum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{min} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0 - 1}{\psi \sigma_0}} \sigma_0^{-\frac{1}{\psi}}. \tag{A.21}$$

We also obtain the following limits:

$$\lim_{k \to 0} \frac{\pi(k)}{1-\pi(k)} = +\infty, \qquad \lim_{k \to \infty} \frac{\pi(k)}{1-\pi(k)} = +\infty, \tag{A.22}$$

$$\lim_{k \to 0} \varphi(k) = 0, \qquad \lim_{k \to \infty} \varphi(k) = \varphi_0 e^{\frac{1}{\psi \sigma_0}} < 0, \tag{A.23}$$

$$\lim_{k \to 0} \sigma(k) = 0, \qquad \lim_{k \to \infty} \sigma(k) = +\infty. \tag{A.24}$$

Our empirical analysis (Section 8) suggests that this case of IEES(k) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, we assume that the elasticity of substitution decreases with the factor ratio k. The production function is well-defined, increasing and concave in its domain $k \in [0, +\infty)$. The relative factor share $\frac{\pi(k)}{1-\pi(k)}$ (and thus the capital's share $\pi(k)$ as well) follows a non-monotonic pattern with k, increasing when $k \in (0, \tilde{k})$ and falling for $k > \tilde{k}$. The maximum capital share, obtained at the point $\tilde{k}$, is equal to:

$$\left(\frac{\pi}{1-\pi}\right)_{max} = \frac{\pi(\tilde{k})}{1-\pi(\tilde{k})} = \frac{\pi_0}{1-\pi_0} e^{-\frac{\sigma_0 - 1}{\psi\sigma_0}} \sigma_0^{-\frac{1}{\psi}}. \tag{A.25}$$

We also obtain the following limits:

$$\lim_{k\to 0} \frac{\pi(k)}{1-\pi(k)} = 0, \qquad \lim_{k\to\infty} \frac{\pi(k)}{1-\pi(k)} = 0, \tag{A.26}$$

$$\lim_{k\to 0} \varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0}} < 0, \qquad \lim_{k\to\infty} \varphi(k) = -\infty, \tag{A.27}$$

$$\lim_{k\to 0} \sigma(k) = +\infty, \qquad \lim_{k\to\infty} \sigma(k) = 0. \tag{A.28}$$

A.1.4 IEES(π)

In the following we shall consider four specific cases of IEES(π) functions, delineated by the assumptions made with respect to ψ and σ_0.

Case with $\psi > 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with κ, and consequently also the capital–labor ratio k. Since also the elasticity of substitution increases with the capital share, it follows that the elasticity of substitution increases with k as well. The production function is well-defined, increasing and concave for $\kappa \in [0, \kappa_{max}]$, with $\kappa_{max} = \kappa_0 \left(\frac{\sigma_0 \pi_0^{-\psi} - 1}{\sigma_0 - 1}\right)^{\frac{1}{\psi}}$. We obtain the following limits:

$$\lim_{\kappa\to 0} \pi(\kappa) = \pi_0 \sigma_0^{-\frac{1}{\psi}} < 1, \qquad \lim_{\kappa\to\kappa_{max}} \pi(\kappa) = 1, \tag{A.29}$$

$$\lim_{\kappa\to 0} \sigma(\kappa) = 1, \qquad \lim_{\kappa\to\kappa_{max}} \sigma(\kappa) = \sigma_0 \pi_0^{-\psi} > 1. \tag{A.30}$$

Case with $\psi < 0$ and $\sigma_0 > 1$. In this case, factors of production are always gross substitutes and hence the capital share increases with the capital–labor ratio k. Since the elasticity of substitution, on the other hand, decreases with the capital share, it follows that the elasticity of substitution decreases with k as well. The production function is again well-defined, increasing and concave in its domain.

There are however two subcases, depending on the choice of normalization. If $\pi_0 \sigma_0^{-\frac{1}{\psi}} \leq 1$, then the domain is $\kappa \in [0, +\infty)$ and we obtain the following limits:

$$\lim_{\kappa \to 0} \pi(\kappa) = 0, \qquad \lim_{\kappa \to +\infty} \pi(\kappa) = \pi_0 \sigma_0^{-\frac{1}{\psi}} \leq 1, \tag{A.31}$$

$$\lim_{\kappa \to 0} \sigma(\kappa) = +\infty, \qquad \lim_{\kappa \to +\infty} \sigma(\kappa) = 1. \tag{A.32}$$

If $\pi_0 \sigma_0^{-\frac{1}{\psi}} > 1$ then the domain is $\kappa \in [0, \kappa_{max}]$, with $\kappa_{max} = \kappa_0 \left(\frac{\sigma_0 \pi_0^{-\psi} - 1}{\sigma_0 - 1} \right)^{\frac{1}{\psi}}$. In such a case,

$$\lim_{\kappa \to 0} \pi(\kappa) = 0, \qquad \lim_{\kappa \to \kappa_{max}} \pi(\kappa) = 1, \tag{A.33}$$

$$\lim_{\kappa \to 0} \sigma(\kappa) = +\infty, \qquad \lim_{\kappa \to \kappa_{max}} \sigma(\kappa) = \sigma_0 \pi_0^{-\psi} > 1. \tag{A.34}$$

Case with $\psi > 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio k. Since the elasticity of substitution, on the other hand, increases with the capital share, it follows that the elasticity of substitution falls with k. The production function is well-defined, increasing and concave in its domain.

There are again two subcases, depending on the choice of normalization. If $\pi_0 \sigma_0^{-\frac{1}{\psi}} \leq 1$, then the domain is $\kappa \in [0, \kappa_{max}]$, with $\kappa_{max} = \kappa_0 (1 - \sigma_0)^{-\frac{1}{\psi}}$ and we obtain the following limits:

$$\lim_{\kappa \to 0} \pi(\kappa) = \pi_0 \sigma_0^{-\frac{1}{\psi}} \leq 1, \qquad \lim_{\kappa \to \kappa_{max}} \pi(\kappa) = 0, \tag{A.35}$$

$$\lim_{\kappa \to 0} \sigma(\kappa) = 1, \qquad \lim_{\kappa \to \kappa_{max}} \sigma(\kappa) = 0. \tag{A.36}$$

If $\pi_0 \sigma_0^{-\frac{1}{\psi}} > 1$, then the domain is $\kappa \in [\kappa_{min}, \kappa_{max}]$, with $\kappa_{min} = \kappa_0 \left(\frac{\sigma_0 \pi_0^{-\psi} - 1}{\sigma_0 - 1} \right)^{\frac{1}{\psi}}$ and $\kappa_{max} = \kappa_0 (1 - \sigma_0)^{-\frac{1}{\psi}}$. In such a case,

$$\lim_{\kappa \to \kappa_{min}} \pi(\kappa) = 1, \qquad \lim_{\kappa \to \kappa_{max}} \pi(\kappa) = 0, \tag{A.37}$$

$$\lim_{\kappa \to \kappa_{min}} \sigma(\kappa) = \sigma_0 \pi_0^{-\psi} < 1, \qquad \lim_{\kappa \to \kappa_{max}} \sigma(\kappa) = 0. \tag{A.38}$$

Case with $\psi < 0$ and $\sigma_0 < 1$. In this case, factors of production are always gross complements and hence the capital share is inversely related to the capital–labor ratio k. Since also the elasticity of substitution is inversely related to the capital share, it follows that the elasticity of substitution increases with k. The production function is well-defined, increasing and concave for $\kappa \in [\kappa_{min}, +\infty)$, where $\kappa_{min} = \kappa_0 \left(\frac{\sigma_0 \pi_0^{-\psi} - 1}{\sigma_0 - 1} \right)^{\frac{1}{\psi}}$. We obtain the following limits:

$$\lim_{\kappa \to \kappa_{min}} \pi(\kappa) = 1, \qquad \lim_{\kappa \to \infty} \pi(\kappa) = \pi_0 \sigma_0^{-\frac{1}{\psi}} < 1, \tag{A.39}$$

$$\lim_{\kappa \to \kappa_{min}} \sigma(\kappa) = \sigma_0 \pi_0^{-\psi} < 1, \qquad \lim_{\kappa \to \infty} \sigma(\kappa) = 1. \tag{A.40}$$

Our empirical analysis (Section 8) suggests that this case of IEES(π) functions is preferred by the data on aggregate production in the post-war US economy.

A.1.5 IEES(r)

In the following we shall consider two specific cases of IEES(r) functions, delineated by the assumptions made with respect to ψ. We restrict ourselves to the choice of normalization constants where $\pi_0 e^{-\frac{\sigma_0-1}{\psi}} \sigma_0^{\frac{1}{\psi}} < 1$.

Case with $\psi > 0$. In this case, the elasticity of substitution decreases with the marginal product of capital ($r = f'(k)$) and thus – by concavity of f – increases with the capital–labor ratio k. The production function is well-defined, increasing and concave in its support. However, due to restrictions in the range of $F(K, L)$, the support of κ is restricted to $\kappa \in [\kappa_{min}, \kappa_{max}]$ where κ_{min} and κ_{max} are the two solutions to the equation $\pi(\kappa) = 1$. The capital's share $\pi(\kappa)$ follows a non-monotonic pattern with κ, declining if $\kappa \in (\kappa_{min}, \tilde{\kappa})$ and increasing for $\kappa \in (\tilde{\kappa}, \kappa_{max})$. The minimum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{min} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0-1}{\psi}} \sigma_0^{\frac{1}{\psi}} \in (0, 1). \tag{A.41}$$

Our empirical analysis (Section 8) suggests that this case of IEES(r) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, the elasticity of substitution increases with the marginal product of capital $f'(k)$ and thus falls with k. The production function is well-defined, increasing and concave for $\kappa \in [0, \kappa_{max}]$, where $\kappa_{max} = \kappa_0 e^{-\sigma_0/\psi}$. The capital share $\pi(\kappa)$ follows a non-monotonic pattern with κ, increasing when $\kappa \in (0, \tilde{\kappa})$ and falling for $\kappa \in (\tilde{\kappa}, \kappa_{max})$. The maximum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{max} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0 - 1}{\psi}} \sigma_0^{\frac{1}{\psi}} \in (0, 1). \tag{A.42}$$

We also obtain the following limits:

$$\lim_{\kappa \to 0} \pi(\kappa) = 0, \qquad \lim_{\kappa \to \kappa_{max}} \pi(\kappa) = 0. \tag{A.43}$$

$$\tag{A.44}$$

A.1.6 IEES(k/y)

In the following we shall consider two specific cases of IEES(κ) functions, delineated by the assumptions made with respect to ψ. We restrict ourselves to the choice of normalization constants where $\pi_0 e^{-\frac{\sigma_0 - 1}{\psi \sigma_0}} \sigma_0^{-\frac{1}{\psi}} < 1$.

