Appendix

A Proof of Lemma 1

As shown in Figure 5, the intersection of $\varepsilon(x_t)$ and $\eta(\varphi_{t-1})$ determines the value of x_t, which satisfies (19) for a given φ_{t-1}. When φ_{t-1} increases, the function $\eta(\varphi_{t-1})$ increases, as depicted by the broken line in Figure 5. Thus, x_t rises correspondingly. As a result, the $\varphi_t = \varphi_{t-1}$ locus can be depicted as an upward-sloping curve on the (x_t, φ_{t-1}) plane in Figure 1. Note that the definition of φ_t implies $\varphi_{t-1} \in (0, 1)$. Let us define $\tilde{\varphi}$ as $\tilde{\varphi} \equiv \eta^{-1}(\varepsilon(0))$. Since $\lim_{\varphi_{t-1}\to 1} \eta(\varphi_{t-1}) = +\infty$ and $\lim_{\varphi_{t-1}\to 0} \eta(\varphi_{t-1}) = -\infty$, the $\varphi_t = \varphi_{t-1}$ locus has an asymptote $\varphi_{t-1} = 1$ when $x_t \to \infty$ and φ_{t-1} has a lower limit $\tilde{\varphi}$ when $x_t = 0$.[18]

Figure 5: Derivation of the $x_{t+1} = x_t$ locus.

B Proof of Lemma 2

We derive the $x_{t+1} = x_t$ locus. Differentiating $\zeta(x_t)$ with respect to x_t yields

$$\begin{aligned}\zeta'(x_t) =& \frac{\left[\gamma(1+x_t)\left(1+\frac{\lambda\Gamma}{x_t}\right) + (1+\gamma x_t)\left(1+\frac{\lambda\Gamma}{x_t}\right) - (1+\gamma x_t)(1+x_t)\frac{\lambda\Gamma}{x_t^2}\right]}{\beta(1+x_t)[1+\gamma(x_t+\mu_1)]} \\ & - \frac{\left[(1+\gamma x_t)(1+x_t)\left(1+\frac{\lambda\Gamma}{x_t}\right) - \bar{\alpha}(1-\gamma)\mu_1\right][1+\gamma(1+2x_t+\mu_1)]}{\beta(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2}. \end{aligned} \tag{B.1}$$

We rearrange (B.1) as follows:

$$\begin{aligned}\zeta'(x_t) =& \frac{(1+x_t)^2\left[\gamma^2\mu_1\left(1+\frac{\lambda\Gamma}{x_t}\right) - (1+\gamma x_t)(1+\gamma x_t+\gamma\mu_1)\frac{\lambda\Gamma}{x_t^2}\right]}{\beta(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2} \\ & + \frac{\bar{\alpha}(1-\gamma)\mu_1\left[1+\gamma(1+2x_t+\mu_1)\right]}{\beta(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2}, \\ =& \frac{\Gamma(1+x_t)^2\left[\gamma^2(1-g)x_t^2 - \lambda\{1+\gamma(2x_t+\mu_1)\} + \bar{\alpha}(1-\gamma)\mu_1 x_t^2\left[1+\gamma(1+2x_t+\mu_1)\right]\right]}{\beta x_t^2(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2}, \\ =& \frac{\sigma(x_t)}{\beta x_t^2(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2}, \end{aligned}$$

where

$$\sigma(x_t) \equiv \Gamma(1+x_t)^2\left[\gamma^2(1-g)x_t^2 - \lambda\{1+\gamma(2x_t+\mu_1)\}\right] + \bar{\alpha}(1-\gamma)\mu_1 x_t^2\left[1+\gamma(1+2x_t+\mu_1)\right].$$

We then differentiate $\sigma(x_t)$ with respect to x_t as follows:

$$\sigma'(x_t) = \mu_2 x_t^3 + \mu_3 x_t^2 + \mu_4 x_t + \mu_5,$$

where

$$\begin{aligned}
\mu_2 &\equiv 4\Gamma\gamma^2(1-g) + 2\gamma\bar{\alpha}(1-\gamma)\mu_1 > 0,\\
\mu_3 &\equiv 6\Gamma\gamma\{\gamma(1-g)-\lambda\} + 2\bar{\alpha}(1-\gamma)\mu_1(1+2\gamma+\gamma\mu_1),\\
\mu_4 &\equiv 2\Gamma\gamma^2(1-g) + 2\gamma\bar{\alpha}(1-\gamma)\mu_1 - 8\lambda\Gamma\gamma - 2\lambda\Gamma(1+\gamma\mu_1),\\
\mu_5 &\equiv -2\lambda\Gamma(1+\gamma+\gamma\mu_1) < 0.
\end{aligned}$$

