Online Appendix of the article "Relative consumption, relative wealth, and long-run growth: When and why is the standard analysis prone to incorrect conclusions?"

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A The model (Section 2)

A.1 The expressions for V_{cc} and $V_{cc}V_{aa} - (V_{ca})^2$

The definition $V(c, C, a, A) \equiv u(c, c/C, a/A)$ implies that

$$V_{cc} = u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)}, \quad \text{(A.1)}$$

$$\begin{aligned} V_{cc}V_{aa} - (V_{ca})^2 &= A^{-2}[u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)}]u_{(a/A)(a/A)} \\ &\quad -A^{-2}[u_{c(a/A)} + C^{-1}u_{(c/C)(a/A)}]^2. \end{aligned} \quad \text{(A.2)}$$

Proof: The validity of (A.1) and (A.2) is easily verified by using the following results:

$$\begin{aligned} V_c &= u_c + C^{-1}u_{c/C}, \\ V_{cc} &= u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)}, \\ V_{ca} &= A^{-1}[u_{c(a/A)} + C^{-1}u_{(c/C)(a/A)}], \\ V_a &= A^{-1}u_{(a/A)}, \\ V_{aa} &= A^{-2}u_{(a/A)(a/A)}. \quad \blacksquare \end{aligned}$$

A.2 The derivation of (8)

The use of the alternative representation of the instantaneous utility function $V(c, C, a, A) \equiv u(c, c/C, a/A)$ implies that the current-value Hamiltonian is given by

$$H = V(c, C, a, A) + \lambda(ra + wl - c).$$

The necessary optimality conditions for an interior equilibrium, $H_c = 0$ and $\dot{\lambda} = \rho\lambda - H_a$, can be written as

$$\begin{aligned} \lambda &= V_c(c, C, a, A), \\ \dot{\lambda} &= -[\lambda r + V_a(c, C, a, A) - \rho\lambda]. \end{aligned} \quad \text{(A.3)}$$

From these two first-order conditions, it follows that

$$\dot{\lambda}/\lambda = -\{r + [V_a(c, C, a, A)/V_c(c, C, a, A)] - \rho\}. \quad \text{(A.4)}$$

Obviously, Equations (A.3) and (A.4) are identical to the two equations that are given in (8).

A.3 The derivation of (12)

A simple transformation of the FOC (4),

$$\lambda = u_c(c, c/C, a/A) + u_{c/C}(c, c/C, a/A)C^{-1},$$

yields the following representation:

$$\lambda = u_c(c, c/C, a/A)\{1 + [(c/C)/c] \times [u_{c/C}(c, c/C, a/A)/u_c(c, c/C, a/A)]\}.$$

Using the definition of $m^{c/C}$ that is given by (11),

$$m^{c/C}(c, c/C, a/A) \equiv [(c/C)/c] \times [u_{c/C}(c, c/C, a/A)/u_c(c, c/C, a/A)],$$

we obtain Equation (12):

$$\lambda = u_c(c, c/C, a/A)[1 + m^{c/C}(c, c/C, a/A)]. \quad \blacksquare$$

A.4 The derivation of the transversality condition (13)

Applying simple transformations, Equation (7),

$$\dot{\lambda}/\lambda = -\left[r + \frac{u_{a/A}(c, c/C, a/A)A^{-1}}{u_c(c, c/C, a/A) + u_{c/C}(c, c/C, a/A)C^{-1}} - \rho\right],$$

can be rewritten as

$$\dot{\lambda}/\lambda = -\left[r + \frac{[(a/A)/c] \times [u_{a/A}(c, c/C, a/A)/u_c(c, c/C, a/A)]}{1 + [(c/C)/c] \times [u_{c/C}(c, c/C, a/A)/u_c(c, c/C, a/A)]} \times \frac{c}{a} - \rho\right].$$

Using the definition of m^x for $x = c/C$ and a/A, given by (11),

$$m^x = m^x(c, c/C, a/A) \equiv (x/c) \times [u_x(c, c/C, a/A)/u_c(c, c/C, a/A)],$$

we obtain

$$\dot{\lambda}/\lambda = -\left[r + \frac{m^{a/A}(c, c/C, a/A)}{1 + m^{c/C}(c, c/C, a/A)} \times \frac{c}{a} - \rho\right]. \tag{A.5}$$

Integration of (A.5) yields

$$\lambda(t) = \lambda(0)e^{\rho t} \exp\left\{-\int_0^t \left[r(v) + \frac{m^{a/A}(v)}{1 + m^{c/C}(v)} \times \frac{c(v)}{a(v)}\right] dv\right\}, \tag{A.6}$$

where $m^x(v) = m^x(c(v), c(v)/C(v), a(v)/A(v))$ for $x = c/C$ and a/A. The assumptions made in (2) that $u_c > 0$ and $u_{c/C} \geq 0$ together with the first-order condition (4), $\lambda = u_c + u_{c/C}C^{-1}$, imply that $\lambda(t) > 0$ for $t \geq 0$. Since $\lambda(0) > 0$, it follows from (A.6) that the transversality condition (6),

$$\lim_{t\to\infty} e^{-\rho t}\lambda a = 0,$$

is equivalent to the representation given by (13),

$$\lim_{t\to\infty} \exp\left\{-\int_0^t \left[r(v) + \frac{m^{a/A}(v)}{1+m^{c/C}(v)} \times \frac{c(v)}{a(v)}\right] dv\right\} a(t) = 0. \qquad \blacksquare$$

A.5 Properties of the production function and the derivation of (14)

By assumption, each firm $i \in [0,1]$ employs the same technology so that $y_i = f(k_i, Bl_i)$ for $i \in [0,1]$. Since, also by assumption, the common production function f exhibits constant returns to scale, the following equations hold for $i \in [0,1]$ (all results are well-known from intermediary microeconomics):

$$y_i = f(k_i, Bl_i) = k_i f(1, Bl_i/k_i), \tag{A.7}$$

$$f_k(k_i, Bl_i) = f_k(1, Bl_i/k_i), \qquad f_{(Bl)}(k_i, Bl_i) = f_{(Bl)}(1, Bl_i/k_i), \tag{A.8}$$

$$f(k_i, Bl_i) = f_k(k_i, Bl_i)k_i + f_{(Bl)}(k_i, Bl_i)Bl_i. \tag{A.9}$$

The equations given in (A.8) follow from the fact that the marginal products of capital f_k and effective labor $f_{(Bl)}$ are homogeneous of degree zero. Equation (A.9) results from the Euler theorem.

Real profits of firm $i \in [0,1]$ denoted by π_i are given by $\pi_i = f(k_i, Bl_i) - rk_i - wl_i$. It can be verified at first glance that the necessary optimality conditions are given by

$$r = f_k(k_i, Bl_i), \qquad w = f_{(Bl)}(k_i, Bl_i)B, \qquad i \in [0,1]. \tag{A.10}$$

Taking into account that $B = K$ holds by assumption, it is obvious from (A.10) that the first-order conditions of the representative firm can be written in the form given by (14):

$$r = f_k(k, Kl), \qquad w = f_{(Bl)}(k, Kl)K. \quad \blacksquare$$

B The decentralized solution – Part I (Section 3)

B.1 Derivation of (15)

Using (A.8) and taking into account that $B = K$, the necessary optimality conditions (A.10),

$$r = f_k(k_i, Bl_i), \qquad w = f_{(Bl)}(k_i, Bl_i)B, \qquad i \in [0,1],$$

can be rewritten as

$$r = f_k(1, Kl_i/k_i), \qquad w = f_{(Bl)}(1, Kl_i/k_i)K, \qquad i \in [0,1]. \tag{B.1}$$

The equations given in (B.1) imply that in a macroeconomic equilibrium each firm will choose the same capital-labor ratio. It is easily verified that

$$k_i/l_i = K/L, \qquad i \in [0,1], \tag{B.2}$$

where K and L denote both the aggregate and the average values of capital and labor input, respectively. Substituting (B.2) into (B.1) we obtain the two equations given in (15):

$$r = f_k(1, L), \qquad w = f_{(Bl)}(1, L)K. \quad \blacksquare$$

B.2 Derivation of the Euler equation for aggregate consumption as given by (17)–(19)

Substituting $f_k(1, L) = r$ [see (15)] as well as $c = C$ and $a = A = K$ [see (16)] into (A.5),

$$\dot{\lambda}/\lambda = -\left[r + \frac{m^{a/A}(c, c/C, a/A)}{1 + m^{c/C}(c, c/C, a/A)} \times \frac{c}{a} - \rho\right],$$

we obtain

$$\dot{\lambda}/\lambda = -\left[f_k(1, L) + \frac{m^{a/A}(C, 1, 1)}{1 + m^{c/C}(C, 1, 1)} \times \frac{C}{K} - \rho\right]. \tag{B.3}$$

Substitution of $c = C$ and $a = A$ [see (16)] into (12),

$$\lambda = u_c(c, c/C, a/A)[1 + m^{c/C}(c, c/C, a/A)],$$

yields

$$\lambda = u_c(C, 1, 1)[1 + m^{c/C}(C, 1, 1)]. \tag{B.4}$$

Differentiating (B.4) with respect to time t, we obtain

$$\dot{\lambda} = \{u_{cc}(C, 1, 1)[1 + m^{c/C}(C, 1, 1)] + u_c(C, 1, 1)m_c^{c/C}(C, 1, 1)\}\dot{C}. \tag{B.5}$$

Using (B.4) and (B.5), we obtain

$$\dot{\lambda}/\lambda = \left[\frac{Cu_{cc}(C, 1, 1)}{u_c(C, 1, 1)} + \frac{m^{c/C}(C, 1, 1)}{1 + m^{c/C}(C, 1, 1)} \times \frac{m_c^{c/C}(C, 1, 1)C}{m^{c/C}(C, 1, 1)}\right](\dot{C}/C). \tag{B.6}$$

Using the elasticities of the marginal utility of absolute consumption u_c and of the percentage MRS $m^{c/C}$ with respect to absolute consumption c,

$$\begin{aligned}
\varepsilon^{u_c,c}(c, c/C, a/A) &\equiv u_{cc}(c, c/C, a/A) \times [c/u_c(c, c/C, a/A)], \\
\varepsilon^{m^{c/C},c}(c, c/C, a/A) &\equiv m_c^{c/C}(c, c/C, a/A) \times [c/m^{c/C}(c, c/C, a/A)],
\end{aligned}$$

(B.6) can be written as

$$\dot{\lambda}/\lambda = \left[\varepsilon^{u_c,c}(C, 1, 1) + \frac{m^{c/C}(C, 1, 1)}{1 + m^{c/C}(C, 1, 1)}\varepsilon^{m^{c/C},c}(C, 1, 1)\right](\dot{C}/C). \tag{B.7}$$

Substituting (B.7) into (B.3) and solving the resulting equation for $\dot{C}/C$, we obtain the Euler equation for aggregate consumption C:

$$\begin{aligned}\dot{C}/C &= -\left[\varepsilon^{u_c,c}(C,1,1) + \frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)}\varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1} \times \\ &\quad \times \left[f_k(1,L) + \frac{m^{a/A}(C,1,1)}{1+m^{c/C}(C,1,1)} \times \frac{C}{K} - \rho\right]. \end{aligned} \tag{B.8}$$

Introducing the definitions of $\sigma^D(C)$ and $\eta^D(C)$ given by (18) and (19),

$$\begin{aligned}\sigma^D(C) &\equiv -\left[\varepsilon^{u_c,c}(C,1,1) + \frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)} \times \varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1}, \\ \eta^D(C) &\equiv m^{a/A}(C,1,1)/[1+m^{c/C}(C,1,1)],\end{aligned} \tag{B.9}$$

the Euler equation (B.8) can be written in the form given by (17):

$$\dot{C}/C = \sigma^D(C)[f_k(1,L) + \eta^D(C) \times (C/K) - \rho].$$

Applying the following simple rules for the calculation of elasticities,

$$\varepsilon^{z^1 \times z^2,x} = \varepsilon^{z^1,x} + \varepsilon^{z^2,x}, \qquad \varepsilon^{z^1+z^2,x} = [z^1/(z^1+z^2)]\varepsilon^{z^1,x} + [z^2/(z^1+z^2)]\varepsilon^{z^2,x},$$

it is easily verified that

$$\begin{aligned}\varepsilon^{\{u_c(C,1,1)\times[1+m^{c/C}(C,1,1)]\},C} &= \varepsilon^{u_c(C,1,1),C} + \varepsilon^{[1+m^{c/C}(C,1,1)],C} \\ &= \varepsilon^{u_c,c}(C,1,1) + \\ &\quad + \frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)} \times \varepsilon^{m^{c/C},c}(C,1,1).\end{aligned} \tag{B.10}$$

From (B.9) and (B.10) it then follows that $\sigma^D(C)$ is the the reciprocal of the magnitude of the elasticity of the total marginal utility of own consumption,

$$\sigma^D(C) = \left[-\varepsilon^{\{u_c(C,1,1)\times[1+m^{c/C}(C,1,1)]\},C}\right]^{-1}. \qquad \blacksquare$$

B.3 Derivation of the Euler equation for individual consumption of the representative household

From Equation (12),

$$\lambda = u_c(c, c/C, a/A)[1 + m^{c/C}(c, c/C, a/A)],$$

that gives an alternative representation of the FOC for the optimal choice of own consumption given by (4), $\lambda = u_c(c, c/C, a/A) + u_{c/C}(c, c/C, a/A)C^{-1}$, it follows that

$$\begin{aligned} \dot{\lambda}/\lambda &= \{\varepsilon^{u_c,c} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},c}\}(\dot{c}/c) \\ &\quad +\{\varepsilon^{u_c,c/C} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},c/C}\}[(\dot{c}/c) - (\dot{C}/C)] \\ &\quad +\{\varepsilon^{u_c,a/A} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},a/A}\}[(\dot{a}/a) - (\dot{A}/A)], \end{aligned} \tag{B.11}$$

where

$$\begin{aligned} m^{c/C} &= m^{c/C}(c, c/C, a/A), \\ \varepsilon^{u_c,x} &= \varepsilon^{u_c,x}(c, c/C, a/A), \quad \varepsilon^{m^{c/C},x} = \varepsilon^{m^{c/C},x}(c, c/C, a/A), \qquad x = c, c/C, a/A. \end{aligned}$$

While the derivation of Equation (B.11) seems to be complicated at first glance, its validity is easily verified by applying the following simple rules for the calculation of elasticities:

$$\varepsilon^{z^1 \times z^2,x} = \varepsilon^{z^1,x} + \varepsilon^{z^2,x}, \qquad \varepsilon^{z^1+z^2,x} = [z^1/(z^1+z^2)]\varepsilon^{z^1,x} + [z^2/(z^1+z^2)]\varepsilon^{z^2,x}.$$

Substituting the expression for $\dot{\lambda}/\lambda$ given by (B.11) into (A.5),

$$\dot{\lambda}/\lambda = -\{r + [m^{a/A}/(1+m^{c/C})] \times (c/a) - \rho\},$$

where $m^{c/C} = m^{c/C}(c, c/C, a/A)$ and $m^{a/A} = m^{a/A}(c, c/C, a/A)$, and solving for $\dot{c}/c$ we obtain the following representation of the Euler equation for the individual consumption of the representative household:

$$\begin{aligned} (\dot{c}/c) &= \sigma^h \times \{r + [m^{a/A}/(1+m^{c/C})] \times (c/a) - \rho \\ &\quad -(\varepsilon^{u_c,c/C} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},c/C})(\dot{C}/C) \\ &\quad +(\varepsilon^{u_c,a/A} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},a/A})[(\dot{a}/a) - (\dot{A}/A)]\}, \end{aligned} \tag{B.12}$$

where

$$\sigma^h \equiv -\{\varepsilon^{u_c,c} + \varepsilon^{u_c,c/C} + [m^{c/C}/(1+m^{c/C})] \times (\varepsilon^{m^{c/C},c} + \varepsilon^{m^{c/C},c/C})\}^{-1} \tag{B.13}$$

denotes the elasticity of intertemporal substitution at the level of the individual household. Please note that $\sigma^h = \sigma^h(c, c/C, a/A)$.