Case with $\psi > 0$. In this case, the elasticity of substitution increases with the capital–output ratio κ. The production function is well-defined, increasing and concave in its domain. However, due to restrictions in the range of $F(K, L)$, the support of κ is restricted to $\kappa \in [\kappa_{min}, \kappa_{max}]$ where κ_{min} and κ_{max} are the two solutions to the equation $\pi(\kappa) = 1$. The capital's share $\pi(\kappa)$ follows a non-monotonic pattern with κ, declining if $\kappa \in (\kappa_{min}, \tilde{\kappa})$ and increasing for $\kappa \in (\tilde{\kappa}, \kappa_{max})$. The minimum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{min} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0 - 1}{\psi \sigma_0}} \sigma_0^{-\frac{1}{\psi}} \in (0, 1). \tag{A.45}$$

Our empirical analysis (Section 8) suggests that this case of IEES(κ) functions is preferred by the data on aggregate production in the post-war US economy. We also find $\sigma_0 < 1$.

Case with $\psi < 0$. In this case, we assume that the elasticity of substitution decreases with the capital–output ratio κ. With the current choice of normalization constants, the production function is well-defined, increasing and concave in its domain $\kappa \in [0, +\infty)$. The capital's share $\pi(k)$ follows a non-monotonic pattern with κ, increasing when $\kappa \in (0, \tilde{\kappa})$ and falling for $\kappa > \tilde{\kappa}$. The maximum capital share, obtained at the point $\tilde{\kappa}$, is equal to:

$$\pi_{max} = \pi(\tilde{\kappa}) = \pi_0 e^{-\frac{\sigma_0 - 1}{\psi \sigma_0}} \sigma_0^{-\frac{1}{\psi}} \in (0, 1), \tag{A.46}$$

with the following limits:

$$\lim_{\kappa \to 0} \pi(\kappa) = 0, \qquad \lim_{\kappa \to \infty} \pi(\kappa) = 0. \tag{A.47}$$

A.2 Self-Duality of IEES Functions: Analytical Method and Detailed Results

To demonstrate self-duality of IEES functions, we proceed as follows. First, we note that perfect competition in factor markets implies that factors of production are remunerated to the amount of their marginal products. We also observe that the dual cost function obtains homogeneity (constant returns to scale) from its primal counterpart. Keeping this in mind, we derive the dual cost function $TC(r,w) = rK^* + wL^*$ associated with a given (primal) production function $Y = F(K,L)$ using the Shephard's lemma (Shephard, 1953). We obtain:

$$\frac{A}{1-A} \equiv \frac{\frac{\partial TC(r,w)}{\partial r}\frac{r}{TC(r,w)}}{\frac{\partial TC(r,w)}{\partial w}\frac{w}{TC(r,w)}} = \frac{rK^*}{wL^*} = \frac{\pi^*}{1-\pi^*} = \Pi^*, \tag{A.48}$$

where A is the partial elasticity of the total cost function with respect to the capital's rate of return r, $1-A$ is the partial elasticity of the total cost function with respect to the wage rate w, and $\pi = \frac{\partial F(K,L)}{\partial K}\frac{K}{F(K,L)}$ and $1-\pi = \frac{\partial F(K,L)}{\partial L}\frac{L}{F(K,L)}$ are the capital and labor's share of output, respectively. Asterisks denote values in the firm's optimum.

Constant returns to scale of the dual cost function imply that we can write $\frac{A}{1-A} = \Pi(k) = \frac{\pi(k)}{1-\pi(k)}$ as a function of the ratio of capital to labor remuneration $\eta = \frac{r}{w}$ only:

$$\frac{TC(r,w)}{w} = TC\left(\frac{r}{w},1\right) \equiv tc(\eta), \qquad \Pi(\eta) = \frac{\eta \cdot tc'(\eta)}{tc(\eta) - \eta \cdot tc'(\eta)} = \frac{rK^*}{wL^*} = \eta k^*. \tag{A.49}$$

Therefore, to obtain $\Pi(\eta)$ and thus identify the dual cost functions associated with IEES(Π), IEES(MRS) and IEES(k) production functions, it suffices to insert $k = \Pi/\eta$ into the respective formula for $\Pi(k)$ and solve the implicit equation for $\Pi(\eta)$.

Under the capital deepening production function representation $y = h(\kappa)$, where $\kappa = k/y$, Shephard's lemma can be used in an analogous manner. We thus identify the dual cost functions associated with IEES(π), IEES(r) and IEES(κ) production functions. Given that $\pi = r\kappa$, it suffices to insert $\kappa = \pi/r$ into the respective formula for $\pi(\kappa)$ and solve the implicit equation for $\pi(r)$.

Detailed results following from applying this analytical procedure are reported in Table A.1.

Table A.1: Primal and Dual IEES Function Representations

Main Results					
	$\sigma=1$ **C-D**	$\frac{\sigma}{\sigma_0}=1$ **CES**	$\frac{\sigma}{\sigma_0}=\left(\frac{\pi}{1-\pi}\frac{1-\pi_0}{\pi_0}\right)^{\psi}$ **IEES**(Π)	$\frac{\sigma}{\sigma_0}=\left(\frac{\varphi(k)}{\varphi_0}\right)^{\psi}$ **IEES(MRS)**	$\frac{\sigma}{\sigma_0}=\left(\frac{k}{k_0}\right)^{\psi}$ **IEES(*k*)**
Primal $\Pi(k)$	$\frac{\pi_0}{1-\pi_0}$	$\frac{\pi_0}{1-\pi_0}\left(\frac{k}{k_0}\right)^{\frac{\sigma_0-1}{\sigma_0}}$	$\frac{\pi_0}{1-\pi_0}\left(\frac{1}{\sigma_0}+\left(1-\frac{1}{\sigma_0}\right)\left(\frac{k}{k_0}\right)^{\psi}\right)^{\frac{1}{\psi}}$	$\frac{\pi_0}{1-\pi_0}\frac{k}{k_0}\left(1+\frac{\psi}{\sigma_0}\ln\left(\frac{k}{k_0}\right)\right)^{-\frac{1}{\psi}}$	$\frac{\pi_0}{1-\pi_0}\frac{k}{k_0}e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}$
Dual $\Pi(\eta)$	$\frac{\pi_0}{1-\pi_0}$	$\frac{\pi_0}{1-\pi_0}\left(\frac{\eta}{\eta_0}\right)^{1-\sigma_0}$	$\frac{\pi_0}{1-\pi_0}\left(\sigma_0+(1-\sigma_0)\left(\frac{\eta}{\eta_0}\right)^{-\psi}\right)^{-\frac{1}{\psi}}$	$\frac{\pi_0}{1-\pi_0}\frac{\eta}{\eta_0}e^{-\frac{\sigma_0}{\psi}\left(1-\left(\frac{\eta}{\eta_0}\right)^{-\psi}\right)}$	$\frac{\pi_0}{1-\pi_0}\frac{\eta}{\eta_0}\left(1+\psi\sigma_0\ln\left(\frac{\eta}{\eta_0}\right)\right)^{-\frac{1}{\psi}}$
Primal $\sigma(k)$	1	σ_0	$1+(\sigma_0-1)\left(\frac{k}{k_0}\right)^{\psi}$	$\sigma_0+\psi\ln\left(\frac{k}{k_0}\right)$	$\sigma_0\left(\frac{k}{k_0}\right)^{\psi}$
Dual $\sigma(\eta)$	1	$\frac{1}{\sigma_0}$	$1+\left(\frac{1}{\sigma_0}-1\right)\left(\frac{\eta}{\eta_0}\right)^{-\psi}$	$\frac{1}{\sigma_0}\left(\frac{\eta}{\eta_0}\right)^{\psi}$	$\frac{1}{\sigma_0}+\psi\ln\left(\frac{\eta}{\eta_0}\right)$
Results for the Capital Deepening Representation (with $\kappa=k/y$)					
	$\sigma=1$ **C-D**	$\frac{\sigma}{\sigma_0}=1$ **CES**	$\frac{\sigma}{\sigma_0}=\left(\frac{\pi}{\pi_0}\right)^{\psi}$ **IEES**(π)	$\frac{\sigma}{\sigma_0}=\left(\frac{r}{r_0}\right)^{-\psi}$ **IEES**(r)	$\frac{\sigma}{\sigma_0}=\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$ **IEES**(κ)
Primal $\pi(\kappa)$	π_0	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma_0-1}{\sigma_0}}$	$\pi_0\left(\frac{1}{\sigma_0}+\left(1-\frac{1}{\sigma_0}\right)\left(\frac{\kappa}{\kappa_0}\right)^{\psi}\right)^{\frac{1}{\psi}}$	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)\left(1+\frac{\psi}{\sigma_0}\ln\left(\frac{\kappa}{\kappa_0}\right)\right)^{-\frac{1}{\psi}}$	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)}$
Dual $\pi(r)$	π_0	$\pi_0\left(\frac{r}{r_0}\right)^{1-\sigma_0}$	$\pi_0\left(\sigma_0+(1-\sigma_0)\left(\frac{r}{r_0}\right)^{-\psi}\right)^{-\frac{1}{\psi}}$	$\pi_0\left(\frac{r}{r_0}\right)e^{-\frac{\sigma_0}{\psi}\left(1-\left(\frac{r}{r_0}\right)^{-\psi}\right)}$	$\pi_0\left(\frac{r}{r_0}\right)\left(1+\psi\sigma_0\ln\left(\frac{r}{r_0}\right)\right)^{-\frac{1}{\psi}}$
Primal $\sigma(\kappa)$	1	σ_0	$1+(\sigma_0-1)\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$	$\sigma_0+\psi\ln\left(\frac{\kappa}{\kappa_0}\right)$	$\sigma_0\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$
Dual $\sigma(r)$	1	$\frac{1}{\sigma_0}$	$1+\left(\frac{1}{\sigma_0}-1\right)\left(\frac{r}{r_0}\right)^{-\psi}$	$\frac{1}{\sigma_0}\left(\frac{r}{r_0}\right)^{\psi}$	$\frac{1}{\sigma_0}+\psi\ln\left(\frac{r}{r_0}\right)$

A.3 Data Construction

Our dataset contains time series for the non-residential business sector in the US economy that spans the period from 1948Q1 to 2013Q4. The basic data source is BEA NIPA.