Assuming that $\gamma(1-g) > \lambda$, we obtain $\mu_4 > 0$. As a result, there is a unique $\bar{\bar{x}} > 0$ that satisfies $\sigma'(\bar{\bar{x}}) = 0$. $\sigma'(x_t) < 0$ holds when $0 < x_t < \bar{\bar{x}}$ and $\sigma'(x_t) > 0$ holds when $x_t > \bar{\bar{x}}$. Moreover, since $\sigma(0) = -\lambda\Gamma(1+\gamma\mu_1) < 0$ and $\lim_{x_t\to\infty}\sigma(x_t) = \infty$, there is a unique $\bar{x} > 0$ that satisfies $\sigma(\bar{x}) = 0$. $\sigma(x_t) < 0$ holds when $0 < x_t < \bar{x}$ and $\sigma(x_t) > 0$ holds when $x_t > \bar{x}$. That is, we obtain

$$\begin{aligned}
\zeta'(x_t) < 0 \quad &\text{if} \quad 0 < x_t < \bar{x},\\
\zeta'(x_t) > 0 \quad &\text{if} \quad x_t > \bar{x}.
\end{aligned}$$

In addition, we obtain the following results:

$$\begin{aligned}
&\lim_{x_t\to 0}\zeta(x_t) = \infty,\\
&\lim_{x_t\to\infty}\zeta(x_t) = \lim_{x_t\to\infty}\left[\frac{\left(\frac{1}{x_t}+\gamma\right)\left(\frac{1}{x_t}+1\right)\left(1+\frac{\lambda\Gamma}{x_t}\right) - \frac{\bar{\alpha}(1-\gamma)\mu_1}{x_t^2}}{\beta\left(\frac{1}{x_t}+1\right)\left(\frac{1+\gamma\mu_1}{x_t}+\gamma\right)} - \alpha_P\right] = \frac{1}{\beta} - \alpha_P.
\end{aligned}$$

By using these results, we obtain a curve, the $x_{t+1} = x_t$ locus, which is U-shaped and has the asymptotes $\varphi_{t-1} = \frac{1}{\alpha_R-\alpha_P}\left(\beta^{-1}-\alpha_P\right) > 1$ when $x_t \to \infty$ and $\varphi_{t-1} \to \infty$ when $x_t \to 0$.

Figure 1 depicts the $x_{t+1} = x_t$ locus.

C Properties of the $K_{t+1}/K_t = 0$ locus

By differentiating $\Lambda(x_t)$ with respect to x_t, we obtain

$$\begin{aligned}
\Lambda'(x_t) =& \frac{\left[(1+\gamma x_t)\left(1+\frac{\lambda\Gamma}{x_t}\right)+\gamma x_t\left(1+\frac{\lambda\Gamma}{x_t}\right)-(1+\gamma x_t)\frac{\lambda\Gamma}{x_t}\right]}{\beta(1+x_t)[1+\gamma(x_t+\mu_1)]} \\
&-\frac{\left[x_t(1+\gamma x_t)\left(1+\frac{\lambda\Gamma}{x_t}\right)-\bar{\alpha}(1-\gamma)\mu_1\right][1+\gamma(1+2x_t+\mu_1)]}{\beta(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2}, \\
=& \frac{(1+2\gamma x_t+\gamma\lambda\Gamma)(1+x_t)[1+\gamma(x_t+\mu_1)]-(1+\gamma x_t)(x_t+\lambda\Gamma)[1+\gamma(1+2x_t+\mu_1)]}{\beta(1+x_t)^2[1+\gamma(x_t+\mu_1)]^2} \\
&+\frac{\bar{\alpha}(1-\gamma)\mu_1\left[1+\gamma(1+2x_t+\mu_1)\right]}{\beta(1+x_t)^2\left[1+\gamma(x_t+\mu_1)\right]^2}, \\
=& \frac{(1-\lambda\Gamma)(1+\gamma x_t)^2+\gamma\mu_1(\gamma x_t^2+2\gamma x_t+1-\lambda\Gamma+\gamma\lambda\Gamma)}{\beta(1+x_t)^2[1+\gamma(x_t+\mu_1)]^2} \\
&+\frac{\bar{\alpha}(1-\gamma)\mu_1\left[1+\gamma(1+2x_t+\mu_1)\right]}{\beta(1+x_t)^2[1+\gamma(x_t+\mu_1)]^2}.
\end{aligned} \tag{C.1}$$