The Euler equation that governs the dynamic evolution of individual consumption at the level of the individual household can be used to derive the Euler equation for aggregate consumption in a symmetric macroeconomic equilibrium in which

$$c = C, \qquad a = A = K, \qquad r = f_k(1, L) \tag{B.14}$$

holds. Substitution of (B.14) into (B.12) and (B.13) yields

$$\begin{aligned}(\dot{C}/C) &= \sigma^h \times \{f_k(1,L) + [m^{a/A}/(1+m^{c/C})] \times (C/K) - \rho \\ &\qquad - (\varepsilon^{u_c,c/C} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},c/C})(\dot{C}/C)\},\end{aligned}$$

where

$$\begin{aligned}m^{c/C} &= m^{c/C}(C,1,1), \quad m^{a/A} = m^{a/A}(C,1,1), \\ \varepsilon^{u_c,x} &= \varepsilon^{u_c,x}(C,1,1), \quad \varepsilon^{m^{c/C},x} = \varepsilon^{m^{c/C},x}(C,1,1), \qquad x = c, c/C, a/A,\end{aligned}$$

so that also $\sigma^h = \sigma^h(C,1,1)$. Solving for $\dot{C}/C$ we obtain

$$\begin{aligned}\dot{C}/C &= -\{\varepsilon^{u_c,c} + [m^{c/C}/(1+m^{c/C})] \times \varepsilon^{m^{c/C},c}\}^{-1} \times \\ &\qquad \times \{f_k(1,L) + [m^{a/A}/(1+m^{c/C})] \times (C/K) - \rho\}.\end{aligned}$$

Obviously, this representation is equivalent to that given by (17),

$$\dot{C}/C = \sigma^D(C)[f_k(1,L) + \eta^D(C) \times (C/K) - \rho],$$

where the definitions of $\sigma^D(C)$ and $\eta^D(C)$ are given by (18) and (19),

$$\begin{aligned}\sigma^D(C) &\equiv -\left[\varepsilon^{u_c,c}(C,1,1) + \frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)} \times \varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1}, \\ \eta^D(C) &\equiv m^{a/A}(C,1,1)/[1+m^{c/C}(C,1,1)].\end{aligned}$$

Please note that four elasticities that are present in the Euler equation for the individual consumption of the representative household, namely $\varepsilon^{u_c,c/C}$, $\varepsilon^{u_c,a/A}$, $\varepsilon^{m^{c/C},c/C}$, and $\varepsilon^{m^{c/C},a/A}$, disappear in the symmetric macroeconomic equilibrium.

B.4 Derivation of (20)

Substitution of $B = K$ and (B.2), $k_i/l_i = K/L$ for $i \in [0,1]$, into $y_i = k_i f(1, Bl_i/k_i)$ [see (A.7)] yields $y_i = k_i f(1,L)$ for $i \in [0,1]$. This, in turn, implies that aggregate output Y is given by

$$Y = f(1,L)K. \tag{B.15}$$

Using the Euler theorem (A.9) and the necessary optimality conditions (A.10), we obtain $y_i = rk_i + wl_i$. Since the adding-up theorem holds at the level of the individual firm, it holds at the aggregate level, too:

$$Y = rK + wL. \tag{B.16}$$

Combining (B.15) and (B.16), we obtain

$$rK + wL = Y = f(1,L)K. \tag{B.17}$$

Substitution of (16), i.e., $c = C$, $a = A = K$, and $l = L$, into the flow budget constraint (1), $\dot{a} = ra + wl - c$, yields

$$\dot{K} = rK + wL - C. \tag{B.18}$$

From (B.17) and (B.18) it then follows that

$$\dot{K} = Y - C = f(1, L)K - C. \tag{B.19}$$

Dividing both sides of (B.19) by K, we obtain (20):

$$\dot{K}/K = f(1, L) - C/K. \quad \blacksquare$$

B.5 Derivation of (22)

Using (A.7)–(A.9),

$$y_i = f(k_i, Bl_i) = k_i f(1, Bl_i/k_i),$$

$$f_k(k_i, Bl_i) = f_k(1, Bl_i/k_i), \qquad f_{(Bl)}(k_i, Bl_i) = f_{(Bl)}(1, Bl_i/k_i),$$

$$f(k_i, Bl_i) = f_k(k_i, Bl_i)k_i + f_{(Bl)}(k_i, Bl_i)Bl_i,$$

the following equation is easily derived:

$$f(1, Bl_i/k_i) = f_k(1, Bl_i/k_i) + f_{(Bl)}(1, Bl_i/k_i)(Bl_i/k_i). \tag{B.20}$$

Substitution of (B.2), $k_i/l_i = K/L$ for $i \in [0, 1]$, and $B = K$ into (B.20) yields

$$f(1, L) = f_k(1, L) + f_{(Bl)}(1, L)L.$$

Taking into account that $f_{(Bl)} > 0$, we obtain (22):

$$f(1, L) > f_k(1, L). \quad \blacksquare$$

B.6 Extended Proof of Proposition 1

In Part I we give a proof for all assertions made in proposition 1. In Part II we derive the conditions for the occurrence of excessive wealth accumulation in the sense that the transversality condition of the standard model is violated. In Part III we show that the decentralized solution has no transitional dynamics.

Part I: Assumption (23),

$$\sigma^D(C) = \hat{\sigma}, \qquad \eta^D(C) = \hat{\eta}, \qquad \forall C > 0, \tag{B.21}$$

where $\hat{\sigma} > 0$ and $\hat{\eta} \geq 0$ are constants, implies that the Euler equation for aggregate consumption (17) simplifies to

$$\dot{C}/C = \hat{\sigma}[f_k(1, L) + \hat{\eta} \times (C/K) - \rho]. \tag{B.22}$$

The differential equation for aggregate capital given by (20) is unaffected by the assumptions made in (23) [= (B.21)]. Consequently, we have

$$\dot{K}/K = f(1, L) - (C/K). \tag{B.23}$$

Taking into account that L is exogenously given and constant over time and that both $\hat{\sigma}$ and $\hat{\eta}$ are constants, it is obvious from (B.22) and (B.23) that a balanced growth path (BGP) exists in which C and K grow at the same constant rate so that C/K remains unchanged over time. The steady-state value of the common growth rate of aggregate consumption and aggregate physical capital denoted by $g^D = (\dot{C}/C)^D = (\dot{K}/K)^D$ and the steady value of the consumption-capital ratio denoted by $(C/K)^D$ are determined by the two equations that are given in (27):

$$g^D = \hat{\sigma}[f_k(1, L) + \hat{\eta} \times (C/K)^D - \rho] \quad \text{and} \quad (C/K)^D = f(1, L) - g^D. \tag{B.24}$$

Solving the two equations given in (B.24) for g^D, we obtain (25):

$$g^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1, L) - \rho + \hat{\eta} f(1, L)]. \tag{B.25}$$

The first equation given in (26), $(C/K)^D = f(1, L) - g^D$, is identical to the second equation given in (B.24). The second equation given in (26), $(\dot{K}/Y)^D = g^D/f(1, L)$, is easily obtained by using the following three facts:

$$(\dot{K}/Y) = (\dot{K}/K)/(Y/K), \qquad Y = f(1, L)K, \qquad (\dot{K}/K)^D = g^D.$$

Substitution of the solution for g^D given by (B.25) into $(C/K)^D = f(1, L) - g^D$ yields

$$(C/K)^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}\{(1/\hat{\sigma})f(1, L) - [f_k(1, L) - \rho]\}. \tag{B.26}$$

From (B.25) it is obvious that

$$g^D > 0 \Leftrightarrow \rho < f_k(1, L) + \hat{\eta} f(1, L) \equiv \rho^g. \tag{B.27}$$

From (B.26) it follows that

$$(C/K)^D > 0 \Leftrightarrow \rho > f_k(1, L) - (1/\hat{\sigma})f(1, L) \equiv \rho^{C/K}. \tag{B.28}$$

Since, by assumption, $\eta^D(C) = \hat{\eta} \geq 0$ holds for $\forall C > 0$, the transversality condition (21) simplifies to

$$\lim_{t\to\infty} \exp\left(-\int_0^t \{f_k(1, L) + \hat{\eta} \times [C(v)/K(v)]\}dv\right) K(t) = 0.$$

Along the BGP, we have $C/K = (C/K)^D$ and $\dot{K}/K = g^D$ at any point in time. Hence, the transversality condition requires that $-[f_k(1, L) + \hat{\eta} \times (C/K)^D] + g^D < 0$. Using the fact that

$$-[f_k(1, L) + \hat{\eta} \times (C/K)^D] + g^D = -[(1/\hat{\sigma}) + \hat{\eta}]^{-1}\{[(1/\hat{\sigma}) - 1]\,[f_k(1, L) + \hat{\eta} f(1, L)] + (1 + \hat{\eta})\rho\},$$

we obtain

$$\begin{aligned} &-[f_k(1,L) + \hat{\eta} \times (C/K)^D] + g^D < 0 \Leftrightarrow \\ &\rho > [1 - (1/\hat{\sigma})](1+\hat{\eta})^{-1}[f_k(1,L) + \hat{\eta} f(1,L)] \equiv \rho^{TC}, \end{aligned} \tag{B.29}$$

where the superscript "TC" stands for "transversality condition". Obviously,

$$\rho^{TC} = [1 - (1/\hat{\sigma})](1+\hat{\eta})^{-1}\rho^g. \tag{B.30}$$

From (B.27) it follows that $\rho^g > 0$. By contrast, both $\rho^{C/K}$ and ρ^{TC} [see (B.28) and (B.29) or (B.30)] may be of either sign. It is easily verified that

$$\rho^{TC} - \rho^{C/K} = (1+\hat{\eta})^{-1}[(1/\hat{\sigma}) + \hat{\eta}][f(1,L) - f_k(1,L)].$$

Taking into account that $f(1,L) > f_k(1,L)$ [see (22)] it is clear that $\rho^{C/K} < \rho^{TC}$. Moreover, we have

$$\rho^g - \rho^{TC} = (1+\hat{\eta})^{-1}[(1/\hat{\sigma}) + \hat{\eta}]\,[f_k(1,L) + \hat{\eta} f(1,L)] > 0.$$

These results imply that $\rho^{C/K} < \rho^{TC} < \rho^g$ holds. Hence, if the condition

$$\rho^{TC} < \rho < \rho^g \tag{B.31}$$

is satisfied (so that also $\rho^{C/K} < \rho$ holds due to the fact that $\rho^{C/K} < \rho^{TC}$), then the solutions given by (B.25) and (B.26) are economically meaningful in the sense that 1) the common growth rate g^D is strictly positive (due to $\rho < \rho^g$), 2) the consumption-capital ratio $(C/K)^D$ is strictly positive (due to $\rho^{C/K} < \rho$), and 3) the transversality condition (21) is satisfied (due to $\rho^{TC} < \rho$). Using (B.27) and (B.30) the condition (B.31) can be written in a form that is identical to the condition (24):

$$[1 - (1/\hat{\sigma})](1+\hat{\eta})^{-1}\rho^g < \rho < \rho^g, \qquad \rho^g \equiv f_k(1,L) + \hat{\eta} f(1,L). \tag{B.32}$$

Please note that if $\hat{\sigma} < 1$ holds, then the lower bound for ρ given by $[1 - (1/\hat{\sigma})](1+\hat{\eta})^{-1}\rho^g$ is negative and, hence, redundant since $\rho > 0$ has to hold by assumption anyway. ■

