A Mathematical Appendix

A.1 Proposition 1

We divide the proof in two parts. First, we show that the policy scheme (q_{ct}, t_{dt}) with $q_{ct} < +\infty$ and $t_{dt} < p_{dt}$ introduced at time $t = T$ is not able to redirect technical change when A_{dT} is too large, then we characterize the unique equilibrium.

In the model, the more the dirty sector is profitable, the more researchers devote effort to innovate therein, the more dirty machines become productive and the relative share of dirty inputs increases. Under the policy scheme (q_{ct}, t_{dt}), the profitability of the two sectors is determined by three elements, namely the productivity ratio, the price of dirty inputs and the carbon tax:

$$\begin{aligned}
\frac{\Pi_{ct}}{\Pi_{dt}} &= (1+q_{ct})\frac{\eta_c}{\eta_d}\left(\frac{p_{dt}-t_d}{p_{dt}}\right)^{-\epsilon}\left(\frac{A_{ct}}{A_{dt}}\right)^{-\varphi-1}\frac{A_{ct-1}}{A_{dt-1}} \\
&= (1+q_{ct})\frac{\eta_c}{\eta_d}\left(\frac{p_{dt}-t_d}{p_{dt}}\right)^{-\epsilon}\left(\frac{1+\gamma\eta_c s_{ct}}{1+\gamma\eta_d s_{dt}}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}, \qquad (17)
\end{aligned}$$

where the second line follows from (5) and where $\varphi = (1-\epsilon)(1-\alpha) < 0$ and $s_{ct} = 1 - s_{dt}$. Let

$$f(s) = (1+q_{ct})\frac{\eta_c}{\eta_d}\left(\frac{p_{dt}-t_d}{p_{dt}}\right)^{-\epsilon}\left(\frac{1+\gamma\eta_c s}{1+\gamma\eta_d(1-s)}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi},$$

where $s = s_{ct} = 1 - s_{dt}$. If $f(0) < 1$, then $s = 0$ is an equilibrium where all scientists devote their effort toward the dirty sector.

By assumption the economy is initially stuck in the bad equilibrium where productivity-improving innovations take place only for dirty machines. A carbon tax on the production of dirty inputs t_{dt}, and a subsidy q_{ct}, introduced at time T, are able to redirect technical change if they guarantee $f(0) > 1$, which corresponds to:

$$(1+q_{cT})\frac{\eta_c}{\eta_d}\left(\frac{p_{dT}-t_{dT}}{p_{dt}}\right)^{-\epsilon}(1+\gamma\eta_d)^{\varphi+1}\left(\frac{A_{cT-1}}{A_{dT-1}}\right)^{-\varphi} > 1. \qquad (18)$$

If q_{cT} is finite, then it exists a $\vartheta \in R$ such that $q_{cT} < \vartheta$; conversely, if $t_{dT} < p_{dT}$, there exist a $\theta \in R$ such that $t_{dT} < \theta < p_{dT}$. We analyze the case where the government is planning to provide the maximum possible subsidy, $q_{cT} = \vartheta^-$. If the tax is not able to redirect technical change in such scenario, then it would not be effective for all $q_{cT} < \vartheta^-$ as well (results do not change if the maximum available tax is fixed and one studies how subsidy affects technical change). The tax is effective in redirecting technical change if the following condition is satisfied:

$$t_{dT} > p_{dT} - \left[(1+\vartheta)\frac{\eta_c}{\eta_d}(1+\eta_d\gamma)^{\varphi+1}\left(\frac{A_{cT-1}}{A_{dT-1}}\right)^{-\varphi}\right]^{\frac{1}{\epsilon}} p_{dT}. \tag{19}$$

Given the productivity of machines in the dirty sector r, define $g(r)$:

$$g(r) := p_{dT} - \left[(1+\vartheta)\frac{\eta_c}{\eta_d}(1+\eta_d\gamma)^{\varphi+1}\left(\frac{A_{cT-1}}{r}\right)^{-\varphi}\right]^{\frac{1}{\epsilon}} p_{dT}.$$

$g(r)$ is a continuous function in $(0;+\infty)$ and satisfies:

$$\lim_{r\to+\infty} g(r) = p_{dT}.$$

Without loss of generality, let $\theta = p_{dT} - \delta$ with $\delta \in \mathbf{R}^+$. Then, using the definition of limit, for all $\delta > 0$, it exists a $\bar{A}_d \ll \infty$, such that for all, $r > \bar{A}_d$, one obtains

$$p_{dT} - g(r) < \delta,$$

which in turns implies $g(r) > p_{dT} - \delta$. Finally, there exists a finite r and a sufficiently low δ such that, in order to redirect technical change, it is required

$$t_{dT} > g(r) > \theta,$$

which is impossible because it contradicts our assumptions.