Real and nominal output is calculated as follows. First, gross domestic product is reduced by government gross value added and gross output in the housing sector. Then, effects of indirect taxation are subtracted from the data. The data are taken from BEA NIPA tables 1.3.5 (Nominal GDP, GVA), 1.3.6 (Real GDP, GVA), 1.12 (Indirect Taxes less Subsidies). Effects of indirect taxation are eliminated from real output by assuming that its share in real output is the same as in nominal output.

The annual real capital stock in the non-residential business sector is taken from NIPA FAT table 4.2. Because there is no available data on quarterly capital stocks, growth rates of nominal private non-residential fixed investment (BEA NIPA table 1.1.5) are used to interpolate the series. The obtained quarterly series displays the same trends as the original annual series.

BEA does not publish data on the labor input at a quarterly frequency. Instead of interpolating the annual series, we use the BLS quarterly series to construct this variable. Our measure of the labor input is a simple sum of the number of employees (FRED code: *USPRIV*) and the self-employed (*LNS12032192*). Since the proposed measure does not take ongoing changes in labor composition into consideration, as a robustness check we use Fernald's (2012) data on quality-adjusted aggregate hours which can be easily converted from annualized growth rates into an index.

To measure factor income shares that are consistent with our definition of output we proceed as follows. The labor share is adjusted by the number of the self-employed in order to deal with the problem of assignment of ambiguous income to either capital or labor (see Mućk et al., 2018, for a wider discussion):

$$\text{Labor share:} \qquad 1 - \pi_t = \frac{w_t L_t}{P_t Y_t} = \frac{CE_t}{Output_t}\left(1 + \frac{SE_t}{E_t}\right) \tag{A.50}$$

where CE_t denotes compensation of employees, $Output_t$ is the above described output in nominal terms, SE_t and E_t stand for the number of the self-employed and employees, respectively. The data on SE_t and E_t are consistent with our measure of the labor input. For consistency in terms of the range of economy, CE_t is calculated as the compensation of employees reduced by wages and salaries in the government sector and supplements to wages in this sector.[1] These series are taken from NIPA table 1.12.

The measurement of the user cost of capital is extremely problematic. We also assume throughout the analysis that the production function has constant returns to scale. Therefore, the capital share is calculated residually:

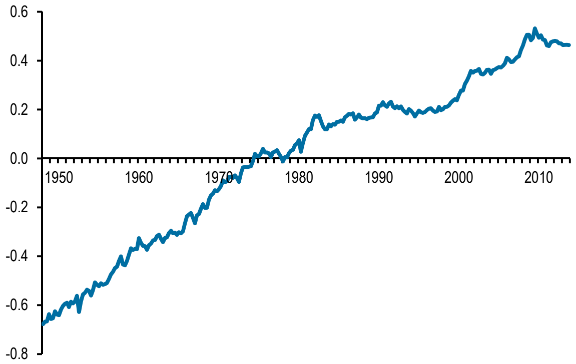
$$\text{Capital share:} \qquad \pi_t = \frac{r_t K_t}{P_t Y_t} = 1 - \frac{w_t L_t}{P_t Y_t}. \tag{A.51}$$

As shown by McAdam and Willman (2013), this agnostic approach of measuring the capital share allows to identify correctly the most critical parameters characterizing the supply side of the postwar US economy.

Figure A.1: The US Time Series: 1948Q1:2013Q4

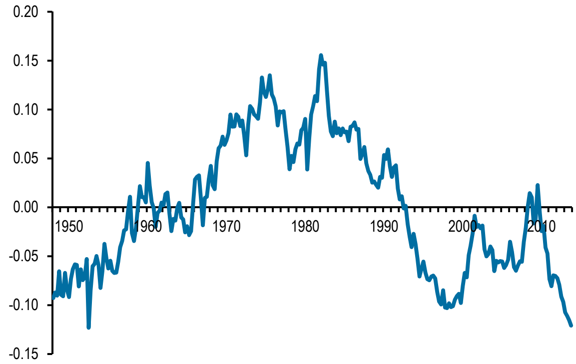
the capital-labor ratio:

$$\ln\left(\frac{k_t}{k_0}\right) = \ln\left(\frac{K_t}{L_t}\frac{L_0}{K_0}\right)$$



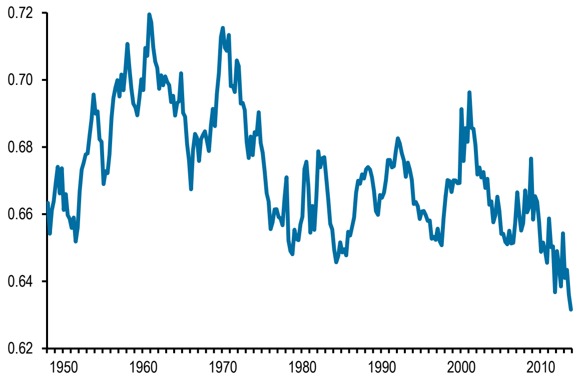
the capital-labor ratio in effective units:

$$\ln\left(\frac{\bar{k}_t}{\bar{k}_0}\right) = \ln\left(\frac{K_t}{L_t}\frac{L_0}{K_0}e^{-\gamma_l(t-\bar{t})}\right), \gamma_l = 0.0045$$



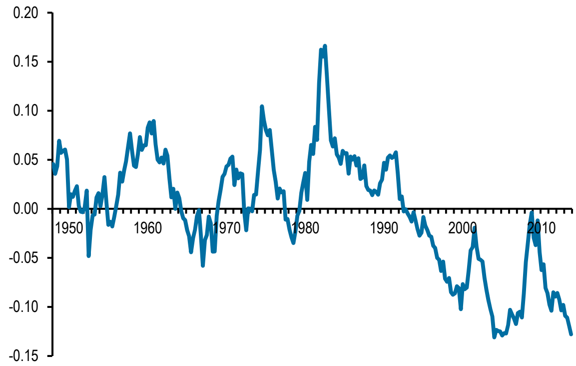
the labor share:

$$1 - \pi_t = \left(\frac{w_t L_t}{P_t Y_t}\right)$$



the capital-output ratio:

$$\ln\left(\frac{\kappa_t}{\kappa_0}\right) = \ln\left(\frac{K_t}{Y_t}\frac{Y_0}{K_0}\right)$$



A.4 Rolling Window Estimation of σ with CES Production

In each of the following figures, we document the estimated values of the elasticity of substitution σ, average capital share π_0, and the pace of labor-augmenting technical change γ_l and capital-augmenting technical change γ_k, with respective 95% confidence intervals. We also present the results of (window-specific) ADF stationarity tests for residuals. Dashed lines represent the 5% significance threshold.

Figure A.2: Rolling window estimates; Baseline (45Y)

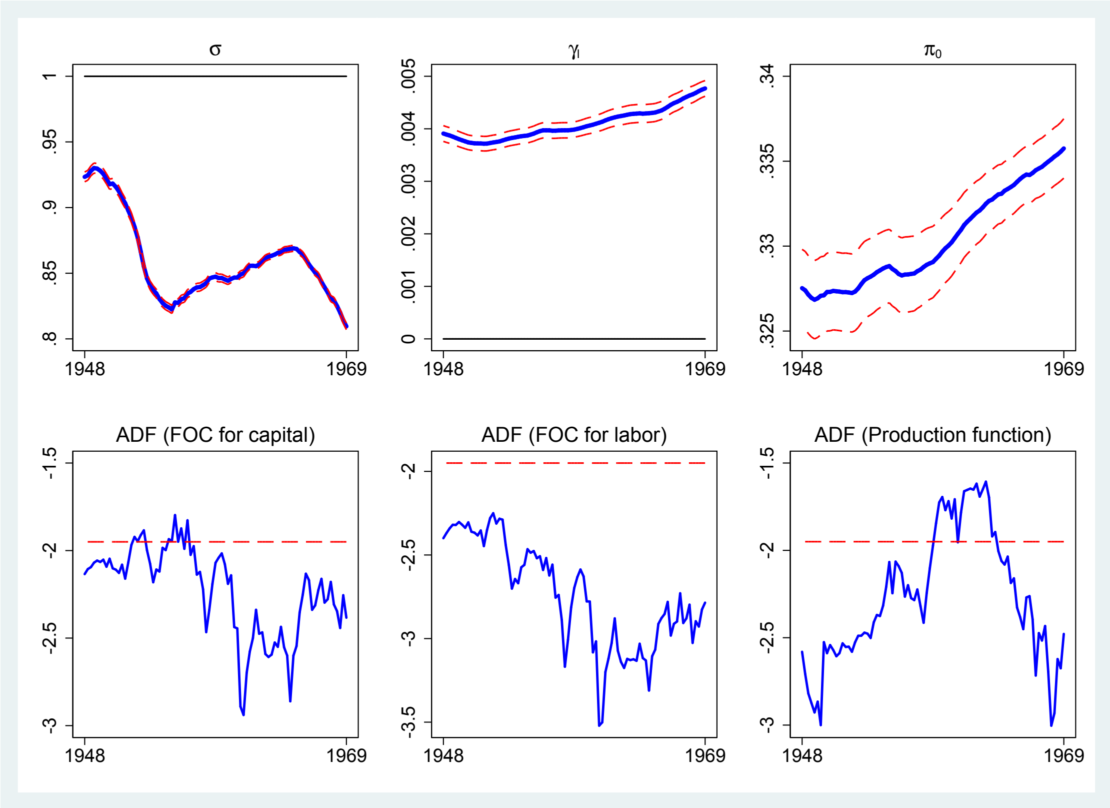


Figure A.3: Rolling window estimates; Quality-Adjusted Labor Input (45Y)