By using $\mu_1 \equiv \Gamma(1+\lambda-g)$, we can rewrite the numerator of (C.1) as

$$\nu_1 x_t^2 + \nu_2 x_t + \nu_3,$$

where

$$\begin{aligned}
\nu_1 &\equiv \gamma^2[1+(1-g)\Gamma], \\
\nu_2 &\equiv 2\gamma\{1+[\gamma\lambda+(1-g)\gamma-\lambda]\Gamma+(1-\gamma)\bar{\alpha}\mu_1\}, \\
\nu_3 &\equiv (1+\gamma\mu_1)\{1+[\bar{\alpha}(1-\gamma)(1+\lambda-g)-\lambda]\Gamma\}+\gamma^2\mu_1\lambda\Gamma+\bar{\alpha}(1-\gamma)\gamma\mu_1.
\end{aligned}$$

Assuming that $\bar{\alpha}(1-\gamma)(1+\lambda-g) > \lambda$ and using the assumption $\gamma(1-g) > \lambda$ in Lemma 2, $\nu_1 > 0$, $\nu_2 > 0$, and $\nu_3 > 0$ hold, and hence $\Lambda'(x_t) > 0$ holds for all $x_t \geq 0$. In addition, we obtain the following property for $\Lambda(x_t)$:

$$\lim_{x_t\to\infty}\Lambda(x_t) = \lim_{x_t\to\infty}\left[\frac{\left(\frac{1}{x_t}+\gamma\right)\left(1+\frac{\lambda\Gamma}{x_t}\right)-\frac{\bar{\alpha}(1-\gamma)\mu_1}{x_t^2}}{\beta\left(\frac{1}{x_t}+1\right)\left(\frac{1+\gamma\mu_1}{x_t}+\gamma\right)}-\alpha_P\right] = \frac{1}{\beta}-\alpha_P.$$

Thus, the $K_{t+1}/K_t = 0$ locus is upward sloping if $\bar{\alpha}(1-\gamma)(1+\lambda-g) > \lambda$ and has an asymptote $\varphi_{t-1} = \frac{1}{\alpha_R - \alpha_P}(\beta^{-1} - \alpha_P) > 1$ when $x_t \to \infty$ on the (x_t, φ_{t-1}) plane.

D Phase diagram

First, we examine whether $\varphi_t > \varphi_{t-1}$ or $\varphi_t < \varphi_{t-1}$ at each point of the (x_t, φ_{t-1}) plane. By using (9), we obtain

$$\varphi_t \gtreqless \varphi_{t-1} \;\Leftrightarrow\; \varepsilon(x_t) \gtreqless \eta(\varphi_{t-1}).$$

Suppose that $(\underline{x}, \underline{\varphi})$ is a combination that satisfies (19); that is, $\varepsilon(\underline{x}) = \eta(\underline{\varphi})$ holds. Moreover, let us define $\underline{\underline{x}}$ by $\underline{x} > \underline{\underline{x}}$. Since $\varepsilon(x_t)$ is increasing in x_t, $\varepsilon(\underline{\underline{x}}) < \eta(\underline{\varphi})$ holds. As a result, we obtain $\varphi_t < \varphi_{t-1}$ on the left of the $\varphi_t = \varphi_{t-1}$ locus. Similarly, we obtain $\varphi_t > \varphi_{t-1}$ on the right of the $\varphi_t = \varphi_{t-1}$ locus.

Next, we investigate whether $x_{t+1} > x_t$ or $x_{t+1} < x_t$ at each point of the (x_t, φ_{t-1}) plane. From (6), we obtain

$$x_{t+1} \gtreqless x_t \;\Leftrightarrow\; \varphi_{t-1} \lesseqgtr \zeta(x_t).$$

Therefore, we obtain $x_{t+1} > x_t$ below the $x_{t+1} = x_t$ locus and $x_{t+1} < x_t$ above the $x_{t+1} = x_t$ locus. By using these results, we can depict a phase diagram, as shown in Figure 1.