Now let us show that the equilibrium where all researchers are employed in the dirty sector ($s = 0$) is also the unique equilibrium when A_{dT-1} is sufficiently large. Two cases must be distinguished.

First, if $1 + \varphi > 0$, then $f(s)$ is strictly decreasing in s and $f(0) < 1$ guarantees that $s = 0$ is the unique equilibrium. The previous condition can be rewritten as follows:

$$f(0) = (1+\vartheta)\frac{\eta_c}{\eta_d}\left(\frac{p_{dT}-t_{dT}}{p_{dT}}\right)^{-\epsilon}(1+\eta_d\gamma)^{\varphi+1}\left(\frac{A_{cT-1}}{A_{dT-1}}\right)^{-\varphi} < 1,$$

which implies

$$\left(\frac{A_{cT-1}}{A_{dT-1}}\right)^{-\varphi} < \frac{\eta_d}{\eta_c(1+\vartheta)}\left(\frac{p_{dT}-t_{dT}}{p_{dT}}\right)^{\epsilon}\left(\frac{1}{1+\eta_d\gamma}\right)^{\varphi+1} = \Psi, \tag{20}$$

where $\Psi > 0$. As $\epsilon > 1$, the left hand side of (20) is a continuous function monotonically decreasing in A_{dT-1} which tends to 0 as the productivity of machines in the dirty sector becomes larger and larger. This proves that for a sufficiently large A_{dT-1}, the unique equilibrium allocation of scientists satisfies $s = 0$.

Now consider the second case where $1+\varphi < 0$. As $f(s)$ is strictly increasing in s, the unique equilibrium is $s = 0$ only if $f(0) < f(1) < 1$, where the first inequality is obviously satisfied. Consider the second inequality:

$$f(1) = (1+\vartheta)\frac{\eta_c}{\eta_d}\left(\frac{p_{dT}-t_{dT}}{p_{dT}}\right)^{-\epsilon}(1+\eta_c\gamma)^{-\varphi-1}\left(\frac{A_{cT-1}}{A_{dT-1}}\right)^{-\varphi} < 1.$$

Accounting for the time the tax is introduced and after some algebra it becomes

$$\left(\frac{A_{cT-1}}{A_{dT-1}}\right)^{-\varphi} < \frac{\eta_d}{\eta_c(1+\vartheta)}\left(\frac{p_{dT}-t_{dT}}{p_{dT}}\right)^{\epsilon}(1+\eta_c\gamma)^{\varphi+1} = \Psi',$$

with $\Psi' > 0$. Analogously to the previous case, it is easy to see that for a sufficiently high A_{dT-1} the equilibrium $s = 0$ is unique. Finally, if $1+\varphi = 1$ then $f(s) \equiv f$ is constant and $f < 1$, surely verified for some large A_{dT-1}, is sufficient to obtain the unique equilibrium $s = 0$.

A.2 Proposition 2

The proof of proposition 2 follows easily from equation (5), which stems in turns from the process of technical change and the law of large numbers.

The evolution of a machine's productivity can be summarized as follows

$$A_{jit} = \begin{cases} (1+\gamma s_{jit})A_{jt-1}, & \text{with probability } \eta_j \\ A_{jit-1}, & \text{with probability } (1-\eta_j) \end{cases}$$

with $j = \{c, d\}$ as usual. Therefore, recalling that scientists targeting sector j are randomly allocated across machines in that sector, the law of large numbers allows writing the average productivity in the dirty sector - where innovation take place - at time t as

$$A_{dt} = (1+\gamma\eta_d s_{dt})A_{dt-1}. \tag{21}$$

Since $s_{dt} = 1$, A_{dt} grows exponentially and deterministically at the rate $\gamma\eta_d > 0$. Hence, there exist a finite time $t = T^*$ such that $A_{dT^*} > \bar{A}_d \geq A_d0$.