Part II: Next, we give the condition for the occurrence of excessive wealth accumulation in the sense that the transversality condition of the standard model as given by

$$\lim_{t\to\infty} \exp\left(-\int_0^t f_k(1,L)dv\right) K(t) = 0$$

is violated. Along the BGP, we have $\dot{K}/K = g^D$ at any point in time. Since L is exogenously given and constant over time, the fulfillment of the standard transversality condition obviously requires that $-f_k(1,L) + g^D < 0$. Using the solution for g^D given by (B.25),

$$g^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1,L) - \rho + \hat{\eta} f(1,L)],$$

we obtain

$$-f_k(1,L) + g^D < 0 \Leftrightarrow \rho > [1 - (1/\hat{\sigma})]f_k(1,L) + \hat{\eta}[f(1,L) - f_k(1,L)] \equiv (\rho^{TC})^{\text{stan}}, \tag{B.33}$$

where the superscripts "TC" and "stan" stand for "transversality condition" and "standard". Please note that $(\rho^{TC})^{\text{stan}}$ may be of either sign. Using (B.33) as well as (B.27), (B.29), and (B.30),

$$\begin{aligned} \rho^g &\equiv f_k(1,L) + \hat{\eta} f(1,L), \\ \rho^{TC} &\equiv [1-(1/\hat{\sigma})](1+\hat{\eta})^{-1}[f_k(1,L)+\hat{\eta} f(1,L)] \\ &= [1-(1/\hat{\sigma})](1+\hat{\eta})^{-1}\rho^g, \end{aligned}$$

it is easily verified that

$$\begin{aligned} (\rho^{TC})^{\text{stan}} - \rho^{TC} &= (1+\hat{\eta})^{-1}\hat{\eta}[\hat{\eta}+(1/\hat{\sigma})][f(1,L)-f_k(1,L)] \geq 0, \\ \rho^g - (\rho^{TC})^{\text{stan}} &= [(1/\hat{\sigma})+\hat{\eta}]f_k(1,L) > 0. \end{aligned}$$

If $\hat{\eta} > 0$, then these results imply that

$$\rho^{TC} < (\rho^{TC})^{\text{stan}} < \rho^g.$$

Consequently, if

$$(\rho^{TC})^{\text{stan}} < \rho < \rho^g$$

holds (in addition to $\rho > 0$), then the decentralized solution is economically meaningful and, in addition, satisfies the standard version of the transversality condition. By contrast, the economically meaningful decentralized solution exhibits excessive wealth accumulation, if

$$0 < \rho^{TC} < \rho \leq (\rho^{TC})^{\text{stan}} \qquad \text{or} \qquad \rho^{TC} < 0 < \rho \leq (\rho^{TC})^{\text{stan}}$$

holds. On the one hand, $\rho \leq (\rho^{TC})^{\text{stan}}$ means that households are sufficiently patient so that the standard transversality condition is violated. On the other hand, $\rho^{TC} < \rho$ implies that agents are sufficiently impatient so that the modified transversality condition holds. ■

Part III: We show that if the condition (23) [= (B.21)] is satisfied so that $\sigma^D(C) = \hat{\sigma} > 0$ and $\eta^D(C) = \hat{\eta} \geq 0$ hold for $C > 0$, then the model has no transitional dynamics. Let $Z \equiv C/K$. Since K is a state variable and C is a control variable, $Z = C/K$ is a control-like variable (this notion is used by Barro and Sala-i-Martin (1995) on p. 162). In contrast to K, both C and $Z = C/K$ can jump at any point in time. Using (B.22), (B.23), and $C/K = Z$, we obtain

$$\dot{C}/C = \hat{\sigma}[f_k(1,L) + \hat{\eta} \times Z - \rho], \tag{B.34}$$
$$\dot{K}/K = f(1,L) - Z, \tag{B.35}$$

which, in turn implies that

$$\dot{Z} = [(\dot{C}/C) - (\dot{K}/K)]Z = -\{f(1,L) - \hat{\sigma}[f_k(1,L) - \rho] - (1+\hat{\sigma}\hat{\eta})Z\}Z \equiv \Phi(Z).$$

Solving $\dot{Z} = \Phi(Z) = 0$ for Z, we obtain $\{Z = 0\}$ and $\{Z = Z^D\}$, where

$$Z^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}\{(1/\hat{\sigma})f(1,L) - [f_k(1,L) - \rho]\}. \tag{B.36}$$

Obviously, Z^D given by (B.36) is identical to $(C/K)^D$ given by (B.26). If the condition (B.32) [= (24)] is satisfied, then $Z^D = (C/K)^D > 0$, so that Z^D is the economically meaningful steady state value of the consumption-capital ratio. From

$$\Phi'(Z) = (1+\hat{\sigma}\hat{\eta})Z - \{f(1,L) - \hat{\sigma}[f_k(1,L) - \rho] - (1+\hat{\sigma}\hat{\eta})Z\}$$

and (B.36) it follows that

$$\Phi'(Z^D) = (1+\hat{\sigma}\hat{\eta})Z^D > 0,$$

because the expression within curly brackets vanishes. $\Phi'(Z^D) > 0$ implies that the economically meaningful steady state of the differential equation $\dot{Z} = \Phi(Z)$ is unstable. Hence, the equilibrium path of Z has no transitional dynamics, i.e., $Z(t) = Z^D$ for $t \geq 0$. The initial value of the jump variable Z has to be chosen in such a way that $Z(0) = Z^D$. From $Z = C/K$ and $Z^D = (C/K)^D$ it then follows that the initial value of the jump variable C has to be chosen according to $C(0) = (C/K)^D \times K_0$, where $(C/K)^D$ is given by (B.26) and K_0 is exogenously given. Since $Z(t) = Z^D$ holds for $t \geq 0$, it then follows from (B.34), (B.35), (B.36), and (B.25) that

$$\begin{aligned}
\dot{C}/C &= \hat{\sigma}[f_k(1,L) + \hat{\eta} \times Z^D - \rho] \\
&= [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1,L) - \rho + \hat{\eta}f(1,L)] = g^D, \\
\dot{K}/K &= f(1,L) - Z^D \\
&= [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1,L) - \rho + \hat{\eta}f(1,L)] = g^D
\end{aligned}$$

hold for $t \geq 0$. The growth rates of C and K are constant over time, identical, and equal to g^D. Consequently, these growth rates have no transitional dynamics. ■

B.7 Proof of Proposition 2

We restrict our attention to a proof of the mathematical results presented in (28)–(30). Taking the partial derivatives of g^D as given by (25) [= (B.25)],

$$g^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1,L) - \rho + \hat{\eta}f(1,L)] > 0,$$

with respect to $\hat{\sigma}$ and $\hat{\eta}$, we obtain

$$\begin{aligned}
\partial g^D/\partial\hat{\sigma} &= \hat{\sigma}^{-2}[(1/\hat{\sigma}) + \hat{\eta}]^{-2}[f_k(1,L) - \rho + \hat{\eta}f(1,L)] \\
&= \hat{\sigma}^{-2}[(1/\hat{\sigma}) + \hat{\eta}]^{-1}g^D > 0, \qquad \text{(B.37)} \\
\partial g^D/\partial\hat{\eta} &= [(1/\hat{\sigma}) + \hat{\eta}]^{-2}\{(1/\hat{\sigma})f(1,L) - [f_k(1,L) - \rho]\} \\
&= [(1/\hat{\sigma}) + \hat{\eta}]^{-1}(C/K)^D > 0. \qquad \text{(B.38)}
\end{aligned}$$

Using (B.37)–(B.38) and taking into account that

$$(C/K)^D = f(1,L) - g^D \qquad \text{and} \qquad (\dot{K}/Y)^D = g^D/f(1,L)$$

hold according to (26), we obtain

$$\begin{aligned}
\partial(\dot{K}/Y)^D/\partial\hat{\sigma} &= [f(1,L)]^{-1}(\partial g^D/\partial\hat{\sigma}) > 0, && \text{(B.39)}\\
\partial(\dot{K}/Y)^D/\partial\hat{\eta} &= [f(1,L)]^{-1}(\partial g^D/\partial\hat{\eta}) > 0, && \text{(B.40)}\\
\partial(C/K)^D/\partial\hat{\sigma} &= -\partial g^D/\partial\hat{\sigma} < 0, && \text{(B.41)}\\
\partial(C/K)^D/\partial\hat{\eta} &= -\partial g^D/\partial\hat{\eta} < 0. && \text{(B.42)}
\end{aligned}$$

Substituting the expression for g^D given by (25) [= (B.25)] into $(C/K)^D = f(1,L) - g^D$ yields the representation of the solution for $(C/K)^D$ as given by (B.26):

$$(C/K)^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}\{(1/\hat{\sigma})f(1,L) - [f_k(1,L) - \rho]\} > 0.$$

The resulting solution for the CIER,

$$\hat{\eta} \times (C/K)^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}\hat{\eta}\{(1/\hat{\sigma})f(1,L) - [f_k(1,L) - \rho]\} \geq 0,$$

has the following properties:

$$\begin{aligned}
\partial[\hat{\eta} \times (C/K)^D]/\partial\hat{\eta} &= \hat{\sigma}^{-1}[(1/\hat{\sigma}) + \hat{\eta}]^{-2}\{(1/\hat{\sigma})f(1,L) - [f_k(1,L) - \rho]\}\\
&= \hat{\sigma}^{-1}[(1/\hat{\sigma}) + \hat{\eta}]^{-1}(C/K)^D > 0, && \text{(B.43)}
\end{aligned}$$

$$\begin{aligned}
\partial[\hat{\eta} \times (C/K)^D]/\partial\hat{\sigma} &= -\hat{\sigma}^{-2}[(1/\hat{\sigma}) + \hat{\eta}]^{-2}\hat{\eta}[f_k(1,L) - \rho + \hat{\eta}f(1,L)]\\
&= -\hat{\sigma}^{-2}[(1/\hat{\sigma}) + \hat{\eta}]^{-1}\hat{\eta}g^D.
\end{aligned}$$

From the last result and $g^D > 0$ it follows that

$$\text{sgn}(\partial[\hat{\eta} \times (C/K)^D]/\partial\hat{\sigma}) = -\text{sgn}(\hat{\eta}). \qquad \text{(B.44)}$$

The validity of the assertions made in (28),

$$\partial g^D/\partial\hat{\sigma} > 0, \qquad \partial(\dot{K}/Y)^D/\partial\hat{\sigma} > 0, \qquad \partial(C/K)^D/\partial\hat{\sigma} < 0,$$

follows from (B.37), (B.39), and (B.41). Analogously, the validity of the assertions made in (29),

$$\partial g^D/\partial\hat{\eta} > 0, \qquad \partial(\dot{K}/Y)^D/\partial\hat{\eta} > 0, \qquad \partial(C/K)^D/\partial\hat{\eta} < 0,$$

is obvious from (B.38), (B.40), and (B.42). Finally, (30),

$$\partial[\hat{\eta} \times (C/K)^D]/\partial\hat{\eta} > 0, \qquad \text{sgn}(\partial[\hat{\eta} \times (C/K)^D]/\partial\hat{\sigma}) = -\text{sgn}(\hat{\eta}),$$

is obtained by using (B.43) and (B.44). ■

B.8 Proof of Proposition 3

Proof of (32)

In (31) we make the following assumptions:

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \forall C > 0,$$

where $\hat{m}^{c/C} \geq 0$, $\hat{m}^{a/A} \geq 0$ (with $\hat{m}^{c/C} > 0 \vee \hat{m}^{a/A} > 0$), and $\hat{\varepsilon}^{u_c,c} < 0$ are constants. From $m^{c/C}(C,1,1) = \hat{m}^{c/C}$, $\forall C > 0$, it then follows that $m_c^{c/C}(C,1,1) = 0$, $\forall C > 0$. Taking into account that $\varepsilon^{m^{c/C},c} \equiv m_c^{c/C} \times (c/m^{c/C})$, we also have $\varepsilon^{m^{c/C},c}(C,1,1) = 0$, $\forall C > 0$. Substituting the latter result and the assumptions made in (31) into the definitions of $\sigma^D(C)$ and $\eta^D(C)$ given by (18) and (19),

$$\begin{aligned}
\sigma^D(C) &\equiv -\left[\varepsilon^{u_c,c}(C,1,1) + \frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)} \times \varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1}, \\
\eta^D(C) &\equiv m^{a/A}(C,1,1)/[1+m^{c/C}(C,1,1)],
\end{aligned}$$

we obtain (32),

$$\sigma^D(C) = 1/|\hat{\varepsilon}^{u_c,c}| \equiv \hat{\sigma} > 0, \quad \eta^D(C) = \hat{m}^{a/A}/(1+\hat{m}^{c/C}) \equiv \hat{\eta} \geq 0, \quad \forall C > 0. \blacksquare$$

Proof of (33)

From Proposition 1 we know that if these results for $\hat{\sigma}$ and $\hat{\eta}$ satisfy condition (24),

$$[1-(1/\hat{\sigma})](1+\hat{\eta})^{-1}\rho^g < \rho < \rho^g, \qquad \rho^g \equiv f_k(1,L) + \hat{\eta} f(1,L),$$

where $\rho > 0$ holds by assumption, then an economically meaningful decentralized BGP exists. To calculate the corresponding BGP growth, we substitute the expressions for $\hat{\sigma}$ and $\hat{\eta}$ into Equation (25) given in Proposition 1,

$$g^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1,L) - \rho + \hat{\eta} f(1,L)].$$

In doing so, we finally obtain (33):

$$g^D = \frac{f_k(1,L) - \rho + [\hat{m}^{a/A}/(1+\hat{m}^{c/C})] \times f(1,L)}{|\hat{\varepsilon}^{u_c,c}| + [\hat{m}^{a/A}/(1+\hat{m}^{c/C})]}. \blacksquare$$

B.9 An extended version of Proposition 5

Recall that we use $\varepsilon^{z,x_i} \equiv (\partial z/\partial x_i) \times (x_i/z)$ to denote the elasticity of z with respect to x_i, where $z = z(x_1, \ldots, x_n)$ is an arbitrary function of arbitrary variables x_i, $i = 1, \ldots, n$.