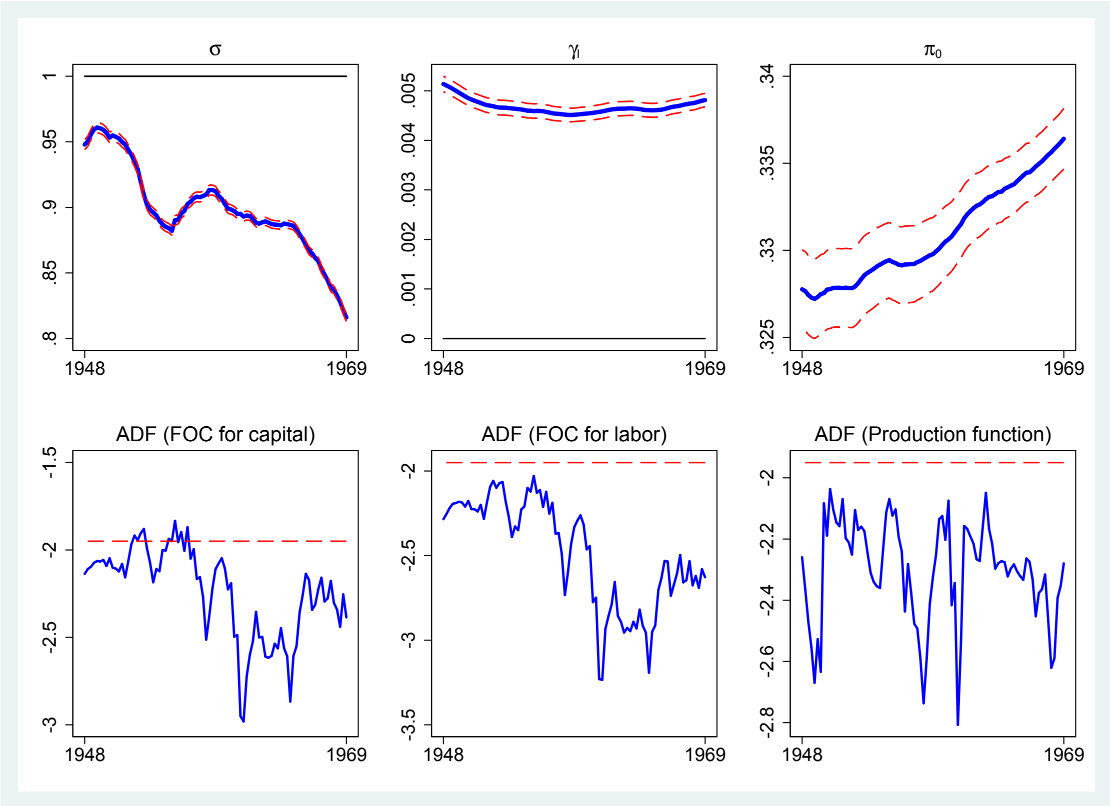


Figure A.4: Rolling window estimates; with KATC (45Y)

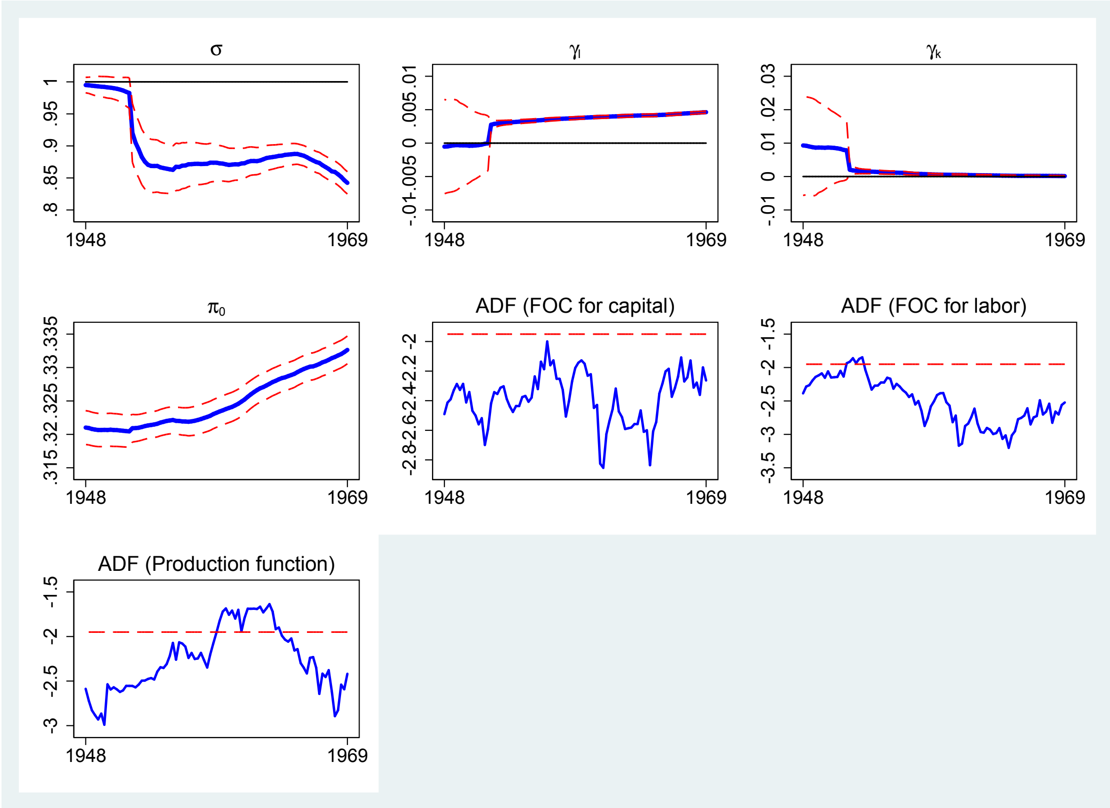


Figure A.5: Rolling window estimates; with KATC & Quality-Adjusted Labor Input (45Y)

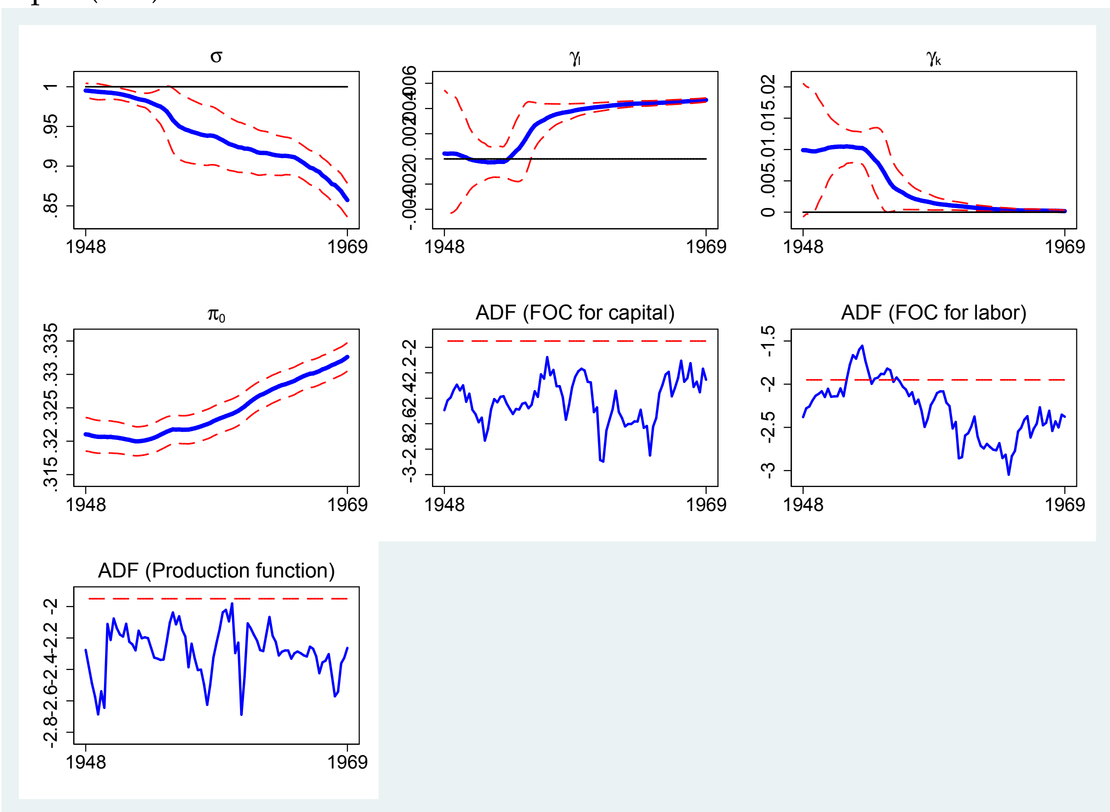


Figure A.6: Rolling window estimates; Baseline (60Y)

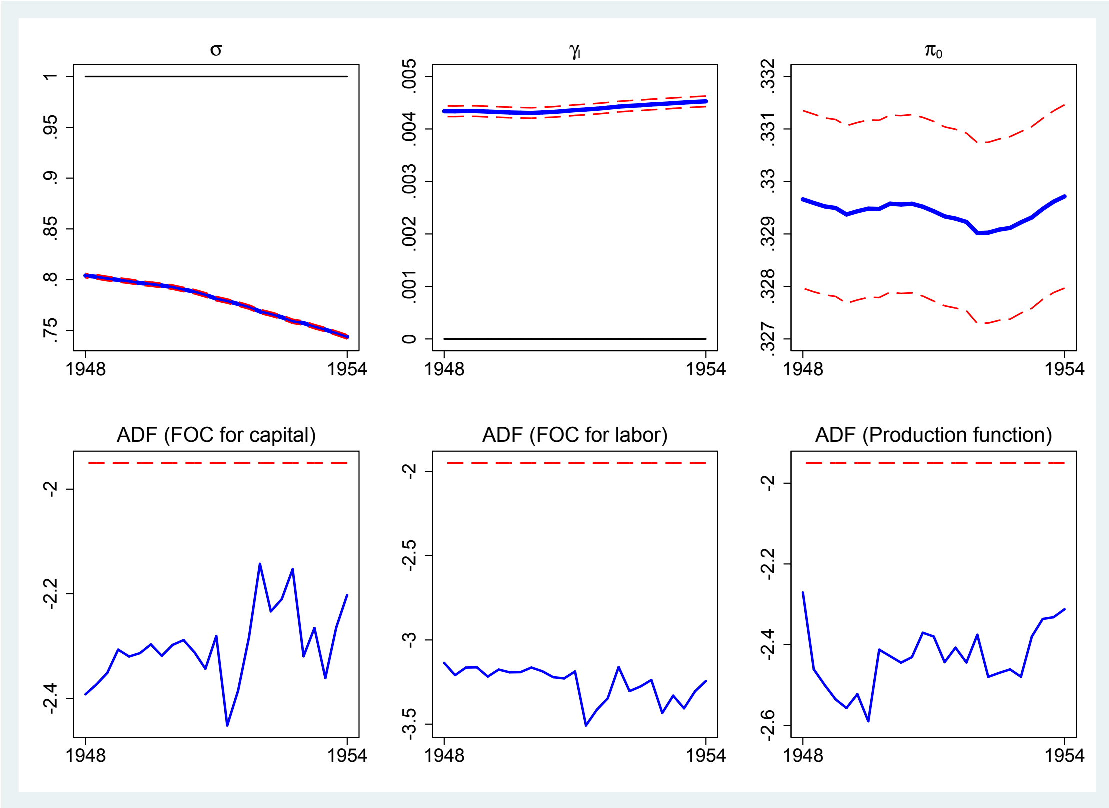


Figure A.7: Rolling window estimates; Baseline (55Y)