A.3 Proposition 3

First, we notice that equation (16) can be expressed as

$$g(s) = \frac{\eta_c}{\eta_d}\kappa\left(\frac{1+\eta_c s}{1+\eta_d(1-s)}\right)^{-(2-\alpha)}\left(\frac{A_{ct}}{A_{dt}}\right)^{-(1-\alpha)}, \tag{22}$$

where $s = s_{ct} = 1 - s_{dt}$. As the economy is stuck in the bad equilibrium, in order to redirect technical change, a command-and-control policy κ should satisfy $g(0) > 1$. This implies that $s = 1$ is the unique equilibrium allocation of scientists. Imposing previous condition ($g(0) > 1$) one obtains:

$$\frac{\eta_c}{\eta_d}\kappa\,(1+\eta_c)^{-(2-\alpha)}\left(\frac{A_{ct}}{A_{dt}}\right)^{-(1-\alpha)} > 1. \tag{23}$$

The latter condition can be easily expressed as

$$\begin{aligned} &\kappa > \bar{\kappa} \\ &\bar{\kappa} = \left(\tfrac{\eta_d(1+\eta_c)^{2-\alpha}}{\eta_c}\right)\left(\tfrac{A_{ct}}{A_{dt}}\right)^{1-\alpha}. \end{aligned} \tag{24}$$

Notice that $\bar{\kappa}$ is finite and strictly positive. In addition as $\kappa \in (0, \infty)$, there will always be a C&C policy $\kappa > \bar{\kappa}$ that successfully redirects technical change. Further, as the economy is stacked in a bad equilibrium, A_{ct} remains constant over time and A_{dt} grows (on average), implying that technical change actually favours C&C policies.

A.4 Proposition 4

Let us start noticing that Assumption 1 implies the command-and-control policy scheme to be binding, that is $Y_{dt} = \hat{\kappa}$. Therefore, by solving the model in section 4.3, one obtains that (similarly to proposition 3's proof) technical change is redirected towards the "green" equilibrium $s_{ct} = 1$ and $s_{dt} = 0$ if

$$\frac{Y_{ct}}{\hat{\kappa}} > \left(\frac{\eta_d(1+\eta_c)}{\eta_c}\right)^{\frac{\epsilon}{\epsilon-1}}, \tag{25}$$

which easily translates in

$$\hat{\kappa} < Y_{ct}\frac{\eta_c}{\eta_d(1+\eta_c)}^{\frac{\epsilon}{\epsilon-1}}. \tag{26}$$

Since Y_{ct}, η_c, η_d and ϵ are strictly positive, there always exists an absolute C&C policy $\hat{\kappa} > 0$ able to redirect technical change.

An environmental disaster is avoided if S_t is permanently positive after policy intervention. Since Y_{dt} is bounded from above, it suffices that $\hat{\kappa} < S_{t-1}/\xi$ to prevent an environmental catastrophe. Therefore, any C&C policy such that

$$\begin{aligned} &\hat{\kappa} < \hat{\kappa}' \\ &\hat{\kappa}' = \min\left(S_{t-1}/\xi,\, Y_{ct}\tfrac{\eta_c}{\eta_d(1+\eta_c)}^{\frac{\epsilon}{\epsilon-1}}\right) \end{aligned} \tag{27}$$

always guarantees the redirection of technical change towards the clean sector and the avoidance of environmental disaster.

A.5 Theorem of section 4.2

Here we show that (i) C&C policies are favoured by path dependence in technological change while (ii) M-B policies are hindered by path dependence. In particular, we show that technological progress dynamically reduces (increases) the strength required to C&C (M-B) policies to induce a transition. (i) and (ii) follow from propositions 3 and 1 respectively.

Let's start by considering the condition for a M-B policy to redirect technological change at time t. Equation (18) implies

$$t_{dt} > \Omega_t = \left(1 - \left[(1+\vartheta)\frac{\eta_c}{\eta_d}(1+\eta_d\gamma)^{\varphi+1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}\right]^{\frac{1}{\epsilon}}\right) p_{dt}, \tag{28}$$

where the right hand side just depends on the history of innovations. Since $\varphi < 0$, Ω_t is increasing in A_{dt} and, hence, it is expected to raise over time as innovations occur in the dirty sector only. This proves (ii).

Now, let us consider the condition for a C&C policy to redirect technological change at time t. Equation (24) implies

$$\kappa > \bar{\Omega}_t = \left(\frac{\eta_d(1+\eta_c)^{2-\alpha}}{\eta_c}\right)\left(\frac{A_{ct}}{A_{dt}}\right)^{1-\alpha}. \tag{29}$$

Since $0 < \alpha < 1$, $\bar{\Omega}_t$ is decreasing in A_{dt} and, hence, it is expected to shrink over time as innovations occur in the dirty sector only. This proves (i).