Proposition 8. *(Extended version of Proposition 5)*

Let the instantaneous utility function u result from the transformation T of a multiplicatively separable function v,

$$u(c, c/C, a/A) = T[v(c, c/C, a/A)], \qquad v(c, c/C, a/A) = P(c)Q(c/C, a/A), \tag{B.45}$$

where the functions $T(v)$, $P(c)$, and $Q(c/C, a/A)$ satisfy the following assumptions over their *corresponding domains:*

$$T' > 0, \quad T'' < 0; \quad P > 0, \quad P' > 0, \quad \varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c} < 0, \tag{B.46}$$

$$\begin{aligned} & Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0; \\ & \text{if } Q_{c/C} > 0, \text{ then } \varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C} < 0; \\ & \text{if } Q_{a/A} > 0, \text{ then } \varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A} < 0. \end{aligned} \tag{B.47}$$

A) The assumptions made in (B.46) and (B.47) are sufficient for the well-behavedness of the utility function (B.45) in the sense that it satisfies all assumptions made in (2).

B) The representation $V(c, C, a, A) \equiv u(c, c/C, a/A)$ is well-behaved in the sense that all assumptions made in (3) are satisfied, if, in addition to (B.46) and (B.47), the conditions

$$\begin{aligned} 0 \;>\; & \varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2\varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\ & + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C}), \end{aligned} \tag{B.48}$$

$$\begin{aligned} 0 \;<\; & \varepsilon^{Q,a/A} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}) \times \\ & \times \{\varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2 \times \varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\ & + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C})\} \\ & - [\varepsilon^{P,c}\varepsilon^{Q,a/A} \times (\varepsilon^{T',v} + 1) + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{c/C},a/A})]^2 \end{aligned} \tag{B.49}$$

are satisfied, where (B.49) is only relevant in case that relative wealth matters for utility so that $Q_{a/A} > 0$.

C) The instantaneous utility function given by (B.45) has the property that

$$m^{c/C}(C, 1, 1) = \hat{m}^{c/C}, \quad m^{a/A}(C, 1, 1) = \hat{m}^{a/A}, \quad \eta^D(C) = \hat{\eta}, \quad \forall C > 0, \tag{B.50}$$

where $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, and $\hat{\eta}$ are constants, if and only if the function $P(c)$ has the form

$$P(c) = \xi_0 c^{\xi_1}, \qquad \text{for } c > 0. \tag{B.51}$$

The assumptions $P > 0$ and $P' > 0$ listed in (B.46) require that

$$\xi_0 > 0, \qquad \xi_1 > 0. \tag{B.52}$$

D) Let the instantaneous utility function have the form

$$u(c, c/C, a/A) = T(\xi_0 c^{\xi_1} Q(c/C, a/A)), \qquad \xi_0 > 0, \quad \xi_1 > 0, \tag{B.53}$$

that is obtained by substituting (B.51) into (B.45) and taking into account (B.52). The specification of $u = u(c, c/C, a/A)$ given by (B.53) has the property that

$$\varepsilon^{u_c,c}(C, 1, 1) = \hat{\varepsilon}^{u_c,c}, \quad \sigma^D(C) = \hat{\sigma}, \qquad \forall C > 0, \tag{B.54}$$

where $\hat{\varepsilon}^{u_c,c}$ and $\hat{\sigma}$ are constants, if and only if the function $T(v)$ has the form

$$T(v) = \kappa_0 + \kappa_1(1-\theta)^{-1}(v^{1-\theta}-1), \qquad \text{for } v > 0. \tag{B.55}$$

The assumptions $T' > 0$, $T'' < 0$, and $\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c} < 0$ made in (B.46) require that

$$\kappa_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\xi_1 > 0. \tag{B.56}$$

E) Let the instantaneous utility function have the form

$$u(c, c/C, a/A) = \kappa_0 + \kappa_1(1-\theta)^{-1}\{[\xi_0 c^{\xi_1} Q(c/C, a/A)]^{1-\theta} - 1\}, \tag{B.57}$$

that results from the substitution of (B.51) and (B.55) into (B.45). Let the parameters satisfy the conditions given in (B.52) and (B.56),

$$\kappa_1 > 0, \quad \theta > 0, \quad \xi_0 > 0, \quad \xi_1 > 0, \quad 1 + (\theta-1)\xi_1 > 0, \tag{B.58}$$

and the function $Q(c/C, a/A)$ have the property that

$$\begin{array}{l} Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0, \\ \text{if } Q_{c/C} > 0, \text{ then } \varepsilon^{Q_{c/C},c/C} - \theta\varepsilon^{Q,c/C} < 0, \\ \text{if } Q_{a/A} > 0, \text{ then } \varepsilon^{Q_{a/A},a/A} - \theta\varepsilon^{Q,a/A} < 0, \end{array} \tag{B.59}$$

where the second and the third line are obtained by substituting $\varepsilon^{T',v} = -\theta$ into the corresponding lines of (B.47).

i) The instantaneous utility function $u = u(c, c/C, a/A)$ given by (B.57) is well-behaved in the sense that all assumptions made in (2) are satisfied.

ii) The representation $V(c, C, a, A) \equiv u(c, c/C, a/A)$ is well-behaved in the sense that all assumptions made in (3) are satisfied, if, in addition to (B.58) and (B.59), the conditions

$$0 < [1 + \xi_1(\theta-1)]\xi_1 + \varepsilon^{Q,c/C}[2(\theta-1)\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}], \tag{B.60}$$

$$\begin{array}{rcl} 0 & < & \{[1 + \xi_1(\theta-1)]\xi_1 + \varepsilon^{Q,c/C}[2(\theta-1)\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}]\} \times \\ & & \times \varepsilon^{Q,a/A}(\theta\varepsilon^{Q,a/A} - \varepsilon^{Q_{a/A},a/A}) \\ & & -\{[(1-\theta)\xi_1 - \theta\varepsilon^{Q,c/C}]\varepsilon^{Q,a/A} + \varepsilon^{Q,c/C}\varepsilon^{Q_{c/C},a/A}\}^2 \end{array} \tag{B.61}$$

are satisfied, where (B.61) is only relevant in case that relative wealth matters for utility so that $Q_{a/A} > 0$.

iii) The conditions given by (31) in Proposition 3 are satisfied, since $m^{c/C}(C,1,1) = \hat{m}^{c/C}$, $m^{a/A}(C,1,1) = \hat{m}^{a/A}$, and $\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}$ hold for $C > 0$, where

$$\hat{m}^{c/C} \equiv \hat{\varepsilon}^{Q,c/C}/\xi_1 \geq 0, \quad \hat{m}^{a/A} \equiv \hat{\varepsilon}^{Q,a/A}/\xi_1 \geq 0, \quad \hat{\varepsilon}^{u_c,c} \equiv -[1 + (\theta-1)\xi_1] < 0, \tag{B.62}$$

with

$$\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1,1), \quad \hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1,1). \tag{B.63}$$

iv) The conditions given in (23) in Proposition 1 are satisfied, because $\sigma^D(C) = \hat{\sigma}$ *and* $\eta^D(C) = \hat{\eta}$ *hold for* $C > 0$*, where*

$$\hat{\sigma} = 1/[1 + (\theta - 1)\xi_1] > 0, \quad \hat{\eta} = (\hat{\varepsilon}^{Q,a/A}/\xi_1)/[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)] \geq 0. \tag{B.64}$$

If these constants $\hat{\sigma}$ *and* $\hat{\eta}$ *satisfy the condition (24),*

$$[1 - (1/\hat{\sigma})](1 + \hat{\eta})^{-1}\rho^g < \rho < \rho^g, \qquad \rho^g \equiv f_k(1, L) + \hat{\eta} f(1, L),$$

where $\rho > 0$ *holds by assumption, then an economically meaningful decentralized BGP exists. The corresponding constant common growth rate is given by*

$$g^D = \frac{f_k(1, L) - \rho + (\hat{\varepsilon}^{Q,a/A}/\xi_1)[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)]^{-1} \times f(1, L)}{1 + (\theta - 1)\xi_1 + (\hat{\varepsilon}^{Q,a/A}/\xi_1)[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)]^{-1}} > 0. \tag{B.65}$$

v) In the instantaneous utility function given by (B.57), we can set, without loss of generality, $\kappa_0 = 0$*,* $\kappa_1 = 1$*, and* $\xi_0 = 1$ *so that*

$$u(c, c/C, a/A) = (1 - \theta)^{-1}\{[c^{\xi_1} Q(c/C, a/A)]^{1-\theta} - 1\}. \tag{B.66}$$

Please note that the representation of the instantaneous utility function given by (B.66) is identical to the specification that is used in Proposition 5 [see (37)]. Moreover, it is easily verified that all assumptions and assertions made in Proposition 5 are elements of item E).

Proof

Preliminaries

The specification of the instantaneous utility function $u = u(c, c/C, a/A)$ given by (B.45),

$$u(c, c/C, a/A) = T[v(c, c/C, a/A)], \qquad v(c, c/C, a/A) = P(c)Q(c/C, a/A),$$

implies that

$$\begin{aligned}
u_c &= T' \times P' \times Q, && \text{(B.67)}\\
u_{c/C} &= T' \times P \times Q_{c/C}, && \text{(B.68)}\\
u_{a/A} &= T' \times P \times Q_{a/A}, && \text{(B.69)}\\
u_{cc} &= T'' \times (P' \times Q)^2 + T' \times P'' \times Q, && \text{(B.70)}\\
u_{c(c/C)} &= T'' \times P \times Q_{c/C} \times P' \times Q + T' \times P' \times Q_{c/C}, && \text{(B.71)}\\
u_{c(a/A)} &= T'' \times P \times Q_{a/A} \times P' \times Q + T' \times P' \times Q_{a/A}, && \text{(B.72)}\\
u_{(c/C)(c/C)} &= T'' \times (P \times Q_{c/C})^2 + T' \times P \times Q_{(c/C)(c/C)}, && \text{(B.73)}\\
u_{(c/C)(a/A)} &= T'' \times P \times Q_{a/A} \times P \times Q_{c/C} + T' \times P \times Q_{(c/C)(a/A)}, && \text{(B.74)}\\
u_{(a/A)(a/A)} &= T'' \times (P \times Q_{a/A})^2 + T' \times P \times Q_{(a/A)(a/A)}, && \text{(B.75)}
\end{aligned}$$

where i) P and P' are functions of c, ii) Q, $Q_{c/C}$, $Q_{a/A}$, $Q_{(c/C)(c/C)}$, $Q_{(c/C)(a/A)}$, and $Q_{(a/A)(a/A)}$ are functions of $(c/C, a/A)$, and iii) T' and T'' are functions of $v = v(c, c/C, a/A)$.

Using the elasticities

$$\begin{aligned}
\varepsilon^{P,c} &\equiv P' \times (c/P), \qquad \varepsilon^{P',c} \equiv P'' \times (c/P'), && \text{(B.76)}\\
\varepsilon^{T',v} &\equiv T'' \times (v/T'), && \text{(B.77)}\\
\varepsilon^{Q,c/C} &\equiv Q_{c/C} \times [(c/C)/Q], \qquad \varepsilon^{Q,a/A} \equiv Q_{a/A} \times [(a/A)/Q], && \text{(B.78)}\\
\varepsilon^{Q_{c/C},c/C} &\equiv Q_{(c/C)(c/C)} \times [(c/C)/Q_{c/C}], && \text{(B.79)}\\
\varepsilon^{Q_{c/C},a/A} &\equiv Q_{(c/C)(a/A)} \times [(a/A)/Q_{c/C}], && \text{(B.80)}\\
\varepsilon^{Q_{a/A},a/A} &\equiv Q_{(a/A)(a/A)} \times [(a/A)/Q_{a/A}], && \text{(B.81)}
\end{aligned}$$

where i) $\varepsilon^{P,c}$ and $\varepsilon^{P',c}$ are functions of c, ii) $\varepsilon^{Q,c/C}$, $\varepsilon^{Q,a/A}$, $\varepsilon^{Q_{c/C},c/C}$, $\varepsilon^{Q_{c/C},a/A}$ and $\varepsilon^{Q_{a/A},a/A}$ are functions of $(c/C, a/A)$, and iii) $\varepsilon^{T',v}$ is a function of $v = v(c, c/C, a/A) = P(c)Q(c/C, a/A)$, Equations (B.67)–(B.75) can be rewritten as follows:

$$\begin{aligned}
u_c &= c^{-1} \times T' \times P \times Q \times \varepsilon^{P,c}, && \text{(B.82)}\\
u_{c/C} &= (c/C)^{-1} \times T' \times P \times Q \times \varepsilon^{Q,c/C}, && \text{(B.83)}\\
u_{a/A} &= (a/A)^{-1} \times T' \times P \times Q \times \varepsilon^{Q,a/A}, && \text{(B.84)}\\
u_{cc} &= c^{-2} \times T' \times P \times Q \times \varepsilon^{P,c} \times (\varepsilon^{T',v} \times \varepsilon^{P,c} + \varepsilon^{P',c}), && \text{(B.85)}\\
u_{c(c/C)} &= c^{-1}(c/C)^{-1} \times T' \times P \times Q \times \varepsilon^{P,c} \times \varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1), && \text{(B.86)}\\
u_{c(a/A)} &= c^{-1}(a/A)^{-1} \times T' \times P \times Q \times \varepsilon^{P,c} \times \varepsilon^{Q,a/A} \times (\varepsilon^{T',v} + 1), && \text{(B.87)}\\
u_{(c/C)(c/C)} &= (c/C)^{-2} \times T' \times P \times Q \times \varepsilon^{Q,c/C} \times \\
&\qquad \times (\varepsilon^{T',v} \times \varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C}), && \text{(B.88)}\\
u_{(c/C)(a/A)} &= (c/C)^{-1} \times (a/A)^{-1} \times T' \times P \times Q \times \varepsilon^{Q,c/C} \times \\
&\qquad \times (\varepsilon^{T',v} \times \varepsilon^{Q,a/A} + \varepsilon^{Q_{c/C},a/A}), && \text{(B.89)}\\
u_{(a/A)(a/A)} &= (a/A)^{-2} \times T' \times P \times Q \times \varepsilon^{Q,a/A} \times \\
&\qquad \times (\varepsilon^{T',v} \times \varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}). && \text{(B.90)}
\end{aligned}$$