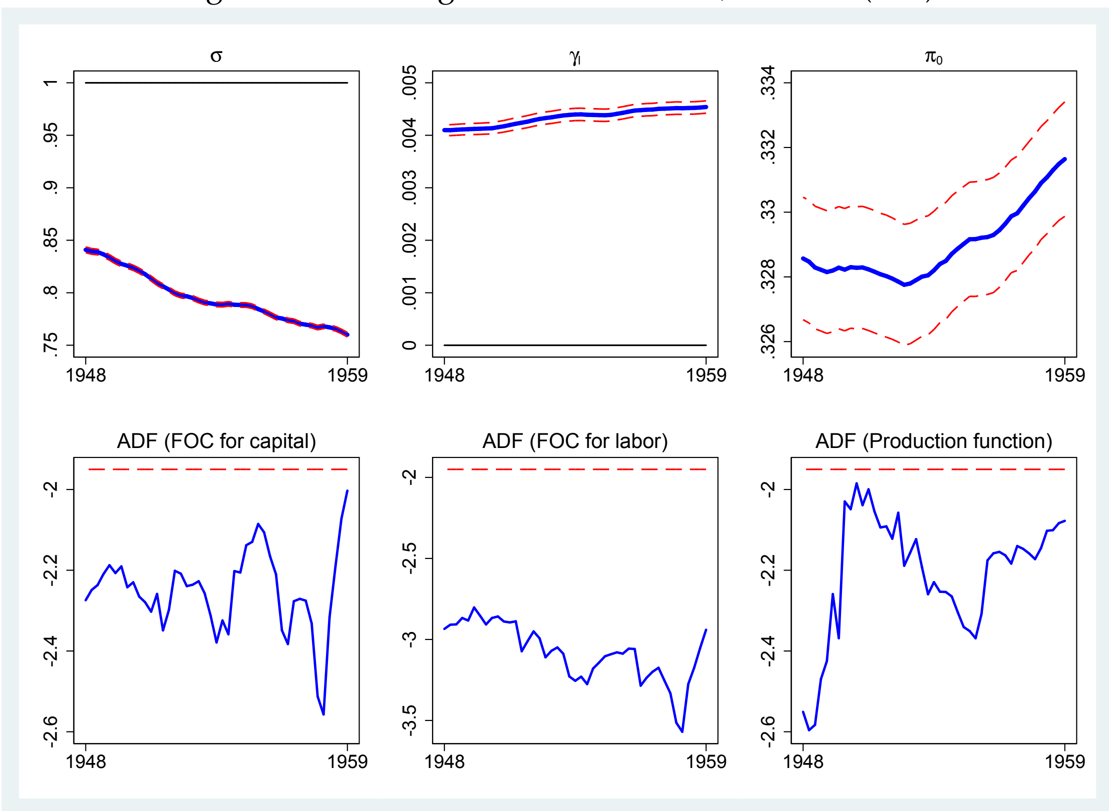


Figure A.8: Rolling window estimates; Baseline (40Y)

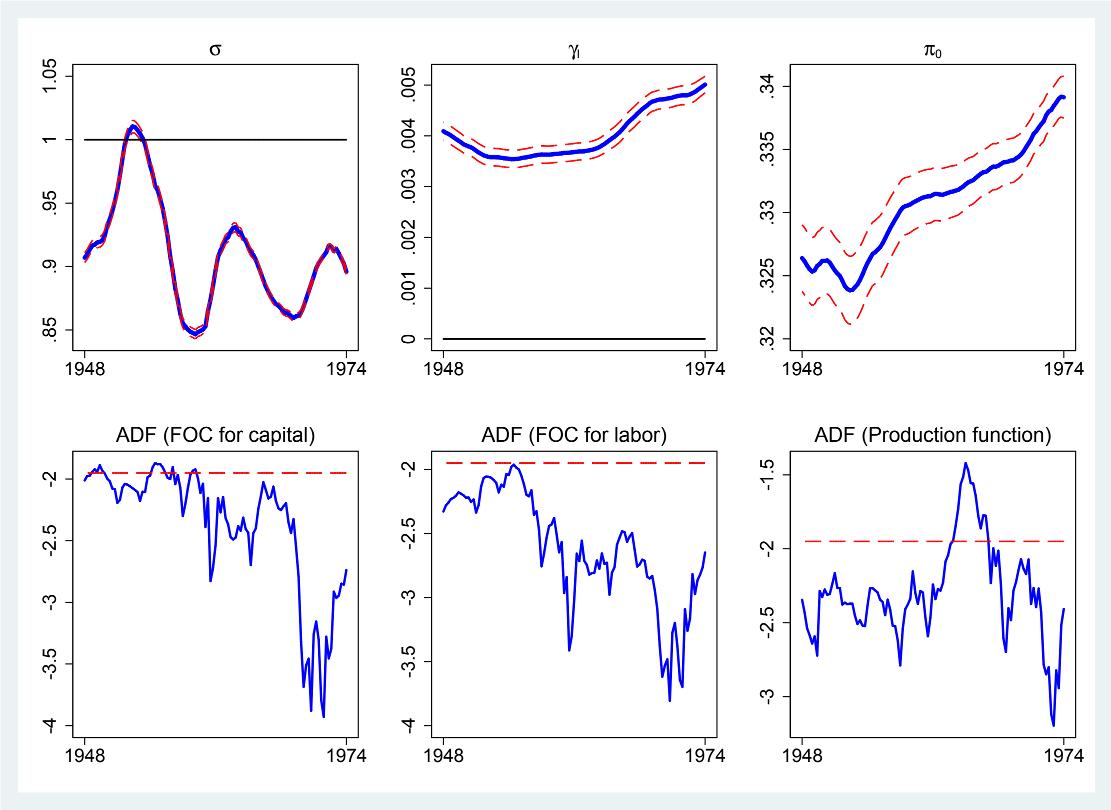
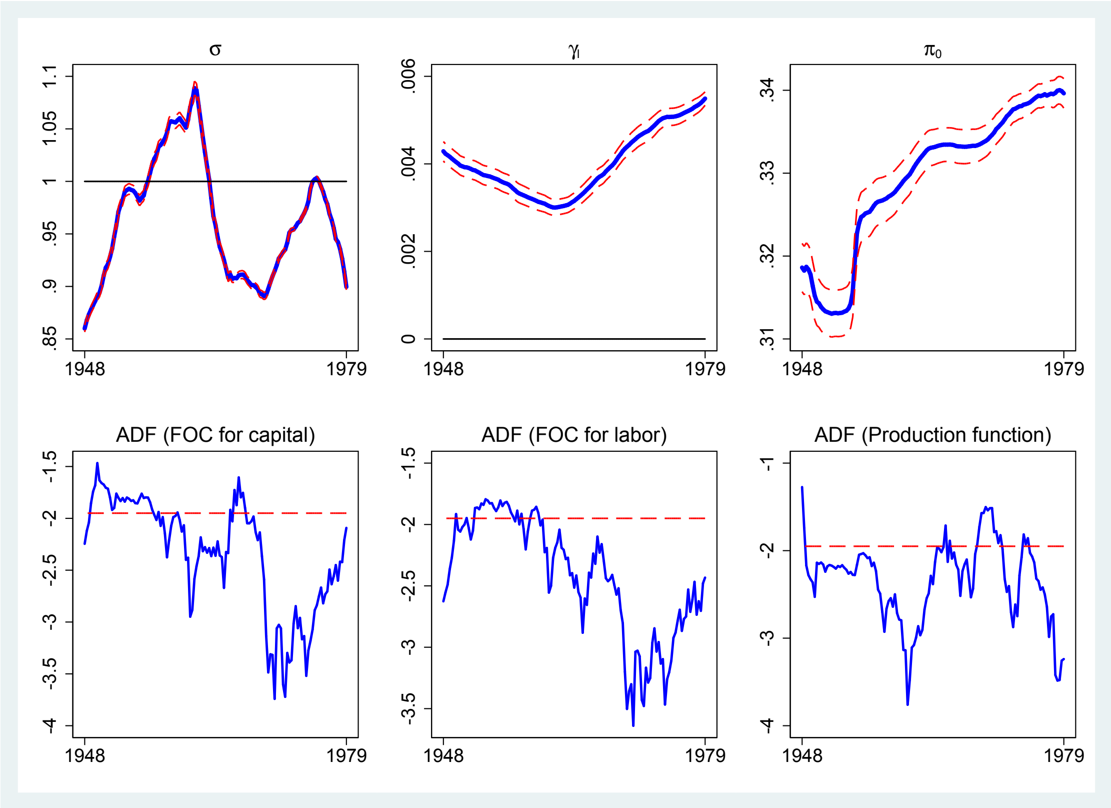


Figure A.9: Rolling window estimates; Baseline (35Y)



A.5 Robustness Checks: Detailed Results

Table A.2: Summary of Estimates of IEES Production Functions: Quality-Adjusted Labor Input