E Local stability around the steady states

By approximating (17) and (18) linearly around the steady state k ($k \in \{S, U\}$), we obtain

$$\begin{pmatrix} \varphi_t - \varphi_k^* \\ x_{t+1} - x_k^* \end{pmatrix} = \begin{pmatrix} J_{\varphi\varphi}^k & J_{\varphi x}^k \\ J_{xx}^k & J_{x\varphi}^k \end{pmatrix} \begin{pmatrix} \varphi_{t-1} - \varphi_k^* \\ x_t - x_k^* \end{pmatrix}, \tag{E.1}$$

where

$$
\begin{aligned}
J^k_{\varphi\varphi} =& \frac{\varphi^*_k G^R_\varphi(x^*_k,\varphi^*_k) + G^R(x^*_k,\varphi^*_k) - \varphi^*_k G^A_\varphi(x^*_k,\varphi^*_k)}{G^A(x^*_k,\varphi^*_k)}, \\
J^k_{\varphi x} =& \frac{\varphi^*_k G^R_x(x^*_k,\varphi^*_k) - \varphi^*_k G^A_x(x^*_k,\varphi^*_k)}{G^A(x^*_k,\varphi^*_k)}, \\
J^i_{xx} =& \frac{G^A(x^*_k,\varphi^*_k)(1-\lambda\Gamma) - (1+x^*_k)(x^*_k+\lambda\Gamma)G^A_x(x^*_k,\varphi^*_k)}{[(1+x^*_k)G^A(x^*_k,\varphi^*_k) - (x^*_k+\lambda\Gamma)]^2}, \\
J^k_{x\varphi} =& -\frac{(x^*_k+\lambda\Gamma)(1+x^*_k)G^A_\varphi(x^*_k,\varphi^*_k)}{[(1+x^*_k)G^A(x^*_k,\varphi^*_k) - (x^*_k+\lambda\Gamma)]^2},
\end{aligned}
$$

where $G^R_z(x^*_k,\varphi^*_k) \equiv \frac{\partial G^R(x_t,\varphi_{t-1})}{\partial z}\big|_{(x_t,\varphi_{t-1})=(x^*_k,\varphi^*_k)}$ and $G^A_z(x^*_k,\varphi^*_k) \equiv \frac{\partial G^A(x_t,\varphi_{t-1})}{\partial z}\big|_{(x_t,\varphi_{t-1})=(x^*_k,\varphi^*_k)}$ for $z \in \{x_t, \varphi_{t-1}\}$. Note that $G^A(x^*_k,\varphi^*_k) = G^R(x^*_k,\varphi^*_k)$ holds. Let us denote the two eigenvalues of the Jacobian matrix of the linearized system as e^k_1 and e^k_2. These eigenvalues are the roots of the characteristic polynomial: $P(e) = e^2 - (J^k_{\varphi\varphi} + J^k_{xx})e + (J^k_{xx}J^k_{\varphi\varphi} - J^k_{\varphi x}J^k_{x\varphi})$. To consider this, we conduct a numerical analysis. We adopt the following benchmark parameters: $\gamma = 0.2$, $\Gamma = 12$, $g = 0.2$, $\delta = 0.5$, $\beta = 0.3$, $\alpha_R = 0.45$, and $\alpha_P = 0.25$. Table 3 shows the two eigenvalues for each steady state.

λ	(e^S_1, e^S_2)	(e^U_1, e^U_2)
0.01	(0.098, 0.378)	(0.219, 2.566)
0.02	(0.104, 0.439)	(0.207, 2.227)
0.03	(0.113, 0.527)	(0.193, 1.874)
0.04	(0.126, 0.678)	(0.174, 1.467)

Table 3: Eigenvalues at the steady states S and U.

From Table 3, we find that both e^S_1 and e^S_2 take real positive values and satisfy $0 < e^S_1 < e^S_2 < 1$, and then the steady state S is a sink. We further find that both e^U_1 and e^U_2 take real positive values and satisfy $0 < e^U_1 < 1 < e^U_2$, and then the steady state U is a saddle point.

F Introduction of redistributive policy

Under the redistributive policy, we rewrite the following optimal conditions of individuals:

$$b_{t+1}^R = \frac{\beta\left[1+(1-\tau_{t+1})r_{t+1}\right]s_t^R}{1+\tau^b}, \tag{F.1}$$

$$c_t^{1P} = (1-\alpha_P)\left[(1-\tau_t)w_t + b_t^P + T_t\right], \tag{F.2}$$

$$s_t^P = \alpha_P\left[(1-\tau_t)w_t + b_t^P + T_t\right], \tag{F.3}$$

The other optimal conditions are the same as in (4a)–(4d). In addition, (5) becomes

$$s_t^R = \alpha_R\left[(1-\tau_t)w_t + \frac{\beta\{1+(1-\tau_t)r_t\}}{1+\tau^b}s_{t-1}^R\right], \tag{F.4}$$

$$s_t^P = \alpha_P\left[(1-\tau_t)w_t + T_t + \beta\{1+(1-\tau_t)r_t\}s_{t-1}^P\right]. \tag{F.5}$$