Moreover, using (A.1), (A.2), and (B.82)–(B.90) it can be shown by tedious calculations that

$$\begin{aligned}
V_{cc} &= c^{-2} \times T' \times P \times Q \times \Psi, && \text{(B.91)}\\
V_{cc}V_{aa} - (V_{ca})^2 &= (c^{-1}a^{-1} \times T' \times P \times Q)^2 \times \Phi, && \text{(B.92)}
\end{aligned}$$

where

$$\begin{aligned}
\Psi &\equiv \varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2\varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\
&+\varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C})\}, \qquad \text{(B.93)}\\
\Phi &\equiv \varepsilon^{Q,a/A} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}) \times \\
&\quad \times\{\varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2 \times \varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\
&\quad\quad +\varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C})\} - \\
&-[\varepsilon^{P,c}\varepsilon^{Q,a/A} \times (\varepsilon^{T',v} + 1) + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{c/C},a/A})]^2. \qquad \text{(B.94)}
\end{aligned}$$

Proof of A)

Using (B.67)–(B.69), (B.70), (B.73), (B.75), and (B.76)–(B.81) it can be shown that

$$\begin{aligned}
u_c &> 0 \Leftrightarrow T' \times P' \times Q > 0,\\
u_{c/C} &\geq 0 \Leftrightarrow T' \times P \times Q_{c/C} \geq 0,\\
u_{a/A} &\geq 0 \Leftrightarrow T' \times P \times Q_{a/A} \geq 0,\\
u_{cc} &< 0 \Leftrightarrow P' \times Q \times T' \times (\varepsilon^{T',v} \times \varepsilon^{P,c} + \varepsilon^{P',c}) < 0,\\
u_{(c/C)(c/C)} &< 0 \Leftrightarrow P \times Q_{c/C} \times T' \times (\varepsilon^{T',v} \times \varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C}) < 0,\\
u_{(a/A)(a/A)} &< 0 \Leftrightarrow P \times Q_{a/A} \times T' \times (\varepsilon^{T',v} \times \varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}) < 0.
\end{aligned}$$

Using these results and the assumptions made in (B.46) and (B.47),

$$T' > 0, \quad T'' < 0; \quad P > 0, \quad P' > 0, \quad \varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c} < 0,$$

$$\begin{aligned}
&Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0,\\
&\text{if } Q_{c/C} > 0, \text{ then } \varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C} < 0,\\
&\text{if } Q_{a/A} > 0, \text{ then } \varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A} < 0,
\end{aligned}$$

we obtain $u_c > 0$, $u_{cc} < 0$, $u_{c/C} \geq 0$, $u_{a/A} \geq 0$, $u_{c/C} > 0 \vee u_{a/A} > 0$, and

$$u_{c/C} > 0 \Rightarrow Q_{c/C} > 0 \wedge \varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C} < 0 \Rightarrow u_{(c/C)(c/C)} < 0,$$

$$u_{a/A} > 0 \Rightarrow Q_{a/A} > 0 \wedge \varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A} < 0 \Rightarrow u_{(a/A)(a/A)} < 0.$$

Consequently, the instantaneous utility function (B.45) is well-behaved in the sense that it satisfies all assumptions made in (2),

$$\begin{aligned}
&u_c > 0, \quad u_{cc} < 0; \qquad u_{c/C} \geq 0, \quad u_{a/A} \geq 0, \quad u_{c/C} > 0 \vee u_{a/A} > 0;\\
&\text{if } u_{c/C} > 0, \text{ then } u_{(c/C)(c/C)} < 0; \qquad \text{if } u_{a/A} > 0, \text{ then } u_{(a/A)(a/A)} < 0. \qquad \blacksquare
\end{aligned}$$

Proof of B)

Since $T' > 0$, $P > 0$, and $Q > 0$ holds due to the assumptions made in (B.46) and (B.47), it follows from (B.91) and (B.93) that $V_{cc} < 0$ holds if and only if $\Psi < 0$, i.e.,

$$\begin{aligned} 0 &> \varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2\varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\ &\quad + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C}). \end{aligned}$$

This condition is identical to the condition given by (B.48).

Relative wealth matters for utility if and only if $Q_{a/A} > 0$. Since $T' > 0$, $P > 0$, and $Q > 0$ holds due to the assumptions made in (B.46) and (B.47), it follows from (B.92) and (B.94) that if $Q_{a/A} > 0$, then $V_{cc}V_{aa} - (V_{ca})^2 > 0$ holds if and only if $\Phi > 0$, i.e.,

$$\begin{aligned} 0 &< \varepsilon^{Q,a/A} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}) \times \\ &\quad \times \{\varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2 \times \varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\ &\quad + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C})\} \\ &\quad - [\varepsilon^{P,c}\varepsilon^{Q,a/A} \times (\varepsilon^{T',v} + 1) + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{c/C},a/A})]^2. \end{aligned}$$

This condition is identical to the condition given by (B.49). ■

Proof of C)

Substituting (B.82)–(B.84) into the definitions of $m^{c/C}$ and $m^{a/A}$ given in (11), we obtain

$$m^{c/C}(c, c/C, a/A) \equiv \frac{c/C}{c} \times \frac{u_{c/C}(c, c/C, a/A)}{u_c(c, c/C, a/A)} = \frac{\varepsilon^{Q,c/C}(c/C, a/A)}{\varepsilon^{P,c}(c)},$$

$$m^{a/A}(c, c/C, a/A) \equiv \frac{a/A}{c} \times \frac{u_{a/A}(c, c/C, a/A)}{u_c(c, c/C, a/A)} = \frac{\varepsilon^{Q,a/A}(c/C, a/A)}{\varepsilon^{P,c}(c)}.$$

Consequently, in symmetric situations, in which $c = C$ and $a = A$ hold, we have

$$m^{c/C}(C, 1, 1) = \hat{\varepsilon}^{Q,c/C}/\varepsilon^{P,c}(C), \tag{B.95}$$

$$m^{a/A}(C, 1, 1) = \hat{\varepsilon}^{Q,a/A}/\varepsilon^{P,c}(C), \tag{B.96}$$

where the constants $\hat{\varepsilon}^{Q,c/C}$ and $\hat{\varepsilon}^{Q,a/A}$ give the values that the elasticities $\varepsilon^{Q,c/C}$ and $\varepsilon^{Q,a/A}$ take at $(c/C, a/A) = (1, 1)$:

$$\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1, 1), \qquad \hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1, 1). \tag{B.97}$$

The assumptions $Q > 0$, $Q_{c/C} \geq 0$, $Q_{a/A} \geq 0$, and $Q_{c/C} > 0 \vee Q_{a/A} > 0$, made in (B.47), imply that

$$\hat{\varepsilon}^{Q,c/C} \geq 0, \qquad \hat{\varepsilon}^{Q,a/A} \geq 0, \qquad \hat{\varepsilon}^{Q,c/C} > 0 \vee \hat{\varepsilon}^{Q,a/A} > 0. \tag{B.98}$$

Substituting (B.95) and (B.96) into the definition of the CIER factor given by (19), we obtain

$$\eta^D(C) \equiv \frac{m^{a/A}(C, 1, 1)}{1 + m^{c/C}(C, 1, 1)} = \frac{\hat{\varepsilon}^{Q,a/A}/\varepsilon^{P,c}(C)}{1 + [\hat{\varepsilon}^{Q,c/C}/\varepsilon^{P,c}(C)]} = \frac{\hat{\varepsilon}^{Q,a/A}}{\varepsilon^{P,c}(C) + \hat{\varepsilon}^{Q,c/C}}. \tag{B.99}$$

From (B.95), (B.96), and (B.99) it is obvious that $m^{c/C}(C, 1, 1)$, $m^{a/A}(C, 1, 1)$, and $\eta^D(C)$ are constant functions of C if and only if the elasticity of the function $P(c)$ with respect to c, $\varepsilon^{P,c}(c)$, is a constant function of c. It is easily verified that $\varepsilon^{P,c}(c)$ is a constant function of c if and only if the function $P(c)$ has the form given by (B.51),

$$P(c) = \xi_0 c^{\xi_1}, \qquad c > 0, \tag{B.100}$$

where ξ_0 and ξ_1 are constants. These considerations prove the validity of the first assertion made in item B): The functions $m^{c/C}$, $m^{a/A}$, and η^D that result from the specification of the instantaneous utility function $u = u(c, c/C, a/A)$ given by (B.45) have the properties described in (B.50),

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \eta^D(C) = \hat{\eta}, \quad \forall C > 0,$$

if and only if the function $P(c)$ has the form given by (B.100) [= (B.51)].

Next, we derive the parameter restrictions given in (B.52). In (B.46) it is assumed that both $P(c) > 0$ and $P'(c) > 0$ hold for $c > 0$. From $P(c) = \xi_0 c^{\xi_1}$ and $P'(c) = \xi_0 \xi_1 c^{\xi_1 - 1}$ it is obvious that we have to introduce the following two assumptions with respect to its parameters:

$$\xi_0 > 0, \qquad \xi_1 > 0. \tag{B.101}$$

Obviously, the assumptions made in (B.101) coincide with those made in (B.52).

From (B.100) and (B.101) it follows that

$$\varepsilon^{P,c}(c) = \xi_1 > 0, \qquad \forall c > 0. \tag{B.102}$$

Using (B.102), (B.95), (B.96), (B.98), and (B.99), it is easily verified that

$$m^{c/C}(C,1,1) = \hat{\varepsilon}^{Q,c/C}/\xi_1 \equiv \hat{m}^{c/C} \geq 0, \qquad \forall C > 0, \tag{B.103}$$
$$m^{a/A}(C,1,1) = \hat{\varepsilon}^{Q,a/A}/\xi_1 \equiv \hat{m}^{a/A} \geq 0, \qquad \forall C > 0, \tag{B.104}$$
$$\eta^D(C) = (\hat{\varepsilon}^{Q,a/A}/\xi_1)/[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)] \equiv \hat{\eta} \geq 0, \qquad \forall C > 0, \tag{B.105}$$

and $\hat{m}^{c/C} > 0 \vee \hat{m}^{a/A} > 0$, where the definitions of $\hat{\varepsilon}^{Q,c/C}$ and $\hat{\varepsilon}^{Q,a/A}$ are given by (B.97). The last three results play an essential role in the proofs of D) and E). ■

Proof of D)

Substituting (B.100) [= (B.51)] into (B.45) and taking into account (B.101) [= (B.52)], we obtain the instantaneous utility function (B.53)

$$u(c, c/C, a/A) = T(\xi_0 c^{\xi_1} Q(c/C, a/A)), \qquad \xi_0 > 0, \quad \xi_1 > 0. \tag{B.106}$$

Since, according to (B.103), $m^{c/C}(C,1,1)$ is a constant function of C, we have $m_c^{c/C}(C,1,1) = 0$, $\forall C > 0$. Hence, we also have

$$\varepsilon^{m^{c/C},c}(C,1,1) = m_c^{c/C}(C,1,1) \times [C/m^{c/C}(C,1,1)] = 0, \qquad \forall C > 0.$$

Substituting the last result into the definition of the effective elasticity of intertemporal substitution given by (18),

$$\sigma^D(C) \equiv -\left[\varepsilon^{u_c,c}(C,1,1) + \frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)} \times \varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1},$$

we obtain

$$\sigma^D(C) = -1/\varepsilon^{u_c,c}(C,1,1). \tag{B.107}$$

From (B.107) it follows that $\sigma^D(C)$ is a constant function of C if and only if $\varepsilon^{u_c,c}(C,1,1)$ is a constant function of C. From (B.82) and (B.85) it follows that the elasticity of the marginal utility of absolute consumption u_c with respect to c can be expressed in the following form:

$$\begin{aligned}\varepsilon^{u_c,c}(c,c/C,a/A) &\equiv u_{cc}(c,c/C,a/A) \times [c/u_c(c,c/C,a/A)] \\ &= \varepsilon^{P,c}(c)\varepsilon^{T',v}(P(c)Q(c/C,a/A)) + \varepsilon^{P',c}(c).\end{aligned}$$

The elasticities $\varepsilon^{P,c}(c)$, $\varepsilon^{P',c}(c)$, and $\varepsilon^{T',v}(v)$ are defined in (B.76) and (B.77). Obviously, the elasticity $\varepsilon^{T',v}(v)$ is evaluated at $v = P(c)Q(c/C,a/A)$. The specification of $P(c)$ given by (B.100) [= (B.51)], $P(c) = \xi_0 c^{\xi_1}$, implies that $\varepsilon^{P,c}(c) = \xi_1$ and $\varepsilon^{P',c}(c) = \xi_1 - 1$ hold for $c > 0$. Using these results, we obtain

$$\varepsilon^{u_c,c}(c,c/C,a/A) = \xi_1\varepsilon^{T',v}(\xi_0 c^{\xi_1}Q(c/C,a/A)) + \xi_1 - 1.$$

In symmetric situations, in which $(c,c/C,a/A) = (C,1,1)$ holds, we thus have

$$\varepsilon^{u_c,c}(C,1,1) = \xi_1\varepsilon^{T',v}(\xi_0 C^{\xi_1}Q(1,1)) + \xi_1 - 1. \tag{B.108}$$

Obviously, $\varepsilon^{u_c,c}(C,1,1)$ and $\sigma^D(C) = -[\varepsilon^{u_c,c}(C,1,1)]^{-1}$ [see (B.107)] are constant functions of C if and only if $\varepsilon^{T',v}(\xi_0 C^{\xi_1}Q(1,1))$ is a constant function of C. Since $\xi_0 > 0$ and $\xi_1 > 0$ [see (B.101)] and $Q(c/C,a/A) > 0$ holds over the domain of Q [see (B.47)], $\varepsilon^{T',v}(\xi_0 C^{\xi_1}Q(1,1))$ is a constant function of C for $C > 0$ if and only if $\varepsilon^{T',v}(v)$ is a constant function of v for $v > 0$. The assumptions made in (B.46) require that $T' > 0$ and $T'' < 0$. Hence, admissible transformations T have the property that $\varepsilon^{T',v}(v) \equiv T''(v) \times [v/T'(v)] < 0$ holds for $v > 0$.