	CES	IEES(Π)	IEES(*MRS*)	IEES($\bar{k}$)	IEES(π)	IEES(*r*)	IEES(κ)
				Single-Equation NLS ($\gamma_l = 0.005$)			
π_0	0.327***	0.324***	0.323***	0.323***	0.363***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.02)	(0.001)	(0.001)
σ_0	0.706***	0.773***	0.777***	0.767***	0.757***	0.751***	0.745***
	(0.024)	(0.036)	(0.036)	(0.035)	(0.041)	(0.034)	(0.029)
ψ		−5.79***	2.121***	3.196***	−8.958**	1.879**	2.676***
		(1.385)	(0.372)	(0.645)	(4.225)	(0.744)	(0.726)
σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.012]	[0.034]	[0.000]
ADF	−2.735***	−2.969***	−2.978***	−2.973***	−2.648***	−2.674***	−2.688***
				Two-Step ($\sigma_0 = 0.775$ and $\gamma_l = 0.005$)			
π_0		0.324***	0.323***	0.323***	0.357***	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(0.008)	(0.001)	(0.001)
ψ		−5.872***	2.102***	3.322***	−10.51***	2.227***	3.172***
		(0.546)	(0.179)	(0.404)	(2.275)	(0.542)	(0.591)
σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF		−3.395***	−3.401***	−3.398***	−2.973***	−3.001***	−3.014***
				System Approach			
π_0	0.331***	0.324***	0.323***	0.323***	0.325***	0.325***	0.325***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.775***	0.887***	0.862***	0.846***	0.888***	0.868***	0.856***
	(0.001)	(0.019)	(0.02)	(0.019)	(0.023)	(0.023)	(0.02)
ξ	1.001***	1.000***	1.000***	1.000***	1.002***	1.002***	1.002***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
γ_l	0.005***	0.005***	0.005***	0.005***	0.005***	0.005***	0.005***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
ψ		−7.061***	1.399***	1.654***	−10.219***	1.842***	2.045***
		(0.905)	(0.161)	(0.216)	(1.9)	(0.32)	(0.327)
σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF_K	−2.681***	−3.14***	−3.133***	−3.1***	−3.012***	−3.02***	−3.002***
ADF_L	−3.4***	−3.552***	−3.517***	−3.505***	−3.528***	−3.529***	−3.514***
ADF_Y	−2.665***	−2.653***	−2.657***	−2.652***	−2.616***	−2.642***	−2.643***

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Table A.3: Summary of Baseline Estimates of IEES Production Functions: Box-Cox Labor-Augmenting Technical Change

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES(κ)
	Single-Equation NLS ($\gamma_l = 0.005$ and $\lambda_l = 1.148$)						
π_0	0.325***	0.327***	0.327***	0.326***	0.363***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.02)	(0.001)	(0.001)
σ_0	0.634***	0.61***	0.607***	0.609***	0.757***	0.751***	0.745***
	(0.019)	(0.021)	(0.021)	(0.023)	(0.041)	(0.034)	(0.029)
ψ		3.339**	−1.001*	−1.318	−8.958**	1.879**	2.676***
		(1.567)	(0.567)	(0.897)	(4.225)	(0.744)	(0.726)
σ_t		$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.033]	[0.077]	[0.142]	[0.012]	[0.034]	[0.000]
ADF	−3.37***	−3.738***	−3.697***	−3.619***	−2.648***	−2.674***	−2.688***
	Two-Step ($\sigma_0 = 0.775$, $\gamma_l = 0.005$ and $\lambda_l = 1.148$)						
π_0		0.323***	0.323***	0.323***	0.364***	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(0.012)	(0.001)	(0.001)
ψ		−5.34***	1.719***	2.334***	−8.816***	1.929***	2.833***
		(0.572)	(0.239)	(0.505)	(2.348)	(0.548)	(0.587)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF		−3.562***	−3.418***	−3.292***	−2.965***	−2.99***	−3.004***
	System Approach						
π_0	0.33***	0.323***	0.323***	0.323***	0.324***	0.325***	0.325***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.755***	0.861***	0.837***	0.826***	0.836***	0.812***	0.805***
	(0.001)	(0.016)	(0.015)	(0.015)	(0.021)	(0.019)	(0.015)
ξ	0.988***	0.987***	0.985***	0.985***	0.991***	0.991***	0.991***
	(0.004)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.004)
γ_l	0.005***	0.005***	0.005***	0.005***	0.005***	0.005***	0.005***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
λ_l	1.148***	1.188***	1.205***	1.203***	1.17***	1.168***	1.165***
	(0.031)	(0.026)	(0.026)	(0.027)	(0.029)	(0.029)	(0.03)
ψ		−5.913***	1.159***	1.376***	−6.943***	1.261***	1.456***
		(0.673)	(0.149)	(0.215)	(1.294)	(0.264)	(0.243)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF_K	−2.715***	−3.132***	−2.856***	−2.83***	−2.968***	−2.916***	−2.903***
ADF_L	−3.646***	−3.578***	−3.519***	−3.52***	−3.672***	−3.613***	−3.601***
ADF_Y	−2.494**	−2.47**	−2.487**	−2.489**	−2.474**	−2.488**	−2.49**

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Table A.4: Summary of Baseline Estimates of IEES Production Functions: Box-Cox Labor-Augmenting Technical Change & Quality-Adjusted Labor Input

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES(κ)
	Single-Equation NLS ($\gamma_l = 0.005$ and $\lambda_l = 0.883$)						
π_0	0.328***	0.325***	0.325***	0.325***	0.363***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.02)	(0.001)	(0.001)
σ_0	0.69***	0.744***	0.73***	0.716***	0.757***	0.751***	0.745***
	(0.025)	(0.04)	(0.034)	(0.03)	(0.041)	(0.034)	(0.029)
ψ		−6.598***	2.29***	3.399***	−8.958**	1.879**	2.676***
		(1.898)	(0.545)	(0.9)	(4.225)	(0.744)	(0.726)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.001]	[0.000]	[0.000]	[0.012]	[0.034]	[0.000]
ADF	−2.787***	−2.968***	−2.947***	−2.934***	−2.648***	−2.674***	−2.688***
	Two-Step ($\sigma_0 = 0.767$, $\gamma_l = 0.005$ and $\lambda_l = 0.883$)						
π_0		0.325***	0.324***	0.324***	0.359***	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(0.009)	(0.001)	(0.001)
ψ		−7.599***	2.831***	4.438***	−9.799***	2.103***	3.031***
		(0.686)	(0.289)	(0.697)	(2.307)	(0.545)	(0.589)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF		−3.36***	−3.336***	−3.302***	−2.969***	−2.996***	−3.009***
	System Approach						
π_0	0.332***	0.324***	0.324***	0.324***	0.325***	0.326***	0.326***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.767***	0.891***	0.84***	0.825***	0.874***	0.844***	0.834***
	(0.001)	(0.02)	(0.019)	(0.018)	(0.025)	(0.023)	(0.019)
ξ	1.012***	1.007***	1.006***	1.006***	1.01***	1.01***	1.01***
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)
γ_l	0.005***	0.005***	0.005***	0.005***	0.005***	0.005***	0.005***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
λ_l	0.883***	0.924***	0.934***	0.931***	0.917***	0.914***	0.911***
	(0.019)	(0.019)	(0.019)	(0.019)	(0.02)	(0.02)	(0.02)
ψ		−8.534***	1.345***	1.54***	−9.261***	1.514***	1.609***
		(1.274)	(0.200)	(0.256)	(1.921)	(0.345)	(0.344)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF_K	−2.693***	−3.13***	−3.064***	−3.021***	−2.995***	−2.974***	−2.946***
ADF_L	−3.316***	−3.478***	−3.408***	−3.384***	−3.469***	−3.429***	−3.404***
ADF_Y	−2.693***	−2.67***	−2.669***	−2.668***	−2.642***	−2.662***	−2.665***

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Table A.5: Summary of Estimates of IEES Production Functions: Break in LATC

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES(κ)
	Single-Equation NLS ($\gamma_l = 0.0045$, $\gamma_{l,\mathcal{B}} = 0.001$ and $\mathcal{B} = 2003$)						
π_0	0.325***	0.325***	0.325***	0.325***	0.363***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.02)	(0.001)	(0.001)
σ_0	0.633***	0.633***	0.633***	0.633***	0.757***	0.751***	0.745***
	(0.017)	(0.034)	(0.018)	(0.017)	(0.041)	(0.034)	(0.029)
ψ		−0.001	0.000	0.000	−8.958**	1.879**	2.676***
		(1.444)	(0.069)	(0.001)	(4.225)	(0.744)	(0.726)
σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.972]	[0.899]	[0.764]	[0.012]	[0.034]	[0.000]
ADF	−3.302***	−3.302***	−3.302***	−3.302***	−2.648***	−2.674***	−2.688***
	Two-Step ($\sigma_0 = 0.742$, $\gamma_l = 0.0045$, $\gamma_{l,\mathcal{B}} = 0.001$ and $\mathcal{B} = 2003$)						
π_0		0.324***	0.324***	0.324***	0.369***	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(0.015)	(0.001)	(0.001)
ψ		−3.827***	1.293***	1.932***	−7.785***	1.744***	2.621***
		(0.541)	(0.147)	(0.28)	(2.385)	(0.549)	(0.588)
σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.002]	[0.001]	[0.000]
ADF		−3.656***	−3.639***	−3.626***	−2.958***	−2.98***	−2.996***
	System Approach						
π_0	0.33***	0.324***	0.324***	0.323***	0.324***	0.324***	0.325***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.742***	0.831***	0.824***	0.817***	0.824***	0.807***	0.799***
	(0.001)	(0.017)	(0.016)	(0.016)	(0.023)	(0.021)	(0.016)
ξ	0.991***	0.992***	0.991***	0.991***	0.993***	0.993***	0.993***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
γ_l	0.004***	0.004***	0.004***	0.004***	0.004***	0.004***	0.004***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\gamma_{l,\mathcal{B}}$	0.001***	0.001***	0.001***	0.001***	0.001***	0.001***	0.001***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
ψ		−4.663***	1.181***	1.542***	−6.462***	1.414***	1.733***
		(0.552)	(0.134)	(0.197)	(1.31)	(0.263)	(0.224)
σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
ADF_K	−2.729***	−3.161***	−3.147***	−3.128***	−3.007***	−3.02***	−3.016***
ADF_L	−4.103***	−4.134***	−4.132***	−4.137***	−4.159***	−4.163***	−4.161***
ADF_Y	−2.882***	−2.889***	−2.897***	−2.898***	−2.882***	−2.89***	−2.89***

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Table A.6: Summary of Estimates of IEES Production Functions: Break in LATC & Quality-Adjusted Labor Input