By substituting (F.1) into the government's redistributive policy $\delta N\tau^b b_t^R = (1-\delta)NT_t$, we obtain

$$T_t = \left(\frac{\delta}{1-\delta}\right)\left(\frac{\tau^b}{1+\tau^b}\right)\beta[1+(1-\tau_t)r_t]s_{t-1}^R. \tag{F.6}$$

The asset market-clearing condition (6), equations determined in the production sector ((7a), (7b), and (7c)), and those in the public sector ((8), (9), and (10)) remain unchanged. Furthermore, the derivations of the other equations are conducted in the same manner as described in Section 3. Some algebra rewrites $G^A(x_t, \varphi_{t-1})$ (in (13), (16), (17), and (18)), $G^K(x_t, \varphi_{t-1})$ (in (16) and (18)), and $G^R(x_t, \varphi_{t-1})$ (in (14) and (17)) into

$$G^A(x_t, \varphi_{t-1}; \tau^b) = \frac{\bar{\alpha}(1-\gamma)\mu_1}{(1+\gamma x_t)(1+x_t)} + \beta\left(1+\frac{\gamma\mu_1}{1+\gamma x_t}\right)\left[\frac{\alpha_R-\alpha_P}{1+\tau^b}\varphi_{t-1} + \alpha_P\right], \tag{F.7}$$

$$G^K(x_t, \varphi_{t-1}; \tau^b) = (1+x_t)G^A(x_t, \varphi_{t-1}; \tau^b) - (x_t + \lambda\Gamma), \tag{F.8}$$

and

$$G^R(x_t, \varphi_{t-1}; \tau^b) = \frac{\alpha_R}{\varphi_{t-1}}\left[\frac{\delta(1-\gamma)\mu_1}{(1+\gamma x_t)(1+x_t)} + \frac{\beta}{1+\tau^b}\left(1+\frac{\gamma\mu_1}{1+\gamma x_t}\right)\varphi_{t-1}\right], \tag{F.9}$$

respectively. Translating these into (17) and (18) transforms the dynamic systems as

$$\begin{aligned}\frac{\varphi_t}{\varphi_{t-1}} &= \frac{G^R(x_t, \varphi_{t-1}; \tau^b)}{G^A(x_t, \varphi_{t-1}; \tau^b)} \\ \frac{x_{t+1}}{x_t} &= \frac{1 + \lambda\Gamma/x_t}{(1 + x_t)G^A(x_t, \varphi_{t-1}; \tau^b) - (x_t + \lambda\Gamma)}.\end{aligned} \tag{F.10}$$

Setting $\varphi_t = \varphi_{t-1}$ and $x_t = x_{t+1}$ in (F.10) yields (27) and (28).

G $\varphi_t = \varphi_{t-1}$ locus under the redistributive policy

We rearrange (27) as follows:

$$\tilde{\eta}(\varphi_{t-1}) = \frac{(1 + \tau^b)(1 - \gamma)}{[\alpha_R - (1 + \tau^b)\alpha_P] - (\alpha_R - \alpha_P)\varphi_{t-1}} \left(\bar{\alpha} - \frac{\delta\alpha_R}{\varphi_{t-1}} \right).$$

Here, we assume $\alpha_R > (1 + \tau^b)\alpha_P$. Under this assumption, when φ_{t-1} increases, the function $\tilde{\eta}(\varphi_{t-1})$ increases. Similar to the analysis in Appendix A, the $\varphi_t = \varphi_{t-1}$ locus can be depicted as an upward-sloping curve on the (x_t, φ_{t-1}) plane. Moreover, we define $\hat{\varphi}$ and $\tilde{\varphi}(\tau^b)$ as $\hat{\varphi} \equiv \frac{1+\tau^b}{\alpha_R - \alpha_P} \left(\frac{\alpha_R}{1+\tau^b} - \alpha_P \right) < 1$ and $\tilde{\varphi}(\tau^b) \equiv \tilde{\eta}^{-1}(\varepsilon(0))$, respectively. Since $\lim_{\varphi_{t-1} \to \hat{\varphi}} \tilde{\eta}(\varphi_{t-1}) = +\infty$ and $\lim_{\varphi_{t-1} \to 0} \tilde{\eta}(\varphi_{t-1}) = -\infty$, the $\varphi_t = \varphi_{t-1}$ locus has an asymptote $\varphi_{t-1} = \hat{\varphi}$ when $x_t \to \infty$ and φ_{t-1} has a lower limit $\tilde{\varphi}(\tau^b)$ when $x_t = 0$.[19]