We can summarize these considerations as follows: If the transformation $T(v)$ is admissible in the sense that $T'(v) > 0$ and $T''(v) < 0$ hold for $v > 0$, then the instantaneous utility function (B.106), $u(c,c/C,a/A) = T(\xi_0 c^{\xi_1}Q(c/C,a/A))$, has the property that $\varepsilon^{u_c,c}(C,1,1)$ and $\sigma^D(C) = -[\varepsilon^{u_c,c}(C,1,1)]^{-1}$ are constant functions of C for $C > 0$ if and only if the function T satisfies the condition

$$\varepsilon^{T',v}(v) = -\theta, \qquad v > 0, \tag{B.109}$$

where θ is an arbitrary strictly positive constant, $\theta > 0$. It is well-known that $\varepsilon^{T',v}(v) = -\theta < 0$ holds for $v > 0$ if and only if the function $T(v)$ is of the CRRA type, i.e.,

$$T(v) = \kappa_0 + \kappa_1(1-\theta)^{-1}(v^{1-\theta} - 1), \qquad v > 0, \tag{B.110}$$

where κ_0 and κ_1 are constants. These considerations prove the validity of the first assertion made in item D): The functions $\varepsilon^{u_c,c}(C,1,1)$ and $\sigma^D(C)$ that result from the specification of the instantaneous utility function $u(c,c/C,a/A) = T(\xi_0 c^{\xi_1} Q(c/C,a/A))$ given by (B.53) [= (B.106)] have the properties described in (B.54),

$$\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \sigma^D(C) = \hat{\sigma}, \qquad \forall C > 0,$$

if and only if the function $T(v)$ has the form given by (B.110) [= (B.55)].

Next, we derive the parameter restrictions given in (B.56). The assumptions made in (B.52), $\xi_0 > 0$ and $\xi_1 > 0$, imply that $P > 0$ and $P' > 0$. To ensure that the remaining three assumptions given in (B.46), $T' > 0$, $T'' < 0$, and $\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c} < 0$, where

$$T'(v) = \kappa_1 v^{-\theta}, \quad T''(v) = -\kappa_1 \theta v^{-1-\theta}, \quad \varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c} = -\theta\xi_1 + \xi_1 - 1,$$

are also satisfied, we assume that

$$\kappa_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\xi_1 > 0 \tag{B.111}$$

holds in addition to (B.52). Obviously, the assumptions made in (B.111) are identical to those given in (B.56).

Using (B.108), (B.109), (B.111), and (B.107), we obtain

$$\varepsilon^{u_c,c}(C,1,1) = -[1 + (\theta - 1)\xi_1] \equiv \hat{\varepsilon}^{u_c,c} < 0, \quad \forall C > 0, \tag{B.112}$$

$$\sigma^D(C) = -1/\varepsilon^{u_c,c}(C,1,1) = 1/[1 + (\theta - 1)\xi_1] \equiv \hat{\sigma} > 0, \quad \forall C > 0. \tag{B.113}$$

The last two results play an important role in the following proof of E). ■

Proof of E-i)

As shown above, the assumptions made in (B.58),

$$\kappa_1 > 0, \quad \xi_0 > 0, \quad \xi_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\xi_1 > 0,$$

ensure that all assumptions made in (B.46) are satisfied. Moreover, since $\varepsilon^{T',v} = -\theta$, the assumptions made in (B.59),

$$\begin{aligned}
&Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0, \\
&\text{if } Q_{c/C} > 0, \text{ then } \varepsilon^{Q_{c/C},c/C} - \theta\varepsilon^{Q,c/C} < 0, \\
&\text{if } Q_{a/A} > 0, \text{ then } \varepsilon^{Q_{a/A},a/A} - \theta\varepsilon^{Q,a/A} < 0,
\end{aligned}$$

imply that all conditions given in (B.47) are satisfied. Hence, it follows directly from item A) of the proposition that the instantaneous utility function (B.57),

$$u(c,c/C,a/A) = \kappa_0 + \kappa_1(1-\theta)^{-1}\{[\xi_0 c^{\xi_1} Q(c/C,a/A)]^{1-\theta} - 1\},$$

is well-behaved in the sense that all assumptions made in (2) are satisfied.

Proof of E-ii)

Substitution of

$$\varepsilon^{P,c} = \xi_1, \qquad \varepsilon^{P',c} = \xi_1 - 1, \qquad \varepsilon^{T',v} = -\theta$$

into (B.48) and (B.49),

$$\begin{aligned} 0 \;>\; & \varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2\varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) \\ & + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C}), \end{aligned}$$

$$\begin{aligned} 0 \;<\; & \varepsilon^{Q,a/A} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}) \times \\ & \times\{\varepsilon^{P,c} \times (\varepsilon^{T',v}\varepsilon^{P,c} + \varepsilon^{P',c}) + 2 \times \varepsilon^{P,c}\varepsilon^{Q,c/C} \times (\varepsilon^{T',v} + 1) + \\ & + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C})\} \\ & - [\varepsilon^{P,c}\varepsilon^{Q,a/A} \times (\varepsilon^{T',v} + 1) + \varepsilon^{Q,c/C} \times (\varepsilon^{T',v}\varepsilon^{Q,a/A} + \varepsilon^{Q_{c/C},a/A})]^2, \end{aligned}$$

yields (B.60) and (B.61):

$$0 < [1 + \xi_1(\theta - 1)]\xi_1 + \varepsilon^{Q,c/C}[2(\theta - 1)\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}],$$

$$\begin{aligned} 0 \;<\; & \{[1 + \xi_1(\theta - 1)]\xi_1 + \varepsilon^{Q,c/C}[2(\theta - 1)\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}]\} \times \\ & \times \varepsilon^{Q,a/A}(\theta\varepsilon^{Q,a/A} - \varepsilon^{Q_{a/A},a/A}) \\ & - \{[(1 - \theta)\xi_1 - \theta\varepsilon^{Q,c/C}]\varepsilon^{Q,a/A} + \varepsilon^{Q,c/C}\varepsilon^{Q_{c/C},a/A}\}^2, \end{aligned}$$

where the second condition is only relevant in case that relative wealth matters for utility so that $Q_{a/A} > 0$. Hence, if the last two conditions hold in addition to (B.58) and (B.59), then $V(c, C, a, A) \equiv u(c, c/C, a/A)$ is well-behaved since all assumptions made in (3) are satisfied.

Proof of E-iii)

From (B.103), (B.104), (B.97), and (B.112) it follows that the conditions given by (31) in Proposition 3 are satisfied, since $m^{c/C}(C, 1, 1) = \hat{m}^{c/C}$, $m^{a/A}(C, 1, 1) = \hat{m}^{a/A}$, and $\varepsilon^{u_c,c}(C, 1, 1) = \hat{\varepsilon}^{u_c,c}$ hold for $C > 0$, where

$$\hat{m}^{c/C} \equiv \hat{\varepsilon}^{Q,c/C}/\xi_1 \geq 0, \quad \hat{m}^{a/A} \equiv \hat{\varepsilon}^{Q,a/A}/\xi_1 \geq 0, \quad \hat{\varepsilon}^{u_c,c} \equiv -[1 + (\theta - 1)\xi_1] < 0. \tag{B.114}$$

The constants $\hat{\varepsilon}^{Q,c/C}$ and $\hat{\varepsilon}^{Q,a/A}$ denote the values that the elasticities of the function $Q(c/C, a/A)$ with respect to c/C and a/A, $\varepsilon^{Q,c/C}(c/C, a/A)$ and $\varepsilon^{Q,a/A}(c/C, a/A)$, take in symmetric situations, i.e., at $(c/C, a/A) = (1, 1)$:

$$\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1, 1), \quad \hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1, 1). \tag{B.115}$$

These results given in (B.114) and (B.115) prove the validity of (B.62) and (B.63).

Proof of E-iv)

From (B.105), (B.97), and (B.113) it follows that the conditions given in (23) in Proposition

1 are satisfied, because $\sigma^D(C) = \hat{\sigma}$ and $\eta^D(C) = \hat{\eta}$ hold for $C > 0$, where

$$\hat{\sigma} = 1/[1 + (\theta - 1)\xi_1] > 0, \quad \hat{\eta} = (\hat{\varepsilon}^{Q,a/A}/\xi_1)/[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)] \geq 0. \tag{B.116}$$

These results given in (B.116) prove the validity of (B.64).

We know from Proposition 1 that if these constants $\hat{\sigma}$ and $\hat{\eta}$ satisfy the condition (24),

$$[1 - (1/\hat{\sigma})](1 + \hat{\eta})^{-1}\rho^g < \rho < \rho^g, \qquad \rho^g \equiv f_k(1, L) + \hat{\eta} f(1, L),$$

where $\rho > 0$ holds by assumption, then an economically meaningful decentralized BGP exists. Substituting the results for $\hat{\sigma}$ and $\hat{\eta}$ given by (B.116) into equation (25) [see Proposition 1],

$$g^D = [(1/\hat{\sigma}) + \hat{\eta}]^{-1}[f_k(1, L) - \rho + \hat{\eta} f(1, L)],$$

we obtain Equation (B.65):

$$g^D = \frac{f_k(1, L) - \rho + (\hat{\varepsilon}^{Q,a/A}/\xi_1)[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)]^{-1} \times f(1, L)}{1 + (\theta - 1)\xi_1 + (\hat{\varepsilon}^{Q,a/A}/\xi_1)[1 + (\hat{\varepsilon}^{Q,c/C}/\xi_1)]^{-1}} > 0. \tag{B.117}$$

Proof of E-v)

The results for $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, $\hat{\varepsilon}^{u_c,c}$, $\hat{\sigma}$, $\hat{\eta}$, and g^D given in (B.62), (B.64), and (B.65) are independent of the parameters κ_0, κ_1, and ξ_0. The well-behavedness of the instantaneous utility function $u = u(c, c/C, a/A)$ given by (B.57) and its alternative representation $V(c, C, a, A) \equiv u(c, c/C, a/A)$ depends on the signs of ξ_0 and κ_1 [i.e., $\xi_0 > 0$ and $\kappa_1 > 0$ has to hold according to (B.52) and (B.56)], but not on the magnitudes of these two parameters. Hence, we can set, without loss of generality, $\kappa_0 = 0$, $\kappa_1 = 1$, and $\xi_0 = 1$, and employ the simplified representation of the utility function given by (B.66):

$$u(c, c/C, a/A) = (1 - \theta)^{-1}\{[c^{\xi_1} Q(c/C, a/A)]^{1-\theta} - 1\}.$$

B.10 Proof of item A) of Corollary 1

Let the instantaneous utility function take the form given by (43),

$$u(c, c/C, a/A) = (1 - \theta)^{-1}\{[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta} - 1\},$$

where the assumptions with respect to the parameters are given by (44):

$$\theta > 0, \quad \xi_1 > 0, \xi_2 \geq 0, \xi_3 \geq 0, \xi_2 > 0 \vee \xi_3 > 0, \quad (1 - \theta)(\xi_1 + \xi_2 + \xi_3) < 1.$$

Obviously, the alternative representation of the utility function given by $V(c, C, a, A) \equiv u(c, c/C, a/A)$ takes the following form:

$$V(c, C, a, A) = (1 - \theta)^{-1}[(c^{\xi_1+\xi_2} C^{-\xi_2} a^{\xi_3} A^{-\xi_3})^{1-\theta} - 1].$$

The following properties of u and V are easily verified:

$$\begin{aligned}
u_c &= \xi_1 c^{-1}[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta},\\
u_{cc} &= -\xi_1[1+\xi_1(\theta-1)]c^{-2}[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta},\\
u_{c/C} &= \xi_2(c/C)^{-1}[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta},\\
u_{(c/C)(c/C)} &= -\xi_2(c/C)^{-2}[1+\xi_2(\theta-1)][c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta},\\
u_{a/A} &= \xi_3(a/A)^{-1}[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta},\\
u_{(a/A)(a/A)} &= -\xi_3(a/A)^{-2}[1+\xi_3(\theta-1)][c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}]^{1-\theta},\\
V_c &= (\xi_1+\xi_2)c^{-1}(c^{\xi_1+\xi_2}C^{-\xi_2}a^{\xi_3}A^{-\xi_3})^{1-\theta},\\
V_{cc} &= -(\xi_1+\xi_2)[1+(\xi_1+\xi_2)(\theta-1)]c^{-2}(c^{\xi_1+\xi_2}C^{-\xi_2}a^{\xi_3}A^{-\xi_3})^{1-\theta},\\
V_a &= \xi_3 a^{-1}(c^{\xi_1+\xi_2}C^{-\xi_2}a^{\xi_3}A^{-\xi_3})^{1-\theta},\\
V_{aa} &= -\xi_3[1+\xi_3(\theta-1)]a^{-2}(c^{\xi_1+\xi_2}C^{-\xi_2}a^{\xi_3}A^{-\xi_3})^{1-\theta},\\
V_{ca} &= (\xi_1+\xi_2)\xi_3(1-\theta)(ca)^{-1}(c^{\xi_1+\xi_2}C^{-\xi_2}a^{\xi_3}A^{-\xi_3})^{1-\theta},\\
V_{cc}V_{aa}-(V_{ca})^2 &= (\xi_1+\xi_2)\xi_3[1+(\theta-1)(\xi_1+\xi_2+\xi_3)]\times\\
&\quad \times(ca)^{-2}(c^{\xi_1+\xi_2}a^{\xi_3}C^{-\xi_2}A^{-\xi_3})^{2(1-\theta)}.
\end{aligned}$$

First, we prove that all assumptions made in (2) are satisfied. From the results given above it is obvious that

$$\begin{aligned}
&u_c>0\Leftrightarrow\xi_1>0, && u_{cc}<0\Leftrightarrow\xi_1[1+\xi_1(\theta-1)]>0,\\
&u_{c/C}\geq 0\Leftrightarrow\xi_2\geq 0, && u_{(c/C)(c/C)}<0\Leftrightarrow\xi_2[1+\xi_2(\theta-1)]>0,\\
&u_{a/A}\geq 0\Leftrightarrow\xi_3\geq 0, && u_{(a/A)(a/A)}<0\Leftrightarrow\xi_3[1+\xi_3(\theta-1)]>0.
\end{aligned}\tag{B.118}$$

Since by assumption $\xi_1>0$, $\xi_2\geq 0$, $\xi_3\geq 0$, and $\xi_2>0\vee\xi_3>0$ hold, we obtain $u_c>0$, $u_{c/C}\geq 0$, $u_{a/A}\geq 0$, and $u_{c/C}>0\vee u_{a/A}>0$.