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES(κ)
	Single-Equation NLS ($\gamma_l = 0.0056$, $\gamma_{l,\mathcal{B}} = -0.0008$ and $\mathcal{B} = 1964$)						
π_0	0.329***	0.327***	0.327***	0.327***	0.363***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.02)	(0.001)	(0.001)
σ_0	0.671***	0.703***	0.684***	0.676***	0.757***	0.751***	0.745***
	(0.024)	(0.037)	(0.028)	(0.025)	(0.041)	(0.034)	(0.029)
ψ		−5.391***	1.659***	2.328**	−8.958**	1.879**	2.676***
		(1.984)	(0.584)	(0.926)	(4.225)	(0.744)	(0.726)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.007]	[0.005]	[0.012]	[0.012]	[0.034]	[0.000]
ADF	−2.872***	−2.98***	−2.948***	−2.933***	−2.648***	−2.674***	−2.688***
	Two-Step ($\sigma_0 = 0.751$, $\gamma_l = 0.0056$, $\gamma_{l,\mathcal{B}} = -0.0008$ and $\mathcal{B} = 1964$)						
π_0		0.326***	0.325***	0.325***	0.365***	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(0.013)	(0.001)	(0.001)
ψ		−7.594***	2.818***	3.743***	−8.493***	1.872***	2.767***
		(0.745)	(0.326)	(0.876)	(2.36)	(0.548)	(0.587)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.000]	[0.000]	[0.001]	[0.000]
ADF		−3.375***	−3.321***	−3.246***	−2.962***	−2.986***	−3.001***
	System Approach						
π_0	0.333***	0.325***	0.326***	0.326***	0.327***	0.327***	0.327***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.751***	0.872***	0.798***	0.79***	0.821***	0.795***	0.793***
	(0.001)	(0.022)	(0.019)	(0.019)	(0.026)	(0.021)	(0.019)
ξ	1.015***	1.013***	1.013***	1.013***	1.015***	1.015***	1.015***
	(0.002)	(0.003)	(0.003)	(0.003)	(0.002)	(0.002)	(0.003)
γ_l	0.006***	0.006***	0.006***	0.006***	0.006***	0.006***	0.006***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\gamma_{l,\mathcal{B}}$	−0.001***	−0.001***	−0.001***	−0.001***	−0.001***	−0.001***	−0.001***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
ψ		−8.465***	1.007***	1.105***	−5.371***	0.663**	0.745**
		(1.248)	(0.237)	(0.301)	(1.687)	(0.33)	(0.338)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.001]	[0.000]	[0.045]	[0.028]
ADF_K	−2.723***	−3.107***	−2.942***	−2.902***	−2.926***	−2.862***	−2.85***
ADF_L	−3.439***	−3.46***	−3.407***	−3.403***	−3.484***	−3.475***	−3.473***
ADF_Y	−2.922***	−2.964***	−2.963***	−2.967***	−2.952***	−2.97***	−2.972***

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Table A.7: Summary of Estimates of IEES Production Functions: LATC & KATC

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES($\bar{\kappa}$)
	Single-Equation NLS ($\gamma_l = 0.0056$ and $\gamma_k = -0.003$)						
π_0	0.327^{***}	0.325^{***}	0.325^{***}	0.325^{***}	1.11	0.325^{***}	0.325^{***}
	(0.001)	(0.001)	(0.001)	(0.001)	(1.171)	(0.001)	(0.001)
σ_0	0.903^{***}	0.908^{***}	0.906^{***}	0.906^{***}	0.908^{***}	0.907^{***}	0.907^{***}
	(0.008)	(0.01)	(0.009)	(0.008)	(0.009)	(0.008)	(0.009)
ψ		-0.772^{*}	0.088^{*}	0.1^{**}	-0.815	0.081	0.091
		(0.454)	(0.046)	(0.051)	(0.699)	(0.059)	(0.069)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.089]	[0.055]	[0.049]	[0.170]	[0.244]	[0.183]
ADF	-3.268^{***}	-3.267^{***}	-3.248^{***}	-3.245^{***}	-3.152^{***}	-3.144^{***}	-3.143^{***}
	Two-Step ($\sigma_0 = 0.883$, $\gamma_l = 0.0056$ and $\gamma_k = -0.003$)						
π_0		0.326^{***}	0.325^{***}	0.325^{***}	0.324^{***}	0.326^{***}	0.326^{***}
		(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
ψ		-0.290	0.056	0.070	-0.200	0.030	0.037
		(0.327)	(0.04)	(0.047)	(0.396)	(0.054)	(0.059)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.375]	[0.160]	[0.135]	[0.647]	[0.584]	[0.538]
ADF		-3.704^{***}	-3.7^{***}	-3.698^{***}	-3.556^{***}	-3.554^{***}	-3.554^{***}
	System Approach						
π_0	0.326^{***}	0.325^{***}	0.325^{***}	0.325^{***}	0.325^{***}	0.324^{***}	0.325^{***}
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.883^{***}	0.919^{***}	0.922^{***}	0.922^{***}	0.918^{***}	0.912^{***}	0.922^{***}
	(0.001)	(0.012)	(0.011)	(0.011)	(0.024)	(0.021)	(0.024)
ξ	1.004^{***}	1.003^{***}	1.003^{***}	1.003^{***}	1.003^{***}	1.002^{***}	1.003^{***}
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
γ_l	0.006^{***}	0.006^{***}	0.006^{***}	0.006^{***}	0.006^{***}	0.006^{***}	0.006^{***}
	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)	(0.001)
γ_k	-0.003^{***}	-0.004^{***}	-0.004^{***}	-0.004^{***}	-0.004^{***}	-0.004^{***}	-0.004^{**}
	(0.000)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.002)
ψ		-0.423	0.037	0.041	-0.601^{**}	-0.112^{*}	0.055
		(0.288)	(0.025)	(0.028)	(0.302)	(0.067)	(0.04)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.142]	[0.141]	[0.139]	[0.047]	[0.093]	[0.169]
ADF_K	-3.563^{***}	-3.488^{***}	-3.483^{***}	-3.482^{***}	-3.486^{***}	-3.474^{***}	-3.485^{***}
ADF_L	-3.886^{***}	-3.793^{***}	-3.782^{***}	-3.781^{***}	-3.793^{***}	-3.84^{***}	-3.781^{***}
ADF_Y	-2.423^{**}	-2.421^{**}	-2.412^{**}	-2.412^{**}	-2.423^{**}	-2.417^{**}	-2.415^{**}

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Table A.8: Summary of Estimates of IEES Production Functions: LATC & KATC & Quality-Adjusted Labor Input

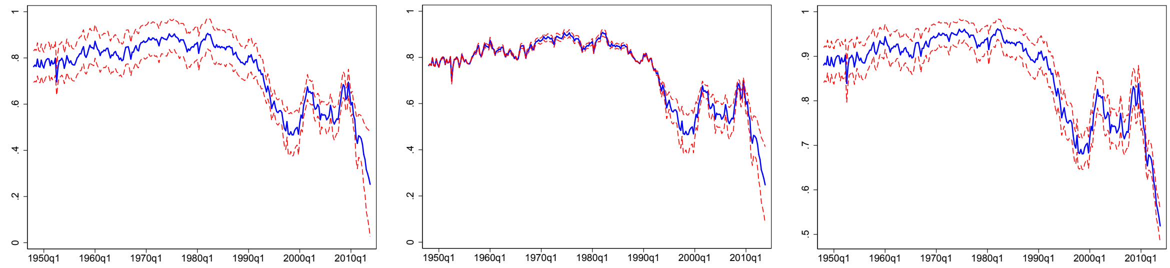
	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES($\bar{\kappa}$)
	Single-Equation NLS ($\gamma_l = 0.013$ and $\gamma_k = -0.017$)						
π_0	0.327***	0.325***	0.324***	0.324***	0.328***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.977***	0.979***	0.978***	0.978***	0.979***	0.978***	0.978***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
ψ		−0.222**	0.007***	0.007***	−0.324**	0.01***	0.01***
		(0.105)	(0.002)	(0.002)	(0.158)	(0.003)	(0.004)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.035]	[0.004]	[0.004]	[0.040]	[0.003]	[0.004]
ADF	−3.209***	−3.224***	−3.192***	−3.191***	−3.215***	−3.184***	−3.182***
	Two-Step ($\sigma_0 = 0.973$, $\gamma_l = 0.013$ and $\gamma_k = -0.017$)						
π_0		0.325***	0.324***	0.324***	85.474	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(300.477)	(0.001)	(0.001)
ψ		−0.115	0.006**	0.006***	−0.179	0.009***	0.01***
		(0.074)	(0.002)	(0.002)	(0.113)	(0.003)	(0.004)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.121]	[0.011]	[0.009]	[0.113]	[0.007]	[0.008]
ADF		−3.659***	−3.649***	−3.648***	−3.633***	−3.621***	−3.619***
	System Approach						
π_0	0.327***	0.326***	0.325***	0.325***	0.326***	0.325***	0.325***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.973***	0.972***	0.972***	0.972***	0.972***	0.969***	0.972***
	(0.002)	(0.008)	(0.009)	(0.009)	(0.008)	(0.01)	(0.008)
ξ	1.013***	1.008***	1.008***	1.008***	1.008***	1.007***	1.008***
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)
γ_l	0.013***	0.012***	0.011***	0.011***	0.012***	0.011***	0.011***
	(0.001)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
γ_k	−0.017***	−0.013***	−0.013***	−0.013***	−0.013***	−0.012***	−0.013***
	(0.001)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)
ψ		−0.08	0.003	0.003	−0.113	−0.007	0.004
		(0.068)	(0.002)	(0.002)	(0.071)	(0.005)	(0.003)
σ_t		$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$\mathcal{H}_0 : \mathcal{CES}$		[0.236]	[0.198]	[0.198]	[0.139]	[0.113]	[0.143]
ADF_K	−3.615***	−3.556***	−3.557***	−3.557***	−3.555***	−3.562***	−3.556***
ADF_L	−3.621***	−3.643***	−3.647***	−3.648***	−3.641***	−3.663***	−3.646***
ADF_Y	−2.61***	−2.65***	−2.646***	−2.646***	−2.651***	−2.649***	−2.647***

Notes: the superscripts ***, ** and * denote rejection of the null about parameters' insignificance at the 1%, 5% and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The superscripts ***, ** and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5% and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively.