Next, we show that

$$1+(\theta-1)\xi_i>0,\qquad i=1,2,3\tag{B.119}$$

holds for $\theta>0$. The proof is simple: Obviously, $\theta\geq 1$ is sufficient (but not necessary) for the validity of (B.119). In the opposite case in which $0<\theta<1$ holds, we make use of the fact that according to (44) both $\xi_2>0\vee\xi_3>0$ and $(1-\theta)(\xi_1+\xi_2+\xi_3)<1$ hold by assumption and thus obtain

$$(1-\theta)\xi_i<(1-\theta)(\xi_1+\xi_2+\xi_3)<1\Rightarrow 1+(\theta-1)\xi_i>0,\quad i=1,2,3.$$

Employing (B.118) and (B.119) we obtain the following: i) From $\xi_1>0$ and $1+\xi_1(\theta-1)>0$ it follows that $u_{cc}<0$. ii) If $u_{c/C}>0$, then $\xi_2>0$ and $\xi_2[1+\xi_2(\theta-1)]>0$, which, in turn, implies that $u_{(c/C)(c/C)}<0$. iii) If $u_{a/A}>0$, then $\xi_3>0$ and $\xi_3[1+\xi_3(\theta-1)]>0$, so that $u_{(a/A)(a/A)}<0$.

The following summary of these results shows that all assumptions made in (2) are satisfied:

$$u_c > 0, \quad u_{cc} < 0; \qquad u_{c/C} \geq 0, \quad u_{a/A} \geq 0, \quad u_{c/C} > 0 \vee u_{a/A} > 0;$$
$$\text{if } u_{c/C} > 0, \text{ then } u_{(c/C)(c/C)} < 0; \qquad \text{if } u_{a/A} > 0, \text{ then } u_{(a/A)(a/A)} < 0.$$

Second, we prove that all assumptions made in (3) are satisfied. Since $\xi_1 > 0$ and $\xi_2 \geq 0$ hold by assumption, it follows from the expression for V_{cc} given above that

$$V_{cc} < 0 \Leftrightarrow 1 + (\theta - 1)(\xi_1 + \xi_2) > 0.$$

It is easily verified that

$$1 + (\theta - 1)(\xi_1 + \xi_2) > 0 \tag{B.120}$$

holds for $\theta > 0$, so that $V_{cc} < 0$ holds for $\theta > 0$. The proof is simple: Obviously, $\theta \geq 1$ is sufficient (but not necessary) for the validity of (B.120). In the opposite case in which $0 < \theta < 1$ holds, we make use of the fact that according to (44) $\xi_3 \geq 0$ and $(1-\theta)(\xi_1 + \xi_2 + \xi_3) < 1$ hold by assumption and thus obtain

$$(1-\theta)(\xi_1 + \xi_2) \leq (1-\theta)(\xi_1 + \xi_2 + \xi_3) < 1 \Rightarrow 1 + (\theta - 1)(\xi_1 + \xi_2) > 0.$$

Finally, we have to show that if $u_{a/A} > 0$ holds, then $V_{cc}V_{aa} - (V_{ca})^2 > 0$. Recall that $u_{a/A} > 0 \Leftrightarrow \xi_3 > 0$. It is obvious from the expression for $V_{cc}V_{aa} - (V_{ca})^2$ given above that if $\xi_3 > 0$ holds in addition to $\xi_1 > 0$ and $\xi_2 \geq 0$, then

$$V_{cc}V_{aa} - (V_{ca})^2 > 0 \Leftrightarrow (1-\theta)(\xi_1 + \xi_2 + \xi_3) < 1.$$

Since the condition given on the right-hand side is one of the assumptions listed in (44), we obtain that if $u_{a/A} > 0$ holds, then $V_{cc}V_{aa} - (V_{ca})^2 > 0$. ■

C The decentralized solution – Part II (Section 4)

C.1 The well-behavedness of specification #4

In specification #4, the function $V = V(c, C)$ takes the form

$$V(c, C) = \frac{1}{1-\theta}\left\{\left[\left(\frac{c^{\varphi} - \kappa C^{\varphi}}{1-\kappa}\right)^{1/\varphi}\right]^{1-\theta} - 1\right\}, \qquad 0 < \kappa < 1, \quad 0 < 1 - \varphi < \theta,$$

where the domain of V is given by $\Theta_V \equiv \{(c, C) | c > 0, C > 0, c^{\varphi} - \kappa C^{\varphi} > 0\}$. From

$$\begin{aligned}
V_c &= \left(\frac{c^{\varphi} - \kappa C^{\varphi}}{1-\kappa}\right)^{(1-\theta-\varphi)/\varphi} \frac{c^{\varphi-1}}{1-\kappa}, \\
V_{cc} &= -\left[\theta - (1-\varphi)\right]\left(\frac{c^{\varphi} - \kappa C^{\varphi}}{1-\kappa}\right)^{(1-\theta-2\varphi)/\varphi}\left(\frac{c^{\varphi-1}}{1-\kappa}\right)^2 \\
&\quad - (1-\varphi)\left(\frac{c^{\varphi} - \kappa C^{\varphi}}{1-\kappa}\right)^{(1-\theta-\varphi)/\varphi} \frac{c^{\varphi-2}}{1-\kappa},
\end{aligned}$$

it follows that the assumptions $0 < \kappa < 1$ and $0 < 1 - \varphi < \theta$ are sufficient for $V_c > 0$ and $V_{cc} < 0$ so that all assumptions made in (3) are satisfied.

The corresponding representation of the function $u = u(c, c/C)$ is given by (58),

$$u(c, c/C) = \frac{1}{1-\theta} \left\{ \left[c \times \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa} \right)^{1/\varphi} \right]^{1-\theta} - 1 \right\}.$$

From

$$\begin{aligned}
u_c &= c^{-\theta} \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa} \right)^{(1-\theta)/\varphi}, \\
u_{cc} &= -\theta c^{-\theta-1} \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa} \right)^{(1-\theta)/\varphi}, \\
u_{c/C} &= \frac{\kappa}{1-\kappa} c^{1-\theta} \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa} \right)^{(1-\theta-\varphi)/\varphi} (c/C)^{-\varphi-1}, \\
u_{(c/C)(c/C)} &= -\frac{\kappa}{1-\kappa} c^{1-\theta} (c/C)^{-(\varphi+2)} \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa} \right)^{(1-\theta-2\varphi)/\varphi} \times \\
&\quad \times \left[[\theta - (1-\varphi)] \frac{\kappa(c/C)^{-\varphi}}{1-\kappa} + (\varphi+1) \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa} \right) \right]
\end{aligned}$$

it follows that the assumptions $0 < \kappa < 1$ and $0 < 1 - \varphi < \theta$ are also sufficient for $u_c > 0$, $u_{cc} < 0$, $u_{c/C} > 0$, and $u_{(c/C)(c/C)} < 0$, so that all assumptions made in (2) are satisfied.

C.2 Specification #7

Specification #7 has the property that the presence of relative consumption and relative wealth in the instantaneous utility function u results from the explicit consideration of status preferences. More precisely, we assume that u can be written as $u(c, c/C, a/A) \equiv \tilde{u}(c, s(c/C, a/A))$, where s stands for status. To ensure that $u(c, c/C, a/A)$ is of the simple form given by (43) and that $\xi_1 + \xi_2 + \xi_3 = 1$ holds, we employ the following specifications of $\tilde{u}(c, s)$ and $s(c/C, a/A)$:

$$\tilde{u}(c, s) = (1-\theta)^{-1}[(c^{1-\beta}s^{\beta})^{1-\theta} - 1], \qquad s(c/C, a/A) = (c/C)^{\gamma}(a/A)^{1-\gamma}, \tag{C.1}$$

where $\theta > 0$, $0 < \beta < 1$, $1 + (\theta - 1)(1 - \beta) > 0$, and $0 < \gamma < 1$. It is easily verified that $\xi_1 = 1-\beta$, $\xi_2 = \beta\gamma$, and $\xi_3 = (1-\gamma)\beta$. A natural extension of our fundamental factor approach implies that the *percentage-MRS* of status s for absolute consumption c, defined by $m^s(c, s) \equiv (s/c) \times [\tilde{u}_s(c, s)/\tilde{u}_c(c, s)]$, represents the appropriate measure of the intensity of the quest for *overall* status as determined by both relative consumption and relative wealth. The simplicity of the specification (C.1) entails two significant drawbacks with respect to the application of the standard analysis: i) Since changes in the parameter γ affect both $\hat{m}^{c/C} = \gamma\beta/(1-\beta)$ and $\hat{m}^{a/A} = (1-\gamma)\beta/(1-\beta)$, the partial derivative $\partial g^D/\partial\gamma$ is unsuited to analyze the effects of ceteris paribus changes in the intensity of the relative consumption motive or the relative wealth motive. ii) The partial derivative $\partial g^D/\partial\beta$ is inappropriate to analyze the effects of a change in the intensity of the quest for overall status. This is due to the following fact: If $\theta \neq 1$, then a change in β affects not only the willingness to substitute status for absolute consumption as measured by $\hat{m}^s = \beta/(1-\beta)$, but also the willingness to substitute absolute consumption intertemporally as determined by $1/|\hat{\varepsilon}^{\tilde{u}_c,c}| = 1/[1+(\theta-1)(1-\beta)]$.

All these problems can be easily avoided by i) eliminating the dependence between the exponents of absolute consumption and status and ii) employing separate parameters for the exponents of relative consumption and relative wealth so that

$$\tilde{u}(c,s) = (1-\theta)^{-1}[(cs^{\beta})^{1-\theta} - 1], \qquad s(c/C, a/A) = (c/C)^{\gamma_1}(a/A)^{\gamma_2}.$$

D The socially planned solution and the inefficiency of the decentralized solution (Section 5)

D.1 The Euler equation (63) and the transversality condition (64)

First, we derive the Euler equation for consumption. The current-value Hamiltonian of the social planner's optimization problem is given by $H = u(C,1,1) + \mu[f(1,L)K - C]$, where the costate variable μ denotes the shadow price of capital. The necessary optimality conditions for an interior equilibrium, $H_C = 0$ and $\dot{\mu} = \rho\mu - H_K$, can be written as

$$\mu = u_c(C,1,1), \tag{D.1}$$

$$\dot{\mu} = -[f(1,L) - \rho]\mu. \tag{D.2}$$

If, in addition, the transversality condition given by

$$\lim_{t\to\infty} e^{-\rho t}\mu K = 0 \tag{D.3}$$

holds, then the necessary optimality conditions are also sufficient, where this property follows from the fact that $u_{cc} < 0$ holds by assumption. Substituting the FOC (D.1) and its derivative with respect to time t,

$$\dot{\mu} = u_{cc}(C,1,1)\dot{C},$$

into (D.2), we obtain

$$u_{cc}(C,1,1)\dot{C} = -[f(1,L) - \rho]u_c(C,1,1).$$

Simple transformations yield the Euler equation for consumption:

$$\dot{C}/C = -\{u_c(C,1,1)/[Cu_{cc}(C,1,1)]\}[f(1,L) - \rho].$$

Using the definition of the elasticity of the marginal utility of absolute consumption u_c with respect to c,

$$\varepsilon^{u_c,c}(c, c/C, a/A) \equiv u_{cc}(c, c/C, a/A) \times [c/u_c(c, c/C, a/A)],$$

the Euler equation can be written in the form given by (63),

$$\dot{C}/C = \sigma^P(C)[f(1,L) - \rho], \qquad \sigma^P(C) \equiv -1/\varepsilon^{u_c,c}(C,1,1).$$

Second, we derive the transversality condition. Since the exogenously given aggregate labor input L is constant over time, integration of (D.2) yields

$$\mu(t) = \mu(0)e^{\rho t}e^{-f(1,L)t}. \tag{D.4}$$

The assumption that $u_c > 0$ together with the FOC (D.1) implies that $\mu(t) > 0$ for $t \geq 0$. Since $\mu(0) > 0$, it follows from (D.4) that the transversality condition (D.3), $\lim_{t\to\infty} e^{-\rho t}\mu K = 0$, is equivalent to the representation given by (64),

$$\lim_{t\to\infty} e^{-f(1,L)t}K(t) = 0.$$

D.2 Proof of Proposition 6

Proof of A) Let the instantaneous utility function u satisfy the conditions (31) that were introduced in the context of the decentralized economy in Proposition 3, i.e.,

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \forall C > 0, \tag{D.5}$$

where $\hat{m}^{c/C} \geq 0$, $\hat{m}^{a/A} \geq 0$ (with $\hat{m}^{c/C} > 0 \vee \hat{m}^{a/A} > 0$), and $\hat{\varepsilon}^{u_c,c} < 0$.

First, we prove the validity of (65). According to Proposition 3, the conditions given in (31) [= (D.5)] imply that

$$\sigma^D(C) = 1/|\hat{\varepsilon}^{u_c,c}|, \qquad \forall C > 0. \tag{D.6}$$

Using the definition of $\sigma^P(C)$ given in (63),

$$\sigma^P(C) \equiv -1/\varepsilon^{u_c,c}(C,1,1),$$

and the assumption that

$$\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c} < 0, \quad \forall C > 0,$$

made in (31) [= (D.5)], we obtain

$$\sigma^P(C) = 1/|\hat{\varepsilon}^{u_c,c}|, \qquad \forall C > 0. \tag{D.7}$$

Combining (D.6) and (D.7), we obtain (65):

$$\sigma^P(C) = \sigma^D(C) = 1/|\hat{\varepsilon}^{u_c,c}| \equiv \hat{\sigma}, \qquad \forall C > 0.$$

Second, we derive the solutions for g^P, $(C/K)^P$, and $(\dot{K}/Y)^P$. Substitution of $\sigma^P(C) = \hat{\sigma}$, $\forall C > 0$, into the Euler equation of aggregate consumption in the socially planned economy that is given by (63), we obtain

$$\dot{C}/C = \hat{\sigma}[f(1,L) - \rho]. \tag{D.8}$$

From the economy's resource constraint $\dot{K} = f(1,L)K - C$ it follows that

$$\dot{K}/K = f(1,L) - (C/K). \tag{D.9}$$

Taking into account that, by assumption, L is exogenously given and constant over time and that $\hat{\sigma}$ is a constant, it is obvious from the last two differential equations that a BGP exists in the socially planned economy along which C and K grow at the same constant rate so that C/K remains unchanged over time. The steady-state value of the common growth rate of aggregate consumption and aggregate physical capital denoted by $g^P = (\dot{C}/C)^P = (\dot{K}/K)^P$ and the steady-state value of the consumption-capital ratio denoted by $(C/K)^P$ are determined by the following system of equations:

$$g^P = \hat{\sigma}[f(1,L) - \rho], \qquad g^P = f(1,L) - (C/K)^P.$$

Solving this system of two equations for g^P and $(C/K)^P$, we obtain

$$g^P = \hat{\sigma}[f(1,L) - \rho], \tag{D.10}$$

$$(C/K)^P = f(1,L) - g^P \tag{D.11}$$

$$= (1 - \hat{\sigma})f(1,L) + \hat{\sigma}\rho. \tag{D.12}$$

The solutions for g^P and $(C/K)^P$ given by (D.10) and (D.11) are identical to those given in Proposition 6. The validity of $(\dot{K}/Y)^P = g^P/f(1,L)$ is obtained by using the following facts:

$$(\dot{K}/Y) = (\dot{K}/K)/(Y/K), \qquad Y = f(1,L)K, \qquad (\dot{K}/K)^P = g^P.$$

Third, we derive condition (66). Using (D.10), we obtain

$$g^P > 0 \Leftrightarrow \rho < f(1,L). \tag{D.13}$$

From (D.12) it follows that

$$(C/K)^P > 0 \Leftrightarrow \rho > (\hat{\sigma} - 1)(\hat{\sigma})^{-1}f(1,L). \tag{D.14}$$

In case that $\hat{\sigma} < 1$, condition (D.14) is redundant because $\rho > 0$ holds by assumption.

Along the BGP, we have $\dot{K}/K = g^P$ at any point in time. Hence, the transversality condition (64),

$$\lim_{t\to\infty} e^{-f(1,L)t}K(t) = 0,$$

requires that

$$-f(1,L) + g^P = -[(1 - \hat{\sigma})f(1,L) + \hat{\sigma}\rho] = -(C/K)^P < 0. \tag{D.15}$$

Obviously, the condition that $\rho > (\hat{\sigma} - 1)(\hat{\sigma})^{-1}f(1,L)$ given in (D.14) implies not only that $(C/K)^P > 0$, but also ensures that the transversality condition is satisfied.

The results given by (D.13), (D.14), and (D.15) can be summarized as follows: If the condition

$$(\hat{\sigma} - 1)\hat{\sigma}^{-1}f(1,L) < \rho < f(1,L) \tag{D.16}$$

is satisfied (where $\rho > 0$ holds by assumption), then the BGP is economically meaningful in the sense that the growth rate and the consumption-capital ratio are strictly positive, $g^P > 0$, $(C/K)^P > 0$, and, in addition, the transversality condition is fulfilled. Obviously, condition

(D.16) is identical to condition (66). ■

Proof of B)

The validity of all assertions made in B) is verified at first glance. Hence, we skip a detailed proof.

Extra Proof – The socially planned solution has no transitional dynamics

Finally, we show that if the condition (31) [= (D.5)] is satisfied, then the model has no transitional dynamics. Let $Z \equiv C/K$. Since K is a state variable and C is a control variable, $Z = C/K$ is a control-like variable (this notion is used by Barro and Sala-i-Martin (1995) on p. 162). In contrast to K, both C and $Z = C/K$ can jump at any point in time. Using (D.8), (D.9), and $C/K = Z$, we obtain the following differential equation:

$$\begin{aligned} \dot{Z} &= [(\dot{C}/C) - (\dot{K}/K)]Z \\ &= \{\hat{\sigma}[f(1,L) - \rho] - [f(1,L) - Z]\}Z \\ &= \{Z - [(1-\hat{\sigma})f(1,L) + \hat{\sigma}\rho]\}Z \equiv \Phi(Z). \end{aligned}$$

Solving $\dot{Z} = \Phi(Z) = 0$ for Z, we obtain $\{Z = 0\}$ and $\{Z = Z^P\}$, where

$$Z^P = (1-\hat{\sigma})f(1,L) + \hat{\sigma}\rho. \tag{D.17}$$

Obviously, Z^P given by (D.17) is identical to $(C/K)^P$ given by (D.12). If (66) [= (D.16)] holds, then $Z^P = (C/K)^P > 0$, so that Z^P is the economically meaningful steady-state value of the consumption-capital ratio. Rewriting $\Phi(Z)$ as $\Phi(Z) = (Z - Z^P)Z$, it is easily verified that

$$\Phi'(Z) = Z + (Z - Z^P), \qquad \Phi'(Z^P) = Z^P > 0.$$

$\Phi'(Z^P) > 0$ implies that the economically meaningful steady state of the differential equation $\dot{Z} = \Phi(Z)$ is unstable. Hence, the perfect-foresight equilibrium path of Z has no transitional dynamics, i.e., $Z(t) = Z^P$ for $t \geq 0$. The initial value of the jump variable Z has to be chosen in such a way that $Z(0) = Z^P$. From $Z = C/K$ and $Z^P = (C/K)^P$ it then follows that the initial value of the jump variable C has to be chosen according to $C(0) = (C/K)^P \times K_0$, where $(C/K)^P = Z^P$ is given by (D.12) or (D.17) and K_0 is exogenously given.

From $Z(t) = Z^P$ for $t \geq 0$, (D.8), (D.9), $Z = C/K$, (D.10), and (D.17) it then follows that

$$\begin{aligned} \dot{C}/C &= \hat{\sigma}[f(1,L) - \rho] = g^P > 0, \\ \dot{K}/K &= f(1,L) - Z^P = \hat{\sigma}[f(1,L) - \rho] = g^P > 0, \end{aligned}$$

hold for $t \geq 0$. The growth rates of consumption and capital are constant over time, identical, and equal to g^P. Consequently, the growth rates of C and K have no transitional dynamics. ■

D.3 Proof of Proposition 7

Proof of i) and iii)

The validity of i) and iii) is easily verified by 1) taking into account that, according to Proposition 6, g^P is independent of both $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$, and 2) recalling that $\partial g^D/\partial \hat{m}^{a/A} > 0$

and $\mathrm{sgn}(\partial g^D/\partial \hat{m}^{c/C}) = -\mathrm{sgn}(\hat{m}^{a/A})$ hold according to Proposition 4. Consequently, we have $\partial(g^P - g^D)/\partial \hat{m}^{a/A} < 0$ and $\mathrm{sgn}[\partial(g^P - g^D)/\partial \hat{m}^{c/C}] = \mathrm{sgn}(\hat{m}^{a/A})$.

Proof of ii) We assume that the conditions given by (24) and (66),

$$[1-(1/\hat{\sigma})](1+\hat{\eta})^{-1}\rho^g < \rho < \rho^g, \qquad \rho^g \equiv f_k(1,L) + \hat{\eta} f(1,L),$$

$$(\hat{\sigma}-1)\hat{\sigma}^{-1} f(1,L) < \rho < f(1,L), \tag{D.18}$$

are satisfied so that in both the decentralized economy and the socially planned economy an economically meaningful BGP exists. The corresponding solutions for g^P and g^D are given by [see (65) and (67) as well as (25) and (33)]:

$$\begin{aligned} g^P &= \hat{\sigma}[f(1,L)-\rho] \\ &= [f(1,L)-\rho]/|\hat{\varepsilon}^{u_c,c}| > 0, \qquad \text{(D.19)} \\ g^D &= [(1/\hat{\sigma})+\hat{\eta}]^{-1}[f_k(1,L)-\rho+\hat{\eta} f(1,L)] \\ &= \frac{f_k(1,L)-\rho+[\hat{m}^{a/A}/(1+\hat{m}^{c/C})]\times f(1,L)}{|\hat{\varepsilon}^{u_c,c}|+[\hat{m}^{a/A}/(1+\hat{m}^{c/C})]} > 0. \qquad \text{(D.20)} \end{aligned}$$

Using (D.19) and (D.20), the growth rate gap $g^P - g^D$ can be expressed as a function of the fundamental factors:

$$g^P - g^D = \frac{f(1,L)-\rho}{|\hat{\varepsilon}^{u_c,c}|} - \frac{f_k(1,L)-\rho+[\hat{m}^{a/A}/(1+\hat{m}^{c/C})]\times f(1,L)}{|\hat{\varepsilon}^{u_c,c}|+[\hat{m}^{a/A}/(1+\hat{m}^{c/C})]}. \tag{D.21}$$

Using (D.21) it is easily verified that $g^P - g^D = 0$ if and only if

$$\hat{m}^{a/A} = (\hat{m}^{a/A})^{\text{crit}} \equiv \frac{[f(1,L)-f_k(1,L)](1+\hat{m}^{c/C})}{[1-(1/|\hat{\varepsilon}^{u_c,c}|)]f(1,L)+(1/|\hat{\varepsilon}^{u_c,c}|)\rho} > 0.$$

The positive sign of $(\hat{m}^{a/A})^{\text{crit}}$ can be verified as follows: The numerator is strictly positive because $f(1,L) > f_k(1,L)$ [see (22)] and $\hat{m}^{c/C} \geq 0$. The denominator is positive for the following reasons: The conditions for the existence of an economically meaningful BGP in the socially planned economy given in (66) [= (D.18)] require that both $\rho < f(1,L)$ and $(\hat{\sigma}-1)\hat{\sigma}^{-1} f(1,L) < \rho$ hold, where the latter inequality can be rewritten as

$$(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho > 0.$$

Substituting $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$, this condition can be expressed as

$$[1-(1/|\hat{\varepsilon}^{u_c,c}|)]f(1,L)+(1/|\hat{\varepsilon}^{u_c,c}|)\rho > 0,$$

where the left-hand side is identical to the denominator of $(\hat{m}^{a/A})^{\text{crit}}$.

From (D.21) it follows that

$$g^P - g^D\big|_{\hat{m}^{a/A}=0} = [f(1,L)-f_k(1,L)]/|\hat{\varepsilon}^{u_c,c}| > 0.$$

Taking into account that

$$g^P - g^D\big|_{\hat{m}^{a/A}=(\hat{m}^{a/A})^{\text{crit}}} = 0$$

and that, according to item i) of the proposition, $\partial(g^P - g^D)/\partial\hat{m}^{a/A} < 0$ holds for $\hat{m}^{a/A} \geq 0$, we obtain the following properties of $g^P - g^D$: 1) If $0 \leq \hat{m}^{a/A} < (\hat{m}^{a/A})^{\text{crit}}$, then $g^D < g^P$. 2) If $\hat{m}^{a/A} = (\hat{m}^{a/A})^{\text{crit}}$, then $g^D = g^P$. 3) If $\hat{m}^{a/A} > (\hat{m}^{a/A})^{\text{crit}}$, then $g^D > g^P$. These properties can be summarized in the following compact way:

$$\text{sgn}(g^P - g^D) = \text{sgn}[(\hat{m}^{a/A})^{\text{crit}} - \hat{m}^{a/A}]. \qquad \blacksquare$$

D.4 Illustration: Erroneous conclusions of the standard analysis with respect to the growth rate gap

In the absence of the relative wealth motive, we have

$$g^P - g^D = \hat{\sigma}[f(1,L) - f_k(1,L)] > 0, \quad \hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}| = 1/[1 + (\theta - 1)\xi_1], \tag{D.22}$$

irrespective of whether the instantaneous utility function is of the general type (37) or the simple type (43). Consequently, the variation in a parameter p_i leads to a change in the strictly positive growth rate gap $g^P - g^D$ if and only if it affects the willingness to substitute absolute consumption intertemporally as measured by $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$. The standard approach is unaware of (D.22) and might therefore question the assertion made in item iii) of Proposition 7 that – in the absence of relative wealth preferences – the strength of the relative consumption motive does not affect the growth gap $g^P - g^D$. For instance, it might employ the geometric weighted average specification #1 in which $\xi_2 = \beta$, $\xi_1 = 1 - \beta$, and $\xi_3 = 0$ holds and point out that

$$g^P - g^D = \frac{f(1,L) - f_k(1,L)}{1 + (\theta - 1)(1 - \beta)} \Rightarrow \text{sgn}[\partial(g^P - g^D)/\partial\beta] = \text{sgn}(\theta - 1).$$

If $\theta > 1$, then a rise in β causes both g^D and g^P to increase, where the rise in g^P exceeds that of g^D so that $g^P - g^D$ increases. Analogously, if $\theta < 1$, then a rise in β causes both g^D and g^P to decrease, where the fall in g^P exceeds that of g^D so that the gap $g^P - g^D$ decreases but remains strictly positive. Our analysis makes it clear that the (ambiguous) dependence of $g^P - g^D$ on β that exists for $\theta \neq 1$ cannot be used to reject our results. From (55) it is obvious that changes in β affect the two fundamental factors $\hat{m}^{c/C} = \beta/(1-\beta)$ and $|\hat{\varepsilon}^{u_c,c}| = 1 + (\theta - 1)(1 - \beta)$. However, only the change in $|\hat{\varepsilon}^{u_c,c}|$ exerts an effect on $g^P - g^D$, namely via the (ambiguous) reaction of the effective elasticity of intertemporal substitution $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$. By contrast, the change in the strength of the relative consumption motive as measured by $\hat{m}^{c/C}$ affects neither g^P nor g^D, and, hence, is irrelevant for $g^P - g^D$. Consequently, the variation in the willingness to substitute absolute consumption intertemporally that results from the change in the exponent of absolute consumption $\xi_1 = 1 - \beta$ explains 100 percent of the reaction of $g^P - g^D$.