Figure A.10: Implied σ_t: Baseline Specification; Quality-Adjusted Labor Input

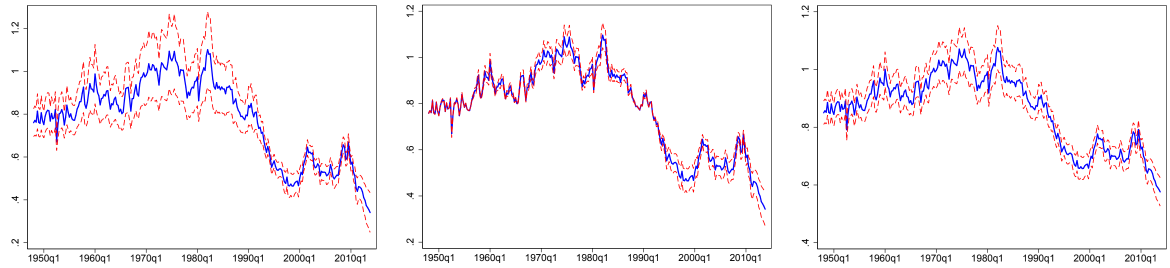
: IEES(Π)

Single-Equation NLS **Two-Step** **System Approach**



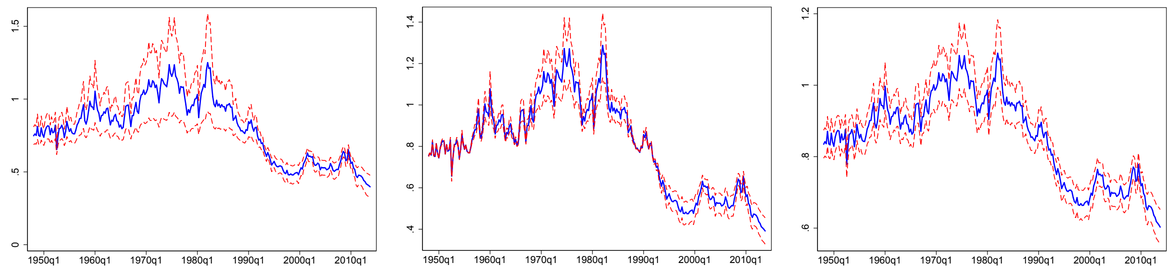
: IEES(MRS)

Single-Equation NLS **Two-Step** **System Approach**



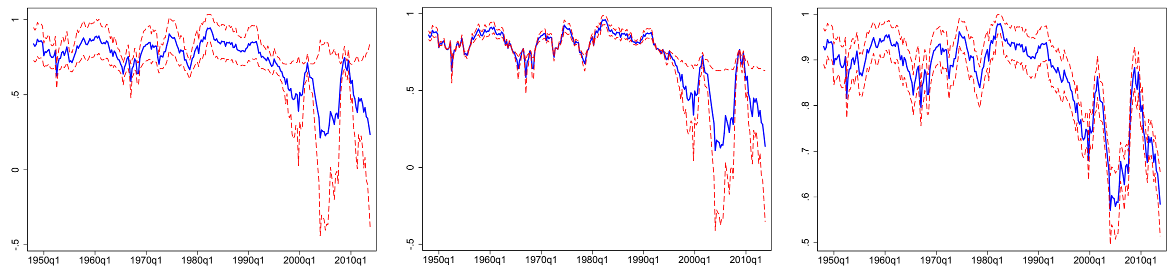
: IEES(k)

Single-Equation NLS **Two-Step** **System Approach**

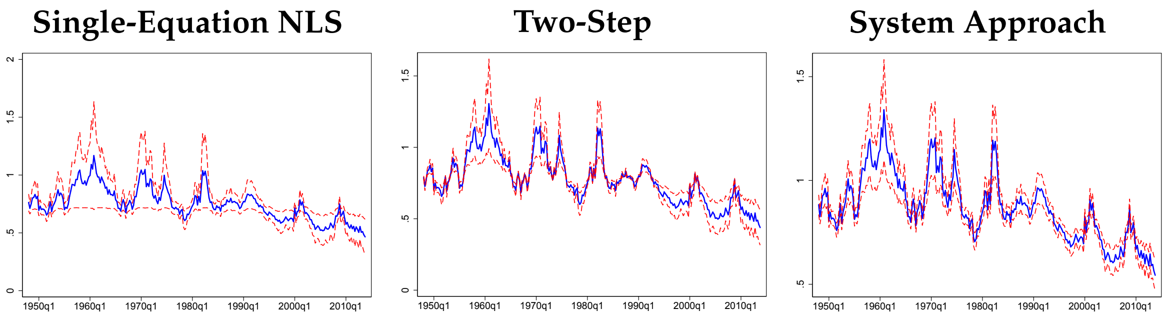


: IEES(π)

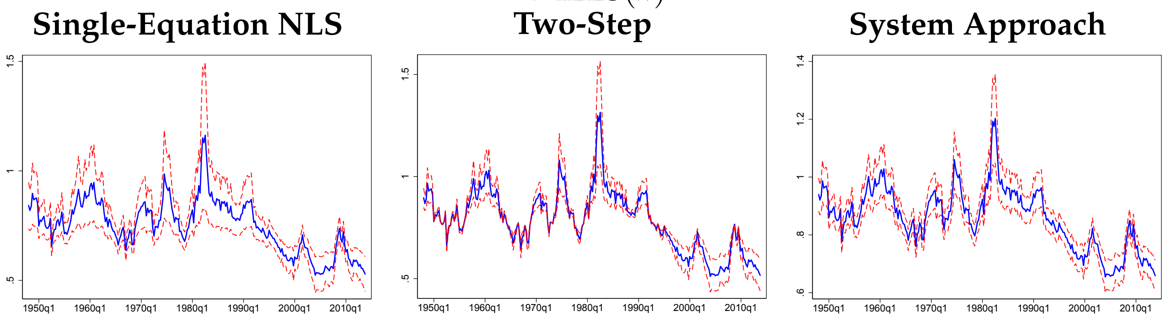
Single-Equation NLS **Two-Step** **System Approach**



: IEES(r)



: IEES(κ)



Note: Dashed lines represent 95% confidence intervals computed with the delta method. In the case of two-step estimates (middle panels), the assessment of variance of estimates may be downward biased because the delta method has been applied to the second step only, taking the estimates from the first step as fixed numbers. Observe that σ_t is a nonlinear function of ψ, as in equations (48)–(53). This implies that (i) around the normalization point, $\bar{k}_t \approx \bar{k}_0$ and $\kappa_t \approx \kappa_0$, $\sigma_t \approx \sigma_0$ regardless of the value of ψ. Then the upper and lower bounds converge to the point estimate of σ_t; (ii) in some cases the p-value of the ψ parameter estimate is only slightly below 0.05. Then at the bound of the 95% confidence interval, $\psi \approx 0$ and thus $\sigma_t \approx \sigma_0$ implying that the given bound is almost flat (even if the point estimate of σ_t and the other bound is not).

: Summary of Goodness-of-Fit Measures

	CES	IEES (Π)	IEES (MRS)	IEES (k)	CES	IEES (π)	IEES (r)	IEES (κ)
Baseline								
$\mathcal{BIC}^1$	−659.126	−664.400	−663.235	−661.752	−814.720	−815.989	−816.548	−816.821
$\mathcal{BIC}^2$	−659.126	−640.478	−634.156	−631.612	−814.720	−821.565	−822.096	−822.256
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.991	0.835	0.990	1.000	0.990	0.835	0.989
Quality-Adjusted Labor Input								
$\mathcal{BIC}^1$	−616.727	−623.481	−624.566	624.712	−814.720	−815.989	−816.548	−816.821
$\mathcal{BIC}^2$	−616.727	−629.052	−630.140	−630.246	−814.720	−821.665	−821.351	−821.543
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.992	0.993	0.992	1.000	0.993	0.992	0.992
Box-Cox Labor-Augmenting Technical Change								
$\mathcal{BIC}^1$	−660.251	−657.381	−656.599	−656.223	−814.720	−815.989	−816.548	−816.821
$\mathcal{BIC}^2$	−660.251	−650.652	−647.243	−645.418	−814.720	−822.113	−821.563	−822.301
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.752	0.752	0.996	1.000	0.990	0.988	0.988
Box-Cox Labor-Augmenting Technical Change and Quality-Adjusted Labor input								
$\mathcal{BIC}^1$	−612.938	−615.961	−616.280	−616.258	−814.720	−815.989	−816.548	−816.821
$\mathcal{BIC}^2$	−612.938	−621.226	−621.035	−619.876	−814.720	−821.498	−821.924	−821.939
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.987	0.986	0.988	1.000	0.988	0.987	0.986
Labor- and Capital-Augmenting Technical Change								
$\mathcal{BIC}^1$	−656.968	−657.823	−658.608	−658.747	−854.580	−859.977	−859.677	−860.026
$\mathcal{BIC}^2$	−656.968	−654.628	−655.500	−655.722	−854.580	−861.841	−856.382	−856.411
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.959	0.955	0.959	1.000	0.962	0.973	0.964
Labor- and Capital-Augmenting Technical Change and Quality-Adjusted Labor Input								
$\mathcal{BIC}^1$	−654.907	−655.471	−658.018	−658.120	−860.427	−863.570	−860.910	−863.678
$\mathcal{BIC}^2$	−654.907	−653.105	−656.484	−656.683	−860.427	−859.122	−862.593	−862.797
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.993	0.995	0.994	1.000	0.989	1.091	0.988
Break in Labor-Augmenting Technical Change								
$\mathcal{BIC}^1$	−647.181	−651.605	−651.605	−651.605	−814.720	−815.989	−816.821	−816.821
$\mathcal{BIC}^2$	−647.181	−652.073	−651.174	−650.580	−814.720	−822.039	−822.384	−821.402
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.979	0.969	0.976	1.000	0.976	0.975	0.975
Break in Labor-Augmenting Technical Change and Quality-Adjusted Labor input								
$\mathcal{BIC}^1$	−612.158	−618.912	−618.361	−618.095	−814.720	−815.989	−816.548	−816.821
$\mathcal{BIC}^2$	−612.158	−623.218	−620.552	−617.936	−814.720	−822.124	−821.542	−822.364
$SSE^i/SSE^{\mathcal{CES}}$	1.000	0.988	0.988	0.989	1.000	0.989	0.988	0.988

Notes: $\mathcal{BIC}^1$ is the Schwarz criterion for the single-equation NLS estimates, $\mathcal{BIC}^2$ denotes the Schwarz criterion for the two-step estimates while $SSE^i/SSE^{\mathcal{CES}}$ is the relative (to the CES estimates) residual sum of squares (SSE) for the production function equation in system estimation.